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Topological expansion of the Bethe ansatz, and non-commutative algebraic geometry

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Abstract:
In this article, we define a non-commutative deformation of the "symplectic invariants" (introduced in [13]) of an algebraic hyperelliptical plane curve. The necessary condition for our definition to make sense is a Bethe ansatz. The commutative limit reduces to the symplectic invariants, i.e. algebraic geometry, and thus we define non-commutative deformations of some algebraic geometry quantities. In particular our non-commutative Bergmann kernel satisfies a Rauch variational formula. Those non-commutative invariants are inspired from the large N expansion of formal non-hermitian matrix models. Thus they are expected to be related to the enumeration problem of discrete non-orientable surfaces of arbitrary topologies.

1 Introduction

In [13], the notion of symplectic invariants of a spectral curve was introduced. For any given algebraic plane curve (called spectral curve) of equation:

$$0 = \mathcal{E}(x,y) = \sum_{i,j} \mathcal{E}_{i,j}\ x^i\, y^j \tag{1.1}$$

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an infinite sequence of numbers

$$F^{(g)}(\mathcal{E}) \qquad , \ g = 0, 1, 2, \ldots, \infty \tag{1.2}$$

and an infinite sequence of multilinear meromorphic forms $W_n^{(g)}$ (meromorphic on the algebraic Riemann surface of equation $\mathcal{E}(x, y) = 0$) were defined.

Their definition was inspired from hermitian matrix models, i.e. in the case where $\mathcal{E} = \mathcal{E}_{\text{M.M.}}$ is the spectral curve ($y(x)$ is the equilibrium density of eigenvalues) of a formal hermitian matrix integral $Z_{\text{M.M.}} = \int dM \, \mathrm{e}^{-N \operatorname{Tr} V(M)}$, the $F^{(g)}$ were such that:

$$\ln Z_{\text{M.M.}} = \sum_{g=0}^{\infty} N^{2-2g} F^{(g)}(\mathcal{E}_{\text{M.M.}}) \tag{1.3}$$

The $F^{(g)}$'s have many remarkable properties (see [13]), in particular invariance under symplectic deformations of the spectral curve, homogeneity (of degree $2 - 2g$), holomorphic anomaly equations (modular transformations), stability under singular limits, ... An important property also, is that the following formal series

$$\tau(\mathcal{E}) = \mathrm{e}^{\sum_g N^{2-2g} F^{(g)}(\mathcal{E})} \tag{1.4}$$

is the "formal" τ function of an integrable hierarchy.

Although those notions were first developed for matrix models, they extend beyond matrix models, and they make sense for spectral curves which are not matrix models spectral curves. For instance the (non-algebraic) spectral curve $\mathcal{E}_{\text{WP}}(x, y) = (2\pi y)^2 - (\sin(2\pi\sqrt{x}))^2$ is such that $F^{(g)}(\mathcal{E}_{\text{WP}}) = \text{Vol}(\overline{\mathcal{M}}_g)$ is the Weyl-Petersson volume of moduli space of Riemann surfaces of genus g (see [11, 12]). It is conjectured [3] that the $F^{(g)}$'s are deeply related to Gromov-Witten invariants, Hurwitz numbers [4] and topological strings [3]. In particular they are related to the Kodaira-Spencer field theory [8].

There were many attempts to compute also non-hermitian matrix integrals, and an attempt to extend the method of [13] was first made in [7], and here in this paper we deeply improve the result of [7]. The aim of the construction we present here, is to define $F^{(g)}$'s for a "non-commutative spectral curve", i.e. a non commutative polynomial:

$$\mathcal{E}(x, y) = \sum_{i,j} \mathcal{E}_{i,j} \ x^i \, y^j \qquad , \qquad [y, x] = \hbar \tag{1.5}$$

For instance we can view y as $y = \hbar \, \partial/\partial x$, and $\mathcal{E}$ is a differential operator, which encodes a linear differential equation.

In this article we choose $\mathcal{E}(x,y)$ of degree 2 in the variable y, i.e. the case of a second order linear differential equation, i.e. Schroedinger equation, and we leave to a further work the general case.

Here, in this article, we define some $F^{(g)}(\mathcal{E})$, which reduce to those of [13] in the limit $\hbar \to 0$, and which compute non-hermitian matrix model topological expansions.

For instance consider a formal matrix integral:

$$Z = \int_{E_{2\beta,N}} dM \mathrm{e}^{-N\sqrt{\beta}\,\mathrm{Tr}\,V(M)} = \mathrm{e}^{\sum_g N^{2-2g} F^{(g)}} \tag{1.6}$$

where $E_{2\beta,N}$ is one of the Wigner matrix ensembles [16] of rank N: $E_{1,N}$ is the set of real symmetric matrices, $E_{2,N}$ is the set of hermitian matrices, and $E_{4,N}$ is the set of self-dual quaternion matrices (see [16] for a review). We define:

$$\hbar = \frac{1}{N}\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right) \tag{1.7}$$

Notice that $\hbar = 0$ for hermitian matrices, i.e. the hermitian case is the classical limit $[y,x]=0$. Notice also that the expected duality $\beta \leftrightarrow 1/\beta$ (cf [17, 6]) corresponds to $\hbar \leftrightarrow -\hbar$, i.e. we expect it to correspond to the duality $x \leftrightarrow y$ (for $\hbar = 0$, the $x \leftrightarrow y$ duality was proved in [14]).

Let us also mention that the topological expansion of non-hermitian matrix integrals is known to be related to the enumeration of unoriented discrete surfaces, and we expect that our $F^{(g)} = \sum_k \hbar^k F^{(g,k)}$ can be interpreted as generating functions of such unoriented surfaces.

So, in this article, we provide a method for computing $F^{(g,k)}$ for any g and k (which is more consise than [7]).

Outline of the article

- In section 2, we introduce our recursion kernel $K(x,x')$, and we show that the mere existence of this kernel is equivalent to the Bethe ansatz condition.
- In section 3, we define the $W_n^{(g)}$'s and the $F^{(g)}$'s, and we study their main properties, for instance that $W_n^{(g)}$ is symmetric.
- In section 4, we study the classical limit $\hbar \to 0$, and we show that we recover the algebro-geometric construction of [13].
- This inspires a notion of non-commutative algebraic geometry in section 5.
- In section 6, we study the application to the topological expansion of non-hermitian matrix integrals.

- In section 7, we study the application to the Gaudin model.
- Section 8 is the conclusion.
- All the technical proofs are written in appendices for readability.

2 Definitions, kernel and Bethe ansatz

Let $V'(x)$ be a rational function (possibly a polynomial), and we call $V(x)$ the **potential**. Let α_i be the poles of $V'(x)$ (one of the poles may be at ∞).

For example, the following potential is called **Gaudin potential** (see section 7):

$$V'_{\text{Gaudin}}(x) = x + \sum_{i=1}^{\overline{n}} \frac{S_i}{x - \alpha_i} \tag{2.1}$$

As another example, we will consider formal matrix models in section 6, for which $V'(x)$ is a polynomial.

However, many other choices can be made.

2.1 The problem

Our problem is to find m complex numbers $s_1, \dots, s_m$, as well as two functions $G(x_0, x)$ and $K(x_0, x)$ with the following properties:

1. $G(x_0, x)$ is a rational function of x with poles at $x = s_i$, and a simple pole of residue $+1$ at $x = x_0$, and which behaves as $O(1/x)$ at $x \to \infty$.
2. $G(x_0, x)$ is a rational function of x_0 with (possibly multiple) poles at $x_0 = s_i$, and a simple pole at $x_0 = x$, and $G(x_0, x)$ behaves like $O(1/x_0)$ at $x_0 \to \infty$.
3. $B(x_0, x) = -\frac{1}{2}\frac{\partial}{\partial x} G(x_0, x)$ is symmetric: $B(x_0, x) = B(x, x_0)$.
4. K and G are related by the following differential equation:
$$\left(2\hbar \sum_{i=1}^{m} \frac{1}{x - s_i} - V'(x) - \hbar \frac{\partial}{\partial x} \right) K(x_0, x) = G(x_0, x) \tag{2.2}$$
5. $K(x_0, x)$ is analytical when $x \to s_i$ for all $i = 1, \dots, m$.

We shall see below that those 5 conditions determine K, G, and the s_i's. In fact condition 5 is the most important one in this list, it amounts to a **no-monodromy condition**, and we shall see below that it implies that the s_i's must obey the **Bethe-ansatz equation**.

2.2 Analytical structure of the kernel G

The 4th and 5th conditions imply that $G(x_0, x)$ has at most simple poles at $x = s_i$. Then condition 3 implies that $G(x_0, x)$ has at most double poles at $x_0 = s_i$.

The first 3 conditions imply that there exists a symmetric matrix $A_{i,j}$ such that $G(x_0, x)$ can be written:

$$G(x_0, x) = \frac{1}{x - x_0} + 2 \sum_{i,j=1}^{m} \frac{A_{i,j}}{(x - s_i)(x_0 - s_j)^2} \tag{2.3}$$

and therefore:

$$B(x_0, x) = \frac{1}{2} \frac{1}{(x - x_0)^2} + \sum_{i,j=1}^{m} \frac{A_{i,j}}{(x - s_i)^2 (x_0 - s_j)^2} \tag{2.4}$$

We will argue in section 5, that B can be viewed as a non=commutative deformation of the algebraic geometry's Bergmann kernel.

2.3 Bethe ansatz and monodromies

First, we study the conditions under which the differential equation eq. (2.2) has no monodromies around s_i, in other words the condition under which $K(x_0, x)$ is analytical when $x \to s_i$, $\forall i$:

$$K(x_0, s_i + \epsilon) = K(x_0, s_i) + \epsilon K'(x_0, s_i) + \frac{\epsilon^2}{2} K''(x_0, s_i) + \frac{\epsilon^3}{6} K'''(x_0, s_i) + \dots \tag{2.5}$$

Equating the coefficient of ϵ^{-1} in eq. (2.2), we get:

$$\hbar K(x_0, s_i) = \sum_j \frac{A_{i,j}}{(x_0 - s_j)^2} \tag{2.6}$$

equating the coefficient of ϵ^0 in eq. (2.2), we get:

$$\hbar K'(x_0, s_i) = \frac{-1}{x_0 - s_i} + V'(s_i) K(x_0, s_i) - 2\hbar \sum_{j \neq i} \frac{K(x_0, s_i) - K(x_0, s_j)}{s_i - s_j} \tag{2.7}$$

and equating the coefficient of ϵ^1 in eq. (2.2), we get:

$$\begin{aligned} & 2\hbar \sum_{j \neq i} \frac{K'(x_0, s_i)}{s_i - s_j} - 2\hbar \sum_{j \neq i} \frac{K(x_0, s_i)}{(s_i - s_j)^2} + V''(s_i) K(x_0, s_i) \\ = \quad & V'(s_i) K'(x_0, s_i) - \frac{1}{(s_i - x_0)^2} - 2 \sum_{j \neq i} \sum_k \frac{A_{j,k}}{(s_i - s_j)^2 (x_0 - s_k)^2} \end{aligned} \tag{2.8}$$

Notice from eq. (2.6), that $K(x_0, s_i)$ has only double poles in x_0, with no residue:

$$\operatorname*{Res}_{x_0 \to s_k} K(x_0, s_i) = 0 \tag{2.9}$$

Then, taking the residue at $x_0 \to s_k$ in eq. (2.7), we see that:

$$\hbar \operatorname*{Res}_{x_0 \to s_k} K'(x_0, s_i) = -\delta_{i,k} \tag{2.10}$$

Then, taking the residue when $x_0 \to s_i$ in eq. (2.8), implies that the s_i's are Bethe roots, i.e. they must obey the **Bethe equation**:

$$\boxed{\forall\, i = 1, \dots, m\,, \qquad 2\hbar \sum_{j \neq i} \frac{1}{s_i - s_j} = V'(s_i)} \tag{2.11}$$

Then eq. (2.8) becomes:

$$\frac{1}{(s_i - x_0)^2} = V''(s_i) K(x_0, s_i) + 2\hbar \sum_{j \neq i} \frac{K(x_0, s_i)}{(s_i - s_j)^2} - 2 \sum_{j \neq i} \sum_k \frac{A_{j,k}}{(s_i - s_j)^2 (x_0 - s_k)^2} \tag{2.12}$$

i.e. by comparing the coefficient of $1/(x_0 - s_k)^2$ on both sides:

$$\boxed{\delta_{i,k} = \frac{1}{\hbar} V''(s_i) A_{i,k} + 2 \sum_{j \neq i} \frac{A_{i,k} - A_{j,k}}{(s_i - s_j)^2}} \tag{2.13}$$

i.e. A is the inverse of the Hessian matrix T:

$$A = T^{-1} \qquad , \qquad \begin{cases} T_{i,i} = \frac{1}{\hbar} V''(s_i) + 2 \sum_{j \neq i} \frac{1}{(s_i - s_j)^2} \\ T_{i,j} = -\frac{2}{(s_i - s_j)^2} \end{cases} \tag{2.14}$$

$$T_{i,j} = \frac{1}{\hbar} \frac{\partial^2}{\partial s_i \partial s_j} \left(\sum_k V(s_k) - \hbar \sum_{k \neq l} \ln (s_k - s_l) \right) \tag{2.15}$$

Therefore the Bethe ansatz equations eq. (2.11) (as well as eq. (2.13)) are the necessary conditions for $K(x_0, x)$ to be analytical when $x \to s_i$. Those conditions are necessary, but also sufficient conditions, as one can see by solving explicitely the linear ODE for K.

$$K(x_0, x) = \int_c^x dx' G(x_0, x')\ \mathrm{e}^{\frac{1}{\hbar}(V(x') - V(x))}\ \prod_i \frac{(x - s_i)^2}{(x' - s_i)^2} \tag{2.16}$$

Remark 2.1 Notice that $K(x_0, x)$ is not analytical everywhere, it has a logarithmic singularity at $x = x_0$, and it has essential singularities at the poles of V'.

Remark 2.2 Notice that if one solution of the ODE is analytical near all s_i's, then all solutions have that property. Indeed, all the solutions differ by a solution of the homogeneous equation, i.e. by:

$$\prod_i (x-s_i)^2 \ \mathrm{e}^{-\frac{1}{\hbar}V(x)} \tag{2.17}$$

which is clearly analytical near the s_i's.

So, for the moment, the requirements 1–5 determine $G(x_0,x)$ uniquely, but $K(x_0,x)$ is not unique. Let us choose one possible $K(x_0,x)$, and we prove below in theorem 3.4, that the objects we are going to define, do not depend on the choice of K.

Remark 2.3 In what follows, it is useful to compute the Taylor expansion of K near a root s_i. We write:

$$K(x_0,x) = \sum_{k=0}^{\infty} K_{i,k}(x_0)\ (x-s_i)^k \tag{2.18}$$

The coefficients $K_{i,k}(x_0)$ are themselves rational fractions of x_0, and are computed in appendix A.

2.4 Schroedinger equation

It is well known that the Bethe condition can be rewritten as a Schroedinger equation [1, 2]. We rederive it here for completeness.

Define the wave function:

$$\psi(x) = \prod_{i=1}^{m}(x-s_i)\ \ \mathrm{e}^{-\frac{1}{2\hbar}V(x)} \qquad , \qquad \omega(x) = \hbar\sum_{i=1}^{m}\frac{1}{x-s_i} \tag{2.19}$$

$$Y(x) = -2\hbar\frac{\psi'(x)}{\psi(x)} = V'(x) - 2\omega(x) = V'(x) - 2\hbar\sum_i \frac{1}{x-s_i} \tag{2.20}$$

then compute:

$$\begin{aligned} U(x) &= Y^2 - 2\hbar Y'(x) = 4\hbar^2\frac{\psi''(x)}{\psi(x)} \\ &= V'(x)^2 - 2\hbar V''(x) + 4(\omega(x)^2 - V'(x)\omega(x) + \hbar\omega'(x)) \end{aligned} \tag{2.21}$$

We have:

$$\begin{aligned} \omega(x)^2 + \hbar\omega'(x) &= \hbar^2\sum_{i,j}\frac{1}{(x-s_i)(x-s_j)} - \hbar^2\sum_i\frac{1}{(x-s_i)^2} \\ &= \hbar^2\sum_{i\neq j}\frac{1}{(x-s_i)(x-s_j)} \end{aligned} \tag{2.22}$$

which is a rational fraction with only simple poles at the s_i's. The residue at s_i is $2\hbar^2\sum_{j\neq i}\frac{1}{s_i-s_j} = \hbar V'(s_i)$, and thus:

$$\omega(x)^2 + \hbar\omega'(x) = \hbar\sum_i\frac{V'(s_i)}{(x-s_i)} \tag{2.23}$$

which implies:

$$\omega(x)^2 - V'(x)\omega(x) + \hbar\omega'(x) = -\hbar \sum_i \frac{V'(x) - V'(s_i)}{(x - s_i)} \tag{2.24}$$

and thus:

$$U(x) = V'(x)^2 - 2\hbar V''(x) - 4\hbar \sum_{i=1}^{m} \frac{V'(x) - V'(s_i)}{x - s_i} \tag{2.25}$$

Therefore $U(x)$ is a rational fraction with poles at the poles of V' (of degree at most those of V'^2), in particular it has no poles at the s_i's.

U is the potential for the Schroedinger equation for ψ:

$$\boxed{4\hbar^2\psi'' = U\,\psi} \tag{2.26}$$

As announced in the introduction, this equation can be encoded in a D-module element:

$$\mathcal{E}(x,y) = y^2 - \frac{1}{4}U(x) \qquad , \qquad y = \hbar\frac{\partial}{\partial x} \qquad , \qquad [y,x] = \hbar \tag{2.27}$$

i.e.

$$\mathcal{E}(x,y).\psi = 0 \tag{2.28}$$

Notice that the Schroedinger equation is equivalent to a Ricatti equation for $Y = -2\hbar\psi'/\psi$:

$$\boxed{Y^2 - 2\hbar Y' = U} \tag{2.29}$$

2.5 Classical limit

We shall come back in more detail to the classical limit $\hbar \to 0$ in section 4. However, let us already make a few comments.

• In the classical limit, the Ricatti equation becomes an algebraic equation (hyperelliptical), which we call the (classical) spectral curve:

$$Y_{\rm cl}^2 = U(x) \tag{2.30}$$

The function $Y_{\rm cl}(x) = \sqrt{U(x)}$ is therefore a multivalued function of x, and it should be seen as a meromorphic function on a branched Riemann surface (branching points are the zeroes of $U(x)$). We shall see below that in the limit $\hbar \to 0$, the kernel $B(x_0, x)$ tends towards the Bergmann kernel of that Riemann surface.

In other words the classical limit is expressed in terms of **algebraic geometry**.

In fact, in this article we are going to define non-commutative deformations of certain algebraic geometric objects in section 5.

3 Definition of correlators and free energies

In this section, we define the quantum deformations of the symplectic invariants introduced in [10, 13]. The following definitions are inspired from (not hermitian) matrix models. The special case of their application to matrix models will be discussed in section 6.

3.1 Definition of correlators

Definition 3.1 *We define the following functions $W_n^{(g)}(x_1,\dots,x_n)$ (called n-point correlation function of "genus"[3] g) by the recursion:*

$$W_1^{(0)}(x) = \omega(x) = \hbar \sum_{i=1}^{m} \frac{1}{x - s_i} \qquad , \qquad W_2^{(0)}(x_1,x_2) = B(x_1,x_2) \tag{3.1}$$

$$\begin{aligned} & W_{n+1}^{(g)}(x_0, J) \\ = & \sum_{i=1}^{m} \operatorname*{Res}_{x \to s_i} K(x_0,x) \left(\overline{W}_{n+2}^{(g-1)}(x,x,J) + \sum_{h=0}^{g} {\sum_{I\subset J}}' W_{|I|+1}^{(h)}(x,I) W_{n-|I|+1}^{(g-h)}(x, J/I) \right) \end{aligned} \tag{3.2}$$

where J is a collective notation for the variables $J = \{x_1,\dots,x_n\}$, and where $\sum\sum'$ means that we exclude the terms $(h=0, I=\emptyset)$ and $(h=g, I=J)$, and where:

$$\overline{W}_n^{(g)}(x_1,...,x_n) = W_n^{(g)}(x_1,...,x_n) - \frac{\delta_{n,2}\delta_{g,0}}{2} \frac{1}{(x_1-x_2)^2} \tag{3.3}$$

Remark 3.1 This is exactly the same recursion as in [13], the only difference is that the kernel K is not algebraic, but it is solution of the differential equation eq. (2.2). We shall show in section 4, that in the limit $\hbar \to 0$, it indeed reduces to the definition of [13].

Remark 3.2 We say that $W_n^{(g)}$ is the correlation function of genus g with n marked points, and sometimes we say that it has characteristics:

$$\chi = 2 - 2g - n \tag{3.4}$$

By analogy with algebraic geometry, we say that $W_n^{(g)}$ is stable if $\chi < 0$ and unstable if $\chi \geq 0$. We see that all the stable $W_n^{(g)}$'s have a common recursive definition def.3.1, whereas the unstable ones appear as exceptions.

Remark 3.3 In order for the definition to make sense, we must make sure that the behaviour of each term in the vicinity of $x \to s_i$ is indeed locally meromorphic so that we can compute residues, i.e. there must be no log-singularity near s_i. In particular, the requirement of section 2.3 for the kernel K is **necessary**. In other words, a necessary condition for definition eq.3.2 to make sense, is the **Bethe ansatz** !

[3] here g is any given integer, it has nothing to do with the genus of the spectral curve.

3.2 Properties of correlators

The main reason of definition. 3.1, is because the $W_n^{(g)}$'s have many beautiful properties, which generalize those of [13].

We shall prove the following properties:

Theorem 3.1 *Each $W_n^{(g)}$ is a rational function of all its arguments. It has poles only at the s_i's (except $W_2^{(0)}$, which also has a pole at $x_1 = x_2$). In particular it has no poles at the α_i's. Moreover, it vanishes as $O(1/x_i)$ when $x_i \to \infty$.*

proof:

in appendix B □

Theorem 3.2 *The $W_n^{(g)}$'s satisfy the loop equation, i.e. Virasoro-like constraints. This means that the quantity:*

$$\begin{aligned} P_{n+1}^{(g)}(x; x_1..., x_n) \quad = \quad & -Y(x)\overline{W}_{n+1}^{(g)}(x, x_1, ..., x_n) + \hbar\partial_x \overline{W}_{n+1}^{(g)}(x, x_1..., x_n) \\ & + \sum_{I \subset J} \overline{W}_{|I|+1}^{(h)}(x, x_I)\overline{W}_{n-|I|+1}^{(g-h)}(x, J/I) + \overline{W}_{n+2}^{(g-1)}(x, x, J) \\ & + \sum_j \partial_{x_j} \left(\frac{\overline{W}_n^{(g)}(x, J/\{j\}) - \overline{W}_n^{(g)}(x_j, J/\{j\})}{(x - x_j)} \right) \end{aligned} \tag{3.5}$$

is a rational fraction of x (possibly a polynomial), with no pole at $x = s_i$. The only possible poles of $P_{n+1}^{(g)}(x; x_1..., x_n)$ are at the poles of $V'(x)$, with degree less than the degree of V'.

proof:

in appendix C □

Theorem 3.3 *Each $W_n^{(g)}$ is a symmetric function of all its arguments.*

proof:

in appendix D, with the special case of $W_3^{(0)}$ in appendix F. □

Theorem 3.4 *The correlation functions $W_n^{(g)}$ are independent of the choice of kernel K, provided that K is solution of the equation eq. (2.2).*

proof:

in appendix E □

Theorem 3.5 *The 3 point function $W_3^{(0)}$ can also be written:*

$$W_3^{(0)}(x_1,x_2,x_3) = 4\sum_i \operatorname*{Res}_{x\to s_i} \frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)} \tag{3.6}$$

(In section 5, we interpret this equation as a non-commutative version of Rauch variational formula).

proof:

in appendix F □

Theorem 3.6 *Under an infinitesimal variation of the potential $V \to V+\delta V$, we have:*

$$\forall n\geq 0, g\geq 0\ , \quad \delta W_n^{(g)}(x_1,\dots,x_n) = -\sum_i \operatorname*{Res}_{x\to s_i} W_{n+1}^{(g)}(x,x_1,\dots,x_n)\,\delta V(x) \tag{3.7}$$

proof:

in appendix G □

This theorem suggest the definition of the "loop operator":

Definition 3.2 *The loop operator δ_x computes the variation of $W_n^{(g)}$ under a formal variation $\delta_x V(x') = \frac{1}{x-x'}$:*

$$\delta_{x_{n+1}} W_n^{(g)}(x_1,\dots,x_n) = W_{n+1}^{(g)}(x_1,\dots,x_n,x_{n+1}) \tag{3.8}$$

The loop operator is a derivation: $\delta_x(uv) = u\delta_x v + v\delta_x u$, and we have $\delta_{x_1}\delta_{x_2} = \delta_{x_2}\delta_{x_1}$, $\delta_{x_1}\partial_{x_2} = \partial_{x_2}\delta_{x_1}$.

Theorem 3.7 *For $n\geq 1$, $W_n^{(g)}$ satify the equation:*

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\overline{W}_n^{(g)}(x_1,\dots,x_n) = -\sum_i \operatorname*{Res}_{x_{n+1}\to s_i} V'(x_{n+1})\,\overline{W}_{n+1}^{(g)}(x_1,\dots,x_n,x_{n+1}) \tag{3.9}$$

and

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} x_i\,\overline{W}_n^{(g)}(x_1,\dots,x_n) = -\sum_i \operatorname*{Res}_{x_{n+1}\to s_i} x_{n+1}\,V'(x_{n+1})\,\overline{W}_{n+1}^{(g)}(x_1,\dots,x_n,x_{n+1}) \tag{3.10}$$

proof:

in appendix H □

Theorem 3.8 *For $n\geq 1$, $W_n^{(g)}$ satify the equation:*

$$(2-2g-n-\hbar\frac{\partial}{\partial\hbar})\,\overline{W}_n^{(g)}(x_1,\dots,x_n) = -\sum_i \operatorname*{Res}_{x_{n+1}\to s_i} V(x_{n+1})\,\overline{W}_{n+1}^{(g)}(x_1,\dots,x_n,x_{n+1}) \tag{3.11}$$

proof:

We give a "long" proof in appendix I.

There is also a short cut:

If one changes $\hbar \to \lambda\hbar$, and $V \to \lambda V$, the s_i's don't change, B and G don't change, and K changes to $\frac{1}{\lambda} K$, thus $W_n^{(g)}$ changes by $\lambda^{2-2g-n} W_n^{(g)}$. The theorem is obtained by computing $\frac{\lambda\partial}{\partial\lambda}\lambda^{2g-2+n} W_n^{(g)} = \sum_k \frac{t_k \partial}{\partial t_k} W_n^{(g)}$, and computing the RHS with theorem 3.6, i.e. $\delta V = V$.

□

3.3 Definition of free energies

So far, we have defined $W_n^{(g)}$ with $n \geq 1$. Now, we define $F^{(g)} = W_0^{(g)}$.

Theorem 3.6, and the symmetry theorem 3.3 imply that:

$$\delta_{x_1} W_1^{(g)}(x_2) = W_2^{(g)}(x_1, x_2) = W_2^{(g)}(x_2, x_1) = \delta_{x_2} W_1^{(g)}(x_1) \tag{3.12}$$

Thus, the symmetry of $W_2^{(g)}$ implies that there exists a "free energy" $F^{(g)} = W_0^{(g)}$ such that:

$$W_1^{(g)}(x) = \delta_x F^{(g)} \tag{3.13}$$

which is equivalent to saying that for any variation δV:

$$\delta F^{(g)} = -\sum_i \operatorname*{Res}_{x \to s_i} W_1^{(g)}(x)\, \delta V(x) \tag{3.14}$$

Therefore, we know that there must exists some $F^{(g)} = W_0^{(g)}$ which satisfy theorem 3.6 for $n = 0$.

Now, let us give a definition of $F^{(g)}$, inspired from theorem 3.8, and which will be proved to satisfy theorem 3.6 for $n = 0$.

Definition 3.3 *We define $F^{(g)} \equiv W_0^{(g)}$ by a solution of the differential equation in $\hbar$:*

$$\forall g \geq 2 \qquad , \qquad (2 - 2g - \hbar \frac{\partial}{\partial \hbar})\, F^{(g)} = -\sum_i \operatorname*{Res}_{x \to s_i} W_1^{(g)}(x)\, V(x) \tag{3.15}$$

more precisely:

$$F^{(g)} = \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \sum_i \operatorname*{Res}_{x \to s_i} V(x) \quad W_1^{(g)}(x) \Big|_{\tilde{\hbar}} \tag{3.16}$$

And the unstable cases $2 - 2g \geq 0$ are defined by:

$$F^{(0)} = \hbar^2 \sum_{i \neq j} \ln(s_i - s_j) - \hbar \sum_i V(s_i) \tag{3.17}$$

$$F^{(1)} = \frac{1}{2}\ln\det A + \ln\left(\Delta(s)^2\right) + \frac{F^{(0)}}{\hbar^2} \tag{3.18}$$

where $\Delta(s) = \prod_{i>j}(s_i - s_j)$ *is the Vandermonde determinant of the* s_i*'s.*

Properties of the $F^{(g)}$'s:

The definition of the $F^{(g)}$'s, is made so that all the theorems for the $W_n^{(g)}$'s, hold for for $n = 0$ as well. Proofs are given in appendices J, K, L.

Explicit computations of the first few $F^{(g)}$'s are given in section 7 and appendix M.

4 Classical limit and WKB expansion

In the $\hbar \to 0$ limit, all quantities can be expanded formally into powers of $\hbar$: Write:

$$W_n^{(g)}(x_1,\dots,x_n) = \sum_k \hbar^k W_n^{(g,k)}(x_1,\dots,x_n) \qquad , \qquad F^{(g)} = \sum_k \hbar^k F^{(g,k)} \tag{4.1}$$

4.1 Classical limit

Here we consider the classical limit $\hbar \to 0$. We noticed in section 2.5, that in that limit, the Ricatti equation

$$Y^2 - 2\hbar Y' = U = V'^2 - 2\hbar V'' - 4P \tag{4.2}$$

where $P(x) = \hbar\sum_i \frac{V'(x)-V'(s_i)}{x-s_i}$, becomes an algebraic hyperelliptical equation:

$$Y_{\rm cl}{}^2 = U(x) = V'(x)^2 - 4P(x) \tag{4.3}$$

i.e.

$$Y(x) \underset{\hbar\to 0}{\sim} Y_{\rm cl}(x) = \sqrt{V'(x)^2 - 4P(x)} \tag{4.4}$$

$Y_{\rm cl}(x)$ is a multivalued function of x, and it should be seen as a meromorphic function on a 2-sheeted Riemann surface, i.e. there is a Riemann surface Σ (of equation $0 = \mathcal{E}_{\rm cl}(x,y) = y^2 - 4U(x)$, such that the solutions of $\mathcal{E}_{\rm cl}(x,y) = 0$ are parametrized by two meromorphic functions on Σ:

$$\mathcal{E}_{\rm cl}(x,y) = 0 \quad \Leftrightarrow \quad \exists z \in \Sigma \begin{cases} x = x(z) \\ y = y(z) \end{cases} \tag{4.5}$$

The Riemann surface Σ has a certain topology[4] characterized by its genus $\overline{g}$. It has a (non-unique) symplectic basis of $2\overline{g}$ non-trivial cycles $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$.

[4]This genus $\overline{g}$ has nothing to do with the index g of $F^{(g)}$ or $W_n^{(g)}$.

The meromorphic forms on Σ are classified as 1st kind (no pole), 3rd kind (only simple poles), and 2nd kind (multiple poles without residues).

There exists a unique 2nd kind differential $B_{\rm cl}$ on Σ, called the Bergmann kernel, such that: $B_{\rm cl}(z_1,z_2)$ has a double pole at $z_1 \to z_2$, and no other pole, without residue and normalized (in any local coordinate z) as:

$$B_{\rm cl}(z_1,z_2) \underset{z_2\to z_1}{\sim} \frac{dz_1 dz_2}{(z_1-z_2)^2} + \text{reg} \qquad , \qquad \forall i = 1,\dots,\overline{g}\,, \ \oint_{\mathcal{A}_i} B_{\rm cl} = 0 \tag{4.6}$$

We define a primitive:

$$G_{\rm cl}(z_0,z) = -2\int^z B_{\rm cl}(z_0,z') \tag{4.7}$$

which is a 3rd kind differential in the variable z_0, it is called $dE_z(z_0)$ in [13].

When $\hbar = 0$, the kernel $K(z_0,z)$ satisfies the equation:

$$K_{\rm cl}(z_0,z) = -\frac{G_{\rm cl}(z_0,z)}{Y_{\rm cl}(z)} = 2\,\frac{\int_c^z B_{\rm cl}(z_0,z')}{Y_{\rm cl}(z)} \tag{4.8}$$

which coincides with the definition of the recursion kernel in [13].

4.2 WKB expansion of the wave function

When $\hbar$ is small but non-zero, we can WKB expand $\psi(x)$, i.e.:

$$\psi(x) \sim \mathrm{e}^{-\frac{1}{2\hbar}\int^x Y_{\rm cl}(x')dx'}\ \frac{1}{\sqrt{Y_{\rm cl}(x)}}\left(1+\sum_k \hbar^k \psi_k(x)\right) \tag{4.9}$$

i.e.

$$Y \sim Y_{\rm cl} + \sum_{k=1}^{\infty} \hbar^k\, Y_k \tag{4.10}$$

The expansion coefficients Y_k can be easily obtained recursively from the Ricatti equation:

$$2Y_{\rm cl}Y_k = 2Y'_{k-1} - \sum_{j=1}^{k-1} Y_j\, Y_{k-j} \tag{4.11}$$

For instance:

$$Y_1 = \frac{{Y_{\rm cl}}'}{Y_{\rm cl}} \qquad , \qquad Y_2 = \frac{Y_1'}{Y_{\rm cl}} - \frac{Y_1^2}{2Y_{\rm cl}} = \frac{{Y_{\rm cl}}''}{{Y_{\rm cl}}^2} - \frac{3}{2}\,\frac{{Y_{\rm cl}}'^2}{{Y_{\rm cl}}^3} \qquad , \qquad \dots \text{ etc} \tag{4.12}$$

4.3 $\hbar$ expansion of correlators and energies

The kernel $K(x_0,x)$ can also be expanded:

$$K(x_0,x) = K_{\rm cl}(x_0,x) + \sum_{k=1}^{\infty} \hbar^k K_{(k)}(x_0,x) \tag{4.13}$$

where $K_{(0)} = K_{\rm cl}$ is the kernel of [13]:

$$K_{\rm cl}(x_0, x) = \frac{dE_{x,o}(x_0)}{Y_{\rm cl}(x)} \tag{4.14}$$

This implies that the correlators $W_n^{(g)}$ can also be expanded:

$$W_n^{(g)}(x_1, \dots, x_n) = \sum_{k=0}^{\infty} \hbar^k \; W_n^{(g,k)}(x_1, \dots, x_n) \tag{4.15}$$

where the $W_n^{(g,k)}$ are obtained by the recursion:

$$\begin{aligned} W_{n+1}^{(g,k)}(x_0, J) \;\; = \;\; & \sum_{l=0}^{k} \sum_{i} \operatorname*{Res}_{x \to s_i} K_{(k-l)}(x_0, x) \Big[\overline{W}_{n+2}^{(g-1,l)}(x, x, J) \\ & + \sum_{h=0}^{g} \sum_{j=0}^{l} {\sum_{I \subset J}}' W_{|I|+1}^{(h,j)}(x, I) W_{n-|I|+1}^{(g-h,l-j)}(x, J/I) \Big] \end{aligned} \tag{4.16}$$

where $J = \{x_1, \dots, x_n\}$.

Therefore, we observe that to leading order in $\hbar$, the $\lim_{\hbar \to 0} W_n^{(g,k)} = W_n^{(g,0)}$ do coincide with the $W_n^{(g)}$ computed with only $K_{\rm cl}$, and thus they coincide with the $W_n^{(g)}$ of [13].

And also, the $\hbar$ expansion must coincide with the diagrammatic rules of [7].

5 Non-commutative algebraic geometry

We have seen that in the limit $\hbar \to 0$, the correlation functions and the various functions we are considering, are fundamental objects of algebraic geometry. For instance B is the Bergmann kernel, and K is the recursion kernel of [13], which generates the symplectic invariants F_g and the correlators $W_n^{(g)}$ attached to the spectral curve $Y_{\rm cl}(x)$.

In this paper, when $\hbar \neq 0$, we have defined deformations of those objects, which have almost the same properties as the classical ones, except that they are no longer algebraic functions.

For instance we have:

- **Spectral curve**

  The algebraic equation of the classical spectral curve is replaced by a linear differential equation:

$$0 = \mathcal{E}(x, y) = \sum_{i,j} \mathcal{E}_{i,j} \, x^i y^j \quad \to \quad 0 = \mathcal{E}(x, \hbar\partial)\psi = \sum_{i,j} \mathcal{E}_{i,j} \, x^i (\hbar\partial)^j \, \psi \tag{5.1}$$

In other words the polynomial $\mathcal{E}(x,y)$ is replaced by a non-commutative polynomial with $y = \hbar\partial_x$, i.e. $[y,x] = \hbar$.

Here, our non-commutative spectral curve is:

$$\mathcal{E}(x,y) = y^2 - U(x) \qquad , \qquad y = \hbar\partial_x \tag{5.2}$$

Notice that it can be factorized as:

$$\mathcal{E}(x,y) = (y - \frac{Y}{2})\,(y + \frac{Y}{2}) \tag{5.3}$$

where $Y(x)$ is solution of $Y^2 - 2\hbar Y' = U$.

- **Bergmann Kernel** $B(x_1,x_2)$

  The non-commutative Bergmann kernel $B(x_1,x_2)$ is closely related to the Inverse of the Hessian T, i.e. to $A = T^{-1}$:

$$B(x_1,x_2) = \frac{1}{2(x_1-x_2)^2} + \sum_{i,j} \frac{A_{i,j}}{(x_1-s_i)^2(x_2-s_j)^2} \tag{5.4}$$

  A property of the classical Bergmann kernel $B_{\rm cl}(x_1,x_2)$ is that it computes derivatives, i.e. for any meromorphic function $f(x)$ defined on the spectral curve we have:

$$df(x) = -\operatorname*{Res}_{x_2 \to \text{poles of } f} B_{\rm cl}(x,x_2)\, f(x_2) \tag{5.5}$$

  Here, this property is replaced by: for any function $f(x)$ defined on the non-commutative spectral curve (i.e. with poles only at the s_i's), we have:

$$f'(x) = -2\sum_i \operatorname*{Res}_{x_2 \to s_i} B(x,x_2)\, f(x_2)\, dx_2 \tag{5.6}$$

  The factor of 2, comes from the fact that the interpretation of x, and thus of derivatives with respect to x, is slightly different. In the classical case, the differentials are computed in terms of local variables, and x is not a local variable near branch-points. A good local variable near a branchpoint a, is $\sqrt{x-a}$. In the non-commutative case, the role of branchpoints seems to be played by the s_i's, and x is a good local variable near s_i.

- **Rauch variational formula**: In classical algebraic geometry, on an algebraic curve of equation $\mathcal{E}(x,y) = \sum_{i,j} \mathcal{E}_{i,j} x^i y^j = 0$, the Bergmann kernel depends only on the location of branchpoints a_i. The branchpoints are the points where the tangent is vertical, i.e. $dx(a_i) = 0$. Their location is $x_i = x(a_i)$. The Bergmann

kernel is only function of the x_i's, and the classical variational Rauch formula reads:

$$\frac{\partial\, B_{\rm cl}(z_1,z_2)}{\partial x_i} = \mathop{\rm Res}_{z\to a_i} \frac{B_{\rm cl}(z,z_1)\, B_{\rm cl}(z,z_2)}{dx(z)} \tag{5.7}$$

Equivalently, we can parametrize the spectral curve as $x(y)$ instead of $y(x)$, and consider the branchpoints of y, i.e. $dy(b_i)=0$, whose location is $y_i = y(b_i)$, and we have:

$$\frac{\partial\, B_{\rm cl}(z_1,z_2)}{\partial y_i} = \mathop{\rm Res}_{z\to b_i} \frac{B_{\rm cl}(z,z_1)\, B_{\rm cl}(z,z_2)}{dy(z)} \tag{5.8}$$

Here, in the non-commutative version, theorem 3.5 and theorem 3.6 implies that under a variation of the spectral curve, we have:

$$\delta B(x_1,x_2) = -\frac{1}{2}\sum_i \mathop{\rm Res}_{x\to s_i} \frac{B(x,x_1)B(x,x_2)}{Y'(x)}\, \delta Y(x) \tag{5.9}$$

Consider the branchpoints b_i such that $Y'(b_i)=0$, and define their location as $Y_i = Y(b_i)$, by moving the integration contours we have:

$$\begin{aligned} \delta B(x_1,x_2) &= \frac{1}{2}\sum_i \mathop{\rm Res}_{x\to b_i} \frac{B(x,x_1)B(x,x_2)}{Y'(x)}\, \delta Y(x)\, dx \\ &= \frac{1}{2}\sum_i \delta Y_i \mathop{\rm Res}_{x\to b_i} \frac{B(x,x_1)B(x,x_2)}{Y'(x)}\, dx \end{aligned} \tag{5.10}$$

i.e.:

$$\frac{\partial\, B(x_1,x_2)}{\partial Y_i} = \frac{1}{2}\mathop{\rm Res}_{x\to b_i} \frac{B(x,x_1)\, B(x,x_2)}{Y'(x)}\, dx \tag{5.11}$$

which is thus the quantum version of the Rauch variational formula eq. (5.8).

Those properties can be seen as the beginning of a dictionary giving the deformations of classical algebraic geometry into non-commutative algebraic geometry.

Conjecture about the symplectic invariants

The F_g's of [13] are the symplectic invariants of the classical spectral curve, which means that they are invariant under any cannonical change of the spectral curve which conserves the symplectic form $dx\wedge dy$. For instance they are invariant under $x\to y, y\to -x$.

Here, we conjecture that we may define some non-commutative $F^{(g)}$'s which are invariant under any cannonical transformation which conserves the commutator $[y,x]=\hbar$. This duality should also correspond to the expected duality $\beta\to 1/\beta$ in matrix models, cf [17, 6].

However, to check the validity of this conjecture, one needs to extend our work to differential operators of any order in y, and not only order 2. We plan to do this in a forthcoming work.

6 Application: non-hermitian Matrix models

The initial motivation for the work of [13], as well as this present work, was initially random matrix models. The classical case corresponds to hermitian matrix models, and here, we show that $\hbar \neq 0$ corresponds in some sense to non-hermitian matrix models [5, 6, 9].

In this section, we show that non-hermitian matrix models satisfy the loop equation eq. (C.1) of theorem 3.2.

We define the matrix integral over $E_{m,2\beta}$ =set of $m \times m$ matrices of Wigner–type 2β ($E_{m,1}$ = real symmetric matrices, $E_{m,2}$ = hermitean matrices, $E_{m,4}$ = real quaternion self-dual matrices, see [16]):

$$Z = \int_{E_{m,2\beta}} dM \ \mathrm{e}^{-N\sqrt{\beta}\ \mathrm{Tr}\, V(M)} \tag{6.1}$$

where N is some arbitrary constant, not necessarily related to the matrix size m.

It is more convenient to rewrite it in terms of eigenvalues of M (see [16]):

$$Z = \int_{\mathcal{C}^m} d\lambda_1 \dots d\lambda_m \ \prod_{i>j} (\lambda_j - \lambda_i)^{2\beta} \ \prod_i \mathrm{e}^{-N\sqrt{\beta} V(\lambda_i)} \tag{6.2}$$

This last expression is well defined for any β, and not only $1/2, 1, 2$, and for any contour of integration $\mathcal{C}$ on which the integral is convergent.

We also define the correlators:

$$\begin{aligned} \overline{W}_n(x_1, \dots, x_n) &= < \mathrm{Tr}\, \frac{1}{x_1 - M} \dots \mathrm{Tr}\, \frac{1}{x_n - M} >_c \\ &= \left(N\sqrt{\beta}\right)^{-n} \ \frac{\partial}{\partial V(x_1)} \dots \frac{\partial}{\partial V(x_n)} \ \ln Z \end{aligned} \tag{6.3}$$

i.e. in terms of eigenvalues:

$$\overline{W}_n(x_1, \dots, x_n) = < \sum_{i_1} \frac{1}{x_1 - \lambda_{i_1}} \dots \sum_{i_n} \frac{1}{x_n - \lambda_{i_n}} >_c \tag{6.4}$$

In order to match with the notations of section 3, we prefer to shift $\overline{W}_2$ by a second order pole, and we define:

$$W_n(x_1, \dots, x_n) = \overline{W}_n(x_1, \dots, x_n) + \frac{\delta_{n,2}}{2(x_1 - x_2)^2} \tag{6.5}$$

We are interested in a case where Z has a large N expansion of the form:

$$\ln Z \sim \sum_{g=0}^{\infty} N^{2-2g} \ F_g \tag{6.6}$$

and for the correlation functions we assume:

$$W_n(x_1, \dots, x_n) = \frac{1}{\beta^{n/2}} \ \sum_{g=0}^{\infty} N^{2-2g-n} W_n^{(g)}(x_1, \dots, x_n) \tag{6.7}$$

6.1 Loop equations

The loop equations can be obtained by integration by parts, or equivalently, they follow from the invariance of an integral under a change of variable. By considering the infinitesimal change of variable:

$$\lambda_i \to \lambda_i + \epsilon \frac{1}{x - \lambda_i} + O(\epsilon^2) \tag{6.8}$$

we obtain:

$$\begin{aligned} & N\sqrt{\beta}(V'(x)\,\overline{W}_{n+1}(x,x_1,\ldots,x_n) - P_{n+1}(x;x_1,\ldots,x_n)) \\ = \;& \beta \sum_{J\subset L} \overline{W}_{1+|J|}(x,J)\,\overline{W}_{1+n-|J|}(x,L/J) \\ & +\beta\overline{W}_{n+2}(x,x,x_1,\ldots,x_n) \\ & -(1-\beta)\,\frac{\partial}{\partial x}\,\overline{W}_{n+1}(x,x_1,\ldots,x_n) \\ & +\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\,\frac{\overline{W}_n(x,L/\{x_j\}) - \overline{W}_n(x_j,L/\{x_j\})}{x-x_j} \end{aligned} \tag{6.9}$$

where $P_{n+1}(x;x_1,\ldots,x_n))$ is a polynomial in its first variable x, of degree $\delta_{n,1}+\deg V-2$.

If we expand this equation into powers of N using eq. (6.7), we have $\forall\, n, g$:

$$\begin{aligned} & V'(x)\,\overline{W}^{(g)}_{n+1}(x,x_1,\ldots,x_n) - P^{(g)}_{n+1}(x;x_1,\ldots,x_n)) \\ = \;& \sum_{g'=0}^{g} \sum_{J\subset L} \overline{W}^{(g')}_{1+|J|}(x,J)\,\overline{W}^{(g-g')}_{1+n-|J|}(x,L/J) \\ & +\beta\overline{W}^{(g-1)}_{n+2}(x,x,x_1,\ldots,x_n) \\ & +\hbar\,\frac{\partial}{\partial x}\,\overline{W}^{(g)}_{n+1}(x,x_1,\ldots,x_n) \\ & +\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\,\frac{\overline{W}^{(g)}_n(x,L/\{x_j\}) - \overline{W}^{(g)}_n(x_j,L/\{x_j\})}{x-x_j} \end{aligned} \tag{6.10}$$

where

$$\hbar = \frac{\sqrt{\beta} - \frac{1}{\sqrt{\beta}}}{N} \tag{6.11}$$

Those loop equations coincide with the loop equations eq. (3.5) of theorem 3.2.

Moreover we have:

$$\overline{W}^{(g)}_n = \frac{\partial \overline{W}^{(g)}_{n-1}}{\partial V} \tag{6.12}$$

and near $x \to \infty$:

$$\sqrt{\beta}\; W_1(x) \sim \frac{m}{x}[N\hbar - \sum_{g=1}^{\infty} (-1)^g \frac{(2g-2)!}{g!(g-1)!}\;(N\hbar)^{1-2g}] \tag{6.13}$$

i.e.

$$W_1^{(0)}(x) \sim \frac{m\hbar}{x} + O(1/x^2) \qquad , \qquad W_1^{(g)}(x) \sim -\frac{m\hbar}{x}\ \hbar^{-2g}\ \frac{(2g-2)!}{g!(g-1)!} + O(1/x^2) \quad (6.14)$$

One should notice that the loop equations are independent of the contour $\mathcal{C}$ of integration of eigenvalues. The contour $\mathcal{C}$ is in fact encoded in the polynomial $P_{n+1}(x; x_1, \ldots, x_n)$.

6.2 Solution of loop equations

To order $g = 0, n = 1$ we have:

$$V'(x)\, W_1^{(0)}(x) - P_1^{(0)}(x) = W_1^{(0)}(x)^2 + \hbar\, \frac{\partial}{\partial x}\, W_1^{(0)}(x) \quad (6.15)$$

which is the same as the Ricatti equation eq. (2.21).

As we said above, the contour $\mathcal{C}$ is in fact encoded in the polynomial $P_1^{(0)}(x)$. From now on, we choose a contour $\mathcal{C}$, i.e. a polynomial $P_1^{(0)}(x)$ such that the solution of the Ricatti equation is rational:

$$W_1^{(0)}(x) = \hbar \sum_{i=1}^{m} \frac{1}{x - s_i} \quad (6.16)$$

It also has the correct behaviour at ∞: $W_1^{(0)}(x) \sim \frac{m\hbar}{x}$. This corresponds to a certain contour $\mathcal{C}$ which we do not determine here.

Since $W_1^{(0)}(x) = \omega(x)$ satisfies the Ricatti equation, i.e. the Bethe ansatz, the kernel K exists, and we can define the functions $K(x_0, x)$, $G(x_0, x)$ and $B(x_0, x)$.

Then, from eq. (6.12), we see that every $\overline{W}_n^{(g)}$ is going to be a rational fraction of x, with poles only at the s_i's. In particular, Cauchy theorem implies:

$$\overline{W}_{n+1}^{(g)}(x_0, x_1, \ldots, x_n) = \operatorname*{Res}_{x \to x_0} G(x_0, x)\, \overline{W}_{n+1}^{(g)}(x, x_1, \ldots, x_n) \quad (6.17)$$

and since both $G(x_0, x)$ and $\overline{W}_{n+1}^{(g)}(x, x_1, \ldots, x_n)$ are rational fractions, which vanish sufficientely at ∞, we may change the integration contour to the other poles of the integrand, namely:

$$\begin{aligned}
& \overline{W}_{n+1}^{(g)}(x_0, x_1, \ldots, x_n) \\
= \quad & -\sum_i \operatorname*{Res}_{x \to s_i} G(x_0, x)\, \overline{W}_{n+1}^{(g)}(x, x_1, \ldots, x_n) \\
= \quad & -\sum_i \operatorname*{Res}_{x \to s_i} \overline{W}_{n+1}^{(g)}(x, x_1, \ldots, x_n)\ (2\omega(x) - V'(x) - \hbar\partial_x) K(x_0, x) \\
= \quad & -\sum_i \operatorname*{Res}_{x \to s_i} K(x_0, x)\ (2\omega(x) - V'(x) + \hbar\partial_x) \overline{W}_{n+1}^{(g)}(x, x_1, \ldots, x_n)
\end{aligned}$$

(6.18)

Now, we insert loop equation eq. (6.10) in the right hand side, and we notice that the term $P_{n+1}^{(g)}$ and $\frac{\partial}{\partial x_j}\frac{W_n^{(g)}(x_j,L/\{x_j\})}{x-x_j}$ do not have poles at the s_i's, so they don't contribute. We thus get:

$$\begin{aligned}
&\overline{W}_{n+1}^{(g)}(x_0,x_1,\dots,x_n)\\
=\ &\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\overline{W}_{n+2}^{(g-1)}(x,x,x_1,\dots,x_n)\right.\\
&\left.+\sum_{g'=0}^{g}\sum_{J\subset L} W_{1+|J|}^{(g')}(x,J)W_{1+n-|J|}^{(g-g')}(x,L/J)\right)
\end{aligned} \tag{6.19}$$

i.e. we find the correlators of def 3.1.

Special care is needed for $W_2^{(0)}$. We have:

$$\begin{aligned}
&\overline{W}_2^{(0)}(x_0,x_1,\dots,x_n)\\
=\ &-\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\ (2\omega(x)-V'(x)+\hbar\partial_x)\overline{W}_2^{(0)}(x,x_1)\\
=\ &\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\ \frac{\omega(x)}{(x-x_1)^2}\\
=\ &\hbar\sum_i \frac{K(x_0,s_i)}{(s_i-x_1)^2}\\
=\ &\sum_{i,j}\frac{A_{i,j}}{(s_i-x_1)^2(s_j-x_0)^2}
\end{aligned} \tag{6.20}$$

which also agrees with def 3.1.

7 Application: Gaudin model

The Gaudin model's Bethe ansatz is obtained for the potential:

$$V'_{\text{Gaudin}}(x) = x + \sum_{i=1}^{\overline{n}} \frac{S_i}{x-\alpha_i} \tag{7.1}$$

i.e. it corresponds to a Gaussian matrix model with sources:

$$Z = \int_{E_{m,2\beta}} dM\ \mathrm{e}^{-\frac{N\sqrt{\beta}}{2}\operatorname{Tr} M^2}\ \prod_i \det(\alpha_i - M)^{-NS_i\sqrt{\beta}} \tag{7.2}$$

with $\hbar = \frac{\sqrt{\beta}-1/\sqrt{\beta}}{N}$.

Z can also be written in eigenvalues:

$$Z = \int d\lambda_1\dots d\lambda_m\ \frac{\prod_{i=1}^m \mathrm{e}^{-\frac{N\sqrt{\beta}}{2}\lambda_i^2}}{\prod_{i=1}^m\prod_{j=1}^{\overline{n}}(\alpha_j-\lambda_i)^{N\sqrt{\beta}S_j}}\ \prod_{i>j}(\lambda_i-\lambda_j)^{2\beta} \tag{7.3}$$

7.1 Example

Consider:

$$V'(x) = x - \frac{s^2}{x} \qquad , \qquad V(x) = \frac{x^2}{2} - s^2 \ln x \tag{7.4}$$

With only 1 root $m = 1$, the solution of the Bethe equation $V'(x) = 0$ is $x = s$.

Thus we have:

$$\omega(x) = \frac{\hbar}{x - s} \tag{7.5}$$

$$B(x_1, x_2) = \frac{1}{2(x_1 - x_2)^2} + \frac{\hbar}{2(x_1 - s)^2 (x_2 - s)^2} \tag{7.6}$$

We find:

$$W_3^{(0)}(x_1, x_2, x_3) = \frac{\hbar}{2(x_1 - s)^2 (x_2 - s)^2 (x_3 - s)^2} \left(\frac{1}{x_1 - s} + \frac{1}{x_2 - s} + \frac{1}{x_3 - s} + \frac{1}{2s} \right) \tag{7.7}$$

$$W_1^{(1)}(x) = \frac{1}{\hbar (x - s)} + \frac{1}{4s(x - s)^2} + \frac{1}{2(x - s)^3} \tag{7.8}$$

For the free energies we have:

$$F^{(0)} = \frac{\hbar\, s^2}{2} (\ln s^2 - 1) \tag{7.9}$$

$$F^{(1)} = \frac{1}{2} \ln (\frac{\hbar}{2}) + \frac{F^{(0)}}{\hbar^2} \tag{7.10}$$

$$F^{(2)} = -\frac{1}{12 \hbar s^2} - \frac{F^{(0)}}{\hbar^4} \tag{7.11}$$

$$F^{(3)} = \frac{1}{12 \hbar^3 s^2} + \frac{2 F^{(0)}}{\hbar^6} \tag{7.12}$$

and

$$Z = \mathrm{e}^{\sum_g N^{2-2g} F^{(g)}} = \mathrm{e}^{-N \sqrt{\beta} V(s)} \frac{1}{\sqrt{2\hbar}} \left(1 - \frac{1}{12 s^2 N^2 \hbar^2} + \ldots\right) \tag{7.13}$$

which is indeed the beginning of the saddle point expansion of:

$$Z = \int dx\ \mathrm{e}^{-N \sqrt{\beta}\, V(x)} \tag{7.14}$$

8 Conclusion

In this article, we have defined a special case of non-commutative deformation of the symplectic invariants of [13]. Many of the fundamental properties of [13] are conserved or only slightly modified.

The main difference, is that the recursion kernel, instead of beeing an algebraic function, is given by the solution of a differential equation, otherwise the recursion is the same.

The main drawback of our definition, is that it concerns only a very restrictive subset of possible non-commutative spectral curves. Namely, we considered here only non commutative polynomials $\mathcal{E}(x,y) = \sum_{i,j} \mathcal{E}_{i,j}\ x^i y^j$ with $y = \hbar\partial_x$, of degree 2 in y, and such that the differential equation $\mathcal{E}(x, \hbar\partial).\psi = 0$ has a "polynomial" solution of the form $\psi(x) = \prod_{i=1}^{m}(x - s_i)\ \mathrm{e}^{-V(x)/2\hbar}$.

It should be possible to extend our definitions to other "non-polynomial" solutions ψ (with an infinite number of zeroes $m = \infty$ for instance), and/or to higher degrees in y. In other words, what we have so far, is only a glimpse on more general structure yet to be discovered.

For example, it is not yet clear how our definitions are related to matrix integrals. We have said that the integration contour for the eigenvalues should be chosen so that the solution of the Schroedinger equation is polynomial of degree m, however, it is not known how to find explicitly such integration contours. Conversely, the usual matrix integrals with eigenvalues on the real axis, do probably not correspond to polynomial solutions of the Schroedinger equation. Similarly, it is not clear what the relationship between our definitions and the number of unoriented ribbon graphs is, for the same reason. The solution of the Schroedinger equation for ribbon graphs, should be chosen such that all the $W_n^{(g,k)}$'s are power series in t, and it is not known which integration contour it corresponds to, and which solution of the Schroedinger equation it corresponds to.

Therefore it seems necessary to extend our definitions to arbitrary solutions, i.e. to arbitrary integration contours for the matrix integrals. A possibility could be to obtain non-polynomial solutions as limits of polynomial ones.

The extension to higher degree in y, can be obtained from multi-matrix integrals, and extension seems rather easy for polynomial solutions again.

Finally, like the symplectic invariants of [13], we expect those "to be defined" non-commutative symplectic invariants, to play a role in several applications to enumerative geometry, and to topological string theory like in [3]. In other words, we expect our $F^{(g)}$'s to be generating functions for intersection numbers in some non-commutative moduli spaces of unoriented Riemann surfaces, whatever it means...

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A Appendix: Expansion of K

Since we have to compute residues at the s_i's, we need to compute the Taylor expansion of $K(x_0, x)$ when $x \to s_i$:

$$K(x_0, x) = \sum_k (x - s_i)^k \, K_{i,k}(x_0) \tag{A.1}$$

For instance we find:

$$K_{i,0} = \frac{1}{\hbar} \sum_j \frac{A_{i,j}}{(x_0 - s_j)^2} \tag{A.2}$$

$$\hbar K_{i,1}(x_0) = -\frac{1}{(x_0 - s_i)} - 2 \sum_{a \neq i} \sum_j \frac{A_{a,j}}{(s_a - s_i)\,(x_0 - s_j)^2} \tag{A.3}$$

$$\begin{aligned} \hbar K_{i,3} = & -\hbar \Big(2 \sum_{a \neq i} \frac{1}{(s_a - s_i)^2} + \frac{1}{\hbar} V''(s_i) \Big) K_{i,1} \\ & -\hbar \Big(2 \sum_{a \neq i} \frac{1}{(s_a - s_i)^3} + \frac{1}{\hbar} \frac{V'''(s_i)}{2} \Big) K_{i,0} \\ & + \frac{1}{(x_0 - s_i)^3} + 2 \sum_{a \neq i} \sum_j \frac{A_{a,j}}{(s_a - s_i)^3 \, (x_0 - s_j)^2} \end{aligned} \tag{A.4}$$

Thanks to property eq. (E.4), we may assume (but it is not necessary) that:

$$K_{i,2} = 0 \tag{A.5}$$

Then, we have the recursion for $k \geq 0$:

$$\begin{aligned} & \hbar \Big((1-k) K_{i,k+1} - 2 \sum_{a \neq i} \sum_{l=0}^{k} \frac{K_{i,k-l}}{(s_a - s_i)^{l+1}} - \frac{1}{\hbar} \sum_{l=0}^{k} \frac{V^{(l+1)}(s_i)}{l!} K_{i,k-l} \Big) \\ = & -\frac{1}{(x_0 - s_i)^{k+1}} - 2 \sum_{a \neq i} \sum_j \frac{A_{a,j}}{(s_a - s_i)^{k+1} \, (x_0 - s_j)^2} \end{aligned} \tag{A.6}$$

This proves that each $K_{i,k}(x_0)$ is a rational fraction of x_0, with poles at the s_j's.

A.1 Rational fraction of x_0

Thus we write:

$$K_{i,k}(x_0) = \sum_{j,l} \frac{1}{(x_0 - s_j)^{k'}} \; K_{i,k;j,k'} \tag{A.7}$$

For instance we have:

$$K_{i,0;j,k'} = \frac{A_{i,j}}{\hbar} \, \delta_{k',2} \tag{A.8}$$

$$\hbar K_{i,1;j,k'} = -\delta_{k',1}\delta_{i,j} - 2\delta_{k',2} \sum_{a \neq i} \frac{A_{a,j}}{s_a - s_i} \tag{A.9}$$

For higher k we have the recursion:

$$\begin{aligned} & \hbar\Big((1-k)K_{i,k+1;j,k'} - 2\sum_{a\neq i}\sum_{l=1}^{k} \frac{K_{i,k-l;j,k'}}{(s_a - s_i)^{l+1}} - \frac{1}{\hbar}\sum_{l=1}^{k} \frac{V^{(l+1)}(s_i)}{l!} K_{i,k-l;j,k'}\Big) \\ = \; & -\delta_{i,j}\,\delta_{k',k+1} - 2\delta_{k',2} \sum_{a\neq i} \frac{A_{a,j}}{(s_a - s_i)^{k+1}} \end{aligned} \tag{A.10}$$

In particular, it shows that if $k' > 2$, then $K_{i,k;i,k'}$ is proportional to $\delta_{i,j}$.

A.2 Generating functions

We introduce generating functions:

$$R_{i;j,k'}(x) = \sum_i K_{i,k;j,k'} \, (x - s_i)^k \tag{A.11}$$

We have:

$$\hbar \left(2\frac{\psi'(x)}{\psi(x)} - \partial_x\right) R_{i;j,k'}(x) = -\delta_{i,j}(x - s_i)^{k'-1} + 2\delta_{k',2} \sum_a \frac{A_{a,j}}{x - s_a} \tag{A.12}$$

i.e.

$$-\hbar\psi^2(x)\,\partial_x\Big(\frac{R_{i;j,k'}(x)}{\psi^2(x)}\Big) = -\delta_{i,j}(x - s_i)^{k'-1} + \delta_{k',1}c_j + 2\delta_{k',2} \sum_a \frac{A_{a,j}}{x - s_a} \tag{A.13}$$

In particular with $k' = 1$ we find:

$$R_{i;j,1}(x) = \frac{\delta_{i,j}}{\hbar} \, \psi(x)\phi(x) \tag{A.14}$$

where

$$\phi(x) = \psi(x) \int^x \frac{dx'}{\psi(x')^2} \qquad , \qquad \phi'(x)\psi(x) - \psi'(x)\phi(x) = 1 \tag{A.15}$$

B Appendix: Proof of theorem 3.1

Theorem 3.1 *Each $W_n^{(g)}$ is a rational function of all its arguments. If $2g+n-2>0$, it has poles only at the s_i's. In particular it has no poles at the α_i's, and it vanishes as $O(1/x_i)$ when $x_i \to \infty$.*

proof:

It is easy to check that $W_1^{(0)}$, $W_2^{(0)}$ satisfy the theorem.

We will now make a recursion over $-\chi = 2g-2+n$ to prove the result for every (n,g). We write:

$$W_{n+1}^{(g)}(x_0,x_1,\dots,x_n) = \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\, U_{n+1}^{(g)}(x,x_1,\dots,x_n) \tag{B.1}$$

where $J=\{x_1,\dots,x_n\}$, and

$$U_{n+1}^{(g)}(x,J) = \overline{W}_{n+2}^{(g-1)}(x,x,J) + \sum_{h=0}^{g}\sum_{I\subset J} W_{|I|+1}^{(h)}(x,I) W_{n-|I|+1}^{(g-h)}(x,J/I) \tag{B.2}$$

First, the recursion hypothesis clearly implies that $U_{n+1}^{(g)}(x,x_1,\dots,x_n)$ is a rational fraction in all its variables $x,x_1,...x_n$.

Then we Taylor expand $K(x_0,x)$ as in eq. (A.1) or eq. (A.7)

$$\begin{aligned} W_{n+1}^{(g)}(x_0,x_1,\dots,x_n) &= \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\, U_{n+1}^{(g)}(x,x_1,\dots,x_n) \\ &= \sum_i \sum_k K_{i,k}(x_0) \operatorname*{Res}_{x\to s_i} (x-s_i)^k\, U_{n+1}^{(g)}(x,x_1,\dots,x_n) \end{aligned} \tag{B.3}$$

Since $U_{n+1}^{(g)}(x,x_1,\dots,x_n)$ is a rational fraction of x, the sum over k is finite, and therefore, $W_{n+1}^{(g)}(x_0,x_1,\dots,x_n)$ is a finite sum of rational fractions of x_0, with poles at the s_j's, therefore it is a rational fraction of x_0 with poles at the s_j's.

It is also clear that $W_{n+1}^{(g)}(x_0,x_1,\dots,x_n)$ is a rational fraction of the other variables $x_1,\dots,x_n$. The poles in those variables are necessarily at the s_j's, because as long as the residues can be computed, $W_{n+1}^{(g)}(x_0,x_1,\dots,x_n)$ is finite. The residue cannot be computed everytime an integration contour gets pinched, and since the integration contours are small circles around the s_i's, the only singularities may occur at the s_i's.

It remains to prove that each $W_n^{(g)}$ behaves like $O(1/x_i)$ at ∞. The proof follows the same line: each $K_{i,k}(x_0)$ behaves like $O(1/x_0)$, and by an easy recursion the result holds for all other variables. $\square$

C Appendix: Proof of theorem 3.2

In this subsection we prove theorem 3.2, that all $W_n^{(g)}$'s satisfy the loop equation.

Theorem 3.2 *The $W_n^{(g)}$'s satisfy the loop equation, i.e. the following quantity $P_{n+1}^{(g)}(x;x_1...,x_n)$*

$$\begin{aligned} P_{n+1}^{(g)}(x;x_1...,x_n) &= -Y(x)\overline{W}_{n+1}^{(g)}(x,x_1,...,x_n) + \hbar\partial_x\overline{W}_{n+1}^{(g)}(x,x_1...,x_n) \\ &\quad + \sum_{I\subset J}\overline{W}_{|I|+1}^{(h)}(x,x_I)\overline{W}_{n-|I|+1}^{(g-h)}(x,J/I) + \overline{W}_{n+2}^{(g-1)}(x,x,J) \\ &\quad + \sum_j \partial_{x_j}\left(\frac{\overline{W}_n^{(g)}(x,J/\{j\}) - \overline{W}_n^{(g)}(x_j,J/\{j\})}{(x-x_j)}\right) \end{aligned} \tag{C.1}$$

is a rational fraction of x (possibly a polynomial), with no pole at $x = s_i$. The only possible poles of $P_{n+1}^{(g)}(x;x_1...,x_n)$ are at the poles of $V'(x)$, and their degree is less than the degree of V'.

proof:

First, from theorem 3.1, we easily see that $P_{n+1}^{(g)}(x;x_1...,x_n)$ is indeed a rational function of x. Moreover it clearly has no pole at coinciding points $x = x_j$.

Then we write Cauchy's theorem for $W_{n+1}^{(g)}$:

$$\begin{aligned} W_{n+1}^{(g)}(x_0,...,x_n) &= \operatorname*{Res}_{x\to x_0}\frac{1}{x-x_0}W_{n+1}^{(g)}(x,x_1,...,x_n) \\ &= \operatorname*{Res}_{x\to x_0}G(x_0,x)\,W_{n+1}^{(g)}(x,x_1,...,x_n) \end{aligned} \tag{C.2}$$

and using again theorem 3.1, i.e. that $W_{n+1}^{(g)}$ has poles only at the s_i's, and that both $W_{n+1}^{(g)}$ and $G(x_0,x)$ behave as $O(1/x)$ for large x, we may move the integration contours:

$$W_{n+1}^{(g)}(x_0,...,x_n) = -\sum_i \operatorname*{Res}_{x\to s_i} G(x_0,x)\; W_{n+1}^{(g)}(x,x_1,...,x_n) \tag{C.3}$$

Then we use the definition of K, and integrate by parts:

$$\begin{aligned} W_{n+1}^{(g)}(x_0,...,x_n) &= \sum_i \operatorname*{Res}_{x\to s_i}(Y(x)K(x_0,x) + \hbar K'(x_0,x))W_{n+1}^{(g)}(x,x_1,...,x_n) \\ &= \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(Y(x)W_{n+1}^{(g)}(x,x_1,...,x_n)\right. \\ &- \left.\hbar\partial_x W_{n+1}^{(g)}(x,x_1,...,x_n)\right) \end{aligned} \tag{C.4}$$

From the definition we have also

$$W_{n+1}^{(g)}(x_0,...,x_n)$$

$$= \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\sum_{h=0}^{g}\sum_{I\subset J} W^{(h)}_{|I|+1}(x,I)W^{(g-h)}_{n-|I|+1}(x,J/I)+\overline{W}^{(g-1)}_{n+2}(x,x,J)\right)$$

$(C.5)$

then we shift $W^{(g)}_n$ to $\overline{W}^{(g)}_n$ in the RHS, i.e.:

$$\begin{aligned}
& W^{(g)}_{n+1}(x_0,...,x_n)\\
= & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(\sum_{h=0}^{g}\sum_{I\subset J} \overline{W}^{(h)}_{|I|+1}(x,I)\overline{W}^{(g-h)}_{n-|I|+1}(x,J/I)+\overline{W}^{(g-1)}_{n+2}(x,x,J)\\
& +\sum_{j=1}^{n}\frac{\overline{W}^{(g)}_n(x,J/\{j\})}{(x-x_j)^2}\Big)\\
= & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(\sum_{h=0}^{g}\sum_{I\subset J} \overline{W}^{(h)}_{|I|+1}(x,I)\overline{W}^{(g-h)}_{n-|I|+1}(x,J/I)+\overline{W}^{(g-1)}_{n+2}(x,x,J)\\
& +\sum_{j=1}^{n}\partial_{x_j}\left(\frac{\overline{W}^{(g)}_n(x,J/\{j\})}{x-x_j}\right)\Big)\\
= & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(\sum_{h=0}^{g}\sum_{I\subset J} \overline{W}^{(h)}_{|I|+1}(x,I)\overline{W}^{(g-h)}_{n-|I|+1}(x,J/I)+\overline{W}^{(g-1)}_{n+2}(x,x,J)\\
& +\sum_{j=1}^{n}\partial_{x_j}\left(\frac{\overline{W}^{(g)}_n(x,J/\{j\})-\overline{W}^{(g)}_n(x_j,J/\{j\})}{x-x_j}\right)\Big)
\end{aligned}$$

$(C.6)$

in the last line we have added for free, the term $\overline{W}^{(g)}_n(x_j,J/\{j\})$ because it has no pole at $x=s_i$.

Therefore we have:

$$\begin{aligned}
0 = & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(-Y(x)W^{(g)}_{n+1}(x,x_1,...,x_n)+\hbar\partial_x W^{(g)}_{n+1}(x,x_1,...,x_n)\\
& +\sum_{h=0}^{g}\sum_{I\subset J} \overline{W}^{(h)}_{|I|+1}(x,I)\overline{W}^{(g-h)}_{n-|I|+1}(x,J/I)+\overline{W}^{(g-1)}_{n+2}(x,x,J)\\
& +\sum_{j=1}^{n}\partial_{x_j}\left(\frac{\overline{W}^{(g)}_n(x,J/\{j\})-\overline{W}^{(g)}_n(x_j,J/\{j\})}{x-x_j}\right)\Big)\\
= & \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)P^{(g)}_{n+1}(x;x_1,...,x_n)\\
= & \sum_i\sum_k K_{i,k}(x_0)\ \operatorname*{Res}_{x\to s_i}(x-s_i)^k P^{(g)}_{n+1}(x;x_1,...,x_n)
\end{aligned}$$

$(C.7)$

Notice that this equation holds for any x_0. Since $K_{i,k}(x_0)$ is a rational fraction with a pole of degree $k+1$ in $x_0=s_i$, the $K_{i,k}(x_0)$ are linearly independent functions, and thus we must have:

$$\forall k,i \qquad 0=\operatorname*{Res}_{x\to s_i}(x-s_i)^k\,P^{(g)}_{n+1}(x;x_1,...,x_n) \tag{C.8}$$

this means that $P^{(g)}_{n+1}$ has no pole at $x = s_i$.

One easily sees that $P^{(g)}_{n+1}(x; x_1, \dots, x_n)$ is a rational fraction of x, and its poles are at most those of $Y(x)$, i.e. at the poles of $V'(x)$. $\square$

D Appendix: Proof of theorem 3.3

Theorem 3.3 *Each $W^{(g)}_n$ is a symmetric function of all its arguments.*

proof:

The special case of $W^{(0)}_3$ is proved in appendix F above. It is obvious from the definition that $W^{(g)}_{n+1}(x_0, x_1, \dots, x_n)$ is symmetric in $x_1, x_2, \dots, x_n$, and therefore we need to show that (for $n \geq 1$):

$$W^{(g)}_{n+1}(x_0, x_1, J) - W^{(g)}_{n+1}(x_1, x_0, J) = 0 \tag{D.1}$$

where $J = \{x_2, \dots, x_n\}$. We prove it by recursion on $-\chi = 2g - 2 + n$.

Assume that every $W^{(h)}_k$ with $2h + k - 2 \leq 2g + n$ is symmetric. We have:

$$\begin{aligned} & W^{(g)}_{n+1}(x_0, x_1, J) \\ = \quad & \sum_i \operatorname*{Res}_{x \to s_i} K(x_0, x) \Big(W^{(g-1)}_{n+2}(x, x, x_1, J) + 2\ B(x, x_1) W^{(g)}_n(x, J) \\ & + 2 \sum_{h=0}^{g} {\sum_{I \in J}}' \ W^{(h)}_{2+|I|}(x, x_1, I) W^{(g-h)}_{n-|I|}(x, J/I) \Big) \end{aligned} \tag{D.2}$$

where $\sum'$ means that we exclude the terms $(I = \emptyset, h = 0)$ and $(I = J, h = g)$. Notice also that $\overline{W}^{(g-1)}_{n+2} = W^{(g-1)}_{n+2}$ because $n \geq 1$. Then, using the recursion hypothesis, we have:

$$\begin{aligned} & W^{(g)}_{n+1}(x_0, x_1, J) \\ = \quad & 2 \sum_i \operatorname*{Res}_{x \to s_i} K(x_0, x)\ B(x, x_1) W^{(g)}_n(x, J) \\ & + \sum_{i,j} \operatorname*{Res}_{x \to s_i} \operatorname*{Res}_{x' \to s_j} K(x_0, x) K(x_1, x') \Big(W^{(g-2)}_{n+3}(x, x, x', x', J) \\ & + 2 \sum_h {\sum_I}' W^{(h)}_{2+|I|}(x', x, I) W^{(g-1-h)}_{1+n-|I|}(x', x, J/I) \\ & + 2 \sum_h {\sum_I}' W^{(h)}_{3+|I|}(x', x, x, I) W^{(g-1-h)}_{n-|I|}(x', J/I) \\ & + 2 \sum_h {\sum_{I \in J}}' \ W^{(g-h)}_{n-|I|}(x, J/I) \Big[W^{(h-1)}_{3+|I|}(x, x', x', I) \end{aligned}$$

$$+2\sum_{h'}\sum_{I'\subset I}{}' W^{(h')}_{2+|I'|}(x',x,I')W^{(h-h')}_{1+|I|-|I'|}(x',I/I')\Big]\Big) \tag{D.3}$$

Now, if we compute $W^{(g)}_{n+1}(x_1,x_0,J)$, we get the same expression, with the order of integrations exchanged, i.e. we have to integrate x' before integrating x. Notice, by moving the integration contours, that:

$$\operatorname*{Res}_{x\to s_i}\operatorname*{Res}_{x'\to s_j} - \operatorname*{Res}_{x'\to s_j}\operatorname*{Res}_{x\to s_i} = -\delta_{i,j}\operatorname*{Res}_{x\to s_i}\operatorname*{Res}_{x'\to x} \tag{D.4}$$

Moreover, the only terms which have a pole at $x=x'$ are those containing $B(x,x')$. Therefore:

$$\begin{aligned}
& W^{(g)}_{n+1}(x_0,x_1,J) - W^{(g)}_{n+1}(x_1,x_0,J) \\
= \quad & 2\sum_i \operatorname*{Res}_{x\to s_i}\left(K(x_0,x)\ B(x,x_1) - K(x_1,x)\ B(x,x_0)\right)\ W^{(g)}_n(x,J) \\
& -2\sum_i \operatorname*{Res}_{x\to s_i}\operatorname*{Res}_{x'\to x}\ K(x_0,x)K(x_1,x')\ B(x,x')\ \Big(\\
& 2W^{(g-1)}_{1+n}(x',x,J) + 2\sum_h\sum_{I\in J}{}'\ W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x',I)\Big)
\end{aligned} \tag{D.5}$$

The residue $\operatorname{Res}_{x'\to x}$ can be computed:

$$\begin{aligned}
& W^{(g)}_{n+1}(x_0,x_1,J) - W^{(g)}_{n+1}(x_1,x_0,J) \\
= \quad & 2\sum_i \operatorname*{Res}_{x\to s_i}\left(K(x_0,x)\ B(x,x_1) - K(x_1,x)\ B(x,x_0)\right)\ W^{(g)}_n(x,J) \\
& -\sum_i \operatorname*{Res}_{x\to s_i}\ K(x_0,x)\frac{\partial}{\partial x'}\Big(K(x_1,x')\ \Big(\\
& 2W^{(g-1)}_{1+n}(x',x,J) + 2\sum_h\sum_{I\in J}{}'\ W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x',I)\Big)\ \Big)_{x'=x} \\
= \quad & 2\sum_i \operatorname*{Res}_{x\to s_i}\left(K(x_0,x)\ B(x,x_1) - K(x_1,x)\ B(x,x_0)\right)\ W^{(g)}_n(x,J) \\
& -\sum_i \operatorname*{Res}_{x\to s_i}\ K(x_0,x)K'(x_1,x)\ \Big(\\
& 2W^{(g-1)}_{1+n}(x,x,J) + 2\sum_h\sum_{I\in J}{}'\ W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x,I)\Big) \\
& -\sum_i \operatorname*{Res}_{x\to s_i}\ K(x_0,x)K(x_1,x)\frac{\partial}{\partial x'}\Big(\\
& 2W^{(g-1)}_{1+n}(x',x,J) + 2\sum_h\sum_{I\in J}{}'\ W^{(g-h)}_{n-|I|}(x,J/I)W^{(h)}_{1+|I|}(x',I)\ \Big)_{x'=x} \\
= \quad & 2\sum_i \operatorname*{Res}_{x\to s_i}\left(K(x_0,x)\ B(x,x_1) - K(x_1,x)\ B(x,x_0)\right)\ W^{(g)}_n(x,J)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)K'(x_1,x)\ \Big(\\
&2W_{1+n}^{(g-1)}(x,x,J)+2\sum_h {\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I)W_{1+|I|}^{(h)}(x,I)\Big)\\
&-\frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)K(x_1,x)\frac{\partial}{\partial x}\Big(\\
&2W_{1+n}^{(g-1)}(x,x,J)+2\sum_h {\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I)W_{1+|I|}^{(h)}(x,I)\ \Big)
\end{aligned}
\tag{D.6}
$$

The last term can be integrated by parts, and we get:

$$
\begin{aligned}
&W_{n+1}^{(g)}(x_0,x_1,J)-W_{n+1}^{(g)}(x_1,x_0,J)\\
=\quad & 2\sum_i \operatorname*{Res}_{x\to s_i}\left(K(x_0,x)\ B(x,x_1)-K(x_1,x)\ B(x,x_0)\right)W_n^{(g)}(x,J)\\
&+\frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i}\Big(K'(x_0,x)K(x_1,x)-K(x_0,x)K'(x_1,x)\Big)\ \Big(\\
&2W_{1+n}^{(g-1)}(x,x,J)+2\sum_h {\sum_{I\in J}}' W_{n-|I|}^{(g-h)}(x,J/I)W_{1+|I|}^{(h)}(x,I)\Big)
\end{aligned}
\tag{D.7}
$$

Then we use theorem 3.2:

$$
\begin{aligned}
&W_{n+1}^{(g)}(x_0,x_1,J)-W_{n+1}^{(g)}(x_1,x_0,J)\\
=\quad & 2\sum_i \operatorname*{Res}_{x\to s_i}\left(K(x_0,x)\ B(x,x_1)-K(x_1,x)\ B(x,x_0)\right)W_n^{(g)}(x,J)\\
&+\sum_i \operatorname*{Res}_{x\to s_i}\Big(K'(x_0,x)K(x_1,x)-K(x_0,x)K'(x_1,x)\Big)\ \Big(P_n^{(g)}(x,J)\\
&+(Y(x)-\hbar\partial_x)W_n^{(g)}(x,J)+\sum_j \partial_{x_j}\Big(\frac{W_{n-1}^{(g)}(x_j,J/\{x_j\})}{x-x_j}\Big)\Big)
\end{aligned}
\tag{D.8}
$$

Since $P_n^{(g)}(x,J)$ and $W_{n-1}^{(g)}(x_j,J/\{x_j\})$ have no poles at the s_i's, we have:

$$
\begin{aligned}
&W_{n+1}^{(g)}(x_0,x_1,J)-W_{n+1}^{(g)}(x_1,x_0,J)\\
=\quad & 2\sum_i \operatorname*{Res}_{x\to s_i}\left(K(x_0,x)\ B(x,x_1)-K(x_1,x)\ B(x,x_0)\right)W_n^{(g)}(x,J)\\
+\quad & \sum_i \operatorname*{Res}_{x\to s_i}\Big(K'(x_0,x)K(x_1,x)-K(x_0,x)K'(x_1,x)\Big)\\
&(Y(x)-\hbar\partial_x)W_n^{(g)}(x,J)
\end{aligned}
\tag{D.9}
$$

Notice that:

$$
K_0'K_1-K_0K_1'=-\frac{1}{\hbar}\left(G_0K_1-K_0G_1\right) \tag{D.10}
$$

and $B = -\frac{1}{2}\,G'$, therefore:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J) - W_{n+1}^{(g)}(x_1,x_0,J) \\ = \quad & -\sum_i \operatorname*{Res}_{x\to s_i} (K_0G_1' - K_1G_0')\; W_n^{(g)}(x,J) \\ & -\frac{1}{\hbar}\sum_i \operatorname*{Res}_{x\to s_i} \Big(G_0K_1 - K_0G_1\Big)\; (Y(x) - \hbar\partial_x)W_n^{(g)}(x,J) \end{aligned} \tag{D.11}$$

we integrate the first line by parts:

$$\begin{aligned} & W_{n+1}^{(g)}(x_0,x_1,J) - W_{n+1}^{(g)}(x_1,x_0,J) \\ = \quad & \sum_i \operatorname*{Res}_{x\to s_i} (K_0'G_1 - K_1'G_0)\; W_n^{(g)}(x,J) \\ & +\sum_i \operatorname*{Res}_{x\to s_i} (K_0G_1 - K_1G_0)\; W_n^{(g)}(x,J)' \\ & -\frac{1}{\hbar}\sum_i \operatorname*{Res}_{x\to s_i} \Big(G_0K_1 - K_0G_1\Big)\; (Y(x) - \hbar\partial_x)W_n^{(g)}(x,J) \end{aligned} \tag{D.12}$$

Notice that:

$$K_0'G_1 - G_0K_1' = -\frac{Y}{\hbar}\,(K_0G_1 - G_0K_1) \tag{D.13}$$

So we find

$$W_{n+1}^{(g)}(x_0,x_1,J) - W_{n+1}^{(g)}(x_1,x_0,J) = 0 \tag{D.14}$$

E Appendix: Proof of theorem 3.4

Theorem E.1 *The correlation functions $W_n^{(g)}$ are independent of the choice of kernel K, provided that K is solution of the equation eq. (2.2).*

proof:

Any two solutions of eq. (2.2), differ by a homogeneous solution, i.e. by $\psi^2(x)$. Therefore, what we have to prove is that the following quantity vanishes:

$$\sum_i \operatorname*{Res}_{x\to s_i} \psi^2(x)\left[W_{n+2}^{(g-1)}(x,x,J) + \sum_h {\sum_{I\subset J}}' W_{1+|I|}^{(h)}(x,I)W_{1+n-|I|}^{(g-h)}(x,J/I)\right] \tag{E.1}$$

Using theorem 3.2, we have:

$$\begin{aligned} & \operatorname*{Res}_{x\to s_i} \psi^2(x)\left[W_{n+2}^{(g-1)}(x,x,J) + \sum_h {\sum_{I\subset J}}' W_{1+|I|}^{(h)}(x,I)W_{1+n-|I|}^{(g-h)}(x,J/I)\right] \\ = \; & \operatorname*{Res}_{x\to s_i} \psi^2(x)\Big(Y(x)W_n^{(g)}(x,J) - \hbar\partial_x W_n^{(g)}(x,J) + P_n^{(g)}(x;J)\Big) \end{aligned} \tag{E.2}$$

Then we notice that $P_n^{(g)}$ gives no residue, and then we use $Y = -2\hbar\psi'/\psi$, and we integrate by parts:

$$\begin{aligned} &= -\hbar \operatorname*{Res}_{x\to s_i} \psi^2(x)\Big(2\,\frac{\psi'}{\psi}\,W_n^{(g)} + \partial_x W_n^{(g)}\Big) \\ &= -\hbar \operatorname*{Res}_{x\to s_i} \partial_x\left(\psi^2\ W_n^{(g)}\right) \\ &= 0 \end{aligned} \tag{E.3}$$

This means that adding to $K(x_0,x)$ a constant times $\psi^2(x)$ doesnot change the $W_n^{(g)}$'s. In fact we may chose a different constant near each s_i, or in other words, we may assume that

$$K_{i,2}(x_0) = 0 \tag{E.4}$$

□

F Appendix: Proof of theorem 3.5

Theorem 3.1 *The 3 point function $W_3^{(0)}$ is symmetric and we have:*

$$W_3^{(0)}(x_1,x_2,x_3) = 4\sum_i \operatorname*{Res}_{x\to s_i} \frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)} \tag{F.1}$$

proof:
The definition of $W_3^{(0)}$ is:

$$\begin{aligned} & W_3^{(0)}(x_0,x_1,x_2) \\ = \quad & 2\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)B(x,x_1)B(x,x_2) \\ = \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K_0\, G_1'\, G_2' \\ = \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K_0\,((\hbar K_1'' + YK_1' + Y'K_1)(\hbar K_2'' + YK_2' + Y'K_2)) \\ = \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K_0\,(\,\hbar^2 K_1''K_2'' + \hbar Y(K_1'K_2'' + K_1''K_2') + \hbar Y'(K_1''K_2 + K_2''K_1) \\ & + Y^2K_1'K_2' + YY'(K_1K_2' + K_1'K_2) + Y'^2K_1K_2\,) \end{aligned} \tag{F.2}$$

where we have written for short $K_i = K(x_i,x)$, $G_i = G(x_i,x)$, and derivative are w.r.t. x.

Since $K(x_i,x)$ has no pole when $x \to s_i$, the first term vanishes. Using the Ricatti equation $Y^2 = 2\hbar Y' + U$ (where U has no pole at s_i), we may replace Y^2 by $2\hbar Y'$ and YY' by $\hbar Y''$ without changing the residues, i.e.:

$$\begin{aligned}
& W_3^{(0)}(x_0,x_1,x_2) \\
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K_0\,(\hbar Y(K_1'K_2''+K_1''K_2') + \hbar Y'(K_1''K_2+K_2''K_1) \\
& +2\hbar Y'K_1'K_2' + \hbar Y''(K_1K_2'+K_1'K_2) + Y'^2K_1K_2\,) \\
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} K_0\,(\hbar Y(K_1'K_2')' + \hbar Y'(K_1K_2)'' + \hbar Y''(K_1K_2)' + Y'^2K_1K_2\,) \\
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} Y'^2K_0K_1K_2 + \hbar(Y''K_0(K_1K_2)' - (YK_0)'K_1'K_2' - (Y'K_0)'(K_1K_2)') \\
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} Y'^2K_0K_1K_2 - \hbar((YK_0)'K_1'K_2' + Y'K_0'(K_1K_2)') \\
= \quad & \frac{1}{2}\sum_i \operatorname*{Res}_{x\to s_i} Y'^2K_0K_1K_2 - \hbar YK_0'K_1'K_2' - \hbar Y'(K_0K_1'K_2' + K_0'K_1K_2' + K_0'K_1'K_2)
\end{aligned}$$

$(F.3)$

This expression is clearly symmetric in x_0,x_1,x_2 as claimed in theorem 3.3.

Let us give an alternative expression, in the form of the Verlinde or Krichever formula [15]:

$$W_3^{(0)}(x_1,x_2,x_3) = 4\sum_i \operatorname*{Res}_{x\to s_i} \frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)} \tag{F.4}$$

proof:

In order to prove formula F.4, compute:

$$B(x,x_i) = -\frac{1}{2}G'(x,x_i) = -\frac{1}{2}G_i' = \frac{1}{2}(\hbar K_i'' + YK_i' + Y'K_i) \tag{F.5}$$

thus:

$$\begin{aligned}
& \sum_i \operatorname*{Res}_{x\to s_i} \frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)} \\
= \quad & \frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i} \frac{1}{Y'(x)}\,(\hbar K_0'' + YK_0' + Y'K_0)(\hbar K_1'' + YK_1' + Y'K_1) \\
& \qquad (\hbar K_2'' + YK_2' + Y'K_2) \\
= \quad & \frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i} \frac{\hbar^3}{Y'}K_0''K_1''K_2'' + \hbar^2\frac{Y}{Y'}(K_0'K_1''K_2'' + K_0''K_1'K_2'' + K_0''K_1''K_2') \\
& +\hbar^2(K_0K_1''K_2'' + K_0''K_1K_2'' + K_0''K_1''K_2) \\
& +\hbar\frac{Y^2}{Y'}(K_0''K_1'K_2' + K_0'K_1''K_2' + K_0'K_1'K_2'') \\
& +\hbar Y(K_0K_1'K_2'' + K_0K_1''K_2' + K_0'K_1K_2'' + K_0'K_1''K_2 + K_0''K_1K_2' + K_0''K_1'K_2)
\end{aligned}$$

$$
\begin{aligned}
&+\hbar Y'(K_0''K_1K_2+K_0K_1''K_2+K_0K_1K_2'')+\frac{Y^3}{Y'}K_0'K_1'K_2'\\
&+Y^2(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)\\
&+YY'(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2')+Y'^2K_0K_1K_2
\end{aligned}
\tag{F.6}
$$

Notice that K_i has no pole at the s_i's, and $1/Y'$ has no pole, Y/Y' has no pole, Y^2/Y' has no pole, thus:

$$
\begin{aligned}
&\sum_i \operatorname*{Res}_{x\to s_i}\frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)}\\
=\;&\frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i}\hbar Y(K_0K_1'K_2''+K_0K_1''K_2'+K_0'K_1K_2''+K_0'K_1''K_2+K_0''K_1K_2'\\
&+K_0''K_1'K_2)+\hbar Y'(K_0''K_1K_2+K_0K_1''K_2+K_0K_1K_2'')+\frac{Y^3}{Y'}K_0'K_1'K_2'\\
&+Y^2(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)\\
&+YY'(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2')+Y'^2K_0K_1K_2
\end{aligned}
\tag{F.7}
$$

Notice that $Y^2=2\hbar Y'+U$, thus we may replace Y^3/Y' by $2\hbar Y$, and Y^2 by $2\hbar Y'$ and YY' by $\hbar Y''$, thus:

$$
\begin{aligned}
&\sum_i \operatorname*{Res}_{x\to s_i}\frac{B(x,x_1)B(x,x_2)B(x,x_3)}{Y'(x)}\\
=\;&\frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i}\hbar Y(K_0K_1'K_2''+K_0K_1''K_2'+K_0'K_1K_2''+K_0'K_1''K_2+K_0''K_1K_2'\\
&+K_0''K_1'K_2)+\hbar Y'(K_0''K_1K_2+K_0K_1''K_2+K_0K_1K_2'')+2\hbar Y K_0'K_1'K_2'\\
&+2\hbar Y'(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)+\hbar Y''(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2')\\
&+Y'^2K_0K_1K_2\\
=\;&\frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i}\hbar Y(K_0(K_1'K_2')'+K_1(K_0'K_2')'+K_2(K_0'K_1')')\\
&+2\hbar Y K_0'K_1'K_2'+Y'^2K_0K_1K_2+\hbar(Y'(K_0'K_1K_2+K_0K_1'K_2+K_0K_1K_2'))'\\
=\;&\frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i}\hbar Y(K_0(K_1'K_2')'+K_1(K_0'K_2')'+K_2(K_0'K_1')')\\
&+2\hbar Y K_0'K_1'K_2'+Y'^2K_0K_1K_2\\
=\;&-\frac{1}{8}\sum_i \operatorname*{Res}_{x\to s_i}3\hbar Y K_0'K_1'K_2'+\hbar Y'(K_0K_1'K_2'+K_0'K_1K_2'+K_0'K_1'K_2)\\
&-2\hbar Y K_0'K_1'K_2'-Y'^2K_0K_1K_2\\
=\;&\frac{1}{4}W_3^{(0)}(x_0,x_1,x_2)
\end{aligned}
\tag{F.8}
$$

F.1 Direct computation

We write

$$W_3^{(0)}(z_1,z_2,z_3)$$

$$
\begin{aligned}
= \quad & 2\sum_i \operatorname*{Res}_{z\to s_i} K(z_1,z)B(z_2,z)B(z_3,z) \\
= \quad & \sum_j\sum_i \frac{A_{i,j}}{(z_2-s_j)^2}\operatorname*{Res}_{z\to s_i} K(z_1,z)\frac{1}{(z-s_i)^2(z_3-z)^2} \ + \text{sym.} \\
& +2\sum_i\sum_{i'\neq i}\sum_{j,k}\frac{A_{i,j}A_{i',k}}{(z_2-s_j)^2(z_3-s_k)^2}\operatorname*{Res}_{z\to s_i} K(z_1,z)\frac{1}{(z-s_i)^2(z-s_{i'})^2} \ + \text{sym.} \\
& +2\sum_i\sum_{j,k}\frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2}\operatorname*{Res}_{z\to s_i} K(z_1,z)\frac{1}{(z-s_i)^4} \\
= \quad & \sum_j\sum_i \frac{A_{i,j}}{(z_2-s_j)^2}\Big(\frac{K_{i,1}(z_1)}{(z_3-s_i)^2}+\frac{2K_{i,0}(z_1)}{(z_3-s_i)^3}\Big) \ + \text{sym.} \\
& +2\sum_i\sum_{i'\neq i}\sum_{j,k}\frac{A_{i,j}A_{i',k}}{(z_2-s_j)^2(z_3-s_k)^2}\Big(\frac{K_{i,1}(z_1)}{(s_{i'}-s_i)^2}+\frac{2K_{i,0}(z_1)}{(s_{i'}-s_i)^3}\Big) \ + \text{sym.} \\
& +2\sum_i\sum_{j,k}\frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2}K_{i,3}(z_1) \\
= \quad & \sum_j\sum_i \frac{A_{i,j}}{(z_2-s_j)^2}\Big(\frac{K_{i,1}(z_1)}{(z_3-s_i)^2}+\frac{2K_{i,0}(z_1)}{(z_3-s_i)^3}\Big) \ + \text{sym.} \\
& +2\sum_i\sum_{i'\neq i}\sum_{j,k}\frac{A_{i,j}A_{i',k}}{(z_2-s_j)^2(z_3-s_k)^2}\Big(\frac{K_{i,1}(z_1)}{(s_{i'}-s_i)^2}+\frac{2K_{i,0}(z_1)}{(s_{i'}-s_i)^3}\Big) \ + \text{sym.} \\
& -2\sum_i\sum_{j,k}\frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2}T_{i,i}\,K_{i,1}(z_1) \\
& -2\sum_i\sum_{j,k}\frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2}\Big(\frac{V'''(s_i)}{2\hbar}+2\sum_{i'\neq i}\frac{1}{(s_{i'}-s_i)^3}\Big)K_{i,0}(z_1) \\
& +\frac{2}{\hbar}\sum_i\sum_{j,k}\frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2(z_1-s_i)^3} \\
& +\frac{4}{\hbar}\sum_i\sum_{i'\neq i}\sum_l\sum_{j,k}\frac{A_{i,j}A_{i,k}A_{i',l}}{(z_2-s_j)^2(z_3-s_k)^2(s_{i'}-s_i)^3(z_1-s_l)^2} \\
= \quad & \frac{2}{\hbar}\sum_{i,j,k}\frac{A_{i,j}A_{i,k}}{(z_1-s_i)^3(z_2-s_j)^2(z_3-s_k)^2}+\frac{A_{j,i}A_{j,k}}{(z_1-s_i)^2(z_2-s_j)^3(z_3-s_k)^2} \\
& +\frac{A_{k,i}A_{k,j}}{(z_1-s_i)^2(z_2-s_j)^2(z_3-s_k)^3} \\
& +\sum_{i,j,k}\frac{K_{i,1}(z_1)}{(z_2-s_j)^2(z_3-s_k)^2}\Big(A_{j,k}\delta_{i,j}+A_{j,k}\delta_{i,k}-A_{i,j}\sum_{i'}T_{i,i'}A_{i',k} \\
& -A_{i,k}\sum_{i'}T_{i,i'}A_{i',j}\Big) \\
& +2\sum_i\sum_{i'\neq i}\sum_{j,k}\frac{A_{i,j}A_{i',k}}{(z_2-s_j)^2(z_3-s_k)^2}\frac{2K_{i,0}(z_1)}{(s_{i'}-s_i)^3} \ + \text{sym.} \\
& -2\sum_i\sum_{j,k}\frac{A_{i,j}A_{i,k}}{(z_2-s_j)^2(z_3-s_k)^2}\Big(\frac{V'''(s_i)}{2\hbar}+2\sum_{i'\neq i}\frac{1}{(s_{i'}-s_i)^3}\Big)K_{i,0}(z_1)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4}{\hbar}\sum_i\sum_{i'\neq i}\sum_l\sum_{j,k}\frac{A_{i,j}A_{i,k}A_{i',l}}{(z_2-s_j)^2(z_3-s_k)^2(s_{i'}-s_i)^3(z_1-s_l)^2}\\
= \quad &\frac{2}{\hbar}\sum_{l,j,k}\frac{1}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2}\sum_i\Big(\frac{\delta_{i,l}A_{i,j}A_{i,k}}{(z_1-s_i)}+\frac{\delta_{i,j}A_{i,l}A_{i,k}}{(z_2-s_i)}\\
&+\frac{\delta_{i,k}A_{i,l}A_{i,j}}{(z_3-s_i)}\Big)\\
&+\frac{4}{\hbar}\sum_{l,j,k}\sum_i\sum_{i'\neq i}\frac{A_{i,j}A_{i,k}A_{i',l}+A_{i,j}A_{i',k}A_{i,l}+A_{i,k}A_{i',j}A_{i,l}-A_{i,j}A_{i,k}A_{i,l}}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2(s_{i'}-s_i)^3}\\
&-\frac{1}{\hbar^2}\sum_{l,j,k}\sum_i\frac{A_{i,j}A_{i,k}A_{i,l}V'''(s_i)}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2}
\end{aligned}
\tag{F.9}
$$

Thus we have:

$$
\begin{aligned}
&W_3^{(0)}(z_1,z_2,z_3)\\
= \quad &\frac{2}{\hbar}\sum_{i,j,k,l}\frac{\frac{\delta_{i,l}A_{i,j}A_{i,k}}{(z_1-s_i)}+\frac{\delta_{i,j}A_{i,l}A_{i,k}}{(z_2-s_i)}+\frac{\delta_{i,k}A_{i,l}A_{i,j}}{(z_3-s_i)}}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2}\\
&+\frac{4}{\hbar}\sum_{l,j,k}\sum_i\sum_{i'\neq i}\frac{A_{i,j}A_{i,k}A_{i',l}+A_{i,j}A_{i',k}A_{i,l}+A_{i,k}A_{i',j}A_{i,l}-A_{i,j}A_{i,k}A_{i,l}}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2(s_{i'}-s_i)^3}\\
&-\frac{1}{\hbar^2}\sum_{l,j,k}\sum_i\frac{A_{i,j}A_{i,k}A_{i,l}V'''(s_i)}{(z_1-s_l)^2(z_2-s_j)^2(z_3-s_k)^2}
\end{aligned}
\tag{F.10}
$$

G Appendix: Proof of theorem 3.6

Theorem 3.6 *Under an infinitesimal variation of the potential $V \to V+\delta V$, we have:*

$$
\forall n\geq 0, g\geq 0\ , \quad \delta W_n^{(g)}(x_1,\dots,x_n) = -\sum_i \operatorname*{Res}_{x\to s_i} W_{n+1}^{(g)}(x,x_1,\dots,x_n)\,\delta V(x) \tag{G.1}
$$

G.1 Variation of ω

We have:

$$
\omega(x) = \hbar\sum_i\frac{1}{x-s_i} \tag{G.2}
$$

and

$$
V'(s_i) = 2\hbar\sum_{j\neq i}\frac{1}{s_i-s_j} \tag{G.3}
$$

Thus taking a variation we have:

$$\delta V'(s_i) + \delta s_i V''(s_i) = -2\hbar \sum_{j\neq i} \frac{\delta s_i - \delta s_j}{(s_i - s_j)^2} \tag{G.4}$$

i.e.

$$\delta V'(s_i) = -\hbar \sum_j T_{i,j} \delta s_j \tag{G.5}$$

which implies:

$$\delta s_i = -\frac{1}{\hbar} \sum_j A_{i,j} \delta V'(s_j) \tag{G.6}$$

and therefore:

$$\delta\omega(x) = -\sum_{i,j} \frac{A_{i,j}\delta V'(s_j)}{(x-s_i)^2} \tag{G.7}$$

which can also be written:

$$\begin{aligned}
\delta\omega(x) &= -\sum_k \operatorname*{Res}_{x'\to s_k} \sum_{i,j} \frac{A_{i,j}}{(x-s_i)^2\,(x'-s_j)}\ \delta V'(x') \\
&= -\sum_k \operatorname*{Res}_{x'\to s_k} \sum_{i,j} \frac{A_{i,j}}{(x-s_i)^2\,(x'-s_j)^2}\ \delta V(x') \\
&= -\sum_k \operatorname*{Res}_{x'\to s_k} B(x,x')\ \delta V(x')
\end{aligned} \tag{G.8}$$

and finally we obtain the case $n = 1, g = 0$ of the theorem:

$$\boxed{\delta\omega(x) = -\sum_k \operatorname*{Res}_{x'\to s_k} B(x,x')\ \delta V(x')} \tag{G.9}$$

G.2 Variation of B

Consider:

$$\overline{W}_2^{(0)}(x,x') = B(x,x') - \frac{1}{2}\frac{1}{(x-x')^2} = \sum_{i,j} \frac{A_{i,j}}{(x-s_j)^2(x'-s_i)^2} \tag{G.10}$$

Due to eq. (2.6) we have:

$$\begin{aligned}
\overline{W}_2^{(0)}(x,x') &= \sum_i \frac{\hbar K(x,s_i)}{(x'-s_i)^2} \\
&= \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \frac{\omega(z)}{(z-x')^2} \\
&= \frac{\partial}{\partial x'} \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \frac{\omega(z)-\omega(x')}{z-x'}
\end{aligned}$$

$$(G.11)$$

On the other hand, since $\overline{W}_2^{(0)}(x,x')$ has poles only at the s_i's we have:

$$\begin{aligned}
\overline{W}_2^{(0)}(x,x') &= \operatorname*{Res}_{z\to x} G(x,z)\ \overline{W}_2^{(0)}(z,x') \\
&= -\sum_i \operatorname*{Res}_{z\to s_i} G(x,z)\ \overline{W}_2^{(0)}(z,x') \\
&= -\sum_i \operatorname*{Res}_{z\to s_i} ((2\omega(z)-V'(z)+\hbar\partial_z)K(x,z))\ \overline{W}_2^{(0)}(z,x') \\
&= -\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \left((2\omega(z)-V'(z)-\hbar\partial_z)\ \overline{W}_2^{(0)}(z,x')\right)
\end{aligned}$$

$$(G.12)$$

This implies that $\forall x$:

$$0 = -\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \left((2\omega(z)-V'(z)-\hbar\partial_z)\ \overline{W}_2^{(0)}(z,x') + \frac{\partial}{\partial x'}\frac{\omega(z)-\omega(x')}{z-x'}\right) \tag{G.13}$$

and therefore, $\overline{W}_2^{(0)}(x,x')$ satisfies the loop equation:

$$(2\omega(x)-V'(x)-\hbar\partial_x)\ \overline{W}_2^{(0)}(x,x') + \frac{\partial}{\partial x'}\frac{\omega(x)-\omega(x')}{x-x'} = -P_2^{(0)}(x,x') \tag{G.14}$$

where $P_2^{(0)}(x,x')$ has no pole at $x\to s_i$'s.

Then we take the variation:

$$\begin{aligned}
(2\omega(x)-V'(x)-\hbar\partial_x)\ \delta\overline{W}_2^{(0)}(x,x') &= -(2\delta\omega(x)-\delta V'(x))\ \overline{W}_2^{(0)}(x,x') \\
&\quad -\frac{\partial}{\partial x'}\frac{\delta\omega(x)-\delta\omega(x')}{x-x'} - \delta P_2^{(0)}(x,x')
\end{aligned}$$

$$(G.15)$$

$\delta\overline{W}_2^{(0)}(x,x')$ is a rational fraction of x, with poles only at the s_i's, and $\delta P_2^{(0)}(x,x')$ has no pole at $x\to s_i$'s. We thus write:

$$\begin{aligned}
\delta W_2^{(0)}(x,x') &= \delta\overline{W}_2^{(0)}(x,x') \\
&= \operatorname*{Res}_{z\to x} G(x,z)\ \delta\overline{W}_2^{(0)}(z,x') \\
&= -\sum_i \operatorname*{Res}_{z\to s_i} G(x,z)\ \overline{W}_2^{(0)}(z,x') \\
&= -\sum_i \operatorname*{Res}_{z\to s_i} ((2\omega(z)-V'(z)+\hbar\partial_z)K(x,z))\ \delta\overline{W}_2^{(0)}(z,x') \\
&= -\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \left((2\omega(z)-V'(z)-\hbar\partial_z)\ \delta\overline{W}_2^{(0)}(z,x')\right) \\
&= \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \left((2\delta\omega(z)-\delta V'(z))\ \overline{W}_2^{(0)}(z,x')\right.
\end{aligned}$$

$$
\begin{aligned}
& + \frac{\partial}{\partial x'} \frac{\delta\omega(z) - \delta\omega(x')}{z - x'} + \delta P_2^{(0)}(z, x') \Big) \\
= & \sum_i \operatorname*{Res}_{z \to s_i} K(x, z) \left((2\delta\omega(z) - \delta V'(z)) \, \overline{W}_2^{(0)}(z, x') + \frac{\delta\omega(z)}{(z - x')^2} \right) \\
= & \sum_i \operatorname*{Res}_{z \to s_i} K(x, z) \, (2\delta\omega(z) - \delta V'(z)) \, B(z, x')
\end{aligned}
\tag{G.16}
$$

Then, we use eq. (G.9), and we get:

$$
\begin{aligned}
\delta W_2^{(0)}(x, x') = & -2 \sum_i \operatorname*{Res}_{z \to s_i} \sum_k \operatorname*{Res}_{x'' \to s_k} K(x, z) \, B(z, x'') \delta V(x'') \, B(z, x') \\
& - \sum_i \operatorname*{Res}_{z \to s_i} K(x, z) \, \delta V'(z) \, B(z, x') \\
= & - \sum_i \operatorname*{Res}_{z \to s_i} \sum_k \operatorname*{Res}_{x'' \to s_k} K(x, z) \, G(z, x'') \delta V'(x'') \, B(z, x') \\
& - \sum_i \operatorname*{Res}_{z \to s_i} \operatorname*{Res}_{x'' \to z} K(x, z) \, G(z, x'') \delta V'(x'') \, B(z, x') \\
= & - \sum_k \operatorname*{Res}_{x'' \to s_k} \sum_i \operatorname*{Res}_{z \to s_i} K(x, z) \, G(z, x'') \delta V'(x'') \, B(z, x') \\
= & -2 \sum_k \operatorname*{Res}_{x'' \to s_k} \sum_i \operatorname*{Res}_{z \to s_i} K(x, z) \, B(z, x'') \delta V(x'') \, B(z, x')
\end{aligned}
\tag{G.17}
$$

We thus obtain the case $n = 2, g = 0$ of the theorem:

$$
\boxed{\delta W_2^{(0)}(x, x') = - \sum_k \operatorname*{Res}_{x'' \to s_k} W_3^{(0)}(x, x', x'') \, \delta V(x'')}
\tag{G.18}
$$

G.3 Variation of other higher correlators

We prove by recursion on $2g + n$, that:

$$
\boxed{\delta W_{n+1}^{(g)}(x, L) = - \sum_k \operatorname*{Res}_{x'' \to s_k} \delta V(x'') \, W_{n+2}^{(g)}(z, L, x'')}
\tag{G.19}
$$

where $L = \{x_1, \dots, x_n\}$.

We write:

$$
U_{n+1}^{(g)}(z, L) = \overline{W}_{n+2}^{(g-1)}(z, z, L) + \sum_h {\sum_{J \subset L}}' W_{1+|J|}^{(h)}(z, J) \, W_{1+n-|J|}^{(g-h)}(z, L/J)
\tag{G.20}
$$

By definition we have:

$$
W_{n+1}^{(g)}(x, L) = \sum_i \operatorname*{Res}_{z \to s_i} K(x, z) \, U_{n+1}^{(g)}(z, L)
\tag{G.21}
$$

From the recursion hypothesis, we have:

$$\begin{aligned}\delta U^{(g)}_{n+1}(z,L) &= -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ \Big(W^{(g-1)}_{n+3}(z,z,L,x'') \\ &\quad -2\sum_h {\sum_{J\subset L}}' W^{(h)}_{2+|J|}(z,J,x'')\, W^{(g-h)}_{1+n-|J|}(z,L/J)\Big) \\ &= -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ \Big(U^{(g)}_{n+2}(z,L,x'') - 2B(z,x'')W^{(g)}_{n+1}(z,L)\Big)\end{aligned}$$

$(G.22)$

Thus:

$$\begin{aligned}&\delta W^{(g)}_{n+1}(x,L) \\ =\ & \sum_i \operatorname*{Res}_{z\to s_i} \delta K(x,z)\ U^{(g)}_{n+1}(z,L) - \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ \Big(\\ & U^{(g)}_{n+2}(z,L,x'') - 2B(z,x'')W^{(g)}_{n+1}(z,L)\Big) \\ =\ & \sum_i \operatorname*{Res}_{z\to s_i} \delta K(x,z)\ U^{(g)}_{n+1}(z,L) - \sum_k \operatorname*{Res}_{x''\to s_k} \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \delta V(x'')\ \Big(\\ & U^{(g)}_{n+2}(z,L,x'') - 2B(z,x'')W^{(g)}_{n+1}(z,L)\Big) \\ =\ & \sum_i \operatorname*{Res}_{z\to s_i} \delta K(x,z)\ U^{(g)}_{n+1}(z,L) \\ & +2\sum_k \operatorname*{Res}_{x''\to s_k} \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \delta V(x'')\ B(z,x'')W^{(g)}_{n+1}(z,L) \\ & -\sum_k \operatorname*{Res}_{x''\to s_k} \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \delta V(x'')\ U^{(g)}_{n+2}(z,L,x'') \\ =\ & \sum_i \operatorname*{Res}_{z\to s_i} \delta K(x,z)\ U^{(g)}_{n+1}(z,L) \\ & +2\sum_i \operatorname*{Res}_{z\to s_i} \sum_k \operatorname*{Res}_{x''\to s_k} K(x,z)\ \delta V(x'')\ B(z,x'')W^{(g)}_{n+1}(z,L) \\ & +2\sum_i \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to z} K(x,z)\ \delta V(x'')\ B(z,x'')W^{(g)}_{n+1}(z,L) \\ & -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ W^{(g)}_{n+2}(z,L,x'')\end{aligned}$$

$(G.23)$

We use the loop equation of theorem 3.2, which says that $U^{(g)}_{n+1}(z,L)+(2\omega(z)-V'(z)+\hbar\partial_z)W^{(g)}_{n+1}(z,L)$ has no pole at $z\to s_i$, and thus:

$$\begin{aligned}&\delta W^{(g)}_{n+1}(x,L) \\ =\ & -\sum_i \operatorname*{Res}_{z\to s_i} \delta K(x,z)\ (2\omega(z)-V'(z)+\hbar\partial_z)W^{(g)}_{n+1}(z,L) \\ & +2\sum_i \operatorname*{Res}_{z\to s_i} \sum_k \operatorname*{Res}_{x''\to s_k} K(x,z)\ \delta V(x'')\ B(z,x'')W^{(g)}_{n+1}(z,L) \\ & +2\sum_i \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to z} K(x,z)\ \delta V(x'')\ B(z,x'')W^{(g)}_{n+1}(z,L)\end{aligned}$$

$$
\begin{aligned}
& -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ W_{n+2}^{(g)}(z,L,x'') \\
= \quad & -\sum_i \operatorname*{Res}_{z\to s_i} W_{n+1}^{(g)}(z,L)\ (2\omega(z)-V'(z)-\hbar\partial_z)\delta K(x,z) \\
& +2\sum_i \operatorname*{Res}_{z\to s_i} \sum_k \operatorname*{Res}_{x''\to s_k} K(x,z)\ \delta V(x'')\ B(z,x'')W_{n+1}^{(g)}(z,L) \\
& +2\sum_i \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to z} K(x,z)\ \delta V(x'')\ B(z,x'')W_{n+1}^{(g)}(z,L) \\
& -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ W_{n+2}^{(g)}(z,L,x'')
\end{aligned}
\tag{G.24}
$$

and we have:

$$
(2\omega(z)-V'(z)-\hbar\partial_z)\delta K(x,z) = \delta G(x,z) - (2\delta\omega(z)-\delta V'(z))K(x,z) \tag{G.25}
$$

$$
\begin{aligned}
& \delta W_{n+1}^{(g)}(x,L) \\
= \quad & -\sum_i \operatorname*{Res}_{z\to s_i} W_{n+1}^{(g)}(z,L)\ \delta G(x,z) \\
& +\sum_i \operatorname*{Res}_{z\to s_i} W_{n+1}^{(g)}(z,L)\ (2\delta\omega(z)-\delta V'(z))\ K(x,z) \\
& +2\sum_i \operatorname*{Res}_{z\to s_i} \sum_k \operatorname*{Res}_{x''\to s_k} K(x,z)\ \delta V(x'')\ B(z,x'')W_{n+1}^{(g)}(z,L) \\
& +\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \delta V'(z)\ W_{n+1}^{(g)}(z,L) \\
& -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ W_{n+2}^{(g)}(z,L,x'')
\end{aligned}
\tag{G.26}
$$

We have:

$$
\sum_i \operatorname*{Res}_{z\to s_i} W_{n+1}^{(g)}(z,L)\ \delta G(x,z) = 0 \tag{G.27}
$$

because the integrand is a rational fraction, and we have taken the sum of residues at all poles.

Using eq. (G.9), we are thus left with:

$$
\delta W_{n+1}^{(g)}(x,L) = -\sum_k \operatorname*{Res}_{x''\to s_k} \delta V(x'')\ W_{n+2}^{(g)}(z,L,x'') \tag{G.28}
$$

which proves the recursion hypothesis for $2g+n+1$. QED.

H Appendix: Proof of theorem 3.7

Theorem 3.7

For $k = 0, 1$, $W_n^{(g)}$ satify the equation:

$$\begin{aligned} & \Big(-\sum_{i=1}^{n} x_i^k \frac{\partial}{\partial x_i} \Big) W_n^{(g)}(x_1, \dots, x_n) \\ = & \sum_i \operatorname*{Res}_{x_{n+1} \to s_i} x_{n+1}^k \, V'(x_{n+1}) \; W_{n+1}^{(g)}(x_1, \dots, x_n, x_{n+1}) \end{aligned} \tag{H.1}$$

proof:

Since $W_{n+1}^{(g)}$ has poles only at the s_i's we have (with as usual $J = \{x_1, \dots, x_n\}$):

$$\begin{aligned} & \sum_i \operatorname*{Res}_{x \to s_i} x^k \, V'(x) \; W_{n+1}^{(g)}(J, x) \\ = & \sum_i \operatorname*{Res}_{x \to s_i} x^k \, Y(x) \; W_{n+1}^{(g)}(J, x) \end{aligned} \tag{H.2}$$

Then using theorem 3.2, we have:

$$\begin{aligned} & \sum_i \operatorname*{Res}_{x \to s_i} x^k \, V'(x) \; W_{n+1}^{(g)}(J, x) \\ = & \sum_i \operatorname*{Res}_{x \to s_i} x^k \, Y(x) \; W_{n+1}^{(g)}(J, x) \\ = & \sum_i \operatorname*{Res}_{x \to s_i} x^k \Big[\hbar \partial_x W_{n+1}^{(g)}(J, x) + U_{n+1}^{(g)}(x, J) - P_{n+1}^{(g)}(x; J) - \sum_{j=1}^{n} \partial_{x_j} \frac{W_n^{(g)}(J)}{x - x_j} \Big] \\ = & \sum_i \operatorname*{Res}_{x \to s_i} x^k \Big[\hbar \partial_x W_{n+1}^{(g)}(J, x) + U_{n+1}^{(g)}(x, J) \Big] \end{aligned} \tag{H.3}$$

Notice that if $n \geq 1$, $W_{n+1}^{(g)}(J, x)$ behaves like $O(1/x^2)$ at $x \to \infty$, and thus, if $k \leq 1$, $x^k \, \partial_x W_{n+1}^{(g)}(J, x)$ behaves like $O(1/x^2)$. Since we take the residues at all poles, the sum of residues vanish and thus:

$$\begin{aligned} & \sum_i \operatorname*{Res}_{x \to s_i} x^k \, V'(x) \; W_{n+1}^{(g)}(J, x) \\ = & \sum_i \operatorname*{Res}_{x \to s_i} x^k \, U_{n+1}^{(g)}(x, J) \end{aligned} \tag{H.4}$$

Notice that $U_{n+1}^{(g)}(x, J)$ (defined in eq. (G.20)), behaves at most like $O(1/x^3)$ for large x, and thus, if $k \leq 1$, the product $x^k \, U_{n+1}^{(g)}(x, J)$ is a rational fraction, which behaves like $O(1/x^2)$ for large x. Its only poles can be at $x = s_i$ or at $x = x_j$. Therefore the sum of residues at s_i's, can be replaced by the sum of residues at x_j's:

$$\sum_i \operatorname*{Res}_{x \to s_i} x^k \, V'(x) \; W_{n+1}^{(g)}(J, x)$$

$$= \quad -\sum_{j=1}^{n} \operatorname*{Res}_{x\to x_j} x^k\, U_{n+1}^{(g)}(x,J)$$

$$(H.5)$$

The only terms in $U_{n+1}^{(g)}(x,J)$ which have poles at $x=x_j$, are the terms containing a $B(x,x_j)$, i.e.:

$$\begin{aligned}\sum_i \operatorname*{Res}_{x\to s_i} x^k\, V'(x)\; W_{n+1}^{(g)}(J,x) &= -2\sum_{j=1}^{n} \operatorname*{Res}_{x\to x_j} x^k\, B(x,x_j)\, W_n^{(g)}(x, J/\{x_j\}) \\ &= -\sum_{j=1}^{n} \operatorname*{Res}_{x\to x_j} x^k\, \frac{1}{(x-x_j)^2}\, W_n^{(g)}(x, J/\{x_j\}) \\ &= -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(x_j^k\; W_n^{(g)}(x_1,\dots,x_n)\right)\end{aligned}$$

$$(H.6)$$

□

I Appendix: Proof of theorem 3.8

Theorem 3.8:

For $n\geq 1$, $W_n^{(g)}$ satify the equation:

$$(2-2g-n-\hbar\frac{\partial}{\partial\hbar})\,\overline{W}_n^{(g)}(x_1,\dots,x_n) = -\sum_i \operatorname*{Res}_{x_{n+1}\to s_i} V(x_{n+1})\;\overline{W}_{n+1}^{(g)}(x_1,\dots,x_n,x_{n+1}) \tag{I.1}$$

I.1 $\hbar$ derivatives for $w(z)$

We have:

$$V'(s_i) = 2\hbar \sum_{\neq i} \frac{1}{s_i - s_j}$$

Taking the derivative with respect to $\hbar$ gives:

$$\hbar V''(s_i)\partial_\hbar s_i = V'(s_i) - 2\hbar^2 \sum_{j\neq i} \frac{\partial_\hbar si - \partial_\hbar s_j}{(s_i-s_j)^2}$$

and so

$$V'(s_i) = \hbar\left(V''(s_i)\partial_\hbar s_i + 2\hbar \sum_{j\neq i} \frac{\partial_\hbar si - \partial_\hbar s_j}{(s_i-s_j)^2}\right)$$

We recognize the general term of the matrix T and find:

$$V'(s_i) = \hbar^2 \sum_j T_{i,j}\partial_\hbar s_j$$

Multiplying by the matrix A gives:

$$\boxed{\hbar^2 \partial_\hbar s_i = \sum_j A_{i,j} V'(s_j)} \tag{I.2}$$

We can use this result to compute:

$$\begin{aligned}
\hbar\partial_\hbar \omega(x) &= \omega(x) + \hbar^2 \sum_i \frac{\partial_\hbar s i}{(x-s_i)^2} \\
&= \omega(x) + \sum_{i,j} \frac{A_{i,j} V'(s_j)}{(x-s_i)^2} \\
&= \omega(x) + \sum_k \operatorname*{Res}_{x' \to s_k} \sum_{i,j} \frac{A_{i,j} V'(x')}{(x-s_i)^2 (x'-s_j)} \\
&= \omega(x) + \sum_k \operatorname*{Res}_{x' \to s_k} \sum_{i,j} \frac{A_{i,j} V(x')}{(x-s_i)^2 (x'-s_j)^2} \\
&= \omega(x) + \sum_k \operatorname*{Res}_{x' \to s_k} \overline{W}_2^{(0)}(x,x') V(x') \\
&= \omega(x) + \sum_k \operatorname*{Res}_{x' \to s_k} W_2^{(0)}(x,x') V(x') \\
&\qquad (I.3)
\end{aligned}$$

Thus we have proved the case $n = 1, g = 0$ of the theorem:

$$\boxed{\hbar\partial_\hbar \omega(x) = \omega(x) + \sum_k \operatorname*{Res}_{x' \to s_k} W_2^{(0)}(x,x') V(x')} \tag{I.4}$$

I.2 $\hbar$ derivatives for $W_2^{(0)}(z)$

We have seen in appendix G, eq. (G.14), that $\overline{W}_2^{(0)}(x,x')$ satisfies the loop equation:

$$(2\omega(x) - V'(x) + \hbar\partial_x)\, \overline{W}_2^{(0)}(x,x') + \frac{\partial}{\partial x'} \frac{\omega(x) - \omega(x')}{x - x'} = -P_2^{(0)}(x,x') \tag{I.5}$$

where $P_2^{(0)}(x,x')$ has no pole at $x \to s_i$'s.

Then we take the derivation $\hbar\partial_\hbar$ of this equation:

$$\begin{aligned}
(2\omega(x) - V'(x) + \hbar\partial_x)\, \hbar\partial_\hbar \overline{W}_2^{(0)}(x,x') + \hbar\partial_x \overline{W}_2^{(0)}(x,x') + 2\hbar\partial_\hbar w(x) \overline{W}_2^{(0)}(x,x') \\
= -\frac{\partial}{\partial x'} \frac{\hbar\partial_\hbar \omega(x) - \hbar\partial_\hbar \omega(x')}{x - x'} - \hbar\partial_\hbar P_2^{(0)}(x,x')
\end{aligned} \tag{I.6}$$

$\hbar\partial_\hbar \overline{W}_2^{(0)}(x,x')$ is a rational fraction of x, with poles only at the s_i's, and $\hbar\partial_\hbar P_2^{(0)}(x,x')$ has no pole at $x \to s_i$'s. We thus write:

$$
\begin{aligned}
& \hbar\partial_\hbar W_2^{(0)}(x,x') \\
= \;& \hbar\partial_\hbar \overline{W}_2^{(0)}(x,x') \\
= \;& \operatorname*{Res}_{z\to x} G(x,z)\ \hbar\partial_\hbar \overline{W}_2^{(0)}(z,x') \\
= \;& -\sum_i \operatorname*{Res}_{z\to s_i} G(x,z)\ \hbar\partial_\hbar \overline{W}_2^{(0)}(z,x') \\
= \;& -\sum_i \operatorname*{Res}_{z\to s_i} ((2\omega(z) - V'(z) - \hbar\partial_z)K(x,z))\ \ \hbar\partial_\hbar \overline{W}_2^{(0)}(z,x') \\
= \;& -\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \ \Big((2\omega(z) - V'(z) + \hbar\partial_z)\ \hbar\partial_\hbar \overline{W}_2^{(0)}(z,x')\Big) \\
= \;& \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \Big((2\hbar\partial_\hbar\omega(z))\ \overline{W}_2^{(0)}(z,x') \\
& + \frac{\partial}{\partial x'}\frac{\hbar\partial_\hbar\omega(z) + \hbar\partial_\hbar\omega(x')}{z-x'} + \hbar\partial_\hbar P_2^{(0)}(z,x') + \hbar\partial_z \overline{W}_2^{(0)}(z,x')\Big) \\
= \;& \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \Big(2\overline{W}_2^{(0)}(z,x')\ \hbar\partial_\hbar\omega(z) + \frac{\hbar\partial_\hbar\omega(z)}{(z-x')^2} + \hbar\partial_z \overline{W}_2^{(0)}(z,x')\Big) \\
= \;& \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \Big(2W_2^{(0)}(z,x')\,\hbar\partial_\hbar\omega(z) + \hbar\partial_z W_2^{(0)}(z,x')\Big)
\end{aligned}
$$

$(I.7)$

Then, we use eq. (I.4), and we get:

$$
\begin{aligned}
& \hbar\partial_\hbar W_2^{(0)}(x,x') \\
= \;& \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\ \Big(2W_2^{(0)}(z,x')w(z) + \hbar\partial_z W_2^{(0)}(z,x')\Big) \\
& + 2\sum_{i,k} \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to s_k} K(x,z) W_2^{(0)}(z,x') W_2^{(0)}(z,x'') V(x'') \\
= \;& \sum_i \operatorname*{Res}_{z\to s_i} W_2^{(0)}(z,x')\ \Big(2w(z) - \hbar\partial_z\Big) K(x,z) \\
& + \sum_{i,k} \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to s_k} K(x,z) W_2^{(0)}(z,x') G(z,x'') V'(x'') \\
= \;& \sum_i \operatorname*{Res}_{z\to s_i} W_2^{(0)}(z,x')\ (G(x,z) + V'(z)K(x,z)) \\
& + \sum_{i,k} \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to s_k} K(x,z) W_2^{(0)}(z,x') G(z,x'') V'(x'') \\
= \;& \sum_i \operatorname*{Res}_{z\to s_i} W_2^{(0)}(z,x')\ G(x,z) \\
& + \sum_{i,k} \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to s_k} K(x,z) W_2^{(0)}(z,x') G(z,x'') V'(x'') \\
& + \sum_i \operatorname*{Res}_{z\to s_i} \operatorname*{Res}_{x''\to z} K(x,z) W_2^{(0)}(z,x') G(z,x'') V'(x'') \\
= \;& \sum_i \operatorname*{Res}_{z\to s_i} W_2^{(0)}(z,x')\ G(x,z)
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{i,k} \operatorname*{Res}_{x'' \to s_k} \operatorname*{Res}_{z \to s_i} K(x,z) W_2^{(0)}(z,x') G(z,x'') V'(x'') \\
= & \sum_i \operatorname*{Res}_{z \to s_i} W_2^{(0)}(z,x') \; G(x,z) \\
& + 2 \sum_{i,k} \operatorname*{Res}_{x'' \to s_k} \operatorname*{Res}_{z \to s_i} K(x,z) W_2^{(0)}(z,x') B(z,x'') V(x'') \\
= & \sum_i \operatorname*{Res}_{z \to s_i} B(z,x') \; G(x,z) \\
& + \sum_k \operatorname*{Res}_{x'' \to s_k} W_3^{(0)}(x,x',x'') V(x'')
\end{aligned}
\tag{I.8}
$$

We now use the fact that $G(x,z)$ and $B(z,x')$ are rational fractions whose only poles are s_i's, as well as $z=x$ and $z=x'$, and we write:

$$
\begin{aligned}
& \sum_i \operatorname*{Res}_{z \to s_i} B(z,x') \; G(x,z) \\
= & - \operatorname*{Res}_{z \to x} B(z,x') \; G(x,z) - \operatorname*{Res}_{z \to x'} B(z,x') \; G(x,z) \\
= & - \operatorname*{Res}_{z \to x} B(z,x') \; \frac{1}{z-x} - \frac{1}{2} \operatorname*{Res}_{z \to x'} \frac{1}{(z-x')^2} \; G(x,z) \\
= & - \operatorname*{Res}_{z \to x} B(z,x') \; \frac{1}{z-x} + \operatorname*{Res}_{z \to x'} \frac{1}{z-x'} \; B(x,z) \\
= & -B(x,x') + B(x,x') \\
= & \; 0
\end{aligned}
\tag{I.9}
$$

So that eventually we have proved the case $n=2, g=0$ of the theorem:

$$
\boxed{\hbar \partial_\hbar W_2^{(0)}(x,x') = \sum_k \operatorname*{Res}_{x'' \to s_k} W_3^{(0)}(x,x',x'') \; V(x'')}
\tag{I.10}
$$

I.3 Recursion for higher correlators

We proceed by recursion on $2g+n$.

From theorem 3.2, we have that:

$$
\begin{aligned}
& (Y(x) - \hbar \partial_x) \hbar \partial_\hbar W_{n+1}^{(g)}(x,L) \\
= & \; \hbar \partial_\hbar U_{n+1}^{(g)}(x;L) + \hbar \partial_x W_{n+1}^{(g)}(x,L) - W_{n+1}^{(g)}(x,L) \, \hbar \partial_\hbar Y(x) \\
& - \hbar \partial_\hbar \left(P_{n+1}^{(g)}(x;L) + \sum_{x_j \in L} \frac{\partial}{\partial x_j} \frac{\overline{W}_n^{(g)}(L)}{x - x_j} \right)
\end{aligned}
\tag{I.11}
$$

where the term on the last line has no pole at $x = s_i$. This implies that:

$$
\sum_i \operatorname*{Res}_{x \to s_i} K(x_0,x) \left((Y(x) - \hbar \partial_x) \hbar \partial_\hbar W_{n+1}^{(g)}(x,L) \right)
$$

$$
\begin{aligned}
&= \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(\hbar\partial_\hbar U^{(g)}_{n+1}(x;L)+\hbar\partial_x W^{(g)}_{n+1}(x,L)\\
&\quad -W^{(g)}_{n+1}(x,L)\,\hbar\partial_\hbar Y(x)\Big) \qquad \text{(I.12)}
\end{aligned}
$$

We have:

$$
\begin{aligned}
&\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big((Y(x)-\hbar\partial_x)\hbar\partial_\hbar W^{(g)}_{n+1}(x,L)\Big)\\
&= \sum_i \operatorname*{Res}_{x\to s_i} \hbar\partial_\hbar W^{(g)}_{n+1}(x,L)\,(Y(x)+\hbar\partial_x)K(x_0,x)\\
&= -\sum_i \operatorname*{Res}_{x\to s_i} \hbar\partial_\hbar W^{(g)}_{n+1}(x,L)\,G(x_0,x)\\
&= \operatorname*{Res}_{x\to x_0} \hbar\partial_\hbar W^{(g)}_{n+1}(x,L)\,G(x_0,x)\\
&= \hbar\partial_\hbar W^{(g)}_{n+1}(x_0,L) \qquad \text{(I.13)}
\end{aligned}
$$

and therefore:

$$
\begin{aligned}
&\hbar\partial_\hbar W^{(g)}_{n+1}(x_0,L)\\
&= \sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\Big(\hbar\partial_\hbar U^{(g)}_{n+1}(x;L)+\hbar\partial_x W^{(g)}_{n+1}(x,L)-W^{(g)}_{n+1}(x,L)\,\hbar\partial_\hbar Y(x)\Big)\\
&(I.14)
\end{aligned}
$$

From the recursion hypothesis we have:

$$
\begin{aligned}
&\hbar\partial_\hbar U^{(g)}_{n+1}(x;L)\\
&= \hbar\partial_\hbar W^{(g-1)}_{n+2}(x,x,L)+\sum_{k=0}^{g}{\sum_{J\subset L}}' W^{(k)}_{1+|J|}(x,J)\hbar\partial_\hbar W^{(g-k)}_{1+n-|J|}(x,L/J)\\
&\quad +\sum_{k=0}^{g}{\sum_{J\subset L}}' W^{(g-k)}_{1+n-|J|}(x,L/J)\hbar\partial_\hbar W^{(k)}_{1+|J|}(x,J)\\
&= (2-2(g-1)-(n+2))W^{(g-1)}_{n+2}(x,x,L)+\sum_i \operatorname*{Res}_{x'\to s_i} W^{(g-1)}_{n+3}(x,x,L,x')\,V(x')\\
&\quad +\sum_{k=0}^{g}{\sum_{J\subset L}}'(2-2(g-k)-(1+n-|J|))\,W^{(k)}_{1+|J|}(x,J)\,W^{(g-k)}_{1+n-|J|}(x,L/J)\\
&\quad +\sum_{k=0}^{g}{\sum_{J\subset L}}'(2-2k-(1+|J|))\,W^{(g-k)}_{1+n-|J|}(x,L/J)\,W^{(k)}_{1+|J|}(x,J)\\
&\quad +\sum_i \operatorname*{Res}_{x'\to s_i} V(x')\sum_{k=0}^{g}{\sum_{J\subset L}}' W^{(k)}_{2+|J|}(x,J,x')\,W^{(g-k)}_{1+n-|J|}(x,L/J)\\
&\quad +\sum_i \operatorname*{Res}_{x'\to s_i} V(x')\sum_{k=0}^{g}{\sum_{J\subset L}}' W^{(k)}_{1+|J|}(x,J)\,W^{(g-k)}_{2+n-|J|}(x,L/J,x')\\
&= (2-2g-n)\;U^{(g)}_{n+1}(x;L)\\
&\quad +\sum_i \operatorname*{Res}_{x'\to s_i} V(x')\,(U^{(g)}_{n+2}(x;x',L)-2B(x,x')W^{(g)}_{n+1}(x,L)) \qquad \text{(I.15)}
\end{aligned}
$$

Thus we have:

$$
\begin{aligned}
& \hbar\partial_\hbar W_{n+1}^{(g)}(x_0,L) \\
= \;& (2-2g-n)\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\,U_{n+1}^{(g)}(x;L) \\
& +\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,(U_{n+2}^{(g)}(x;x',L)-2B(x,x')W_{n+1}^{(g)}(x,L)) \\
& +\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_x W_{n+1}^{(g)}(x,L)-W_{n+1}^{(g)}(x,L)\,\hbar\partial_\hbar Y(x)\right) \\
= \;& (2-2g-n)W_{n+1}^{(g)}(x_0,L) \\
& +\sum_j \operatorname*{Res}_{x'\to s_j}\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)V(x')\,(U_{n+2}^{(g)}(x;x',L)-2B(x,x')W_{n+1}^{(g)}(x,L)) \\
& +\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_x W_{n+1}^{(g)}(x,L)-W_{n+1}^{(g)}(x,L)\,\hbar\partial_\hbar Y(x)\right) \\
= \;& (2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L) \\
& -2\sum_j \operatorname*{Res}_{x'\to s_j}\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)V(x')\,B(x,x')W_{n+1}^{(g)}(x,L) \\
& +\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_x W_{n+1}^{(g)}(x,L)-W_{n+1}^{(g)}(x,L)\,\hbar\partial_\hbar Y(x)\right) \\
= \;& (2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L) \\
& -2\sum_i \operatorname*{Res}_{x\to s_i}\sum_j \operatorname*{Res}_{x'\to s_j} K(x_0,x)V(x')\,B(x,x')W_{n+1}^{(g)}(x,L) \\
& -2\sum_i \operatorname*{Res}_{x\to s_i}\operatorname*{Res}_{x'\to x} K(x_0,x)V(x')\,B(x,x')W_{n+1}^{(g)}(x,L) \\
& +\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_x W_{n+1}^{(g)}(x,L)-W_{n+1}^{(g)}(x,L)\,\hbar\partial_\hbar Y(x)\right)
\end{aligned}
\tag{I.16}
$$

Notice that:

$$
\hbar\partial_\hbar Y(x)+2\sum_j \operatorname*{Res}_{x'\to s_j} B(x,x')V(x')+2\operatorname*{Res}_{x'\to x} B(x,x')V(x')=Y(x) \tag{I.17}
$$

therefore:

$$
\begin{aligned}
& \hbar\partial_\hbar W_{n+1}^{(g)}(x_0,L) \\
= \;& (2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L) \\
& +\sum_i \operatorname*{Res}_{x\to s_i} K(x_0,x)\left(\hbar\partial_x W_{n+1}^{(g)}(x,L)-Y(x)W_{n+1}^{(g)}(x,L)\right) \\
= \;& (2-2g-n)W_{n+1}^{(g)}(x_0,L)+\sum_j \operatorname*{Res}_{x'\to s_j} V(x')\,W_{n+2}^{(g)}(x_0,x',L) \\
& -\sum_i \operatorname*{Res}_{x\to s_i} W_{n+1}^{(g)}(x,L)\,(Y(x)+\hbar\partial_x)K(x_0,x)
\end{aligned}
$$

$$
\begin{aligned}
&= (2-2g-n)W_{n+1}^{(g)}(x_0,L) + \sum_j \operatorname*{Res}_{x'\to s_j} V(x')\, W_{n+2}^{(g)}(x_0,x',L) \\
&\quad + \sum_i \operatorname*{Res}_{x\to s_i} W_{n+1}^{(g)}(x,L)\, G(x_0,x) \\
&= (2-2g-n)W_{n+1}^{(g)}(x_0,L) + \sum_j \operatorname*{Res}_{x'\to s_j} V(x')\, W_{n+2}^{(g)}(x_0,x',L) \\
&\quad - \operatorname*{Res}_{x\to x_0} W_{n+1}^{(g)}(x,L)\, G(x_0,x) \\
&= (2-2g-n)W_{n+1}^{(g)}(x_0,L) + \sum_j \operatorname*{Res}_{x'\to s_j} V(x')\, W_{n+2}^{(g)}(x_0,x',L) \\
&\quad - W_{n+1}^{(g)}(x_0,L) \\
&= (2-2g-n-1)W_{n+1}^{(g)}(x_0,L) + \sum_j \operatorname*{Res}_{x'\to s_j} V(x')\, W_{n+2}^{(g)}(x_0,x',L) \qquad \text{(I.18)}
\end{aligned}
$$

i.e. we have proved the theorem for $2g+n+1$.

J Appendix: Free Energies

Here we consider $g \geq 2$.

The free energies defined in eq. (3.16), automatically satisfy theorem 3.8, and thus are homogeneous:

$$F^{(g)}(\lambda V, \lambda\hbar) = \lambda^{2-2g}\ F^{(g)}(V,\hbar) \qquad \text{(J.1)}$$

Here we show that they satisfy theorem 3.6.

We start from the definition:

$$F^{(g)} = \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \sum_i \operatorname*{Res}_{x\to s_i} V(x)\ \ W_1^{(g)}(x)\Big|_{\tilde{\hbar}} \qquad \text{(J.2)}$$

and we compute the loop operator applied to $F^{(g)}$:

$$
\begin{aligned}
\delta_{x_1} F^{(g)} &= \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \sum_i \operatorname*{Res}_{x\to s_i} \Big(V(x)\ W_2^{(g)}(x,x_1) + \delta_{x_1} V(x)\ W_1^{(g)}(x)\Big)_{\tilde{\hbar}} \\
&= \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \sum_i \operatorname*{Res}_{x\to s_i} \left(V(x)\ W_2^{(g)}(x,x_1) + \frac{W_1^{(g)}(x)}{x-x_1} \right)_{\tilde{\hbar}} \\
&= \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \left(\left(\sum_i \operatorname*{Res}_{x\to s_i} V(x)\ W_2^{(g)}(x,x_1) \right) - W_1^{(g)}(x_1) \right)_{\tilde{\hbar}} \\
&= \hbar^{2-2g} \int_0^{\hbar} \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}} \left(\tilde{h}^{2-2g} \frac{d(\tilde{h}^{2g-1} W_1^{(g)}(x_1))}{d\tilde{h}} - W_1^{(g)}(x_1) \right)_{\tilde{\hbar}} \\
&= \hbar^{2-2g} \int_0^{\hbar} \left(\frac{1}{\tilde{h}}\, d\left(\tilde{h}^{2g-1} W_1^{(g)}(x_1) \right) - \frac{d\tilde{\hbar}}{\tilde{\hbar}^{3-2g}}\ W_1^{(g)}(x_1) \right)_{\tilde{\hbar}}
\end{aligned}
$$

$(J.3)$

we integrate by parts, and since $2g-2>0$, there is no boundary term coming from the bound at 0, and thus:

$$\begin{aligned}\delta_{x_1} F^{(g)} &= W_1^{(g)}(x_1) + \hbar^{2-2g} \int_0^{\hbar} \left(\tilde{\hbar}^{2g-3}\, W_1^{(g)}(x_1) - \tilde{\hbar}^{2g-3}\; W_1^{(g)}(x_1) \right)_{\tilde{\hbar}} d\tilde{\hbar} \\ &= W_1^{(g)}(x_1) \end{aligned} \tag{J.4}$$

Therefore we have proved that the loop operator acting on $F^{(g)}$ is indeed $W_1^{(g)}$, i.e. we have proved theorem 3.6.

K Appendix: $F^{(0)}$

We have defined $F^{(0)}$ as:

$$F^{(0)} = -\hbar \sum_i V(s_i) + \hbar^2 \sum_{i \neq j} \ln(s_i - s_j) \tag{K.1}$$

• Proof of theorem 3.6 for $F^{(0)}$:
consider a variation δV, we have:

$$\begin{aligned}\delta F^{(0)} &= -\hbar \sum_i \delta V(s_i) - \hbar \sum_i V'(s_i)\delta s_i + 2\hbar^2 \sum_{j\neq i} \frac{\delta s_i}{s_i - s_j} \\ &= -\hbar \sum_i \delta V(s_i) \\ &= -\sum_i \operatorname*{Res}_{x\to s_i} \omega(x)\, \delta V(x) \end{aligned} \tag{K.2}$$

• Proof of theorem 3.8 for $F^{(0)}$:
we have:

$$\begin{aligned}\hbar\partial_\hbar F^{(0)} &= -\hbar \sum_i V(s_i) + 2\hbar^2 \sum_{i\neq j} \ln(s_i - s_j) \\ &\quad -\hbar^2 \sum_i \frac{\partial s_i}{\partial \hbar} \left(V'(s_i) - 2\hbar \sum_{j\neq i} \frac{1}{s_i - s_j} \right) \\ &= -\hbar \sum_i V(s_i) + 2\hbar^2 \sum_{i\neq j} \ln(s_i - s_j) \\ &= 2F^{(0)} + \hbar \sum_i V(s_i) \\ &= 2F^{(0)} + \sum_i \operatorname*{Res}_{x\to s_i} \omega(x)\, V(x) \end{aligned} \tag{K.3}$$

Therefore:

$$(2 - \hbar\partial_\hbar) F_0 = -\sum_i \operatorname*{Res}_{x\to s_i} V(x) w(x) \tag{K.4}$$

L Appendix: $F^{(1)}$

We have defined $F^{(1)}$ as:

$$\begin{aligned} F^{(1)} &= \frac{1}{2}\ln(\det A) + \frac{F^{(0)}}{\hbar^2} + \ln(\Delta(s)^2) \\ &= \frac{1}{2}\ln(\det A) - \frac{1}{\hbar}\sum_i V(s_i) + \sum_{i\neq j}\ln(s_i - s_j) + \sum_{i\neq j}\ln(s_i - s_j) \\ &= \frac{1}{2}\ln(\det A) - \frac{1}{\hbar}\sum_i V(s_i) + 2\sum_{i\neq j}\ln(s_i - s_j) \end{aligned} \tag{L.1}$$

• Proof of theorem 3.6 for $F^{(1)}$:
Let us start from $W_1^{(1)}$

$$\begin{aligned} W_1^{(1)}(x) &= \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\overline{W}_2(z,z) \\ &= \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\left[\frac{A_{i,i}}{(z-s_i)^4} + 2\sum_{j\neq i}\frac{A_{i,j}}{(z-s_i)^2(z-s_j)^2}\right] \\ &= \sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\frac{A_{i,i}}{(z-s_i)^4} \\ &\quad +2\sum_i\sum_{j\neq i} K'(x,s_i)\,\frac{A_{i,j}}{(s_i-s_j)^2} \\ &\quad -4\sum_i\sum_{j\neq i} K(x,s_i)\,\frac{A_{i,j}}{(s_i-s_j)^3} \end{aligned} \tag{L.2}$$

We have:

$$\begin{aligned} &\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\frac{A_{i,i}}{(z-s_i)^4} \\ =\ &\frac{1}{3}\sum_i \operatorname*{Res}_{z\to s_i} K'(x,z)\,\frac{A_{i,i}}{(z-s_i)^3} \\ =\ &\frac{1}{3}\sum_i \operatorname*{Res}_{z\to s_i}\Big(\frac{2}{z-s_i} + 2\omega_i(z) - \frac{1}{\hbar}V'(z)\Big)K(x,z)\,\frac{A_{i,i}}{(z-s_i)^3} \\ &-\frac{1}{3\hbar}\sum_i \operatorname*{Res}_{z\to s_i} G(x,z)\,\frac{A_{i,i}}{(z-s_i)^3} \end{aligned} \tag{L.3}$$

Therefore:

$$\begin{aligned} &\sum_i \operatorname*{Res}_{z\to s_i} K(x,z)\,\frac{A_{i,i}}{(z-s_i)^4} \\ =\ &\sum_i \operatorname*{Res}_{z\to s_i}\Big(2\omega_i(z) - \frac{1}{\hbar}V'(z)\Big)K(x,z)\,\frac{A_{i,i}}{(z-s_i)^3} \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{\hbar}\sum_i \operatorname*{Res}_{z\to s_i} G(x,z)\,\frac{A_{i,i}}{(z-s_i)^3} \\
= & \sum_i \operatorname*{Res}_{z\to s_i}\Big[\frac{2\omega_i(z)-\frac{1}{\hbar}V'(z)}{z-s_i}\,K(x,z)\Big]\,\frac{A_{i,i}}{(z-s_i)^2} \\
& -\frac{1}{2\hbar}\sum_i \operatorname*{Res}_{z\to s_i} G'(x,z)\,\frac{A_{i,i}}{(z-s_i)^2} \\
= & \sum_i A_{i,i}\Big[\frac{2\omega_i(z)-\frac{1}{\hbar}V'(z)}{z-s_i}\,K(x,z)\Big]'_{z=s_i} \\
& +\frac{1}{\hbar}\sum_i \operatorname*{Res}_{z\to s_i} B(x,z)\,\frac{A_{i,i}}{(z-s_i)^2} \\
= & \frac{1}{2}\sum_i (2\omega_i''(s_i)-\frac{1}{\hbar}V'''(s_i))K(x,s_i)\,A_{i,i} \\
& -\sum_i K'(x,s_i)\,A_{i,i}T_{i,i} \\
& +\frac{1}{\hbar}\sum_i \operatorname*{Res}_{z\to s_i} B(x,z)\,\frac{A_{i,i}}{(z-s_i)^2}
\end{aligned}
\tag{L.4}
$$

Notice that:

$$
\begin{aligned}
\operatorname*{Res}_{x\to s} K(x,s_i)\delta V(x) &= \frac{1}{\hbar}\sum_j \operatorname*{Res}_{x\to s}\frac{A_{i,j}\delta V(x)}{(x-s_j)^2} \\
&= \frac{1}{\hbar}\sum_j A_{i,j}\delta V'(s_j) \\
&= -\delta s_i
\end{aligned}
\tag{L.5}
$$

$$
\operatorname*{Res}_{x\to s} K'(x,s_i)\delta V(x) = -\frac{1}{\hbar}\sum_j \delta_{i,j}\delta V(s_j) - 2\sum_{j\neq i}\frac{\delta s_j}{s_i-s_j} \tag{L.6}
$$

$$
\begin{aligned}
\operatorname*{Res}_{x\to s}\operatorname*{Res}_{z\to s_i}\frac{B(x,z)}{(z-s_i)^2}\,\delta V(x) &= \operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x\to s}\frac{B(x,z)}{(z-s_i)^2}\,\delta V(x) \\
&\quad + \operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x\to z}\frac{B(x,z)}{(z-s_i)^2}\,\delta V(x) \\
&= \operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x\to s}\frac{A_{j,l}}{(x-s_l)^2(z-s_j)^2(z-s_i)^2}\,\delta V(x) \\
&\quad +\frac{1}{2}\operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x\to z}\frac{1}{(x-z)^2(z-s_i)^2}\,\delta V(x) \\
&= \hbar\operatorname*{Res}_{z\to s_i}\operatorname*{Res}_{x\to s}\frac{K(x,s_j)}{(z-s_j)^2(z-s_i)^2}\,\delta V(x) \\
&\quad +\frac{1}{2}\operatorname*{Res}_{z\to s_i}\frac{1}{(z-s_i)^2}\,\delta V'(z) \\
&= -\hbar\operatorname*{Res}_{z\to s_i}\frac{\delta s_j}{(z-s_j)^2(z-s_i)^2}+\frac{1}{2}\,\delta V''(s_i)
\end{aligned}
$$

$$= \quad 2\hbar \frac{\delta s_j}{(s_i - s_j)^3} + \frac{1}{2}\, \delta V''(s_i) \tag{L.7}$$

That gives:

$$\begin{aligned}
& \operatorname*{Res}_x \operatorname*{Res}_{z\to s_i} \frac{K(x,z)\, A_{i,i}}{(x-s_i)^4}\, \delta V(x) \\
= \quad & -\frac{1}{2}(2\omega_i''(s_i) - \frac{1}{\hbar} V'''(s_i))\delta s_i\, A_{i,i} + \frac{1}{\hbar} \sum_j \delta_{i,j} \delta V(s_j)\, A_{i,i} T_{i,i} \\
& +2 \sum_{j\neq i} \frac{\delta s_j}{s_i - s_j}\, A_{i,i} T_{i,i} + 2\hbar \frac{\delta s_j}{(s_i - s_j)^3}\, A_{i,i} + \frac{1}{2}\, \delta V''(s_i)\, A_{i,i} \\
= \quad & \frac{1}{2}\delta(T_{i,i})\, A_{i,i} + \frac{1}{\hbar} \sum_j \delta_{i,j} \delta V(s_j)\, A_{i,i} T_{i,i} + 2 \sum_{j\neq i} \frac{\delta s_j}{s_i - s_j}\, A_{i,i} T_{i,i}
\end{aligned} \tag{L.8}$$

and thus:

$$\begin{aligned}
& \operatorname*{Res}_{x\to s} W_1^{(1)}(x) \delta V(x) \\
= \quad & \sum_i \frac{1}{2}\delta(T_{i,i})\, A_{i,i} + \frac{1}{\hbar} \sum_i \sum_j \delta_{i,j} \delta V(s_j)\, A_{i,i} T_{i,i} + 2 \sum_{j\neq i} \frac{\delta s_j}{s_i - s_j}\, A_{i,i} T_{i,i} \\
& -2 \sum_i \sum_{j\neq i} \frac{\frac{1}{\hbar} \sum_l \delta_{i,l} \delta V(s_l)}{(s_i - s_j)^2}\, A_{i,j} - 4 \sum_i \sum_{j\neq i} \sum_{l\neq i} \frac{\delta s_l}{(s_i - s_l)(s_i - s_j)^2}\, A_{i,j} \\
& +4 \sum_i \sum_{j\neq i} \frac{\delta s_i}{(s_i - s_j)^3}\, A_{i,j} \\
= \quad & \sum_i \frac{1}{2}\delta(T_{i,i})\, A_{i,i} + \frac{1}{\hbar} \sum_i \sum_j \sum_l \delta_{i,j} \delta V(s_j)\, A_{i,l} T_{l,i} + 2 \sum_{j\neq i} \frac{\delta s_j}{s_i - s_j}\, A_{i,i} T_{i,i} \\
& -4 \sum_i \sum_{j\neq i} \sum_{l\neq i} \frac{\delta s_l}{(s_i - s_l)(s_i - s_j)^2}\, A_{i,j} + 4 \sum_i \sum_{j\neq i} \frac{\delta s_i}{(s_i - s_j)^3}\, A_{i,j} \\
= \quad & \frac{1}{2} \operatorname{Tr} A\, \delta T + \frac{1}{\hbar} \sum_i \sum_j \sum_l \delta_{i,j} \delta V(s_j)\, A_{i,l} T_{l,i} + 2 \sum_{j\neq i} \frac{\delta s_j}{s_i - s_j} \\
& +4 \sum_{j\neq i} \sum_{l\neq i} \frac{\delta s_j}{(s_i - s_j)(s_i - s_l)^2}\, A_{i,l} - 4 \sum_{i\neq j\neq l} \frac{\delta s_l}{(s_i - s_l)(s_i - s_j)^2}\, A_{i,j} \\
= \quad & \frac{1}{2} \operatorname{Tr} A\, \delta T + \frac{1}{\hbar} \sum_j \delta V(s_j) - \sum_{j\neq i} \frac{\delta s_i - \delta s_j}{s_i - s_j} \\
= \quad & \frac{1}{2}\delta \ln\det T + \frac{1}{\hbar} \sum_j \delta(V(s_j)) - \frac{1}{\hbar} \sum_j V'(s_j)\delta s_j - \sum_{j\neq i} \frac{\delta s_i - \delta s_j}{s_i - s_j} \\
= \quad & \frac{1}{2}\delta \ln\det T + \frac{1}{\hbar} \sum_j \delta(V(s_j)) - 2 \sum_j \sum_{i\neq j} \frac{\delta s_j}{s_j - s_i} - \sum_{j\neq i} \frac{\delta s_i - \delta s_j}{s_i - s_j} \\
= \quad & \frac{1}{2}\delta \ln\det T + \frac{1}{\hbar} \sum_j \delta(V(s_j)) - \sum_j \sum_{i\neq j} \frac{\delta s_j - \delta s_i}{s_j - s_i} - \sum_{j\neq i} \frac{\delta s_i - \delta s_j}{s_i - s_j} \\
= \quad & \frac{1}{2}\delta \ln\det T + \frac{1}{\hbar} \sum_j \delta(V(s_j)) - 2 \sum_{i\neq j} \frac{\delta s_j - \delta s_i}{s_j - s_i}
\end{aligned} \tag{L.9}$$

That implies:

$$F_1 \;=\; -\frac{1}{2}\ln\det T - \frac{1}{\hbar}\sum_j V(s_j) + 2\sum_{i\neq j}\ln(s_i - s_j) \tag{L.10}$$

$$\boxed{F_1 = \frac{1}{2}\ln\det A - \frac{1}{\hbar}\sum_j V(s_j) + 2\sum_{i\neq j}\ln(s_i - s_j)} \tag{L.11}$$

M Appendix: Example $m = 1$

We choose $s = 0$, and $V'(s) = v_2 s + v_3 s^2 + \sum v_{k+1} s^k$.

We have

$$\omega(x) = \frac{\hbar}{x} \tag{M.1}$$

$$A = \frac{\hbar}{v_2} \tag{M.2}$$

$$K(x_1, x) = \sum_k K_k(x_1)\; x^k \tag{M.3}$$

$$K_0 = \frac{1}{v_2 x_1^2} \qquad , \qquad K_1 = K_2 = 0 \tag{M.4}$$

$$K_3 = \frac{1}{\hbar x_1^3} - \frac{v_3}{\hbar v_2 x_1^2} \tag{M.5}$$

$$B(x_1, x_2) = \frac{1}{2(x_1 - x_2)^2} + \frac{A}{x_1^2 x_2^2} \tag{M.6}$$

$$W_3^{(0)} \;=\; \frac{2\hbar}{v_2^2\, x_1^2\, x_2^2\, x_3^2}\,(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}) - \frac{2\hbar\, v_3}{v_2^3\, x_1^2\, x_2^2\, x_3^2} \tag{M.7}$$

$$\begin{aligned} W_4^{(0)} \;=\;& \frac{6\hbar}{v_2^3\, x_1^2\, x_2^2\, x_3^2}\,(\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2}) \\ &+ \frac{8\hbar}{v_2^3\, x_1^2\, x_2^2\, x_3^2}\,(\frac{1}{x_1 x_2} + \frac{1}{x_1 x_3} + \frac{1}{x_1 x_4} + \frac{1}{x_2 x_3} + \frac{1}{x_2 x_4} + \frac{1}{x_3 x_4}) \\ &- \frac{12\hbar v_3}{v_2^4\, x_1^2\, x_2^2\, x_3^2}\,(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}) + \frac{12\hbar\, v_3^2}{v_2^5\, x_1^2\, x_2^2\, x_3^2} - \frac{6\hbar\, v_4}{v_2^4\, x_1^2\, x_2^2\, x_3^2} \end{aligned} \tag{M.8}$$

$$W_1^{(1)} \;=\; \frac{1}{\hbar x} + \frac{1}{v_2\, x^3} - \frac{v_3}{v_2^2\, x^2} \tag{M.9}$$

$$W_2^{(1)} = \frac{3}{v_2^2\,x_1^2x_2^2}\,(\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{2}{3\,x_1x_2})+\frac{1}{\hbar\,v_2\,x_1^2\,x_2^2}-\frac{4v_3}{v_2^3\,x_1^2x_2^2}\,(\frac{1}{x_1}+\frac{1}{x_2}) +\frac{4v_3^2}{v_2^4\,x_1^2x_2^2}-\frac{3v_4}{v_2^3\,x_1^2x_2^2} \quad \text{(M.10)}$$

$$\begin{aligned} W_3^{(1)} = {} & \frac{12}{v_2^3\,x_1^2x_2^2x_3^2}\,(\frac{1}{x_1^3}+\frac{1}{x_2^3}+\frac{1}{x_3^3}) \\ & +\frac{12}{v_2^3\,x_1^2x_2^2x_3^2}\,(\frac{1}{x_1^2x_2}+\frac{1}{x_2^2x_3}+\frac{1}{x_3^2x_1}+\frac{1}{x_1x_2^2}+\frac{1}{x_2x_3^2}+\frac{1}{x_3x_1^2}) \\ & +\frac{8}{v_2^3\,x_1^3x_2^3x_3^3}+\frac{2}{\hbar v_2^2x_1^2x_2^2x_3^2}\,(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}) \\ & -\frac{24v_3}{v_2^4x_1^2x_2^2x_3^2}\,(\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2}+\frac{1}{x_1x_2}+\frac{1}{x_2x_3}+\frac{1}{x_3x_1})-\frac{2v_3}{\hbar v_2^3x_1^2x_2^2x_3^2} \\ & +\frac{32v_3^2}{v_2^5x_1^2x_2^2x_3^2}\,(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3})-\frac{32v_3^3}{v_2^6x_1^2x_2^2x_3^2}-\frac{18v_4}{v_2^4x_1^2x_2^2x_3^2}\,(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}) \\ & +\frac{42v_3v_4}{v_2^5x_1^2x_2^2x_3^2}-\frac{12v_5}{v_2^4x_1^2x_2^2x_3^2} \end{aligned} \quad \text{(M.11)}$$

$$W_1^{(2)} = -\frac{1}{\hbar^3x}+\frac{3}{\hbar\,v_2^2\,x^5}-\frac{5v_3}{\hbar\,v_2^3\,x^4}+\frac{5v_3^2}{\hbar\,v_2^4\,x^3}-\frac{5v_3^3}{\hbar\,v_2^5\,x^2}-\frac{3v_4}{\hbar\,v_2^3\,x^3} +\frac{8v_3\,v_4}{\hbar\,v_2^4\,x^2}-\frac{3v_5}{\hbar\,v_2^3\,x^2} \quad (M.12)$$

$$\begin{aligned} W_2^{(2)} = {} & \frac{15}{\hbar\,v_2^3\,x_1^2x_2^2}\,(\frac{1}{x_1^4}+\frac{1}{x_2^4}+\frac{1}{x_1^2\,x_2^2})+\frac{12}{\hbar\,v_2^3\,x_1^2x_2^2}\,(\frac{1}{x_1^3x_2}+\frac{1}{x_1x_2^3})-\frac{1}{\hbar^3\,v_2\,x_1^2x_2^2} \\ & -\frac{32v_3}{\hbar\,v_2^4\,x_1^2x_2^2}\,(\frac{1}{x_1^3}+\frac{1}{x_2^3})-\frac{30v_3}{\hbar\,v_2^4\,x_1^2x_2^2}\,(\frac{1}{x_1x_2^2}+\frac{1}{x_1^2x_2})+\frac{45v_3^2}{\hbar\,v_2^5\,x_1^2x_2^2}\,(\frac{1}{x_1^2}+\frac{1}{x_2^2}) \\ & +\frac{40v_3^2}{\hbar\,v_2^5\,x_1^3x_2^3}-\frac{50v_3^3}{\hbar\,v_2^6\,x_1^2x_2^2}\,(\frac{1}{x_1}+\frac{1}{x_2})+\frac{50v_3^4}{\hbar\,v_2^7\,x_1^2x_2^2}-\frac{24v_4}{\hbar\,v_2^4\,x_1^2x_2^2}\,(\frac{1}{x_1^2}+\frac{1}{x_2^2}) \\ & -\frac{18v_4}{\hbar\,v_2^4\,x_1^3x_2^3}+\frac{64v_3\,v_4}{\hbar\,v_2^5\,x_1^2x_2^2}\,(\frac{1}{x_1}+\frac{1}{x_2})-\frac{109v_3^2\,v_4}{\hbar\,v_2^6\,x_1^2x_2^2}+\frac{24v_4^2}{\hbar\,v_2^5\,x_1^2x_2^2} \\ & -\frac{18v_5}{\hbar\,v_2^4\,x_1^2x_2^2}\,(\frac{1}{x_1}+\frac{1}{x_2})+\frac{50v_3\,v_5}{\hbar\,v_2^5\,x_1^2x_2^2}-\frac{15v_6}{\hbar\,v_2^4\,x_1^2x_2^2} \end{aligned} \quad \text{(M.13)}$$

$$W_1^{(3)} = \frac{2}{\hbar^5x}+\frac{15}{\hbar^2\,v_2^3\,x^7}-\frac{3}{\hbar^3\,v_2^2\,x^5}-\frac{35v_3}{\hbar^2\,v_2^4\,x^6}+\frac{5v_3}{\hbar^3\,v_2^3\,x^4}+\frac{50v_3^2}{\hbar^2\,v_2^5\,x^5}-\frac{5v_3^2}{\hbar^3\,v_2^4\,x^3} -\frac{60v_3^3}{\hbar^2\,v_2^6\,x^4}+\frac{5v_3}{\hbar^3\,v_2^5\,x^2}+\frac{60v_3^4}{\hbar^2\,v_2^7\,x^3}-\frac{60v_3^5}{\hbar^2\,v_2^8\,x^2}-\frac{24v_4}{\hbar^2\,v_2^4\,x^5}+\frac{3v_4}{\hbar^3\,v_2^3\,x^3}$$

$$+\frac{75v_3v_4}{\hbar^2\,v_2^5\,x^4}-\frac{8v_3v_4}{\hbar^3\,v_2^4\,x^2}-\frac{125v_3^2v_4}{\hbar^2\,v_2^6\,x^3}+\frac{185v_3^3v_4}{\hbar^2\,v_2^7\,x^2}+\frac{24v_4^2}{\hbar^2\,v_2^5\,x^3}-\frac{99v_3v_4^2}{\hbar^2\,v_2^6\,x^2}$$
$$-\frac{21v_5}{\hbar^2\,v_2^4\,x^4}+\frac{3v_5}{\hbar^3\,v_2^3\,x^2}+\frac{56v_3v_5}{\hbar^2\,v_2^5\,x^3}-\frac{106v_3^2v_5}{\hbar^2\,v_2^6\,x^2}+\frac{45v_4v_5}{\hbar^2\,v_2^5\,x^2}-\frac{15v_6}{\hbar^2\,v_2^4\,x^3}$$
$$+\frac{50v_3v_6}{\hbar^2\,v_2^5\,x^2}-\frac{15v_7}{\hbar^2\,v_2^4\,x^2} \tag{M.14}$$

The free energies are:

$$F_1=\frac{1}{2}\ln\left(v_2/\hbar\right) \tag{M.15}$$

$$F_2=-\frac{5v_3^2}{6\hbar\,v_2^3}+\frac{3v_4}{4\hbar\,v_2^2} \tag{M.16}$$

$$F_3=\frac{5v_3^2}{6\hbar^3\,v_2^3}-\frac{5v_3^4}{\hbar^2\,v_2^6}-\frac{3v_4}{4\hbar^3\,v_2^2}+\frac{25v_3^2v_4}{2\hbar^2\,v_2^5}-\frac{3v_4^2}{\hbar^2\,v_2^4}-\frac{7v_3v_5}{\hbar^2\,v_2^4}+\frac{5v_6}{2\hbar^2\,v_2^3} \tag{M.17}$$

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