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Journal of Modern Physics, 2023, 14, 311-329
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

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How to cite this paper: Meis, C. (2023) Photon Structure and Wave Function from the Vector Potential Quantization. *Journal of Modern Physics*, **14**, 311-329.
https://doi.org/10.4236/jmp.2023.143020

Received: November 28, 2022
Accepted: February 24, 2023
Published: February 27, 2023

Abstract

A photon structure is advanced based on the experimental evidence and the vector potential quantization at a single photon level. It is shown that the photon is neither a point particle nor an infinite wave but behaves rather like a local "wave-corpuscle" extended over a wavelength, occupying a minimum quantization volume and guided by a non-local vector potential real wave function. The quantized vector potential oscillates over a wavelength with circular left or right polarization giving birth to orthogonal magnetic and electric fields whose amplitudes are proportional to the square of the frequency. The energy $\hbar\omega$ and momentum $\hbar\vec{k}$ are carried by the local wave-corpuscle guided by the non-local vector potential wave function suitably normalized.

Keywords

Photons, Photon Wave Function, Vector Potential Quantization, Photon Electric and Magnetic Fields, Photon Structure, Wave-Corpuscle Representation, Photon "Energy-Vector Potential" Equation

1. Introduction

A single photon is a particular relativistic massless wave-particle for which scientists have still major difficulties to attribute a clear physical representation. Historically, the scientific understanding of light's properties, guided each time by the interpretation of the experiments, was a continuous balancing between the wave and particle natures. Before going ahead to the quantum description of the photons and the possible associated wave functions it is of crucial importance to understand the historical evolution of our concepts about the nature of light.

In the seventeenth century, reviving the ideas of ancient Greeks on light's nature, Newton advanced that light is composed of individual corpuscles which travel rectilinearly in a homogeneous medium [1] [2]. On the other side, Huygens developed a refined theory of light [3] [4] based on the wave representation and refuted Newton's corpuscular theory. Almost two centuries later, Young obtained experimentally interference patterns using different light sources. He also went further by explaining some polarization experiments assuming that light oscillations take particular orientations with respect to the propagation axis [4] [5]. By that time it was impossible to explain the experimental diffraction patterns using Newton's corpuscular theory, while Euler and Fresnel advanced easily precise interpretations applying the wave representation. James Clerk Maxwell published his remarkable theory on the electromagnetic waves in 1865. He established, for the first time, the relations between the electric and magnetic fields and advanced that light is composed of electromagnetic waves [4] [6]. Hertz confirmed this experimentally by discovering the long wavelength electromagnetic radiation. Thus, by the end of the nineteenth century, the scientific community replaced Newton's corpuscular representation by the wave theory of light. However, that was not for a long time.

In the beginning of the twentieth century, in order to explain the spectral energy density emitted by a black body, Max Planck reintroduced the notion of the particle nature of light with a particular sense. In fact, he assumed that bodies are composed of "oscillators" having the particularity to emit electromagnetic "packets" each with energy $h\nu$, where ν is the frequency and h Planck's constant [7] [8]. A few years later, based on Planck's works, Einstein proposed an interpretation of the photoelectric effect, first observed by Hertz [9], assuming that the electromagnetic radiation itself is composed of quanta with energy $h\nu$ [10]. Furthermore, in 1923 Compton advanced that only the light quanta could explain the experimental observations of the X-rays scattering on free electrons [11]. Hence, the photoelectric effect and Compton X-rays scattering have been always considered as the strongest arguments in favor of the particle nature of light and Quantum Electrodynamics (QED) theory was developed during the years of thirties to sixties based on the point photon model.

However, it is scarcely quoted in the literature that some important studies in favor of the electromagnetic wave theory have been totally disregarded and fell into oblivion. In fact, Wentzel in 1926 [12] and Beck in 1927 [13], as well as Lamb and Scully in the 1960s [14], demonstrated that the photoelectric effect can be interpreted directly by considering only the electromagnetic wave nature of light and without referring to photons at all. Furthermore, Klein and Nishina in 1929 [15] interpreted fully Compton's scattering by also considering the wave nature of light and without invoking the photon concept. So, still after the second world war, although the majority of scientists joined Bohr's *complementarity principle* according to which *light exhibits both wave and particle natures appearing mutually exclusively in each experiment*, a significant part of the scientific community still had the conviction that the electromagnetic wave representation was sufficient to understand light's nature.

Things radically changed in the years of fifties and sixties. Robinson in 1953 [16] and Hadlock in 1958 [17] carried out systematic experiments using microwaves crossing rectangular or circular apertures with variable dimensions and deduced that no energy is transmitted when the apertures dimensions are smaller than approximately the quarter of the wavelength ($\sim \lambda_\kappa/4$). This was also confirmed later by Hunter and Wadlinger [18] [19] using X-band microwaves. On the other hand, Mandel's experiments in the sixties [20] [21], employing the recently discovered laser technics, concluded that a single photon has circular polarization and cannot be localized in a length shorter than its wavelength λ_κ [22] and more generally it cannot be better localized than within a volume of the order of the cube of its wavelength (λ_κ^3) [23] [24]. Consequently, the experimental evidence conflicts with both the point photon model, upon which QED has been developed, and the classical continuous electromagnetic wave theory, showing that light must be composed of localized "wave packets" traveling in vacuum at the universal velocity c [25] carrying energy $h\nu$ and momentum $h\nu/c$. In reality, the point photon concept has permitted to establish an extremely efficient mathematical approach in QED for describing states before and after an interaction processes [26] [27] [28]. However, it is obvious that it is inappropriate for the description of the real nature of a single photon state.

The development of the revolutionary parametric down converters techniques [29] [30] in the seventies permitted to realize conditions in which, with an excellent statistical confidence, only a single photon is present in the experimental device. Employing these techniques, the double prism experiment [31] carried out in the nineties contradicted Bohr's *mutual exclusiveness principle* showing that a *single photon exhibits both the wave and particle natures in the same experiment*. Furthermore, Grangier *et al.* demonstrated experimentally the indivisibility of photons [28] [32] while the entangled states experiments, first observed by Kosher and Commins [33] and further investigated in the eighties [34] [35], have shown that the photon should be locally an integral entity during the detection procedure but with a real non-local wave function.

Finally, a synthesis of the experimental studies shows that a single k-mode photon is a local, indivisible segment of the electromagnetic field with circular left or right polarization corresponding to spin $\pm h/2\pi$ respectively. It carries a quantum of energy $h\nu$ and has a momentum $h\nu/c$. Its intrinsic spatial length extends over a wavelength λ_k and can only be detected within a volume $\sim \lambda_k^3$, yielding that its radial expansion should be proportional to a fraction of its wavelength. Consequently, it seems to be a local "wave-corpuscle" absorbed and emitted as a whole and guided by a non-local wave function.

All these physical characteristics yield particular difficulties for the description of a real single photon state and consequently for the definition of an appropriate wave function within the quantum mechanics concepts.

In what follows we take into account all the above experimental and theoretical facts in order to obtain a physical picture of the single photon and establish its wave function. We first give a brief presentation of the link between the classical and quantum theory through the vector potential and then we analyze the second quantization process resulting to a quantum description of the electromagnetic field. Then, we consider the spatial properties of a single photon state and enhance the quantization of the vector potential amplitude to a single photon level getting a physical representation conform to the experimental evidence. Next, we advance a photon wave function based on the vector potential quantization satisfying the wave propagation equation, Schrodinger's equation with the relativistic massless particle Hamiltonian and an equivalent equation for the vector potential. Finally, we discuss the characteristic properties as well as the normalization of the established photon wave-function.

2. The Electromagnetic Field Vector Potential in Classical and Quantum Theories

2.1. The Vector Potential: Classical to Quantum Link

Experiments have shown that the vector potential is not a mathematical artefact but a real physical field exerting a direct influence on charges [36] [37] [38] [39]. As that, it represents the fundamental link between the classical electromagnetic wave theory issued from Maxwell's equations and QED [8] [22] [28] [40] [41]. In the classical theory [3] [4] [5] [6] the energy density of a mode k of the electromagnetic wave depends on the square of the modulus of the electric field $\vec{E}_k(\vec{r},t)$ and the magnetic induction $\vec{B}_k(\vec{r},t)$ and writes

$$W_k(\vec{r},t)=\frac{1}{2}\left(\varepsilon_0\left|\vec{E}_k(\vec{r},t)\right|^2+\frac{1}{\mu_0}\left|\vec{B}_k(\vec{r},t)\right|^2\right) \tag{1}$$

with ε_0 and μ_0 being the electric permittivity and magnetic permeability of the vacuum respectively satisfying the relation $\varepsilon_0\mu_0c^2=1$, where c is the speed of light in vacuum.

For a monochromatic plane wave mode k with angular frequency ω_k and vector potential amplitude $a_{0k}(\omega_k)$ we have

$$\vec{E}_k(\vec{r},t)=-2\omega_k a_{0k}(\omega_k)\hat{\varepsilon}\sin\left(\vec{k}\cdot\vec{r}-\omega_k t\right) \tag{2}$$

$$\vec{B}_k(\vec{r},t)=-\frac{1}{c}2\omega_k a_{0k}(\omega_k)\left(\hat{k}\times\hat{\varepsilon}\right)\sin\left(\vec{k}\cdot\vec{r}-\omega_k t\right) \tag{3}$$

with $\hat{\varepsilon}$ the unit vector perpendicular to the propagation axis, $\left|\vec{k}\right|=2\pi/\lambda_k$ the wave-vector along the propagation axis and λ_k the wavelength of the mode k.

Using (2) and (3) in (1), the energy density writes as a function depending uniquely on the square of the vector potential amplitude and on the angular frequency

$$W_k(\vec{r},t)=4\varepsilon_0\omega_k^2\left|a_{0k}(\omega_k)\right|^2\sin^2\left(\vec{k}\cdot\vec{r}-\omega_k t\right) \tag{4}$$

The mean value of the last expression over a period, that is over a wavelength, is

$$\langle W_k\rangle=2\varepsilon_0\omega_k^2\left|a_{0k}(\omega_k)\right|^2 \tag{5}$$

It is important noting that the mean energy density $\langle W_k \rangle$ is independent on any external volume parameter. Hence, in the classical theory a free of cavity electromagnetic radiation mode occupies naturally a minimum volume while in a resonant cavity this volume corresponds roughly to that imposed by the boundary conditions and the cut-off wave vectors for the mode [4] [42].

Now, in the quantum description the energy density in a given volume V for a number $n(\omega_k)$ of k-mode photons, each with angular frequency ω_k and energy $\hbar\omega_k$ is simply

$$W_k = \frac{n(\omega_k)\hbar\omega_k}{V} \tag{6}$$

The link between the classical and the quantum theories is established by imposing the classical mean energy density over a period (5) to be equal to that of the quantum description (6). Thus, considering $n(\omega_k)=1$ we get the vector potential amplitude for a single k-mode photon

$$a_{0k}(\omega_k) = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}} \tag{7}$$

The last relation constitutes the fundamental link between the classical and quantum theories of light and it is applied in QED in order to define the vector potential amplitude operators for a single photon [23] [27] [28]

$$\tilde{a}_{k\lambda} = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}} a_{k\lambda}, \quad \tilde{a}^*_{k\lambda} = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}} a^+_{k\lambda} \tag{8}$$

where $a_{k\lambda}$ and $a^+_{k\lambda}$ are respectively the annihilation and creation non-Hermitian operators for a single k-mode and λ-polarization photon.

Obviously, this procedure introduces an external volume parameter V in the definitions of the single photon vector potential amplitude operators. Consequently, one could draw out that single photons in free space, in other words in a cavity with infinite dimensions, should have zero vector potential amplitudes and thus zero energy. This ambiguity, which is scarcely quoted in the literature, automatically implies that for a single photon the volume V in (8) cannot be an arbitrary external parameter but corresponds roughly to that defined by the boundary conditions in a cavity for the single radiation mode k.

2.2. The Electromagnetic Field Vector Potential in QED

In quantum field theory [23] [28] [40] the radiation vector potential writes

$$\vec{A}(\vec{r},t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k}} \sum_{\lambda=1}^{2} \left[\alpha_{k\lambda}(t) \mathrm{e}^{i\vec{k}\cdot\vec{r}} \hat{\varepsilon}_{k\lambda} + \alpha^+_{k\lambda}(t) \mathrm{e}^{-i\vec{k}\cdot\vec{r}} \hat{\varepsilon}^*_{k\lambda} \right] \tag{9}$$

where the discrete summation runs over two polarizations λ, circular left and right, and $\hat{\varepsilon}_{k\lambda}$ is the polarization complex unit vector.

According to the density of states theory the continuous summation over the modes k in (9) can be transformed to a discrete one over the k-mode photons by considering the electromagnetic field in a cavity of volume V so that we can put

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \to \sqrt{\frac{1}{V}}\sum_k \tag{10}$$

Therein, it is extremely important to note that the last relation is only valid for an ensemble of *k*-mode photons whose wavelengths λ_k are much shorter than the characteristic dimensions of the volume V

$$\lambda_k \ll V^{1/3} \quad (\forall k) \tag{11}$$

Adopting now Heisenberg's representation,

$$\alpha_{k\lambda}(t) = \alpha_{k\lambda}\mathrm{e}^{-i\omega_k t}; \quad \alpha^+_{k\lambda}(t) = \alpha^+_{k\lambda}\mathrm{e}^{i\omega_k t} \tag{12}$$

and using the transformation relation (10) the vector potential of the electromagnetic field writes in QED [23] [28]

$$\vec{A}(\vec{r},t) = \sum_k \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\sum_{\lambda=1}^{2}\left[\alpha_{k\lambda}\mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}_{k\lambda} + \alpha^+_{k\lambda}\mathrm{e}^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}^*_{k\lambda}\right] \tag{13}$$

where we note that the vector potential amplitude for each mode k is defined exactly as in (7).

Considering the scalar potential of the electromagnetic field to be constant in space then the electric field writes

$$\vec{E}(\vec{r},t) = -\partial\vec{A}(\vec{r},t)/\partial t = i\sum_k \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\sum_{\lambda=1}^{2}\left[\alpha_{k\lambda}\mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}_{k\lambda} - \alpha^+_{k\lambda}\mathrm{e}^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}^*_{k\lambda}\right] \tag{14}$$

The amplitudes in (13) and (14) have been obtained using the density of states theory and are valid only on the condition (11).

Now, the boundary conditions in cavities and waveguides impose the wave-vectors k of the modes to be higher than a characteristic cut-off value $k > k_{\text{cut-off}}$ ($\lambda_k < \lambda_{\text{cut-off}}$) depending on the dimensions as well as on the shape of the volume V containing the radiation field [42]. Consequently, for a volume V with finite dimensions the summation in (13) and (14) runs only over the modes k with wave-vectors higher than the minimum cut-off value $k_{\text{cut-off}}(V)$ corresponding to the shape and dimensions of V so that we can write more precisely

$$\vec{A}(\vec{r},t) = \sum_{k>k_{\text{cut-off}}(V)} \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\sum_{\lambda=1}^{2}\left[\alpha_{k\lambda}\mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}_{k\lambda} + \alpha^+_{k\lambda}\mathrm{e}^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}^*_{k\lambda}\right] \tag{15}$$

$$\vec{E}(\vec{r},t) = i\sum_{k>k_{\text{cut-off}}(V)} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\sum_{\lambda=1}^{2}\left[\alpha_{k\lambda}\mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}_{k\lambda} - \alpha^+_{k\lambda}\mathrm{e}^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}^*_{k\lambda}\right] \tag{16}$$

The last equations represent the vector potential and the electric field of a large number of *k*-mode photons in a finite volume V with $\lambda_k \ll V^{1/3}(\forall k)$.

Following the last relations, we can now draw the amplitude of the electric field $|\vec{\varepsilon}_k|$ and magnetic field $|\vec{\beta}_k|$ of a single *k*-mode photon

$$|\vec{\varepsilon}_k| = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}; \quad |\vec{\beta}_k| = \sqrt{\frac{\mu_0\hbar\omega_k}{2V}} \tag{17}$$

These expressions are often used in the literature for the free single photon electric and magnetic fields amplitudes. However, as mentioned above, the expressions (15) and (16) are only valid for a large number of photons in a cavity V under the condition (11) and consequently the attribution of the amplitudes (17) to free of cavity single photons has to be considered with caution.

3. Photon Wave Function from the Electric and Magnetic Fields

In quantum theory, a wave function $\Psi(\vec{r},t)$, depending on space and time, characterizes precisely the quantum state of a given particle. It satisfies Schrödinger's equation with eigenvalue the corresponding energy of the quantum state. However, in the case of the single photon, which is a relativistic massless "particle", the wave functions that have been proposed until now are not in general satisfactory. Nevertheless, some very fruitful studies have already been carried out in order to establish a photon wave function and they are given here in the bibliography [43]-[48] and commented extensively [49] [50] [51] [52] [53]. The most pertinent theoretical developments are based on the complex vector function $\vec{F}(\vec{r},t)$ initially introduced by Riemann

$$\vec{F}(\vec{r},t)=\frac{1}{2\sqrt{\varepsilon_0}}\vec{D}(\vec{r},t)+i\frac{1}{2\sqrt{\mu_0}}\vec{B}(\vec{r},t) \tag{18}$$

where the electric displacement flux density $\vec{D}(\vec{r},t)$ and the magnetic field flux density $\vec{B}(\vec{r},t)$ are defined respectively through the electric and magnetic fields intensities $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ following the expressions

$$\vec{D}_{\alpha\beta}(t)=\hat{\varepsilon}_{\alpha\beta}\cdot\vec{E}_{\alpha\beta}(t);\quad \vec{B}_{\alpha\beta}(t)=\hat{\mu}_{\alpha\beta}\cdot\vec{H}_{\alpha\beta}(t) \tag{19}$$

with the electric permittivity $\hat{\varepsilon}_{\alpha\beta}$ and magnetic permeability $\hat{\mu}_{\alpha\beta}$ being tensors ($\alpha=x,y,z$ and $\beta=x,y,z$) characterizing the intrinsic electric and magnetic nature of the medium in which the electric and magnetic fields propagate.

At that point it is important to mention that the electric field $\vec{E}(\vec{r},t)$ and the magnetic induction $\vec{B}(\vec{r},t)$ are fundamental fields while $\vec{D}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ are fields that include macroscopically the response of the medium. In the case of an isotropic medium, in which the electric and magnetic properties are the same in all directions, the electric field is parallel to the electric displacement while the magnetic field is parallel to the magnetic field flux intensity

$$\vec{D}(\vec{r},t)=\varepsilon\vec{E}(\vec{r},t);\quad \vec{H}(\vec{r},t)=\frac{1}{\mu}\vec{B}(\vec{r},t) \tag{20}$$

Using the complex function $\vec{F}(\vec{r},t)$ Maxwell's equations in an homogeneous medium become

$$i\frac{\partial}{\partial t}\vec{F}(\vec{r},t)=c\vec{\nabla}\times\vec{F}(\vec{r},t) \tag{21}$$

$$\vec{\nabla}\cdot\vec{F}(\vec{r},t)=0 \tag{22}$$

In this way, a suggested photon wave function was defined as

$$\Psi(\vec{r},t)=\begin{pmatrix}\frac{1}{2\sqrt{\varepsilon_0}}\vec{D}(\vec{r},t)+i\frac{1}{2\sqrt{\mu_0}}\vec{B}(\vec{r},t)\\ \frac{1}{2\sqrt{\varepsilon_0}}\vec{D}(\vec{r},t)-i\frac{1}{2\sqrt{\mu_0}}\vec{B}(\vec{r},t)\end{pmatrix} \quad (23)$$

which is a six component vector satisfying Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)=c\begin{pmatrix}-i\hbar\vec{\nabla}\cdot\vec{S} & 0\\ 0 & i\hbar\vec{\nabla}\cdot\vec{S}\end{pmatrix}\Psi(\vec{r},t) \quad (24)$$

where $\vec{S}$ is the spin matrix for a spin 1 particle with the Cartesian components

$$S_x=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad S_y=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix},\quad S_z=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix} \quad (25)$$

The square modulus of the above defined photon wave function $\Psi(\vec{r},t)$ provides the classical electromagnetic field energy density (1) at an instant t and at a given coordinate $\vec{r}$ on the propagation axis

$$\left|\Psi(\vec{r},t)\right|^2=W(\vec{r},t)=\frac{1}{2}\left(\varepsilon_0\left|\vec{E}(\vec{r},t)\right|^2+\frac{1}{\mu_0}\left|\vec{B}(\vec{r},t)\right|^2\right) \quad (26)$$

The energy is obtained directly by integrating in space the bilinear form of the complex vector function $\vec{F}(\vec{r},t)$

$$E=\int\vec{F}^*(\vec{r},t)\cdot\vec{F}(\vec{r},t)\mathrm{d}^3r \quad (27)$$

However, the relations (26) and (27) give respectively the energy density and the energy of the electromagnetic field but not precisely those of a single photon state. Furthermore, in quantum mechanics, the integration in space of the square modulus of the normalized photon wave function, given by (26), should correspond to the probability of localizing a photon and not to the energy of the classical electromagnetic field representation. In a characteristic single photon quantization volume on the propagation axis this quantity should be equal to unity and this is not really the case here for $\Psi(\vec{r},t)$.

Finally, some different versions from the one described above have been advanced [43]-[48] for the photon wave function but without introducing a real progress in the whole problematics.

4. Spatial Properties of a Single Photon and the Vector Potential Quantization

4.1. Lateral Expansion of a Single Photon, Theoretical and Experimental Facts

The moment of inertia I_k that may be attributed to a single k-mode photon with angular frequency ω_k and wavelength λ_k writes

$$I_k=\frac{\hbar}{\omega_k} \quad (28)$$

From the mass-energy equivalence the mass m_k, which is not a rest mass, that may be attributed to a k-mode photon is

$$m_k = \frac{\hbar \omega_k}{c^2} \tag{29}$$

In order to estimate the radial expansion of the photon we may assume here a homogeneous distribution of the field in the single photon state quantization volume and consider the mechanical analogue in two simple cases, that of a cylinder whose axis is along the propagation axis and that of a sphere.

For a cylinder we have

$$I_{\text{cylinder}} = \frac{1}{2} m_k r_{\text{cylinder}}^2 = \frac{1}{2} \frac{\hbar \omega_k}{c^2} r_{\text{cylinder}}^2 = \frac{\hbar}{\omega_k} \tag{30}$$

and consequently

$$r_{\text{cylinder}} = \left(\frac{2c^2}{\omega_k^2} \right)^{1/2} = \frac{\lambda_k}{\pi\sqrt{2}} \sim \frac{\lambda_k}{4.4} \tag{31}$$

In the case of a spherical distribution of the field inside the photon quantization volume we have

$$I_{\text{sphere}} = \frac{2}{5} m_k r_{\text{sphere}}^2 = \frac{2}{5} \frac{\hbar \omega_k}{c^2} r_{\text{sphere}}^2 = \frac{\hbar}{\omega_k} \tag{32}$$

so that the sphere radius is

$$r_{\text{sphere}} = \left(\frac{5}{2} \right)^{1/2} \frac{c}{\omega_k} = \frac{\lambda_k}{\pi\sqrt{8/5}} \sim \frac{\lambda_k}{3.9} \tag{33}$$

Consequently, from the above simple theoretical calculations we may presume that the radius of a single k-mode photon should be a fraction of its wavelength, roughly $\sim \lambda_k/4$.

From the experimental point of view, the fact that a single mode plane electromagnetic wave should have a minimum radius was already suspected since the very first applications of the Faraday grid shielding. As mentioned in the introduction, various experiments carried out by Robinson, Hadlock, Hunter and Wadlingler [16] [17] [18] [19] using microwaves and measuring the transmitted power through rectangular or circular apertures concluded that the photons lateral expansion should be indeed close to the quarter of its wavelength $\sim \lambda_k/4$, confirming the above simple calculations based on the moment of inertia.

On the other hand, it is well established [20] [21] [22] that a single photon cannot be conceived in a length shorter than its wavelength and it cannot be localized within a volume less than $\sim \lambda_k^3$ [21] [23] entailing that the single photon quantization volume should be proportional to λ_k^3, that is proportional to ω_k^{-3}.

4.2. Single Photon Quantization Volume

4.2.1. The Single Photon Quantization Volume from the Density of States Theory

According to the density of states theory, taking into account the two spin values

$\pm\hbar$, corresponding to circular left and right polarization, as well as the two possible directions $\pm z$ along the same propagation axis *z*, the number of *k*-mode photons $\mathrm{d}n(\omega_k)$ in the quantization volume V in the frequency interval between ω_k and $\omega_k + \mathrm{d}\omega_k$ writes [23] [27]

$$\mathrm{d}n(\omega_k) = 4V\frac{\omega_k^2}{\pi^2 c^3}\mathrm{d}\omega_k \tag{34}$$

where we have also considered that all the possible states in the volume V are occupied.

Thus,

$$n(\omega_k) = 4V\frac{\omega_k^3}{3\pi^2 c^3} \tag{35}$$

Considering now $n(\omega_k) = 1$ the corresponding quantization volume that could be attributed to a single photon is

$$V_k = \left(\frac{3}{4}\pi^2 c^3\right)\omega_k^{-3} \tag{36}$$

The last expression is in agreement with the experimental evidence that the single photon quantization volume is proportional to ω_k^{-3}, getting

$$V_k = \frac{3}{32\pi}\lambda_k^3 \simeq 0.029\lambda_k^3 \tag{37}$$

Thus, the minimum quantization volume corresponding to a single photon with a wavelength λ_k is roughly 3% of the volume λ_k^3.

4.2.2. The Single Photon Quantization Volume from the Energy Normalization

We can also obtain the single photon quantization volume in a different way by normalizing the electromagnetic energy of a plane-wave mode to that of a single photon

$$E_k = \int \varepsilon_0 \omega_k^2 \alpha_{0k}^2(\omega_k)\left[\hat{\varepsilon}_{k\lambda}\mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega_k t+\phi)} + \hat{\varepsilon}_{k\lambda}^*\mathrm{e}^{-i(\vec{k}\cdot\vec{r}-\omega_k t+\phi)}\right]^2 \mathrm{d}^3 r = \hbar\omega_k \tag{38}$$

where $\alpha_{0k}(\omega_k)$ represents the single *k*-mode photon vector potential amplitude and $\hat{\varepsilon}_{k\lambda}$ the polarization complex unit vector.

The last equation holds at any instant *t* if the *k*-mode photon polarization unit vector $\hat{\varepsilon}_{k\lambda}$ has two orthogonal components $\hat{e}_1$ and $\hat{e}_2$ such as $\hat{e}_1 \cdot \hat{e}_2 = 0$ and $\hat{\varepsilon}_{k\lambda} = \sigma_{k1}\hat{e}_1 + \sigma_{k2}\hat{e}_2$ with $|\sigma_{k1}|^2 + |\sigma_{k2}|^2 = 1$. Obviously, these conditions are satisfied naturally by circular left (L) and right (R) polarization unit vectors $\hat{\varepsilon}_{L,R} = \frac{1}{\sqrt{2}}(\hat{e}_1 \pm i\hat{e}_2)$, so that we have [49]

$$E_k = \int 2\varepsilon_0 \omega_k^2 \alpha_{0k}^2(\omega_k)\mathrm{d}^3 r = \hbar\omega_k \tag{39}$$

Now, the dimension analysis of the vector potential issued from the general solution of Maxwell's equations yields that it is proportional to an angular frequency [3] [4] [5] [6]

$$\vec{A}(\vec{r},t) = \frac{\mu}{4\pi}\int \frac{\vec{J}\left(\vec{r}',t-\left|\vec{r}-\vec{r}'\right|/c\right)}{\left|\vec{r}-\vec{r}'\right|}\mathrm{d}^3 r \propto \omega \tag{40}$$

where μ is the magnetic permeability and J is the current density ($\mathrm{C{\cdot}m^{-2}{\cdot}s^{-1}}$).

Thus, considering in (5) that the vector potential amplitude is proportional to ω then $\langle W_k \rangle$ is proportional to ω^4 which is confirmed experimentally for the energy density radiated by a dipole [5] [6]. The concluded property for the vector potential amplitude is related to its natural units and consequently is gauge independent.

Hence, we may consider that the vector potential amplitude $\alpha_{0k}(\omega_k)$ of a single k-mode photon can be expressed as follows [49] [50] [51]

$$\alpha_{0k}(\omega_k) = \xi\omega_k \tag{41}$$

where ξ is a constant to be determined through the normalization procedure.

Obviously, when introducing (41) in (39) an appropriate volume V_k has to be considered for the single k-mode photon for the equation to hold

$$E_k = 2\varepsilon_0\xi^2\omega_k^4 V_k = \hbar\omega_k \tag{42}$$

From the last relation we draw that the energy density of a single photon state is

$$W_k = 2\varepsilon_0\xi^2\omega_k^4 \tag{43}$$

It is worth noticing that the last expression depends on the fourth power of the angular frequency as in the case of the energy density radiated by a dipole [5] [6].

We now deduce from (42) that the characteristic volume of a cavity-free single photon writes [51] [53] [54]

$$V_k = \left(\frac{\hbar}{2\varepsilon_0\xi^2}\right)\omega_k^{-3} \tag{44}$$

which is in agreement with the experimental evidence regarding the frequency dependence.

Hence, in order to estimate ξ we can integrate (39) by using some experimental facts. Considering the propagation along the z axis, with ρ the lateral coordinate, and respecting the experimental results by imposing the limits $0 \le z \le \lambda_k$ and $0 \le \rho \le \eta\lambda_k$ [49] where η characterizes the radial extension, (39) gives

$$\varepsilon_0\xi^2\eta^2\omega_k^4\lambda_k^3 = (2\pi c)^3\,\varepsilon_0\xi^2\eta^2\omega_k = \hbar\omega_k \tag{45}$$

Hence,

$$\eta\xi = \pm\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{\hbar}{\varepsilon_0 c^3}} \tag{46}$$

Instead of using the approximate experimental value of $\eta \sim 1/4$ in (46) in order to fix ξ we can try to get a more precise result. In fact, Equation (45) writes with a slight rearrangement

$$4\pi c\left(\frac{1}{4\pi c}\varepsilon_0\omega_k^2\eta^2\alpha_{0k}(\omega_k)\lambda_k^3\right)\alpha_{0k}(\omega_k) = 4\pi c\,Q\,\alpha_{0k}(\omega_k) = \hbar\omega_k \tag{47}$$

where Q has charge units. Using (41) and (46) we get

$$Q = \frac{1}{4\pi c}\varepsilon_0 \omega_k^2 \eta^2 \alpha_0(\omega_k)\lambda_k^3 = \pm\frac{\eta}{2}\sqrt{\varepsilon_0 hc} \quad (48)$$

with h being Planck's constant.

It is of high importance to underline here that when introducing in (48) the approximate experimental value for $\eta \sim 1/4$ we get

$$Q \simeq \pm 1.6\times 10^{-19}\ \text{C} \quad (49)$$

which is the electron-positron charge, a physical constant appearing naturally [51] in the normalization process.

We draw that the physical origin of the elementary charge is strongly related to the photon vector potential [51] [53] [54]. Now, we can define more precisely η and ξ by replacing Q in (48) by the elementary charge e and using the fine structure constant $\alpha = e^2/4\pi\varepsilon_0\hbar c = 1/137$, we get

$$\eta = \sqrt{8\alpha} \quad (50)$$

Introducing (50) in (46) we obtain the vector potential amplitude normalization constant [50] [51] [53] [54]

$$\xi = \pm\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{\hbar}{8\alpha\varepsilon_0 c^3}} = \pm\frac{\hbar}{4\pi|e|c} = \pm 1.747\times 10^{-25}\ \text{Volt}\cdot\text{m}^{-1}\cdot\text{s}^2 \quad (51)$$

Using (51) in (44) the single k-mode photon quantization volume writes now

$$V_k = \left(\frac{\hbar}{2\varepsilon_0\xi^2}\right)\omega_k^{-3} = \frac{\eta^2}{2}\lambda_k^3 = 4\alpha\lambda_k^3 \simeq 0.029\lambda_k^3 \quad (52)$$

which is equivalent to (37).

Consequently, the helicoidally distributed vector potential field of a single photon state over a wavelength occupies roughly 3% of the λ_k^3 volume, as we have seen before, and has a wave front section [54]

$$S_k = 8\pi\alpha\lambda_k^2 \simeq 0.18\lambda_k^2 \quad (53)$$

The last relations are useful for technological applications like wave-guides and fiber optics transmissions or Faraday grid shielding…etc.

Thus, the fundamental properties of the photon, energy E_k, momentum $\vec{p}_k$ and wave-vector $\vec{k}$, are complemented by the vector potential amplitude α_{0k} expressing its intrinsic electromagnetic nature

$$E_k/\hbar = |\vec{p}_k|c/\hbar = |\vec{k}|c = \alpha_{0k}/|\xi| = \omega_k \quad (54)$$

From the relation (54) and Heisenberg's *energy-time* uncertainty a *vector potential-time* uncertainty is drawn

$$\delta E_k \delta t \geq \hbar \rightarrow \delta\alpha_{0k}\delta t \geq |\xi| \quad (55)$$

showing that the particle (energy) and the wave (vector potential) are intrinsic properties of the single photon nature.

5. Single Photon Wave-Particle (Classical-Quantum) Physical Properties

Following the above considerations, the vector potential for a free single k-mode photon with λ-polarization (circular left or right) writes in both classical and quantum formalism respectively [53]

$$\begin{aligned}\tilde{\alpha}_{k\lambda}(\vec{r},t) &= \xi\,\omega_k\left[\hat{\varepsilon}_{k\lambda}\alpha_{k\lambda}\mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega_k t)}+\hat{\varepsilon}^*_{k\lambda}\alpha^+_{k\lambda}\mathrm{e}^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\right]\\ \vec{\alpha}_{k\lambda}(\vec{r},t) &= \xi\,\omega_k\left[\hat{\varepsilon}_{k\lambda}\mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega_k t)}+\hat{\varepsilon}^*_{k\lambda}\mathrm{e}^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\right]\end{aligned} \tag{56}$$

For a free k-mode photon, the volume V_k expressed by (37) and (52) corresponds to the space in which the quantized vector potential rotates around the propagation axis at the angular frequency ω_k over a period (that is over a wavelength λ_k). This precession generates orthogonal electric and magnetic fields whose amplitudes are

$$\left|\vec{\varepsilon}_k\right| = \left|-\partial\vec{\alpha}_{k\lambda}(\vec{r},t)/\partial t\right| \propto |\xi|\omega_k^2;\quad \left|\vec{\beta}_k\right| \propto \sqrt{\varepsilon_0\mu_0}\,|\xi|\,\omega_k^2 \tag{57}$$

It is important to note that last expressions are directly proportional to the square of the angular frequency and do not depend on any external volume parameter [53] [54] [55].

The quantum properties of a single photon, energy, momentum and spin result readily from the integration of the classical electromagnetic expressions over the quantization volume V_k, linking by this way the classical (wave) to the quantum (particle) representations [51] [53] [54]. Indeed, using (41) and (52) the energy writes

$$E_k = \int_{V_k} 2\varepsilon_0\alpha_{0k}^2\omega_k^2\mathrm{d}^3r = \int_{V_k} 2\varepsilon_0\xi^2\omega_k^4\mathrm{d}^3r = 2\varepsilon_0\xi^2\omega_k^4 V_k = \hbar\omega_k \tag{58}$$

Considering circular polarizations for the electric and magnetic fields components and using (57) the momentum is obtained directly

$$\vec{p}_k = \int_{V_k}\varepsilon_0\vec{\varepsilon}_{k\lambda}\times\vec{\beta}_{k\lambda}\mathrm{d}^3r = \varepsilon_0\left(\sqrt{2}\omega_k\alpha_{0k}\right)\left(\sqrt{2}\omega_k\alpha_{0k}/c\right)V_k\vec{k}/\left|\vec{k}\right| = \hbar\vec{k} \tag{59}$$

According to the classical electromagnetic theory the spin writes through the electric and magnetic fields components, hence, using again the circular polarization we get

$$\left|\vec{S}\right| = \int_{V_k}\varepsilon_0\vec{r}\times\left(\vec{\varepsilon}_{k\lambda}\times\vec{\beta}_{k\lambda}\right)\mathrm{d}^3r = \pm\varepsilon_0\left(c/\omega_k\right)\left(\sqrt{2}\omega_k\alpha_{0k}\right)\left(\sqrt{2}\omega_k\alpha_{0k}/c\right)V_k = \pm\hbar \tag{60}$$

where we have considered the mean value of the distance [56] for a single photon state $\langle|\vec{r}|\rangle_k = c/\omega_k$

The last relations show that the quantum properties of the single photon can be obtained from the classical electromagnetic field considered in the quantization volume V_k entailing that the photon has naturally a three dimensional extension and consequently the term "wave-corpuscle" should be more appropriate.

Thus, a photon is not a point particle and Heisenberg's uncertainty relation for the position and momentum is precisely due to its spatial properties and more precisely to the quantization volume V_k. Indeed, replacing V in (8) by V_k we get the single photon vector potential amplitude operators expressed through the annihilation and creation operators [51] [53]

$$\tilde{\alpha}_{0k\lambda} = \xi\omega_k \alpha_{k\lambda}; \quad \tilde{\alpha}^*_{0k\lambda} = \xi\omega_k \alpha^+_{k\lambda} \tag{61}$$

The corresponding position $\tilde{Q}_{k\lambda}$ and momentum $\tilde{P}_{k\lambda}$ Hermitian operators [23] [27] [28] write

$$\tilde{Q}_{k\lambda} = \sqrt{\varepsilon_0 V_k}\left(\tilde{\alpha}_{0k\lambda} + \tilde{\alpha}^*_{0k\lambda}\right); \quad \tilde{P}_{k\lambda} = -i\omega_k\sqrt{\varepsilon_0 V_k}\left(\tilde{\alpha}_{0k\lambda} - \tilde{\alpha}^*_{0k\lambda}\right) \tag{62}$$

Thus, introducing (61) in (62) and using (52) Heisenberg's commutation relation, a fundamental concept in quantum theory, results directly [51] [53] [54]

$$\left[\tilde{Q}_{k\lambda}, \tilde{P}_{k'\lambda'}\right] = -i\varepsilon_0\omega^2_{k'}\omega_k\sqrt{V_k V_{k'}}\left[\left(\xi a_{k\lambda} + \xi a^+_{k\lambda}\right), \left(\xi a_{k'\lambda'} - \xi a^+_{k'\lambda'}\right)\right] = i\hbar\delta_{kk'}\delta_{\lambda\lambda'} \tag{63}$$

The position-momentum uncertainty, as well as the energy and vector potential uncertainties with respect to time are intrinsic physical properties of the nature of the photon.

By this way, the general equation for the vector potential of the electromagnetic wave considered as a superposition of plane wave modes writes

$$\vec{A}(\vec{r},t) = \sum_{k,\lambda}\xi\omega_k\left[\hat{\varepsilon}_{k\lambda}\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)} + \hat{\varepsilon}^*_{k\lambda}\mathrm{e}^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)}\right] \tag{64}$$

and the vector potential operator for a large number of cavity free photons in quantum electrodynamics is

$$\tilde{A} = \sum_{k,\lambda}\xi\omega_k\left[a_{k\lambda}\hat{\varepsilon}_{k\lambda}\mathrm{e}^{i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)} + a^+_{k\lambda}\hat{\varepsilon}^*_{k\lambda}\mathrm{e}^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)}\right] \tag{65}$$

In what follows, the photon wave-particle equation is introduced showing that the vector potential with the quantized amplitude can be naturally a wave function for the photon.

6. Wave-Particle Equation for the Photon and the Vector Potential Wave Function

It can be readily demonstrated that the vector potential $\vec{\alpha}_{k\lambda}(\vec{r},t)$ with the quantized amplitude $\xi\omega_k$ expressed by (41) satisfies Maxwell's wave propagation

$$\vec{\nabla}^2_k\vec{\alpha}_{k\lambda}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\alpha}_{k\lambda}(\vec{r},t) = 0 \tag{66}$$

and the photon *vector potential – energy* (*wave – particle*) equation [51] [53] [54]

$$i\begin{pmatrix}\xi \\ \hbar\end{pmatrix}\frac{\partial}{\partial t}\vec{\alpha}_{k\lambda}(\vec{r},t) = \begin{pmatrix}\tilde{\alpha}_{0k} \\ \tilde{H}_k\end{pmatrix}\vec{\alpha}_{k\lambda}(\vec{r},t) \tag{67}$$

where the operator $\vec{\nabla}_k$ acts upon the mode *k*.

Equation (67) expresses both the first (energy) and second (vector potential) quantization of the electromagnetic field.

Obviously, the relativistic Hamiltonian for a massless particle $\tilde{H}_k = -i\hbar c\vec{\nabla}_k$ as well as the vector potential amplitude operator $\tilde{\alpha}_{0k} = -i\xi c\vec{\nabla}_k$ have the eigenvalues $\hbar\omega_k$ and $\xi\omega_k$ respectively [51] [53] [54] [55] [57] [58] [59]. It is worth noting the symmetry between $\{E_k, \hbar\}$ and $\{\alpha_{0k}, \xi\}$ characterizing respectively the particle (energy) and electromagnetic wave (vector potential) intrinsic natures of a single photon.

Consider now the emission in vacuum of a single *k*-mode photon at the coordinate $\vec{r}$ at an instant *t*. The probability to be localized at the instant *t*′ at the coordinate $\vec{r}' = \vec{r} + c(t'-t)\hat{r}$ can be obtained by the square of the modulus of the vector potential function $\vec{\alpha}_{k\lambda}(\vec{r},t)$

$$\left|\vec{\alpha}_{k\lambda}(z,t)\right|^2 = \xi^2\omega_k^2 \propto \lambda_k^{-2} \tag{68}$$

It is evident that the higher the frequency, the shorter the photon wavelength and the higher the localization probability. This is in agreement with Heisenberg's uncertainty principle as well as with the experimental evidence and consequently the vector potential function $\vec{\alpha}_{k\lambda}(\vec{r},t)$ can be considered as a real wave function for the photon that can be suitably normalized.

The photon momentum according to (59) is $\vec{p}_k = \hbar\vec{k} = (h/\lambda_k)\hat{k}$ and considering the propagation along the *z*-axis Heisenberg's position-momentum uncertainty, expressed by (63), writes now simply

$$\delta z \delta p_{k_z} \geq h \rightarrow \delta z \delta(1/\lambda_k) \geq 1 \tag{69}$$

The physical meaning of the last relation is that the localization uncertainty of a single photon on the propagation axis is of the order of the wavelength λ_k. Furthermore, as already mentioned, a single photon is localizable only within a volume proportional to λ_k^3.

Now, taking into account the above considerations and weighting the vector potential function $\vec{\alpha}_{k\lambda}(\vec{r},t)$ by the factor $(\varepsilon_0\omega_k/\hbar)^{1/2}$ a general wave function for the photon can be defined

$$\Phi_{k,(L,R)}(\vec{r},t) = \left(\frac{\varepsilon_0\omega_k}{\hbar}\right)^{1/2}\left[\alpha_{0k}(\omega_k)\left(\hat{\varepsilon}_{k(L,R)}\mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + \hat{\varepsilon}^*_{k,(L,R)}\mathrm{e}^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\right)\right] \tag{70}$$

where the polarization can be either left (*L*) or right (*R*) circular.

It can be readily demonstrated that $\Phi_{k,(L,R)}(\vec{r},t)$ satisfies the wave propagation equation (66) and the *vector potential – energy* equation (67).

In addition, whatever the circular polarization, the square of the modulus of the normalized general wave function (70) gives the inverse of the single photon quantization volume (52)

$$\left|\Phi_{k,(L,R)}(\vec{r},t)\right|^2 = \frac{\varepsilon_0\omega_k}{\hbar}\left[\xi^2\omega_k^2\left(\left|\hat{\varepsilon}_{k,(L,R)}\right|^2 + \left|\hat{\varepsilon}^*_{k,(L,R)}\right|^2\right)\right] = \frac{2\varepsilon_0\xi^2\omega_k^3}{\hbar} = \frac{1}{V_k} \tag{71}$$

so that

$$\int_{V_k}\left|\Phi_{k,(L,R)}(\vec{r},t)\right|^2 \mathrm{d}^3 r = 1 \tag{72}$$

The probability to find the photon in the volume V_k around the coordinate $\vec{r}$

on the propagation axis equals one.

On the other hand, the mean value of the relativistic massless particle Hamiltonian in the $\Phi_{k,(L,R)}(\vec{r},t)$ photon state is Planck's single photon energy

$$\begin{aligned}&\left|\left\langle \Phi_{k,(L,R)}(\vec{r},t)\right| H_k \left|\Phi_{k,(L,R)}(\vec{r},t)\right\rangle\right| = \left|\left\langle \Phi_{k,(L,R)}(\vec{r},t)\right| -i\hbar c\vec{\nabla}_k \left|\Phi_{k,(L,R)}(\vec{r},t)\right\rangle\right| \\ &= \left|\varepsilon_0\xi^2\omega_k^4\int_{V_k}\left(e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}_{k,(L,R)} + e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}^*_{k,(L,R)}\right)\left(e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}_{k,(L,R)} - e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}^*_{k,(L,R)}\right)\mathrm{d}^3r\right| \\ &= \varepsilon_0\xi^2\omega_k^4\int_{V_k}\left(\left|e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}_{k,(L,R)}\right|^2 + \left|e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\hat{\varepsilon}^*_{k,(L,R)}\right|^2\right)\mathrm{d}^3r \\ &= 2\varepsilon_0\xi^2\omega_k^4 V_k = \hbar\omega_k\end{aligned} \quad (73)$$

while the energy density is obtained directly by

$$W_k = \left|\Phi_{k,(L,R)}(\vec{r},t)\right|^2 \left|\left\langle \Phi_{k,(L,R)}(\vec{r},t)\right| H_k \left|\Phi_{k,(L,R)}(\vec{r},t)\right\rangle\right| = \frac{\hbar\omega_k}{V_k} = 2\varepsilon_0\xi^2\omega_k^4 \quad (74)$$

By the same token the mean value of the momentum operator is

$$\left|\left\langle \Phi_{k,(L,R)}(\vec{r},t)\right| \vec{p}_k \left|\Phi_{k,(L,R)}(\vec{r},t)\right\rangle\right| = \left|\left\langle \Phi_{k,(L,R)}(\vec{r},t)\right| -i\hbar\vec{\nabla}_k \left|\Phi_{k,(L,R)}(\vec{r},t)\right\rangle\right| = \frac{\hbar\omega_k}{c} \quad (75)$$

Consequently, the photon wave function defined in (70) obtained from the vector potential quantization at a single photon level, satisfies the propagation equation (66), the vector potential – energy equation (67), the normalization probability condition (72) and represents a real eigenstate of the relativistic massless particle Hamiltonian and the corresponding momentum operator with eigenvalues $\hbar\omega_k$ and $\hbar|\vec{k}|$ respectively.

7. Epilogue

Based on the experimental confirmation that the vector potential is a real physical entity we advanced theoretical developments complementing the standard formalism for the description of a single photon and the definition of a wave function.

The quantization of the vector potential amplitude of a cavity free *k*-mode photon with angular frequency ω_k is $\alpha_{0k} = \xi\omega_k$, where $\xi = \hbar/4\pi ec$, and yields naturally a *vector potential-energy* (electromagnetic wave-particle) equation expressing the simultaneous wave-particle nature of the photon. A single photon is a local indivisible part of the electromagnetic field extending over a wavelength λ_k and composed of the quantized vector potential rotating at the angular frequency ω_k with circular left or right polarization corresponding respectively to spin $\pm\hbar$. The precession of the vector potential gives birth to orthogonally oscillating electric and magnetic fields whose amplitudes are proportional to the square of the angular frequency $|\xi|\omega_k^2$. The lateral expansion of the photon has been estimated experimentally yielding a photon quantization volume V_k proportional to λ_k^3. The energy and momentum of the photon as well as the spin are obtained from the classical electromagnetic expressions integrated over the volume V_k showing that the photon is not a point particle but rather a "wave-corpuscle". Indeed, it is readily demonstrated that the origin of Heisenberg's uncertainty lays precisely on the photon spatial properties.

The quantized vector potential $\Phi_{k,(L,R)}(\vec{r},t)$, with circular left (*L*) or right (*R*) polarization, defined in Equation (70) satisfies both the propagation equation and the *vector potential-energy* equation behaving as a natural wave function for the photon. In fact, the square modulus of its amplitude is proportional to the square of the angular frequency and consequently can be employed for defining a localization probability yielding that the higher the frequency the better the localization in agreement with the experiments. Furthermore, the normalization factor $(\varepsilon_0\omega_k/\hbar)^{1/2}$ ensures that the summation of $\left|\Phi_{k,(L,R)}(\vec{r},t)\right|^2$ over the quantization volume V_k equals unity. Also, the mean value of the relativistic massless particle Hamiltonian $\left|\left\langle\Phi_{k,(L,R)}(\vec{r},t)\right|-i\hbar c\vec{\nabla}_k\left|\Phi_{k,(L,R)}(\vec{r},t)\right\rangle\right|$ equals the single photon energy $\hbar\omega_k$ while that of the momentum operator $\left|\left\langle\Phi_{k,(L,R)}(\vec{r},t)\right|-i\hbar\vec{\nabla}_k\left|\Phi_{k,(L,R)}(\vec{r},t)\right\rangle\right|$ equals the single photon momentum $\hbar\left|\vec{k}\right|$.

Thus, a photon appears to be a local three-dimensional electromagnetic "wave-corpuscle" carrying the energy quantum $\hbar\omega_k$ and the momentum $\hbar\left|\vec{k}\right|$, guided by the non-local vector potential function $\Phi_{k,(L,R)}(\vec{r},t)$ with circular polarization corresponding to spin $\pm\hbar$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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