IMPROVED STABILITY ESTIMATES FOR SOLVING STOKES PROBLEM WITH FORTIN-SOULIE FINITE ELEMENTS

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Abstract. We propose to analyse the discretization of the Stokes problem with nonconforming finite elements in light of the T-coercivity. First we explicit the stability constants with respect to the shape regularity parameter for order 1 in 2 or 3 dimension, and order 2 in 2 dimension. In this last case, we improve the result of the original Crouzeix-Raviart paper. Second, we illustrate the importance of using a divergence-free velocity reconstruction on some numerical experiments.

Keywords. Stokes problem, T-coercitivity, Fortin-Soulie finite elements, Fortin operator

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1. Introduction

The Stokes problem describes the steady state of incompressible Newtonian flows. They are derived from the Navier–Stokes equations [1]. With regard to numerical analysis, the study of Stokes problem helps to build an appropriate approximation of the Navier–Stokes equations. We consider here a discretization with nonconforming finite elements [2, 3]. We propose to state the discrete inf-sup condition in light of the T-coercivity (cf. [4] for Helmholtz-like problems, see [5], [6] and [7] for the neutron diffusion equation), which allows to estimate the discrete error constant. In Section 2, we recall the T-coercivity theory [4], which is known to be an equivalent reformulation of the Banach–Nečas–Babuška Theorem. In Section 3 we apply it to the continuous Stokes Problem. We give details on the triangulation in Section 4, and we apply the T-coercivity to the discretization of Stokes problem with nonconforming mixed finite elements in Section 5. For the Stokes problem, in the discrete case, this amounts to finding a Fortin operator. In Section 6 (resp. 7), we precise the proof of the well-posedness in the case of order 1 (resp. order 2) nonconforming mixed finite elements. In Section 8, we illustrate the importance of using a divergence-free velocity on some numerical experiments.

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2. T-coercivity

We recall here the T-coercivity theory as written in [4]. Consider first the variational problem, where V and W are two Hilbert spaces and $f \in V'$:

$$\text{Find } u \in V \text{ such that } \forall v \in W,\ a(u,v) = \langle f, v\rangle_V. \tag{2.1}$$

Classically, we know that Problem (2.1) is well-posed if $a(\cdot,\cdot)$ satisfies the stability and the solvability conditions of the so-called Banach–Nečas–Babuška (BNB) Theorem (see a.e. [8, Thm. 25.9]). For some models, one can also prove the well-posedness using the T-coercivity theory (cf. [4] for Helmholtz-like problems, see [5], [6] and [7] for the neutron diffusion equation).

Definition 1. *Let V and W be two Hilbert spaces and $a(\cdot,\cdot)$ be a continuous and bilinear form over $V \times W$. It is T-coercive if*

$$\exists T \in \mathcal{L}(V,W),\ \textit{bijective},\ \exists \alpha > 0,\ \forall v \in V,\ |a(v,Tv)| \geq \alpha \|v\|_V^2. \tag{2.2}$$

It is proved in [4, 9] that the T-coercivity condition is equivalent to the stability and solvability conditions of the BNB Theorem. Whereas the BNB theorem relies on an abstract inf–sup condition, T-coercivity uses explicit inf–sup operators, both at the continuous and discrete levels.

Theorem 1. *(well-posedness) Let $a(\cdot,\cdot)$ be a continuous and bilinear form. Suppose that the form $a(\cdot,\cdot)$ is T-coercive. Then Problem (2.1) is well-posed.*

3. Stokes problem

Let Ω be a connected bounded domain of $\mathbb{R}^d$, $d = 2, 3$, with a polygonal ($d = 2$) or Lipschitz polyhedral ($d = 3$) boundary $\partial\Omega$. We consider Stokes problem:

$$\text{Find } (\mathbf{u}, p) \text{ such that } \left\{ \begin{array}{rcl} -\nu\, \boldsymbol{\Delta}\, \mathbf{u} + \mathbf{grad}\, p & = & \mathbf{f}, \\ \operatorname{div} \mathbf{u} & = & 0. \end{array} \right. \tag{3.1}$$

with Dirichlet boundary conditions for the velocity $\mathbf{u}$ and a normalization condition for the pressure p:

$$\mathbf{u} = 0 \text{ on } \partial\Omega, \qquad \int_\Omega p = 0.$$

The vector field $\mathbf{u}$ represents the velocity of the fluid and the scalar field p represents its pressure divided by the fluid density which is supposed to be constant. The first equation of (3.1) corresponds to the momentum balance equation and the second one corresponds to the conservation of the mass. The constant parameter $\nu > 0$ is the kinematic viscosity of the fluid. The vector field $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ represents a body forces divided by the fluid density.

Before stating the variational formulation of Problem (3.1), we provide some definition and reminders. Let us set $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^d$, $\mathbf{H}^{-1}(\Omega) = (H^{-1}(\Omega))^d$ its dual space and $L^2_{zmv}(\Omega) = \{q \in L^2(\Omega) \,|\, \int_\Omega q = 0\}$. We recall that $\mathbf{H}(\mathrm{div};\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,|\, \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$. Let us first recall Poincaré-Steklov inequality:

$$\exists C_{PS} > 0 \,|\, \forall v \in H_0^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C_{PS} \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}. \tag{3.2}$$

Thanks to this result, in $H_0^1(\Omega)$, the semi-norm is equivalent to the natural norm, so that the scalar product reads $(v,w)_{H_0^1(\Omega)} = (\mathbf{grad}\, v, \mathbf{grad}\, w)_{\mathbf{L}^2(\Omega)}$ and the norm is $\|v\|_{H_0^1(\Omega)} = \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}$. Let $\mathbf{v}$, $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$. We denote by $(v_i)_{i=1}^d$ (resp.

$(w_i)_{i=1}^d$) the components of $\mathbf{v}$ (resp. $\mathbf{w}$), and we set $\mathbf{Grad}\,\mathbf{v} = (\partial_j v_i)_{i,j=1}^d \in \mathbb{L}^2(\Omega)$, where $\mathbb{L}^2(\Omega) = [L^2(\Omega)]^{d\times d}$. We have:

$$(\mathbf{Grad}\,\mathbf{v}, \mathbf{Grad}\,\mathbf{w})_{\mathbb{L}^2(\Omega)} = (\mathbf{v}, \mathbf{w})_{\mathbf{H}_0^1(\Omega)} = \sum_{i=1}^d (v_i, w_i)_{H_0^1(\Omega)}$$

and:

$$\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} = \left(\sum_{j=1}^d \|v_j\|^2_{H_0^1(\Omega)} \right)^{1/2} = \|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}.$$

Let us set $\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0 \right\}$. The vector space $\mathbf{V}$ is a closed subset of $\mathbf{H}_0^1(\Omega)$. We denote by $\mathbf{V}^\perp$ the orthogonal of $\mathbf{V}$ in $\mathbf{H}_0^1(\Omega)$. Let $\nu_p > 0$ be a kinematic viscosity. We recall that [1, cor. I.2.4]:

Proposition 1. *The operator* $\operatorname{div} : \mathbf{H}_0^1(\Omega) \to L^2(\Omega)$ *is an isomorphism of* $\mathbf{V}^\perp$ *onto* $L^2_{zmv}(\Omega)$*. We call* C_{div} *the constant such that:*

$$\forall p \in L^2_{zmv}(\Omega),\ \exists! \mathbf{v} \in \mathbf{V}^\perp \mid \operatorname{div} \mathbf{v} = p \text{ and } \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} \le C_{\mathrm{div}} \|p\|_{L^2(\Omega)}. \tag{3.3}$$

The constant C_{div} depends only on the domain Ω. Notice that we have: $C_{\mathrm{div}} = 1/\beta(\Omega)$ where $\beta(\Omega)$ is the inf-sup condition (or Ladyzhenskaya–Babuška–Brezzi condition):

$$\beta(\Omega) = \inf_{q \in L^2_{zmv}(\Omega)\setminus\{0\}} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)\setminus\{0\}} \frac{(q, \operatorname{div} \mathbf{v})_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)} \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)}}. \tag{3.4}$$

Generally, the value of $\beta(\Omega)$ is not known explicitly. In [10], Bernardi et al established results on the discrete approximation of $\beta(\Omega)$ using conforming finite elements. Recently, Gallistl proposed in [11] a numerical scheme with adaptive meshes for computing approximations to $\beta(\Omega)$. In the case of $d = 2$, Costabel and Dauge [12] established the following bound:

Theorem 2. *Let* $\Omega \subset \mathbb{R}^2$ *be a domain contained in a ball of radius* R*, star-shaped with respect to a concentric ball of radius* ρ*. Then*

$$\beta(\Omega) \ge \frac{\rho}{\sqrt{2}\,R} \left(1 + \sqrt{1 - \frac{\rho^2}{R^2}} \right)^{-1/2} \ge \frac{\rho}{2\,R}. \tag{3.5}$$

Let us detail the bound for some remarkable domains. If Ω is a ball, $\beta(\Omega) \ge \frac{1}{2}$ and if Ω is a square, $\beta(\Omega) \ge \frac{1}{2\sqrt{2}}$. Suppose now that Ω is stretched in some direction by a factor k, then $\beta(\Omega) \ge \frac{1}{2\,k}$. Finally, if Ω is L-shaped (resp. cross-shaped) such that $L = k\,l$, where L is the largest length and l is the smallest length of an edge, then $\beta(\Omega) \ge \frac{1}{2\sqrt{2}\,k}$ (resp. $\beta(\Omega) \ge \frac{1}{4\,k}$).

The variational formulation of Problem (3.1) reads:
Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$ such that

$$\left\{ \begin{array}{rcll} \nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} & = & \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\ ; \\ (q, \operatorname{div} \mathbf{u})_{L^2(\Omega)} & = & 0 & \forall q \in L^2_{zmv}(\Omega). \end{array} \right. \tag{3.6}$$

Classically, one proves that Problem (3.6) is well-posed using Poincaré-Steklov inequality (3.2) and Prop. 1. Check for instance the proof of [1, Thm. I.5.1].

Let us set $\mathcal{X} = \mathbf{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$ which is a Hilbert space which we endow with the following norm:

$$\|(\mathbf{v}, q)\|_{\mathcal{X}} = \left(\|\mathbf{v}\|^2_{\mathbf{H}_0^1(\Omega)} + \nu^{-2} \|q\|^2_{L^2(\Omega)} \right)^{1/2}. \tag{3.7}$$

We consider now the following bilinear symmetric and continuous form:

$$\left\{ \begin{array}{rcl} a_S : \mathcal{X} \times \mathcal{X} & \to & \mathbb{R} \\ (\mathbf{u}', p') \times (\mathbf{v}, q) & \mapsto & \nu(\mathbf{u}', \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p', \operatorname{div} \mathbf{v})_{L^2(\Omega)} - (q, \operatorname{div} \mathbf{u}')_{L^2(\Omega)} \end{array} \right. . \tag{3.8}$$

We can write Problem (3.1) in an equivalent way as follows:

$$\text{Find } (\mathbf{u}, p) \in \mathcal{X} \text{ such that } \quad a_S\left((\mathbf{u}, p), (\mathbf{v}, q)\right) = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} \quad \forall (\mathbf{v}, q) \in \mathcal{X}. \tag{3.9}$$

Let us prove that Problem (3.9) is well-posed using the T-coercivity theory.

Proposition 2. *The bilinear form $a_S(\cdot, \cdot)$ is T-coercive:*

$$\begin{gathered} \exists T \in \mathcal{L}(\mathcal{X}), \ \textit{bijective}\ , \exists \alpha > 0, \forall (\mathbf{u}', p') \in \mathcal{X}, \\ a_S\left((\mathbf{u}', p'), T\left((\mathbf{u}', p') \right) \right) \geq \alpha \, \|(\mathbf{u}', p')\|^2_{\mathcal{X}}. \end{gathered} \tag{3.10}$$

Proof. We follow here the proof given in [13, 14]. Let us consider $(\mathbf{u}', p') \in \mathcal{X}$ and let us build $(\mathbf{v}^\star, q^\star) = T(\mathbf{u}', p') \in \mathcal{X}$ satisfying (2.2) (with $V = \mathcal{X}$). We need three main steps.

1. According to Prop. 1, there exists $\tilde{\mathbf{v}}_{p'} \in \mathbf{H}_0^1(\Omega)$ such that: $\operatorname{div} \tilde{\mathbf{v}}_{p'} = p'$ in Ω and $\|\tilde{\mathbf{v}}_{p'}\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\text{div}} \|p'\|_{L^2(\Omega)}$. Let us set $\mathbf{v}_{p'} = \nu^{-1} \tilde{\mathbf{v}}_{p'}$ so that $\operatorname{div} \mathbf{v}_{p'} = \nu^{-1} p'$ and

$$\|\mathbf{v}_{p'}\|_{\mathbf{H}_0^1(\Omega)} \leq \nu^{-1} C_{\text{div}} \|p'\|_{L^2(\Omega)}. \tag{3.11}$$

   Let us set $(\mathbf{v}^\star, q^\star) := (\gamma\, \mathbf{u}' - \mathbf{v}_{p'}, -\gamma\, p')$, with $\gamma > 0$. We obtain:

$$a_S\left((\mathbf{u}', p'), (\mathbf{v}^\star, q^\star) \right) = \nu\, \gamma \, \|\mathbf{u}'\|^2_{\mathbf{H}_0^1(\Omega)} + \nu^{-1} \|p'\|^2_{L^2(\Omega)} - \nu\, (\mathbf{u}', \mathbf{v}_{p'})_{\mathbf{H}_0^1(\Omega)}. \tag{3.12}$$

2. In order to bound the last term of (3.12), we use Young inequality and then inequality (3.11), so that for all $\eta > 0$:

$$(\mathbf{u}', \mathbf{v}_{p'})_{\mathbf{H}_0^1(\Omega)} \leq \frac{\eta}{2} \|\mathbf{u}'\|^2_{\mathbf{H}_0^1(\Omega)} + \frac{\eta^{-1}}{2} \left(\frac{C_{\text{div}}}{\nu} \right)^2 \|p'\|^2_{L^2(\Omega)}. \tag{3.13}$$

3. Using the bound (3.13) in (3.12) and choosing $\eta = \gamma$, we get:

$$a_S\left((\mathbf{u}', p'), (\mathbf{v}^\star, q^\star) \right) \geq \nu \left(\frac{\gamma}{2} \|\mathbf{u}'\|^2_{\mathbf{H}_0^1(\Omega)} + \nu^{-2} \left(1 - \frac{\gamma^{-1}}{2} (C_{\text{div}})^2 \right) \|p'\|^2_{L^2(\Omega)} \right).$$

   Consider now $\gamma = (C_{\text{div}})^2$. We obtain:

$$a_S\left((\mathbf{u}', p'), (\mathbf{v}^\star, q^\star) \right) \geq \nu\, C_{\min} \|(\mathbf{u}', p')\|^2_{\mathcal{X}} \text{ where } C_{\min} = \frac{1}{2} \min(\, (C_{\text{div}})^2, 1\,).$$

We obtain (3.10) with $\alpha = \nu\, C_{\min}$. The operator T such that $T\,(\,(\mathbf{u}',p')\,) = (\mathbf{v}^\star, q^\star)$ is linear and continuous:

$$\begin{aligned}
\|T\,(\,(\mathbf{u}',p')\,)\,\|_{\mathcal{X}}^2 &:= \|\mathbf{v}^\star\|^2_{\mathbf{H}^1_0(\Omega)} + \nu^{-2}\,\|q^\star\|^2_{L^2(\Omega)} \\
&\leq 2\,\gamma^2\,\|\mathbf{u}'\|^2_{\mathbf{H}^1_0(\Omega)} + 2\,\|\mathbf{v}_{p'}\|^2_{\mathbf{H}^1_0(\Omega)} + \gamma^2\,\nu^{-2}\,\|p'\|^2_{L^2(\Omega)}, \\
&\leq 2\,\gamma^2\,\|\mathbf{u}'\|^2_{\mathbf{H}^1_0(\Omega)} + \,(\,2\,(C_{\mathrm{div}})^2 + \gamma^2)\,\nu^{-2}\,\|p'\|^2_{L^2(\Omega)}, \\
&\leq (C_{\max})^2\,\|(\mathbf{u}',p')\|^2_{\mathcal{X}},
\end{aligned}$$

where $C_{\max} = C_{\mathrm{div}}\,\left(\max(2+(C_{\mathrm{div}})^2, 2\,(C_{\mathrm{div}})^2)\right)^{1/2}$.
Remark that, given $(\mathbf{v}^\star, q^\star) \in \mathcal{X}$, choosing $(\mathbf{u}',p') = (\gamma^{-1}\mathbf{v}^\star - \gamma^{-2}\mathbf{v}_{q^\star}, -\gamma^{-1}q^\star)$ yields $T((\mathbf{u}',p')) = (\mathbf{v}^\star, q^\star)$. Hence, the operator $T \in \mathcal{L}(\mathcal{X})$ is bijective. □

We can now prove the

Theorem 3. *Problem* (3.9) *is well-posed. It admits one and only one solution such that:*

$$\forall \mathbf{f} \in \mathbf{H}^{-1}(\Omega), \quad \left\{ \begin{array}{l} \|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega)} \leq \nu^{-1}\,\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}, \\ \|p\|_{L^2(\Omega)} \leq C_{\mathrm{div}}\,\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}. \end{array} \right. \tag{3.14}$$

Proof. According to Prop. 2, the continuous bilinear form $a_S(\cdot,\cdot)$ is T-coercive. Hence, according to Theorem 1, Problem (3.9) is well-posed. Let us prove (3.14). Consider $(\mathbf{u}, p)$ the unique solution of Problem (3.9). Choosing $\mathbf{v} = 0$, we obtain that $\forall q \in L^2_{zmv}(\Omega)$, $(q, \operatorname{div}\mathbf{u})_{L^2(\Omega)} = 0$, so that $\mathbf{u} \in \mathbf{V}$. Now, choosing $\mathbf{v} = \mathbf{u}$ and using Cauchy-Schwarz inequality, we have: $\nu\,\|\mathbf{u}\|^2_{\mathbf{H}^1_0(\Omega)} = \langle \mathbf{f}, \mathbf{u}\rangle_{\mathbf{H}^1_0(\Omega)} \leq \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega)}$, so that: $\|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega)} \leq \nu^{-1}\,\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$. Next, we choose in (3.9) $\mathbf{v} = \tilde{\mathbf{v}}_p \in \mathbf{V}^\perp$, where $\operatorname{div}\tilde{\mathbf{v}}_p = -p$ (see Prop. 1). Since $\mathbf{u} \in \mathbf{V}$ and $\tilde{\mathbf{v}}_p \in \mathbf{V}^\perp$, we have $(\mathbf{u}, \tilde{\mathbf{v}}_p)_{\mathbf{H}^1_0(\Omega)} = 0$. This gives:

$$-(p, \operatorname{div}\tilde{\mathbf{v}}_p)_{L^2(\Omega)} = \|p\|^2_{L^2(\Omega)} = \langle \mathbf{f}, \tilde{\mathbf{v}}_p\rangle_{\mathbf{H}^1_0(\Omega)} \leq \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|\tilde{\mathbf{v}}_p\|_{\mathbf{H}^1_0(\Omega)} \leq C_{\mathrm{div}}\,\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|p\|_{L^2(\Omega)},$$

so that: $\|p\|_{L^2(\Omega)} \leq C_{\mathrm{div}}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$. □

Remark 1. *We recover the first Banach–Nečas–Babuška condition* [8, Thm. 25.9, (BNB1)]*:*

$$a_S\,(\,(\mathbf{u}',p'),\,(\mathbf{v}^\star,q^\star)\,) \geq \nu\,C_{\min}\,(C_{\max})^{-1}\,\|(\mathbf{u}',p')\|_{\mathcal{X}}\,\|(\mathbf{v}^\star,q^\star)\|_{\mathcal{X}}.$$

Thus, the T-coercivity approach allows to give an estimate of the stability constant $C_{\mathrm{stab}} := \nu\,C_{\min}\,(C_{\max})^{-1}$. In our computations, it depends on the choice of the parameters η and γ, so that it could be further optimized.
If we were using a conforming discretization to solve Problem (3.9) (a.e. Taylor-Hood finite elements [15]), we would use the bilinear form $a_S(\cdot,\cdot)$ to state the discrete variational formulation. Let us call the discrete spaces $\mathbf{X}_{c,h} \subset \mathbf{H}^1_0(\Omega)$ and $Q_{c,h} \subset L^2_{zmv}(\Omega)$. Then to prove the discrete T-coercivity, we would need to state the discrete counterpart to Proposition 1. To do so, we can build a linear operator $\Pi_c : \mathbf{X} \to \mathbf{X}_h$, known as Fortin operator, such that (see a.e. [16, §8.4.1]):

$$\exists C_c \,|\, \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\,\mathbf{Grad}\,\Pi_c\mathbf{v}\|_{\mathbb{L}^2(\Omega)} \leq C_c\|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{3.15}$$

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\operatorname{div}\Pi_c\mathbf{v}, q_h)_{L^2(\Omega)} = (\operatorname{div}\mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q_h \in Q_{c,h}. \tag{3.16}$$

Using a nonconforming discretization, we will not use the bilinear form $a_S(\cdot,\cdot)$ to exhibit the discrete variational formulation, but we will need a similar operator to (3.15)-(3.16) to prove the discrete T-coercivity, which is stated in Theorem 4.

4. Discretization

We call $(O,(x_{d'})_{d'=1}^d)$ the Cartesian coordinates system, of orthonormal basis $(\mathbf{e}_{d'})_{d'=1}^d$. Consider $(\mathcal{T}_h)_h$ a simplicial triangulation sequence of Ω, where h denotes the mesh size. For a triangulation $\mathcal{T}_h$, we use the following index sets:

- $\mathcal{I}_K$ denotes the index set of the elements, such that $\mathcal{T}_h := \bigcup_{\ell\in\mathcal{I}_K} K_\ell$ is the set of elements.
- $\mathcal{I}_F$ denotes the index set of the facets[1], such that $\mathcal{F}_h := \bigcup_{f\in\mathcal{I}_F} F_f$ is the set of facets.
  Let $\mathcal{I}_F = \mathcal{I}_F^i \cup \mathcal{I}_F^b$, where $\forall f \in \mathcal{I}_F^i$, $F_f \in \Omega$ and $\forall f \in \mathcal{I}_F^b$, $F_f \in \partial\Omega$.
- $\mathcal{I}_S$ denotes the index set of the vertices, such that $(S_j)_{j\in\mathcal{I}_S}$ is the set of vertices.
  Let $\mathcal{I}_S = \mathcal{I}_S^i \cup \mathcal{I}_S^b$, where $\forall j \in \mathcal{I}_S^i$, $S_j \in \Omega$ and $\forall j \in \mathcal{I}_S^b$, $S_j \in \partial\Omega$.

We also define the following index subsets:

- $\forall \ell \in \mathcal{I}_K$, $\mathcal{I}_{F,\ell} = \{f \in \mathcal{I}_F \,|\, F_f \in K_\ell\}$, $\quad \mathcal{I}_{S,\ell} = \{j \in \mathcal{I}_S \,|\, S_j \in K_\ell\}$.
- $\forall j \in \mathcal{I}_S$, $\mathcal{I}_{K,j} = \{\ell \in \mathcal{I}_K \,|\, S_j \in K_\ell\}$, $\quad N_j := \mathrm{card}(\mathcal{I}_{K,j})$.

For all $\ell \in \mathcal{I}_K$, we call h_ℓ and ρ_ℓ the diameters of K_ℓ and its inscribed sphere respectively, and we let: $\sigma_\ell = \frac{h_\ell}{\rho_\ell}$. When the $(\mathcal{T}_h)_h$ is a shape-regular triangulation sequence (see a.e. [17, def. 11.2]), there exists a constant $\sigma > 1$, called the shape regularity parameter, such that for all h, for all $\ell \in \mathcal{I}_K$, $\sigma_\ell \le \sigma$. For all $f \in \mathcal{I}_F$, M_f denotes the barycentre of F_f, and by $\mathbf{n}_f$ its unit normal (outward oriented if $F_f \in \partial\Omega$). For all $j \in \mathcal{I}_S$, for all $\ell \in \mathcal{I}_{K,j}$, $\lambda_{j,\ell}$ denotes the barycentric coordinate of S_j in K_ℓ; $F_{j,\ell}$ denotes the face opposite to vertex S_j in element K_ℓ, and $\mathbf{x}_{j,\ell}$ denotes its barycentre. We call $\mathcal{S}_{j,\ell}$ the outward normal vector of $F_{j,\ell}$ and of norm $|\mathcal{S}_{j,\ell}| = |F_{j,\ell}|$.

Let introduce spaces of piecewise regular elements:
We set $\mathcal{P}_h H^1 = \left\{ v \in L^2(\Omega)\,; \quad \forall \ell \in \mathcal{I}_K,\, v_{|K_\ell} \in H^1(K_\ell) \right\}$, endowed with the scalar product :

$$(v,w)_h := \sum_{\ell\in\mathcal{I}_K} (\mathbf{grad}\, v, \mathbf{grad}\, w)_{\mathbf{L}^2(K_\ell)} \quad \|v\|_h^2 = \sum_{\ell\in\mathcal{I}_K} \|\,\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)}.$$

We set $\mathcal{P}_h\mathbf{H}^1 = [\mathcal{P}_h H^1]^d$, endowed with the scalar product :

$$(\mathbf{v},\mathbf{w})_h := \sum_{\ell\in\mathcal{I}_K} (\mathbf{Grad}\,\mathbf{v}, \mathbf{Grad}\,\mathbf{w})_{\mathbb{L}^2(K_\ell)} \quad \|\mathbf{v}\|_h^2 = \sum_{\ell\in\mathcal{I}_K} \|\,\mathbf{Grad}\,\mathbf{v}\|^2_{\mathbb{L}^2(K_\ell)}.$$

Let $f \in \mathcal{I}_F^i$ such that $F_f = \partial K_L \cap \partial K_R$ and $\mathbf{n}_f$ is outward K_L oriented.
The jump (resp. average) of a function $v \in \mathcal{P}_h H^1$ across the facet F_f is defined as follows: $[v]_{F_f} := v_{|K_L} - v_{|K_R}$ (resp. $\{v\}_{F_f} := \frac{1}{2}(v_{|K_L} + v_{|K_R})$). For $f \in \mathcal{I}_F^b$, we set: $[v]_{F_f} := v_{|F_f}$ and $\{v\}_{F_f} := v_{|F_f}$.

[1]The term facet stands for face (resp. edge) when $d = 3$ (resp. $d = 2$).

We set $\mathcal{P}_h\mathbf{H}(\mathrm{div}) = \left\{\mathbf{v}\in\mathbf{L}^2(\Omega)\,;\quad \forall\ell\in\mathcal{I}_K,\, \mathbf{v}_{|K_\ell}\in\mathbf{H}(\mathrm{div};\,K_\ell)\right\}$, and we define the operator div_h such that:

$$\forall\mathbf{v}\in\mathcal{P}_h\mathbf{H}(\mathrm{div}),\,\forall q\in L^2(\Omega),\quad (\mathrm{div}_h\,\mathbf{v},q)=\sum_{\ell\in\mathcal{I}_K}(\mathrm{div}\,\mathbf{v},q)_{L^2(K_\ell)}.$$

We recall classical finite elements estimates [17]. Let $\hat{K}$ be the reference simplex and $\hat{F}$ be the reference facet. For $\ell\in\mathcal{I}_K$ (resp. $f\in\mathcal{I}_F$), we denote by $T_\ell:\hat{K}\to K_\ell$ (resp. $T_f:\hat{F}\to F_f$) the geometric mapping such that $\forall\hat{\mathbf{x}}\in\hat{K}$, $\mathbf{x}_{|K_\ell}=T_\ell(\hat{\mathbf{x}})=\mathbb{B}_\ell\hat{\mathbf{x}}+\mathbf{b}_\ell$ (resp. $\mathbf{x}_{|F_f}=T_f(\hat{\mathbf{x}})=\mathbb{B}_f\hat{\mathbf{x}}+\mathbf{b}_f$), and we set $J_\ell=\det(\mathbb{B}_\ell)$ (resp. $J_f=\det(\mathbb{B}_f)$). There holds:

$$|J_\ell|=d!\,|K_\ell|,\quad \|\mathbb{B}_\ell\|=\frac{h_\ell}{\rho_{\hat{K}}},\quad \|\mathbb{B}_\ell{}^{-1}\|=\frac{h_{\hat{K}}}{\rho_\ell},\quad |J_f|=(d-1)!\,|F_f|. \tag{4.1}$$

For $v\in L^2(K_\ell)$, we set $\hat{v}_\ell=v\circ T_\ell$. For $v\in v^2(F_f)$, we set: $\hat{v}_f=v\circ T_f$. Changing the variable, we get:

$$\|v\|^2_{L^2(K_\ell)}=|J_\ell|\,\|\hat{v}_\ell\|^2_{L^2(\hat{K})},\quad\text{and}\quad \|v\|^2_{L^2(F_f)}=|J_f|\,\|\hat{v}_f\|^2_{L^2(\hat{F})}. \tag{4.2}$$

Let $v\in\mathcal{P}_hH^1$. By changing the variable, $\mathbf{grad}\,v_{|K_\ell}=(\mathbb{B}_\ell{}^{-1})^T\,\mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{v}_\ell$, and it holds:

$$\begin{array}{llll} (i) & \|\,\mathbf{grad}\,v\|^2_{\mathbf{L}^2(K_\ell)} & \le & \|\mathbb{B}_\ell{}^{-1}\|^2\,|K_\ell|\,\|\,\mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{v}_\ell\|^2_{\mathbf{L}^2(\hat{K})},\\ (ii) & \|\,\mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{v}_\ell\|^2_{\mathbf{L}^2(\hat{K})} & \le & \|\mathbb{B}_\ell\|^2\,|K_\ell|^{-1}\,\|\,\mathbf{grad}\,v\|^2_{\mathbf{L}^2(K_\ell)}. \end{array} \tag{4.3}$$

We recall the Poincaré-Steklov inequality in cells [17, Lemma 12.11]:
for all $\ell\in\mathcal{I}_K$ (K_ℓ is a convex set), $\forall v\in H^1(K_\ell)$:

$$\|\underline{v}_\ell\|_{L^2(K_\ell)}\le\pi^{-1}h_\ell\|\,\mathbf{grad}\,v\|_{\mathbf{L}^2(K_\ell)},\quad\text{where } \underline{v}_\ell=v_{|K_\ell}-\frac{\int_{K_\ell}v}{|K_\ell|}. \tag{4.4}$$

For all $D\subset\mathbb{R}^d$, and $k\in\mathbb{N}^*$, we call $P^k(D)$ the set of order k polynomials on D, $\mathbf{P}^k(D)=(P^k(D))^d$, and we consider the broken polynomial space:

$$P^k_{disc}(\mathcal{T}_h)=\left\{q\in L^2(\Omega);\quad \forall\ell\in\mathcal{I}_K,\,q_{|K_\ell}\in P^k(K_\ell)\right\},\quad \mathbf{P}^k_{disc}(\mathcal{T}_h):=(P^k_{disc}(\mathcal{T}_h))^d.$$

We let $P^0(\mathcal{T}_h)$ be the space of piecewise constant functions on $\mathcal{T}_h$.

5. The nonconforming mixed finite element method for Stokes

The nonconforming finite element method was introduced by Crouzeix and Raviart in [2] to solve Stokes Problem (3.1). We approximate the vector space $\mathbf{H}^1(\Omega)$ component by component by piecewise polynomials of order $k\in\mathbb{N}^\star$. Let us consider X_h (resp. $X_{0,h}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order k:

$$\begin{aligned} X_h&=\left\{v_h\in P^k_{disc}(\mathcal{T}_h)\,;\quad \forall f\in\mathcal{I}^i_F,\,\forall q_h\in P^{k-1}(F_f),\,\int_{F_f}[v_h]\,q_h=0\right\};\\ X_{0,h}&=\left\{v_h\in X_h\,;\quad \forall f\in\mathcal{I}^b_F,\,\forall q_h\in P^{k-1}(F_f),\,\int_{F_f}v_h\,q_h=0\right\}. \end{aligned} \tag{5.1}$$

The condition on the jumps of v_h on the inner facets is often called the patch-test condition.

Proposition 3. *The broken norm $v_h \to \|v_h\|_h$ is a norm over $X_{0,h}$.*

Proof. Let $v_h \in X_{0,h}$ such that $\|v_h\|_h = 0$. Then for all $\ell \in \mathcal{I}_K$, $v_{h|K_\ell}$ is a constant. For all $f \in \mathcal{I}_F^i$ the jump $[v_h]_{F_f}$ vanishes, so that v_h is a constant over Ω. We deduce from the discrete boundary condition that $v_h = 0$. □

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order k is $\mathbf{X}_h = (X_h)^d$ (resp. $\mathbf{X}_{0,h} = (X_{0,h})^d$). We set $\mathcal{X}_h := \mathbf{X}_{0,h} \times Q_h$ where $Q_h = P^{k-1}_{disc}(\mathcal{T}_h) \cap L^2_{zmv}(\Omega)$. We deduce from Proposition 3 the

Proposition 4. *The broken norm defined below is a norm on $\mathcal{X}_h$:*

$$\|(\cdot,\cdot)\|_{\mathcal{X}_h} : \left\{ \begin{array}{ccc} \mathcal{X}_h & \mapsto & \mathbb{R} \\ (\mathbf{v}_h, q_h) & \to & \left(\|\mathbf{v}_h\|_h^2 + \nu^{-2} \|q_h\|^2_{L^2(\Omega)} \right)^{1/2} \end{array} \right. . \tag{5.2}$$

Thus, the product space $\mathcal{X}_h$ endowed with the broken norm $\|\cdot\|_{\mathcal{X}_h}$ is a Hilbert space.

Proposition 5. *The following discrete Poincaré–Steklov inequality holds: there exists a constant C^{nc}_{PS} independent of $\mathcal{T}_h$ such that*

$$\forall \mathbf{v}_h \in \mathbf{X}_{0,h}, \quad \|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \le C^{nc}_{PS} \|\mathbf{v}_h\|_h, \tag{5.3}$$

where C^{nc}_{PS} is independent of $\mathcal{T}_h$ and is proportional to the diameter of Ω.

Proof. Inequality (5.3) is stated in [8, Lemma 36.6] for $k = 1$, but one can check that the proof holds true for higher-order, thanks to the patch-test condition. An alternative proof is given in [18, Theorem C.1]. □

We consider the discrete continuous bilinear form $a_{S,h}(\cdot,\cdot)$ such that :

$$\left\{ \begin{array}{ccc} a_{S,h} : \mathcal{X}_h \times \mathcal{X}_h & \to & \mathbb{R} \\ (\mathbf{u}'_h, p'_h) \times (\mathbf{v}_h, q_h) & \mapsto & \nu(\mathbf{u}'_h, \mathbf{v}_h)_h - (\operatorname{div}_h \mathbf{v}_h, p'_h) - (\operatorname{div}_h \mathbf{u}'_h, q_h) \end{array} \right. .$$

Let $\ell_{\mathbf{f}} \in \mathcal{L}(\mathcal{X}_h, \mathbb{R})$ be such that :

$$\forall (\mathbf{v}_h, q_h) \in \mathcal{X}_h, \quad \ell_{\mathbf{f}}\,(\,(\mathbf{v}_h, q_h)\,) = \left\{ \begin{array}{cl} (\mathbf{f}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega) \\ \langle \mathbf{f}, \mathcal{I}_h(\mathbf{v}_h) \rangle_{\mathbf{H}^1_0(\Omega)} & \text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega) \end{array} \right. ,$$

where $\mathcal{I}_h : \mathbf{X}_{0,h} \to \mathbf{Y}_{0,h}$, with $\mathbf{Y}_{0,h} = \{\mathbf{v}_h \in \mathbf{H}^1_0(\Omega)\,; \quad \forall \ell \in \mathcal{I}_K,\, \mathbf{v}_{h|K_\ell} \in \mathbf{P}^k(K_\ell)\}$, is the averaging operator described in [17, §22.4.1]. There exists a constant $C^{nc}_{\mathcal{I}_h} > 0$ independent of $\mathcal{T}_h$ such that :

$$\|\mathcal{I}_h \mathbf{v}_h\|_{\mathbf{H}^1_0(\Omega)} \le C^{nc}_{\mathcal{I}_h} \|\mathbf{v}_h\|_h, \quad \forall \mathbf{v}_h \in \mathbf{X}_{0,h}. \tag{5.4}$$

The nonconforming discretization of Problem (3.9) reads:
Find $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$ such that

$$a_{S,h}\,((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \ell_{\mathbf{f}}\,(\,(\mathbf{v}_h, q_h)\,) \quad \forall (\mathbf{v}_h, q_h) \in \mathcal{X}_h. \tag{5.5}$$

Let us prove that Problem (5.5) is well-posed using the T-coercivity theory.

Theorem 4. *Suppose that there exists a Fortin operator $\Pi_{nc} : \mathbf{H}^1(\Omega) \to \mathbf{X}_h$ such that*

$$\exists C_{nc} \,|\, \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\Pi_{nc}\mathbf{v}\|_h \le C_{nc} \|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{5.6}$$

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\operatorname{div}_h \Pi_{nc}\mathbf{v}, q_h) = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q \in Q_h, \tag{5.7}$$

where the constant C_{nc} does not depend on h. Then Problem (5.5) *is well-posed. Moreover, it admits one and only one solution* $(\mathbf{u}_h, p_h)$ *such that:*

$$\text{if } \mathbf{f} \in \mathbf{L}^2(\Omega): \quad \begin{cases} \|\mathbf{u}_h\|_h & \leq & C_{PS}^{nc}\, \nu^{-1}\, \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \\ \|p_h\|_{L^2(\Omega)} & \leq & 2\, C_{PS}^{nc}\, C_{\text{div}}^{nc}\, \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \end{cases},$$

$$\text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega): \quad \begin{cases} \|\mathbf{u}_h\|_h & \leq & C_{\mathcal{I}_h}^{nc}\, \nu^{-1}\, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \\ \|p_h\|_{L^2(\Omega)} & \leq & 2\, C_{\mathcal{I}_h}^{nc}\, C_{\text{div}}^{nc}\, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \end{cases}, \tag{5.8}$$

where $C_{\text{div}}^{nc} = C_{\text{div}}\, C_{nc}$.

Proof. Let us consider $(\mathbf{u}'_h, p'_h) \in \mathcal{X}_h$ and let us build $(\mathbf{v}^\star_h, q^\star_h) \in \mathcal{X}_h$ satisfying (2.2) (with $V = \mathcal{X}_h$). We follow the three main steps of the proof of Theorem 1.

1. According to Proposition 1, there exists $\tilde{\mathbf{v}}_{p'_h} \in \mathbf{V}^\perp$ such that $\operatorname{div} \tilde{\mathbf{v}}_{p'_h} = p'_h$ in Ω and $\|\tilde{\mathbf{v}}_{p'_h}\|_{\mathbf{H}^1_0(\Omega)} \leq C_{\text{div}} \|p'_h\|_{L^2(\Omega)}$. Let us set $\mathbf{v}_{p'_h} = \nu^{-1} \tilde{\mathbf{v}}_{p'_h}$. Consider $\mathbf{v}_{h,p'_h} = \Pi_{nc} \mathbf{v}_{p'_h}$, for all $q_h \in Q_h$, we have: $(\operatorname{div}_h \mathbf{v}_{h,p'_h}, q_h) = \nu^{-1}\,(p'_h, q_h)_{L^2(\Omega)}$ and
$$\|\mathbf{v}_{h,p'_h}\|_h \leq \nu^{-1}\, C_{\text{div}}^{nc}\, \|p'_h\|_{L^2(\Omega)} \text{ where } C_{\text{div}}^{nc} = C_{nc}\, C_{\text{div}}. \tag{5.9}$$
Let us set $(\mathbf{v}^\star_h, q^\star_h) := (\gamma_{nc} \mathbf{u}'_h - \mathbf{v}_{h,p'_h}, -\gamma_{nc}\, p'_h)$, with $\gamma_{nc} > 0$. We obtain:
$$a_{S,h}\,(\,(\mathbf{u}'_h, p'_h),\, (\mathbf{v}^\star_h, q^\star_h)\,) = \nu\, \gamma_{nc} \|\mathbf{u}'_h\|_h^2 + \nu^{-1} \|p'_h\|^2_{L^2(\Omega)} - \nu (\mathbf{u}'_h, \mathbf{v}_{h,p'_h})_h. \tag{5.10}$$
2. In order to bound the last term of (5.10), we use Young inequality and then inequality (5.9) so that for all $\eta_{nc} > 0$:
$$(\mathbf{u}'_h, \mathbf{v}_{h,p'_h})_h \leq \frac{\eta_{nc}}{2} \|\mathbf{u}'_h\|_h^2 + \frac{\eta_{nc}^{-1}}{2} \left(\frac{C_{\text{div}}^{nc}}{\nu}\right)^2 \|p'_h\|^2_{L^2(\Omega)}. \tag{5.11}$$
3. Using the bound (5.11) in (5.10) and choosing $\eta_{nc} = \gamma_{nc}$, we get:

$$a_{S,h}\,(\,(\mathbf{u}'_h, p'_h),\, (\mathbf{v}^\star_h, q^\star_h)\,) \geq \nu \left(\frac{\gamma_{nc}}{2}\, \|\mathbf{u}'_h\|_h^2 + \nu^{-2} \left(1 - \frac{(\gamma_{nc})^{-1}}{2}\, (C_{\text{div}}^{nc})^2 \right) \|p'_h\|^2_{L^2(\Omega)} \right).$$

Consider now $\gamma_{nc} = (C_{\text{div}}^{nc})^2$. We obtain:
$$a_{S,h}\,(\,(\mathbf{u}'_h, p'_h),\, (\mathbf{v}^\star_h, q^\star_h)\,) \geq \frac{\nu}{2} C_{\min}^{nc}\, \|(\mathbf{u}'_h, p'_h)\|^2_{\mathcal{X}_h},$$

where $C_{\min}^{nc} = \dfrac{1}{2}\, \min(\,(C_{\text{div}}^{nc})^2, 1\,)$.
The operator T_h such that $T_h(\mathbf{u}'_h, p'_h) = (\mathbf{v}^\star_h, p^\star_h)$ is linear and continuous:
$$\|T_h(\mathbf{u}'_h, p'_h)\|^2_{\mathcal{X}_h} = \|\mathbf{v}^\star_h\|_h^2 + \nu^{-2}\, \|q^\star_h\|_{L^2(\Omega)} \leq (C_{\max}^{nc})^2\, \|(\mathbf{u}'_h, p'_h)\|^2_{\mathcal{X}_h}$$
where $C_{\max}^{nc} = C_{\text{div}}^{nc}\, \left(\max\left(\,2 + (C_{\text{div}}^{nc})^2\,,\, 2\,(C_{\text{div}}^{nc})^2\,\right)\right)^{1/2}$. Remark that the operator $T_h \in \mathcal{L}(\mathcal{X}_h)$ is bijective. The discrete continuous bilinear form $a_{S,h}(\cdot,\cdot)$ is then T_h-coercive and according to Theorem 1, Problem (5.5) is well posed. Consider $(\mathbf{u}_h, p_h)$ the unique solution of Problem (5.5). Choosing $\mathbf{v}_h = 0$, we obtain that $\operatorname{div}_h \mathbf{u}_h = 0$. Now, choosing $\mathbf{v}_h = \mathbf{u}_h$ in (5.5) and using Cauchy-Schwarz inequality, we get that:
$$\begin{cases} \|\mathbf{u}_h\|_h & \leq & \nu^{-1}\, C_{PS}^{nc}\, \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega), \text{ using (5.3)}; \\ \|\mathbf{u}_h\|_h & \leq & \nu^{-1}\, C_{\mathcal{I}_h}^{nc}\, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} & \text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega), \text{ using (5.4)}. \end{cases} \tag{5.12}$$

Consider $(\mathbf{v}_h, q_h) = (\mathbf{v}_{h,p_h}, 0)$ in (5.5), where $\mathbf{v}_{h,p_h} = \Pi_{nc}\mathbf{v}_{p_h}$ is built as $\mathbf{v}_{h,p'_h}$ in point 1, setting $p'_h = p_h$. Suppose that $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Using the triangular inequality, Cauchy-Schwarz inequality, Poincaré-Steklov inequality (5.3), Theorem 4, and estimate (5.12), we have:

$$\begin{aligned}
\|p_h\|^2_{L^2(\Omega)} &= \nu\,(\mathbf{u}_h, \mathbf{v}_{h,p_h})_h - (\mathbf{f}, \mathbf{v}_{h,p_h})_{\mathbf{L}^2(\Omega)}\,,\\
&\leq \nu\,\|\mathbf{u}_h\|_h\,\|\mathbf{v}_{h,p_h}\|_h + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|\mathbf{v}_{h,p_h}\|_{\mathbf{L}^2(\Omega)}\\
&\leq 2\,C^{nc}_{PS}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|\mathbf{v}_{h,p_h}\|_h \leq 2\,C^{nc}_{PS}\,C_{nc}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|\,\mathbf{Grad}\,\mathbf{v}_{p_h}\|_{\mathbb{L}^2(\Omega)}\,,\\
&\leq 2\,C^{nc}_{PS}\,C^{nc}_{\mathrm{div}}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|p_h\|_{L^2(\Omega)}.
\end{aligned}$$

Applying the same reasoning when $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, we get that:

$$\left\{\begin{array}{lcll}
\|p_h\|_{L^2(\Omega)} & \leq & 2\,C^{nc}_{PS}\,C^{nc}_{\mathrm{div}}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega),\ \text{using (5.3)}\ ;\\
\|p_h\|_{L^2(\Omega)} & \leq & 2\,C^{nc}_{\mathcal{I}_h}\,C^{nc}_{\mathrm{div}}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} & \text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega),\ \text{using (5.4)}.
\end{array}\right. \tag{5.13}$$

□

Remark 2. *Again, we recover the first Banach–Nečas–Babuška condition* [8, Thm. 25.9, (BNB1)]*:*

$$a_{S,h}\,\left(\,(\mathbf{u}'_h, p'_h),\,(\mathbf{v}^\star_h, q^\star_h)\,\right) \geq \nu\,C^{nc}_{\min}\,(C^{nc}_{\max})^{-1}\,\|(\mathbf{u}'_h, p'_h)\|_{\mathcal{X}_h}\,\|(\mathbf{v}^\star_h, q^\star_h)\|_{\mathcal{X}_h}.$$

As a corollary of Theorem 4, the following a priori error estimate follows [2, Theorem 4]:

Corollary 1. *Under the assumption of Theorem 4, suppose that* $(\mathbf{u}, p) \in \mathbf{H}^{1+k}(\Omega)\times H^k(\Omega)$*, we then have the estimate:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \leq C\sigma^\ell\,h^{k+1}\,\left(|\mathbf{u}|_{\mathbf{H}^{k+1}(\Omega)} + \nu^{-1}\,|p|_{H^k(\Omega)}\right), \tag{5.14}$$

where the constant $C > 0$ *is independent of* h*,* σ *is the shape regularity parameter and the exponent* $\ell \in \mathbb{N}^\star$ *depends on* k*.*

The main issue with nonconforming mixed finite elements is the construction the basis functions. In a recent paper, Sauter explains such a construction in two dimensions [18, Corollary 2.4], and gives a bound to the discrete counterpart $\beta_{\mathcal{T}}(\Omega)$ of $\beta(\Omega)$ defined in (3.4):

$$\beta_{\mathcal{T}}(\Omega) = \inf_{q_h\in Q_h\setminus\{0\}}\ \sup_{\mathbf{v}_h\in\mathbf{X}_{0,h}\setminus\{0\}} \frac{(\mathrm{div}_h\,\mathbf{v}_h, q_h)}{\|q_h\|_{L^2(\Omega)}\,\|\mathbf{v}_h\|_h} \geq c_{\mathcal{T}}\,k^{-\alpha}. \tag{5.15}$$

This bound is in $c_{\mathcal{T}}\,k^{-\alpha}$, where the parameter α is explicit and depends on k and on the mesh topology; and the constant $c_{\mathcal{T}}$ depends only on the shape-regularity of the mesh.

6. Nonconforming Crouzeix-Raviart mixed finite elements

We study the lowest order nonconforming Crouzeix-Raviart mixed finite elements [2]. Let us consider X_{CR} (resp. $X_{0,CR}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order 1:

$$
\begin{aligned}
X_{CR} &= \left\{ v_h \in P^1_{disc}(\mathcal{T}_h)\,; \quad \forall f \in \mathcal{I}^i_F,\ \int_{F_f} [v_h] = 0 \right\}; \\
X_{0,CR} &= \left\{ v_h \in X_{CR}\,; \quad \forall f \in \mathcal{I}^b_F,\ \int_{F_f} v_h = 0 \right\}.
\end{aligned}
\tag{6.1}
$$

The space of nonconforming approximation of of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order 1 is $\mathbf{X}_{CR} = (X_{CR})^d$ (resp. $\mathbf{X}_{0,CR} = (X_{0,CR})^d$). We set $\mathcal{X}_{CR} := \mathbf{X}_{0,CR} \times Q_{CR}$ where $Q_{CR} = P^0(\mathcal{T}_h) \cap L^2_{zmv}(\Omega)$.
We can endow X_{CR} with the basis $(\psi_f)_{f\in\mathcal{I}_F}$ such that: $\forall \ell \in \mathcal{I}_K$,

$$
\psi_{f|K_\ell} = \begin{cases} 1 - d\lambda_{i,\ell} & \text{if } f \in \mathcal{I}_{F,\ell}, \\ 0 & \text{otherwise,} \end{cases}
$$

where S_i is the vertex opposite to F_f in K_ℓ. We then have $\psi_{f|F_f} = 1$, so that $[\psi_f]_{F_f} = 0$ if $f \in \mathcal{I}^i_F$ (i.e. $F_f \in \overset{\circ}{\Omega}$), and $\forall f' \neq f$, $\int_{F_{f'}} \psi_f = 0$.
We have: $X_{CR} = \text{vect}\left((\psi_f)_{f\in\mathcal{I}_F}\right)$ and $X_{0,CR} = \text{vect}\left((\psi_f)_{f\in\mathcal{I}^i_F}\right)$.
The Crouzeix-Raviart interpolation operator π_{CR} for scalar functions is defined by:

$$
\pi_{CR} : \begin{cases} H^1(\Omega) & \rightarrow & X_{CR} \\ v & \mapsto & \displaystyle\sum_{f\in\mathcal{I}_F} \pi_f v\, \psi_f \end{cases}, \text{ where } \pi_f v = \frac{1}{|F_f|}\int_{F_f} v.
$$

Notice that $\forall f \in \mathcal{I}_F$, $\int_{F_f} \pi_{CR} v = \int_{F_f} v$. Moreover, the Crouzeix-Raviart interpolation operator preserves the constants, so that $\pi_{CR}\underline{v}_\Omega = \underline{v}_\Omega$ where $\underline{v}_\Omega = \int_\Omega v/|\Omega|$.
We recall the following result [19, Lemma 2]):

Lemma 1. *The Crouzeix-Raviart interpolation operator π_{CR} is such that:*

$$
\forall v \in H^1(\Omega), \quad \|\pi_{CR} v\|_h \le \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}. \tag{6.2}
$$

Proof. We have, integrating by parts twice and using Cauchy-Schwarz inequality:

$$
\begin{aligned}
\mathbf{grad}\,\pi_{CR} v_{|K_\ell} &= |K_\ell|^{-1}\int_{K_\ell} \mathbf{grad}\,\pi_{CR} v = |K_\ell|^{-1}\sum_{f\in\mathcal{I}_{F,\ell}}\int_{F_f} \pi_{CR} v\, \mathbf{n}_f, \\
&= |K_\ell|^{-1}\sum_{f\in\mathcal{I}_{F,\ell}}\int_{F_f} v\, \mathbf{n}_f = |K_\ell|^{-1}\int_{K_\ell} \mathbf{grad}\, v, \\
|\,\mathbf{grad}\,\pi_{CR} v_{|K_\ell}| &\le |K_\ell|^{-1/2}\,\|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)} \\
\Rightarrow \quad \|\,\mathbf{grad}\,\pi_{CR} v\|^2_{\mathbf{L}^2(K_\ell)} &\le \|\,\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)}.
\end{aligned}
$$

Summing these local estimates over $\ell \in \mathcal{I}_K$, we obtain (6.2). □

For a vector $\mathbf{v} \in \mathbf{H}^1(\Omega)$ of components $(v_{d'})_{d'=1}^d$, the Crouzeix-Raviart interpolation operator is such that: $\Pi_{CR}\mathbf{v} = (\pi_{CR} v_{d'})_{d'=1}^d$. Let us set $\Pi_f \mathbf{v} = (\pi_f v_{d'})_{d'=1}^d$.

Lemma 2. *The Crouzeix-Raviart interpolation operator Π_{CR} can play the role of the Fortin operator:*

$$
\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\Pi_{CR}\mathbf{v}\|_h \le \|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{6.3}
$$

$$
\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\text{div}_h\,\Pi_{CR}\mathbf{v}, q_h) = (\text{div}\,\mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q \in Q_h, \tag{6.4}
$$

Moreover, for all $\mathbf{v} \in \mathbf{P}^1(\Omega)$, $\Pi_{CR}\mathbf{v} = \mathbf{v}$.

Proof. We obtain (6.3) applying Lemma 1 component by component. By integrating by parts, we have $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\forall \ell \in \mathcal{I}_K$:

$$\begin{aligned}\int_{K_\ell} \operatorname{div} \Pi_{CR}\mathbf{v} &= \sum_{f\in\mathcal{I}_{F,\ell}} \int_{F_f} \Pi_{CR}\mathbf{v}\cdot\mathbf{n}_f = \sum_{f\in\mathcal{I}_{F,\ell}} \int_{F_f} \Pi_f\mathbf{v}\cdot\mathbf{n}_f,\\ &= \sum_{f\in\mathcal{I}_{F,\ell}} \int_{F_f} \mathbf{v}\cdot\mathbf{n}_f = \int_{K_\ell} \operatorname{div}\mathbf{v},\end{aligned}$$

so that (6.4) is satisfied. □

We can apply the T-coercivity theory to show the next following result:

Theorem 5. *Let* $\mathcal{X}_h = \mathcal{X}_{CR}$*. Then the continuous bilinear form* $a_{S,h}(\cdot,\cdot)$ *is* T_h*-coercive and Problem* (5.5) *is well-posed.*

Proof. Using estimates (6.3) and (5.3), we apply the proof of Theorem 4. □

Since the constant of the interpolation operator Π_{CR} is equal to 1, we have $C_{min}^{CR} = C_{min}$ and $C_{max}^{CR} = C_{max}$: the stability constant of the nonconforming Crouzeix-Raviart mixed finite elements is independent of the mesh. This is not the case for higher-order (see [20, Theorem 2.2]).
For higher-order, we cannot built the interpolation operator component by component, since higher-order divergence moments must be preserved. Thus, for $k > 1$, we must build Π_{nc} so that for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $\ell \in \mathcal{I}_K$, for all $q \in P^{k-1}(K_\ell)$:

$$\int_{K_\ell} q \operatorname{div} \Pi_{nc}\mathbf{v} = \int_{K_\ell} q \operatorname{div}\mathbf{v}.$$

We recall that by integration by parts, we have:

$$\int_{K_\ell} q \operatorname{div} \Pi_{nc}\mathbf{v} + \int_{K_\ell} \mathbf{grad}\, q \cdot \Pi_{nc}\mathbf{v} = \int_{\partial K_\ell} q\, \Pi_{nc}\mathbf{v}\cdot\mathbf{n}_{|\partial K_\ell}. \tag{6.5}$$

Hence, to obtain a local estimate of $\|\mathbf{Grad}\,\Pi_{nc}\mathbf{v}\|_{\mathbb{L}^2(K_\ell)}$, we will need the following Lemma:

Lemma 3. *Let* $\mathbf{v} \in \mathbf{H}^1(\Omega)$ *and* $q \in P^{k-1}(K_\ell)$*. We set* $\underline{\mathbf{v}}_\ell := \mathbf{v}_\ell - \dfrac{\int_{K_\ell}\mathbf{v}}{|K_\ell|}$*, where* $\mathbf{v}_\ell = \mathbf{v}_{|K_\ell}$*. We have:*

$$\left|\int_{\partial K_\ell} q\,\underline{\mathbf{v}}_\ell\cdot\mathbf{n}_{|\partial K_\ell}\right| \le |K_\ell|^{k/2}\, \|\mathbf{Grad}\,\mathbf{v}_\ell\|_{\mathbb{L}^2(K_\ell)} \tag{6.6}$$

Proof. We have by integration by parts, and then using Cauchy-Schwarz inequality:

$$\begin{aligned}\left|\int_{\partial K_\ell} q\,\underline{\mathbf{v}}_\ell\cdot\mathbf{n}_f\right| &\le \left|\int_{K_\ell} q \operatorname{div}\underline{\mathbf{v}}_\ell\right| + \left|\int_{K_\ell}\mathbf{grad}\, q\cdot\underline{\mathbf{v}}_\ell\right|,\\ &\le \|q\|_{L^2(K_\ell)}\,\|\mathbf{Grad}\,\underline{\mathbf{v}}_\ell\|_{\mathbb{L}^2(K_\ell)} + \|\mathbf{grad}\, q\|_{\mathbf{L}^2(K_\ell)}\,\|\underline{\mathbf{v}}_\ell\|_{\mathbf{L}^2(K_\ell)},\\ &\le |K_\ell|^{k/2}\|\mathbf{Grad}\,\underline{\mathbf{v}}_\ell\|_{\mathbb{L}^2(K_\ell)} + |K_\ell|^{(k-1)/2}\,\|\underline{\mathbf{v}}_\ell\|_{\mathbf{L}^2(K_\ell)},\\ &\lesssim |K_\ell|^{k/2}\|\mathbf{Grad}\,\mathbf{v}_\ell\|_{\mathbb{L}^2(K_\ell)} \text{ using (4.4)}.\end{aligned}$$

□

In the next section, we will see that for $k=2$, we will need Lemma 3. For $k\geq 3$, it could be necessary to bound the tangential components of $\underline{\mathbf{v}}_\ell$. To do so, we would need to preserve curl integrals on K_ℓ. Indeed, by integration by parts, we have:

- For $d=2$, $\mathbf{v}\in\mathbf{H}^1(\Omega)$, $q\in P^{k-1}(K_\ell)$:

$$\int_{K_\ell} q\,(\mathbf{curl}\, q\cdot\mathbf{v}-\mathrm{curl}\,\mathbf{v}\, q)=\int_{\partial K_\ell} q\,\mathbf{v}\times\mathbf{n}_{|\partial K_\ell}. \tag{6.7}$$

- For $d=3$, $\mathbf{v}\in\mathbf{H}^1(\Omega)$, $\mathbf{w}\in\mathbf{P}^{k-1}(K_\ell)$:

$$\int_{K_\ell}(\mathbf{w}\cdot\mathbf{curl}\,\mathbf{v}-\mathbf{curl}\,\mathbf{w}\cdot\mathbf{v})=\int_{\partial K_\ell}(\mathbf{n}_{|\partial K_\ell}\times\mathbf{v}\times\mathbf{n}_{|\partial K_\ell})\cdot(\mathbf{w}\times\mathbf{n}_{|\partial K_\ell}). \tag{6.8}$$

7. Fortin-Soulie mixed finite elements

We consider here the case $d=2$ and we study Fortin-Soulie mixed finite elements [3]. We consider a shape-regular triangulation sequence $(\mathcal{T}_h)_h$.
Let us consider X_{FS} (resp. $X_{0,FS}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order 2:

$$\begin{aligned} X_{FS} &= \left\{ v_h\in P^2_{disc}(\mathcal{T}_h)\,;\quad \forall f\in\mathcal{I}^i_F,\,\forall q_h\in P^1(F_f),\, \int_{F_f}[v_h]\,q_h=0\right\};\\ X_{0,FS} &= \left\{ v_h\in X_{FS}\,;\quad \forall f\in\mathcal{I}^b_F,\,\forall q_h\in P^1(F_f),\, \int_{F_f}v_h\,q_h=0\right\}. \end{aligned} \tag{7.1}$$

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order 2 is $\mathbf{X}_{FS}=(X_{FS})^2$ (resp. $\mathbf{X}_{0,FS}=(X_{0,FS})^2$). We set $\mathcal{X}_{FS}=\mathbf{X}_{0,FS}\times Q_{FS}$ where $Q_{FS}:=P^1_{disc}(\mathcal{T}_h)\cap L^2_{zmv}(\Omega)$.
The building of a basis for $X_{0,FS}$ is more involved than for $X_{0,CR}$ since we cannot use two points per facet as degrees of freedom. Indeed, for all $\ell\in K_\ell$, there exists a polynomial of order 2 vanishing on the Gauss-Legendre points of the facets of the boundary ∂K_ℓ. Let $f\in\mathcal{I}_F$. The barycentric coordinates of the two Gauss-Legendre points $(p_{+,f},p_{-,f})$ on F_f are such that:

$$p_{+,f}=(c_+,c_-),\ p_{-,f}=(c_-,c_+),\ \text{where } c_\pm=(1\pm 1/\sqrt{3})/2.$$

These points can be used to integrate exactly order three polynomials:

$$\forall g\in P^3(F_f),\ \int_{F_f} g=\frac{|F_f|}{2}\left(g(p_{+,f})+g(p_{-,f})\right).$$

For all $\ell\in\mathcal{I}_K$, we define the quadratic function ϕ_{K_ℓ} that vanishes on the six Gauss-Legendre points of the facets of K_ℓ (see Fig. 1):

$$\phi_{K_\ell}:=2-3\sum_{i\in\mathcal{I}_{S,\ell}}\lambda^2_{i,\ell}\ \text{such that}\quad \forall f\in\mathcal{I}_{F,\ell},\,\forall q\in P^1(F),\quad \int_{F_f}\phi_{K_\ell}\,q=0. \tag{7.2}$$

We also define the spaces of P^2-Lagrange functions:

$$\begin{aligned} X_{LG} &:= \left\{v_h\in H^1(\Omega);\quad \forall\ell\in\mathcal{I}_K,\, v_{h|K_\ell}\in P^2(K_\ell)\right\},\\ X_{0,LG} &:= \left\{v_h\in X_{LG};\quad v_{h|\partial\Omega}=0\right\}. \end{aligned}$$

The Proposition below proved in [3, Prop. 1] allows to build a basis for $X_{0,FS}$:

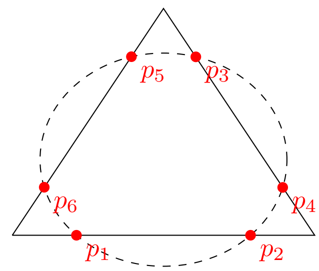

FIGURE 1. The six Gauss-Legendre points of an element K_ℓ and the elliptic function ϕ_{K_ℓ}.

Proposition 6. *We have the following decomposition:* $X_{FS} = X_{LG} + \Phi_h$ *with* $\dim(X_{LG} \cap \Phi_h) = 1$. *Any function of* X_{FS} *can be written as the sum of a function of* X_{LG} *and a function of* Φ_h. *This representation can be made unique by specifying one degree of freedom.*

Notice that $\Phi_h \cap X_{LG} = \text{vect}(v_\Phi)$, where for all $\ell \in \mathcal{I}_K$, $v_{\Phi|K_\ell} = \phi_{K_\ell}$. Then, counting the degrees of freedom, one can show that $\dim(X_{FS}) = \dim(X_{LG}) + \dim(\Phi_h) + 1$. For problems involving Dirichlet boundary conditions we can prove thus that for $X_{0,FS}$ the representation is unique and $X_{0,FS} = X_{0,LG} \oplus \Phi_h$. We have $X_{LG} = \text{vect}\left((\phi_{S_i})_{i\in\mathcal{I}_S}, (\phi_{F_f})_{f\in\mathcal{I}_F}\right)$ where the basis functions are such that:

$$\begin{aligned}&\forall i, j \in \mathcal{I}_S,\ \forall f, g \in \mathcal{I}_F :\\ &\quad \phi_{S_i}(S_j) = \delta_{ij},\ \phi_{S_i}(M_f) = 0, \quad \phi_{M_f}(M_g) = \delta_{fg},\ \phi_{M_f}(S_i) = 0.\end{aligned}$$

For all $\ell \in \mathcal{I}_K$, we will denote by $(\phi_{\ell,j})_{j=1}^6$ the local nodal basis such that:

$$(\phi_{\ell,j})_{j=1}^3 = (\phi_{S_i|K_\ell})_{i\in\mathcal{I}_{S,\ell}} \quad \text{and} \quad (\phi_{\ell,j})_{j=4}^6 = (\phi_{F_f|K_\ell})_{f\in\mathcal{I}_{F,\ell}}.$$

The spaces X_{FS} and $X_{0,FS}$ are such that:

$$\begin{aligned} X_{FS} &= \text{vect}\left((\phi_{S_i})_{i\in\mathcal{I}_S}, (\phi_{F_f})_{f\in\mathcal{I}_F}, (\phi_{K_\ell})_{\ell\in\mathcal{I}_K}\right), \\ X_{0,FS} &= \text{vect}\left((\phi_{S_i})_{i\in\mathcal{I}_S^i}, (\phi_{F_f})_{f\in\mathcal{I}_F^i}, (\phi_{K_\ell})_{\ell\in\mathcal{I}_K}\right). \end{aligned} \tag{7.3}$$

We propose here an alternative definition of the Fortin interpolation operator proposed in [3]. Let us first recall the Scott-Zhang interpolation operator [21, 22]. For all $i \in \mathcal{I}_S$, we choose some $\ell_i \in \mathcal{I}_{K,i}$, and we build the $L^2(K_{\ell_i})$-dual basis $(\tilde{\phi}_{\ell_i,j})_{j=1}^6$ of the local nodal basis such that:

$$\forall j,\ j' \in \{1, \cdots, 6\}, \quad \int_{K_{\ell_i}} \tilde{\phi}_{\ell_i,j}\, \phi_{\ell_i,j'} = \delta_{j,j'}.$$

Let us define the Fortin-Soulie interpolation operator for scalar functions by:

$$\pi_{FS} : \left\{ \begin{array}{rcl} H^1(\Omega) & \to & X_{FS} \\ v & \mapsto & \tilde{\pi} v + \displaystyle\sum_{\ell\in\mathcal{I}_K} v_{K_\ell} \phi_{K_\ell} \end{array} \right., \tag{7.4}$$
$$\text{with} \quad \tilde{\pi} v = \sum_{i\in\mathcal{I}_S} v_{S_i} \phi_{S_i} + \sum_{f\in\mathcal{I}_F} v_{F_f} \phi_{F_f}.$$

- The coefficients $(v_{S_i})_{i\in\mathcal{I}_S}$ are fixed so that: $\forall i \in \mathcal{I}_S$, $v_{S_i} = \displaystyle\int_{K_{\ell,i}} v\,\tilde{\phi}_{\ell_i,j_i}$, where j_i is the index such that $\displaystyle\int_{K_{\ell_i}} \tilde{\phi}_{\ell_i,j_i}\,\phi_{S_i|K_{\ell_i}} = 1$.
- The coefficients $\left(v_{F_f}\right)_{f\in\mathcal{I}_F}$ are fixed so that: $\forall f \in \mathcal{I}_F$, $\displaystyle\int_{F_f} \tilde{\pi} v = \int_{F_f} v$.

For all $\ell \in \mathcal{I}_K$, the coefficient v_{K_ℓ} is fixed so that: $\displaystyle\int_{K_\ell} \pi_{FS} v = \int_{K_\ell} v$.
The definition (7.4) is more general than the one given in [3], which holds for $v \in H^2(\Omega)$.
We set $\mathbf{v}_{S_i} := (\,\tilde{\pi}v_1(S_i), \tilde{\pi}v_2(S_i))^T$ and $\mathbf{v}_{F_f} := (\,\tilde{\pi}v_1(F_f), \tilde{\pi}v_2(F_f)\,)^T$.
We can define two different Fortin-Soulie interpolation operators for vector functions. First, let

$$\tilde{\Pi}_{FS} : \left\{ \begin{array}{rcl} \mathbf{H}^1(\Omega) & \to & \mathbf{X}_{FS} \\ \mathbf{v} & \mapsto & (\pi_{FS}(\mathbf{v})_1, \pi_{FS}(\mathbf{v})_2)^T. \end{array} \right.$$

We remind that the coefficients $(\tilde{\mathbf{v}}_{K_\ell})_{\ell\in\mathcal{I}_K}$ are such that:

$$\forall \ell \in \mathcal{I}_K, \quad \int_{K_\ell} \tilde{\Pi}_{FS}\mathbf{v} = \int_{K_\ell} \mathbf{v}. \tag{7.5}$$

The interpolation operator $\tilde{\Pi}_{FS}$ preserves the local averages, but it doesn't preserve the divergence. We then define a second interpolation operator which preserves the divergence in a weak sense:

$$\Pi_{FS} : \left\{ \begin{array}{rcl} \mathbf{H}^1(\Omega) & \to & \mathbf{X}_{FS} \\ \mathbf{v} & \mapsto & \displaystyle\sum_{i\in\mathcal{I}_S} \mathbf{v}_{S_i}\phi_{S_i} + \sum_{f\in\mathcal{I}_F} \mathbf{v}_{F_f}\phi_{F_f} + \sum_{\ell\in\mathcal{I}_K} \mathbf{v}_{K_\ell}\phi_{K_\ell} \end{array} \right. .$$

For all $\ell \in \mathcal{I}_K$, the vector coefficient $\mathbf{v}_{K_\ell} \in \mathbb{R}^2$ is now fixed so that condition (5.7) is satisfied. We can impose for example that the projection $\Pi_{FS}\mathbf{v}$ satisfies:

$$\int_{K_\ell} T_\ell^{-1}(\mathbf{x})\ \mathrm{div}\,\Pi_{FS}\mathbf{v} = \int_{K_\ell} T_\ell^{-1}(\mathbf{x})\ \mathrm{div}\,\mathbf{v}. \tag{7.6}$$

Notice that due to (7.2), the patch-test condition is still satisfied.

Proposition 7. *Let $\sigma_D > 0$. The Fortin-Soulie interpolation operator Π_{FS} is such for all $\mathbf{v} \in \displaystyle\bigcap_{0<s<\sigma_D} \mathbf{H}^{1+s}(\Omega)$ we have:*

$$\forall\, s \in]0, \sigma_D[,\ \forall \ell \in \mathcal{I}_K,\ \|\,\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim (\sigma_\ell)^2\,(h_\ell)^s\,|\mathbf{v}|_{1+s,K_\ell}, \tag{7.7}$$

$$\forall\, s \in]0, \sigma_D[,\ \exists C_{FS} = \mathcal{O}(\sigma^2), \quad \|\Pi_{FS}\mathbf{v} - \mathbf{v}\|_h \le C_{FS}\,h^s\,|\mathbf{v}|_{1+s,\Omega}. \tag{7.8}$$

Remark 3. *Albeit we are inspired by the proof of* [2, Lemma 4]*, we changed the transition from equation (4.27) to (4.29) there by using only the properties related to the normal component of the velocity, cf* (6.6)*. As a matter of fact, in the original proof, one ends up with either $C_{FS} = \mathcal{O}(\sigma^3)$ with the help of the multiple trace inequality or with $C_{FS} = \mathcal{O}(\sigma^2)$ at the cost of imposing a stronger assumption on the regularity of $\mathbf{v}$ (namely, $\sigma_D > 1/2$). Finally, because we do not split the integral over the boundaries of elements into the sum of $d+1$ integrals over the facets, we obtain purely local estimates, which appear to be new for the Fortin-Soulie element in the case of low-regularity fields $\mathbf{v}$.*

Proof. Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$. By construction, we have:

$$\int_{K_\ell} (\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v}) = 0 \text{ and for all } f \in \mathcal{I}_{F,\ell},\ \int_{F_f} (\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})_{|K_\ell} = 0. \tag{7.9}$$

We have:

$$\begin{aligned} \|\,\mathbf{Grad}\,(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \leq \ & \|\,\mathbf{Grad}\,(\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \\ & + \|\,\mathbf{Grad}\,(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}. \end{aligned} \tag{7.10}$$

Notice that for all $\ell \in \mathcal{I}_K$, $(\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v})_{|K_\ell} = (\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell})\,\phi_{K_\ell}$.
Using (4.3)-(i), we obtain that:

$$\begin{aligned} \|\,\mathbf{Grad}\,(\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v})\|_{\mathbb{L}^2(K_\ell)} & \lesssim |\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|\,\|\,\mathbf{grad}\ \phi_{K_\ell}\|_{\mathbf{L}^2(K_\ell)}, \\ & \lesssim \|\mathbb{B}_\ell{}^{-1}\|\,|K_\ell|^{1/2}\,|\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|, \\ & \lesssim \sigma_\ell\,|\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|. \end{aligned} \tag{7.11}$$

Let us estimate $|\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|$. On the one hand, we have:

$$\begin{aligned} \int_{K_\ell} (\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v}) & = \int_{K_\ell} (\Pi_{FS}\mathbf{v} - \mathbf{v}) \text{ from (7.5)}, \\ & = \int_{\partial K_\ell} \mathbf{x}\,(\Pi_{FS}\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{|\partial K_\ell} \text{ by IBP}, \\ & = \int_{\partial K_\ell} \mathbf{x}\,(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{|\partial K_\ell} \text{ from (7.2)}. \end{aligned}$$

Since (7.9) holds, we can use Lemma 3. We obtain:

$$\left| \int_{K_\ell} (\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v}) \right| \leq |K_\ell|\,\|\,\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}. \tag{7.12}$$

On the other hand, it holds:

$$\int_{K_\ell} (\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v}) = (\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}) \int_{K_\ell} \phi_{K_\ell} = \frac{|K_\ell|}{4}\,(\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}). \tag{7.13}$$

Hence, combining (7.12) and (7.13), we have:

$$|\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}| \leq 4\,\|\,\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \tag{7.14}$$

Using this results in (7.11), we deduce from (7.10) that for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $\ell \in \mathcal{I}_K$ we have:

$$\|\,\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{K_\ell} \lesssim \sigma_\ell\,\|\,\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}. \tag{7.15}$$

For all $\mathbf{v} \in \mathbf{P}^2(K_\ell)$ we have $\tilde{\Pi}_{FS}(\mathbf{v}) = \mathbf{v}$ and $\hat{\tilde{\Pi}}_{FS}\hat{\mathbf{v}}_\ell = \hat{\mathbf{v}}_\ell$. Hence, using Bramble-Hilbert/Deny-Lions Lemma [17, Lemma 11.9], we have:

$$\begin{aligned} \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad & \|\,\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim \sigma_\ell\,|\mathbf{v}|_{1,K_\ell}, \\ \forall \mathbf{v} \in \mathbf{H}^2(\Omega) \quad & \|\,\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim \sigma_\ell\,h_\ell\,|\mathbf{v}|_{2,K_\ell}. \end{aligned}$$

We deduce that:

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\,\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim (\sigma_\ell)^2\,|\mathbf{v}|_{1,K_\ell}, \tag{7.16}$$

$$\forall \mathbf{v} \in \mathbf{H}^2(\Omega) \quad \|\,\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim (\sigma_\ell)^2\,h_\ell\,|\mathbf{v}|_{2,K_\ell}. \tag{7.17}$$

Using interpolation property [23, Lemma 22.2], we obtain (7.7). By summation, we get (7.8). □

We recall that the discrete Poincaré–Steklov inequality (5.3) holds.

Theorem 6. *Let $\mathcal{X}_h = \mathcal{X}_{FS}$. Then the continuous bilinear form $a_{S,h}(\cdot,\cdot)$ is T_h-coercive and Problem* (5.5) *is well-posed.*

Proof. According to Proposition 7, the Fortin-Soulie interpolation operator Π_{FS} satisfies (5.6)-(5.7), so that we can apply the proof of Theorem 4. □

Notice that in the recent paper [24], the inf-sup condition of the mixed Fortin-Soulie finite element is proven directly on a triangle and then using the macro-element technique [25], but it seems difficult to use this technique to build a Fortin operator, which is needed to compute error estimates.
The study can be extended to higher orders for $d = 2$ using the following papers: [26] for $k \geq 4$, k even, [27] for $k = 3$ and [20] for $k \geq 5$, k odd. In [28], the authors propose a local Fortin operator for the lowest order Taylor-Hood finite element [15] for $d = 3$.

8. Numerical results

Consider Problem (3.1) with data $\mathbf{f} = -\,\mathbf{grad}\,\phi$, where $\phi \in H^1(\Omega) \cap L^2_{zmv}(\Omega)$. The unique solution is then $(\mathbf{u}, p) := (0, \phi)$. By integrating by parts, the source term in (3.6) reads:

$$\forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \quad \int_\Omega \mathbf{f}\cdot\mathbf{v} = \int_\Omega \phi\, \operatorname{div} \mathbf{v}. \tag{8.1}$$

Recall that the nonconforming space $\mathbf{X}_h$ defined in (5.1) is a subset of $\mathcal{P}_h\mathbf{H}^1$: using a nonconforming finite element method, the integration by parts must be done on each element of the triangulation, and we have:

$$\forall \mathbf{v} \in \mathcal{P}_h\mathbf{H}^1, \quad \int_\Omega \mathbf{f}\cdot\mathbf{v} = (\operatorname{div}_h \mathbf{v}, \phi) + \sum_{f\in\mathcal{I}_F} \int_{F_f} [\mathbf{v}\cdot\mathbf{n}_f]\,\phi. \tag{8.2}$$

When we apply this result to the right-hand-side of (5.5), we notice that the term with the jumps acts as a numerical source, which numerical influence is proportional to $1/\nu$. Thus, we cannot obtain exactly $\mathbf{u}_h = 0$ (see also (5.14)). Linke proposed in [29] to project the test function $\mathbf{v}_h \in \mathbf{X}_h$ on a discrete subspace of $\mathbf{H}(\operatorname{div};\Omega)$, like Raviart-Thomas or Brezzi-Douglas-Marini finite elements (see [30, 31], or the monograph [16]). Let $\Pi_{\rm div} : \mathbf{X}_{0,h} \to P^k_{disc}(\mathcal{T}_h) \cap \mathbf{H}_0(\operatorname{div};\Omega)$ be some interpolation operator built so that for all $\mathbf{v}_h \in \mathbf{X}_{0,h}$, for all $\ell \in \mathcal{I}_K$, $(\operatorname{div}\Pi_{\rm div}\mathbf{v}_h)_{|K_\ell} = \operatorname{div}\mathbf{v}_{h|K_\ell}$. Integrating by parts, we have for all $\mathbf{v}_h \in \mathbf{X}_{0,h}$:

$$\begin{aligned}\int_\Omega \mathbf{f}\cdot\Pi_{\rm div}\mathbf{v}_h &= \int_\Omega \phi\,\operatorname{div}\Pi_{\rm div}\mathbf{v}_h = \sum_{\ell\in K_\ell}\int_{K_\ell}\phi\,\operatorname{div}\Pi_{\rm div}\mathbf{v}_h,\\ &= \sum_{\ell\in K_\ell}\int_{K_\ell}\phi\,\operatorname{div}\mathbf{v}_h = (\operatorname{div}_h\mathbf{v}_h,\phi).\end{aligned}$$

The projection $\Pi_{\rm div}$ allows to eliminate the terms of the integrals of the jumps in (8.2).
Let us write Problem (5.5) as:
Find $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$ such that

$$a_{S,h}\,((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)) = \ell_{\mathbf{f}}\,(\,(\Pi_{\rm div}\mathbf{v}_h,q_h)\,) \quad \forall(\mathbf{v}_h,q_h)\in\mathcal{X}_h. \tag{8.3}$$

In the case of $\mathcal{X}_h = \mathcal{X}_{CR}$ and a projection on Brezzi-Douglas-Marini finite elements, the following error estimate holds if $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$:

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \leq \widetilde{C}\, h^2\, |\mathbf{u}|_{\mathbf{H}^2(\Omega)}, \tag{8.4}$$

where the constant $\widetilde{C}$ if independent of h. The proof is detailed in [32] for shape-regular meshes and [33] for anisotropic meshes. We remark that the error doesn't depend on the norm of the pressure nor on the ν parameter. We will provide some numerical results to illustrate the effectiveness of this formulation, even with a projection on the Raviart-Thomas finite elements, which, for a fixed polynomial order, are less precise than the Brezzi-Douglas-Marini finite elements.

For all $\ell \in \mathcal{I}_K$, we let $P_H^k(K_\ell)$ be the set of homogeneous polynomials of order k on K_ℓ.

For $k \in \mathbb{N}^\star$, the space of Raviart-Thomas finite elements can be defined as:

$$\mathbf{X}_{RT_k} := \left\{ \mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega);\, \forall \ell \in \mathcal{I}_k,\, \mathbf{v}_{|K_\ell} = \mathbf{a}_\ell + b_\ell \mathbf{x} \,|\, (\mathbf{a}_\ell, b_\ell) \in P^k(K_\ell)^d \times P_H^k(K_\ell) \right\}.$$

Let $k \leq 1$.

The Raviart–Thomas interpolation operator $\Pi_{RT_k} : \mathbf{H}^1(\Omega) \cup \mathbf{X}_h \to \mathbf{X}_{RT_k}$ is defined by: $\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \cup \mathbf{X}_h$,

$$\begin{cases} \forall f \in \mathcal{I}_F, & \displaystyle\int_{F_f} \Pi_{RT_k}\mathbf{v} \cdot \mathbf{n}_f\, q = \int_{F_f} \mathbf{v} \cdot \mathbf{n}_f\, q, \quad \forall q \in P^k(F_f) \\ \text{for } k = 1,\, \forall \ell \in \mathcal{I}_K, & \displaystyle\int_{K_\ell} \Pi_{RT_1}\mathbf{v} = \int_{K_\ell} \mathbf{v} \end{cases}. \tag{8.5}$$

Note that the Raviart–Thomas interpolation operator preserves the constants. Let $\mathbf{v}_h \in \mathbf{X}_h$. In order to compute the left-hand-side of (8.2), we must evaluate $(\Pi_{RT_k}\mathbf{v}_h)_{|K_\ell}$ for all $\ell \in \mathcal{I}_K$. Calculations are performed using the proposition below, which corresponds to [34, Lemma 3.11]:

Proposition 8. *Let $k \leq 1$. Let $\hat{\Pi}_{RT_k} : \mathbf{H}^1(\hat{K}) \to \mathbf{P}^k(\hat{K})$ be the Raviart–Thomas interpolation operator restricted to the reference element, so that: $\forall \hat{\mathbf{v}} \in \mathbf{H}^1(\hat{K})$,*

$$\begin{cases} \forall \hat{F} \in \partial\hat{K}, & \displaystyle\int_{\hat{F}} \hat{\Pi}_{RT_k}\hat{\mathbf{v}} \cdot \mathbf{n}_{\hat{F}}\, \hat{q} = \int_{\hat{F}} \hat{\mathbf{v}} \cdot \mathbf{n}_{\hat{F}}\, \hat{q}, \quad \forall \hat{q} \in P^k(\hat{F}) \\ \text{for } k = 1, & \displaystyle\int_{\hat{K}} \hat{\Pi}_{RT_k}\hat{\mathbf{v}} = \int_{\hat{K}} \hat{\mathbf{v}} \end{cases}. \tag{8.6}$$

We then have: $\forall \ell \in \mathcal{I}_K$,

$$(\Pi_{RT_k}\mathbf{v})_{|K_\ell}(\mathbf{x}) = \mathbb{B}_\ell \left(\hat{\Pi}_{RT_k} \mathbb{B}_\ell{}^{-1} \hat{\mathbf{v}}_\ell \right) \circ T_\ell{}^{-1}(\mathbf{x}) \;\; \textit{where} \;\; \hat{\mathbf{v}}_\ell = \mathbf{v} \circ T_\ell(\hat{\mathbf{x}}). \tag{8.7}$$

The proof is based on the equality of the $\hat{F}$ and $\hat{K}$-moments of $(\Pi_{RT_k}\mathbf{v})_{|K_\ell} \circ T_\ell(\hat{\mathbf{x}})$ and $\mathbb{B}_\ell \left(\hat{\Pi}_{RT_k} \mathbb{B}_\ell{}^{-1} \hat{\mathbf{v}}_\ell \right)(\hat{\mathbf{x}})$. For $k = 0$, setting for $d' \in \{1, \cdots, d\}$: $\boldsymbol{\psi}_{f,d'} := \psi_f\, \mathbf{e}_{d'}$, we obtain that:

$$\forall \ell \in \mathcal{I}_K\,,\, \forall f \in \mathcal{I}_{F,\ell}\,, \quad (\Pi_{RT_0}\boldsymbol{\psi}_{f,d'})_{|K_\ell} = (d\,|K_\ell|)^{-1} \left(\mathbf{x} - \vec{OS}_{f,\ell} \right) \mathcal{S}_{f,\ell} \cdot \mathbf{e}_{d'}\,, \tag{8.8}$$

where $S_{f,\ell}$ is the vertex opposite to F_f in K_ℓ.

For $k = 1$, the vector $(\Pi_{RT_1}\mathbf{v}_h)_{|K_\ell}$ is described by eight unknowns:

$$(\Pi_{RT_1}\mathbf{v}_h)_{|K_\ell} = \mathbb{A}_\ell\, \mathbf{x} + (\mathbf{b}_\ell \cdot \mathbf{x})\, \mathbf{x} + \mathbf{d}_\ell, \text{ where } \mathbb{A}_\ell \in \mathbb{R}^{2\times 2},\, \mathbf{b}_\ell \in \mathbb{R}^2,\, \mathbf{d}_\ell \in \mathbb{R}^2.$$

We compute only once the inverse of the matrix of the linear system (8.6), in $\mathbb{R}^{8\times 8}$. In the Tables 1, 2 and 3, we call $\varepsilon_0^\nu(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} / \|(\mathbf{u}, p)\|_{\mathcal{X}}$ the velocity error

in $\mathbf{L}^2(\Omega)$-norm, where $\mathbf{u}_h$ is the solution to Problem (5.5) (columns $\mathbf{X}_{CR}$ and $\mathbf{X}_{FS}$) or (8.3) (columns $\mathbf{X}_{CR}+\Pi_{RT_0}$ and $\mathbf{X}_{FS}+\Pi_{RT_1}$) and h is the mesh size.
We first consider Stokes Problem (3.1) in $\Omega=(0,1)^2$ with $\mathbf{u}=0$, $p=(x_1)^3+(x_2)^3-0.5$, $\mathbf{f}=\mathbf{grad}\,p=3\,\left((x_1)^2,(x_2)^2\right)^T$. We report in Table 1 $\varepsilon_0^\nu(\mathbf{u})$ for $h=5.00\,e-2$ and for different values of ν.

ν	$\mathbf{X}_{CR}$	$\mathbf{X}_{CR}+\Pi_{RT_0}$	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS}+\Pi_{RT_1}$
$1.00\,e-4$	$7.96\,e-4$	$4.59\,e-17$	$8.81\,e-7$	$1.54\,e-16$
$1.00\,e-5$	$7.96\,e-4$	$4.59\,e-17$	$8.81\,e-7$	$1.54\,e-16$
$1.00\,e-6$	$7.96\,e-4$	$4.59\,e-17$	$8.81\,e-7$	$1.54\,e-16$

TABLE 1. Values of $\varepsilon_0^\nu(\mathbf{u})$ for $h=5.00\,e-2$

Here we have $\|(\mathbf{u},p)\|_{\mathcal{X}}=\nu\|p\|_{L^2(\Omega)}$. Hence, the $\mathbf{L}^2(\Omega)$-norm of the discrete velocity $\|\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}$ is proportional to ν^{-1}. Using the projection, we obtain $\varepsilon_0^\nu(\mathbf{u})=0$ close to machine precision.
We now consider Stokes Problem (3.1) in $\Omega=(0,1)^2$ with:

$$\mathbf{u}=\begin{pmatrix}(1-\cos(2\,\pi\,x_1))\,\sin(2\,\pi\,x_2)\\(\cos(2\,\pi\,x_2)-1)\,\sin(2\,\pi\,x_1)\end{pmatrix},\quad\left\{\begin{array}{rcl}p&=&\sin(2\,\pi\,x_1)\,\sin(2\,\pi\,x_2),\\ \mathbf{f}&=&-\nu\,\Delta\mathbf{u}+\mathbf{grad}\,p.\end{array}\right.$$

We report in Table 2 (resp. 3) the values of $\varepsilon_0^\nu(\mathbf{u})$ in the case $\nu=1.00\,e-3$ (resp. $\nu=1.00\,e-4$) for mesh sizes. We observe that when there is no projection, $\varepsilon_0^\nu(\mathbf{u})$ is independent of ν, whereas using the projection, $\varepsilon_0^\nu(\mathbf{u})$ is proportional to ν.

h	$\mathbf{X}_{CR}$	$\mathbf{X}_{CR}+\Pi_{RT_0}$	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS}+\Pi_{RT_1}$
$5.00\,e-2$	$1.32\,e-3$	$2.74\,e-5$	$4.73\,e-6$	$5.05\,e-7$
$2.50\,e-2$	$3.30\,e-4$	$6.93\,e-6$	$5.06\,e-7$	$6.42\,e-8$
$1.25\,e-2$	$8.25\,e-5$	$1.74\,e-6$	$6.31\,e-8$	$8.10\,e-9$
$6.25\,e-3$	$2.04\,e-5$	$4.35\,e-7$	$7.44\,e-9$	$1.03\,e-9$
Rate	$h^{2.00}$	$h^{1.99}$	$h^{3.08}$	$h^{2.97}$

TABLE 2. Values of $\varepsilon_0^\nu(\mathbf{u})$ for $\nu=1.00\,e-3$

h	$\mathbf{X}_{CR}$	$\mathbf{X}_{CR}+\Pi_{RT_0}$	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS}+\Pi_{RT_1}$
$5.00\,e-2$	$1.32\,e-3$	$2.74\,e-6$	$4.70\,e-6$	$5.05\,e-8$
$2.50\,e-2$	$3.30\,e-4$	$6.93\,e-7$	$5.10\,e-7$	$6.43\,e-9$
$1.25\,e-2$	$8.25\,e-5$	$1.74\,e-7$	$6.37\,e-8$	$8.11\,e-10$
$6.25\,e-3$	$2.04\,e-5$	$4.36\,e-8$	$7.51\,e-9$	$9.77\,e-11$
Rate	$h^{2.00}$	$h^{1.99}$	$h^{3.08}$	$h^{2.99}$

TABLE 3. Values of $\varepsilon_0^\nu(\mathbf{u})$ for $\nu=1.00\,e-4$

Let us consider Stokes Problem (3.1) with a low-regular velocity. Let $\Omega=(0,1)^2$, $S_0=(0.5,0.5)$, and (r,θ) be the polar coordinates centred on S_0. We set:

$$\mathbf{u}=r^\alpha\mathbf{e}_\theta,\ p=r\ \text{ so that }\ \mathbf{f}:=-\nu\,\mathbf{\Delta}\,\mathbf{u}+\mathbf{grad}\,p=\nu\,(1-\alpha^2)\,r^{\alpha-2}\,\mathbf{e}_\theta+\mathbf{e}_r.$$

We report in Table 4 the values of $\varepsilon_0^\nu(\mathbf{u})$ for $\nu=1.00\,e-4$, and for different for mesh sizes, with $\alpha=1$ and $\alpha=0.49$. For $\alpha=1$, $\mathbf{u}=(-y,x)^T\in\mathbf{H}^2(\Omega)$. For $\alpha=0.49$,

$\mathbf{u} \in \bigcap_{0<s<\alpha} \mathbf{H}^{1+s}(\Omega)$, hence $\mathbf{u} \notin \mathbf{H}^2(\Omega)$. It seems that the Raviart-Thomas projection is less efficient in that last case.
In order to enhance the numerical results, one could also use a posteriori error estimators to adapt the mesh near point S_0 (see [35, 36] for $k=1$ and [37] for $k=2$). Alternatively, using the nonconforming Crouzeix-Raviart mixed finite element method, one can build a divergence-free basis, as described in [38]. Notice that using conforming finite elements, the Scott-Vogelius finite elements [39, 40, 41] produce velocity approximations that are exactly divergence free.
The code used to get the numerical results can be downloaded on GitHub [42].

	$\alpha = 1$		$\alpha = 0.45$	
h	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS} + \Pi_{RT_1}$	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS} + \Pi_{RT_1}$
$1.00\,e-1$	$3.03\,e-5$	$2.81\,e-6$	$3.05\,e-5$	$3.94\,e-6$
$5.00\,e-2$	$4.34\,e-6$	$1.54\,e-6$	$4.57\,e-6$	$2.15\,e-6$
$2.50\,e-2$	$4.72\,e-7$	$2.41\,e-8$	$9.70\,e-7$	$8.52\,e-7$
Rate	$h^{3.00}$	$h^{3.43}$	$h^{2.48}$	$h^{1.11}$

TABLE 4. Values of $\varepsilon_0^\nu(\mathbf{u})$, $\nu = 1.00\,e-4$.

9. Conclusion

We analysed the discretization of Stokes problem with nonconforming finite elements in light of the T-coercivity theory, we obtained local stability estimates for order 1 in 2 or 3 dimension without mesh regularity assumption; and for order 1 in 2 dimension in the case of a shape-regular triangulation sequence. This local approach, splitting the normal and the tangential components could help to generalize our results to order $k \geq 3$ (using maybe also other internal moment conservation). This is ongoing work.We then provided numerical results to illustrate the importance of using $\mathbf{H}(\mathrm{div})$-conforming projection. Further, we intend to extend the study to other mixed finite element methods.

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References

[1] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations.* Springer-Verlag, 1986.
[2] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *RAIRO, Sér. Anal. Numér.*, 7(3):33–75, 1973.
[3] M. Fortin and M. Soulie. A non-conforming piecewise quadratic finite element on triangles. *International Journal for Numerical Methods in Engineering*, 19(4):505–520, 1983.
[4] P. Ciarlet Jr. T-coercivity: Application to the discretization of Helmhotz-like problems. *Computers & Mathematics with Applications*, 64(1):22–24, 2012.
[5] E. Jamelot and P. Ciarlet, Jr. Fast non-overlapping Schwarz domain decomposition methods for solving the neutron diffusion equation. *Journal of Computational Physics*, 241:445–463, 2013.
[6] P. Ciarlet Jr., E. Jamelot, and F. D. Kpadonou. Domain decomposition methods for the diffusion equation with low-regularity solution. *Computers & Mathematics with Applications*, 74(10):2369–2384, 2017.

[7] L. Giret. *Non-Conforming Domain Decomposition for the Multigroup Neutron SPN Equation*. PhD thesis, Université Paris-Saclay, 2018.

[8] A. Ern and J.-L. Guermond. *Finite elements II*, volume 73 of *Texts in Applied Mathematics*. Springer, 2021.

[9] L. Chesnel and P. Ciarlet Jr. T-coercivity and continuous Galerkin methods: application to transmission problems with sign changing coefficients. *Numer. Math.*, 124:1–29, 2013.

[10] C. Bernardi, M. Costabel, M. Dauge, and V. Girault. Continuity properties of the inf-sup constant for the divergence. *SIAM J. Math. Anal.*, 48(2):1250–1271, 2016.

[11] D. Gallistl. Rayleigh-Ritz approximation of the inf-sup constant for the divergence. *Mathematics of Computation*, 88(315):73–89, 2019.

[12] M. Costabel and M. Dauge. On the inequalities of Babuška-Aziz, Friedrichs and Horgan-Payne. *Arch. Rational Mech. Anal.*, 217:873–898, 2015.

[13] M. Barré and P. Ciarlet Jr. T-coercivité et problèmes mixtes. working paper or preprint, October 2022.

[14] A.-S. Bonnet-Ben Dhia and P. Ciarlet Jr. Méthodes variationnelles pour l'analyse de problèmes non coercifs, 2022. M.Sc. AMS lecture notes (ENSTA-IPP).

[15] C. Taylor and T. Hood. Numerical solution of the Navier-Stokes equations using the finite element technique. *Computers & Fluids*, 1:73–100, 1973.

[16] D. Boffi, F. Brezzi, and M. Fortin. *Mixed and hybrid finite element methods and applications*. Springer-Verlag, 2013.

[17] A. Ern and J.-L. Guermond. *Finite elements I*, volume 72 of *Texts in Applied Mathematics*. Springer, 2021.

[18] S. Sauter. The inf-sup constant for Crouzeix-Raviart triangular elements of any polynomial order, 2022.

[19] T. Apel, S. Nicaise, and J. Schöberl. Crouzeix-Raviart type finite elements on anisotropic meshes. *Numerische Mathematik*, 89(2):193–223, 2001.

[20] C. Carstensen and S. Sauter. Critical functions and inf-sup stability of Crouzeix-Raviart elements. *Computers and Mathematics with Applications*, 108:12–23, 2022.

[21] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54:483–493, 1990.

[22] P. Ciarlet Jr. Analysis of the Scott–Zhang interpolation in the fractional order Sobolev spaces. *J. Numer. Math.*, 21(3):173—-180, 2013.

[23] L. Tartar. *An introduction to sobolev spaces and interpolation spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer, 2007.

[24] S. Sauter and C. Torres. On the Inf-Sup Stabillity of Crouzeix-Raviart Stokes Elements in 3D, 2022.

[25] R. Stenberg. Error analysis of some finite element methods for the stokes problem. *Math. Comp.*, 54:495–508, 1990.

[26] À. Baran and G. Stoyan. Gauss-Legendre elements: a stable, higher order non-conforming finite element family. *Computing*, 79(1):1–21, 2007.

[27] C. Carstensen and S. Sauter. Crouzeix-Raviart triangular elements are inf-sup stable, 2021. Preprint.

[28] L. Diening, J. Storn, and T. Tscherpel. Fortin operator for Taylor-Hood element. *Numerische Mathematik*, 150(2):671–689, 2022.

[29] A. Linke. On the Role of the Helmholtz-Decomposition in Mixed Methods for Incompressible Flows and a New Variational Crime. *Comput. Methods Appl. Mech. Engrg.*, 268:782–800, 2014.

[30] P.-A. Raviart and J.-M. Thomas. A mixed finite element method for second order elliptic problems. In *Mathematical aspects of finite element methods*, volume 606 of *Lecture Notes in Mathematics*, pages 292–315. Springer, 1977.

[31] F. Brezzi, J. Douglas, and L. D. Marini. Two families of mixed finite elements for second order elliptic problems. *Numerische Mathematik*, 47(2):217–235, 1985.

[32] C. Brennecke, A. Linke, C. Merdon, and J. Schöberl. Optimal and pressure independent L^2 velocity error estimates for a modified Crouzeix-Raviart Stokes element with BDM reconstructions. *Journal of Computational Mathematics*, 33(2):191–208, 2015.

[33] T. Apel, V. Kempf, A. Linke, and C. Merdon. A nonconforming pressure-robust finite element method for the Stokes equations on anisotropic meshes. *IMA Journal of Numerical Analysis*, 42(1):392–416, 2022.

[34] G. N. Gatica. *A Simple Introduction to the Mixed Finite Element Method: Theory and Applications*. SpringerBriefs in Mathematics. Springer, 2014.

[35] E. Dari, R. Durán, and C. Padra. Error estimators for nonconforming finite element approximations of the Stokes problem. *Mathematics of Computation*, 64(211):1017–1033, 1995.

[36] W. Dörfler and M. Ainsworth. Reliable a posteriori error control for nonconforming finite element approximation of Stokes flow. *Mathematics of Computation*, 74(252):1599–1619, 2005.

[37] M. Ainsworth, A. Allendes, G. R. Barrenechea, and R. Rankin. Computable error bounds for nonconforming Fortin–Soulie finite element approximation of the Stokes problem. *IMA Journal of Numerical Analysis*, 32(2):417–447, 2011.

[38] F. Hecht. Construction d'une base de fonctions P_1 non conforme à divergence nulle dans $\mathbb{R}^3$. *RAIRO, Sér. Anal. Numér.*, 15(2):119–150, 1981.

[39] L. R. Scott and M. Vogelius. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *RAIRO, Sér. Anal. Numér.*, 19(1):111–143, 1985.

[40] S. Zhang. A new family of stable mixed finite elements for the 3D Stokes equations. *Mathematics of Computation*, 74:543–554, 2005.

[41] Maurice Fabien, Johnny Guzmán, Michael Neilan, and Ahmed Zytoon. Low-order divergence-free approximations for the stokes problem on worsey–farin and powell–sabin splits. *Computer Methods in Applied Mechanics and Engineering*, 390:114444, 2022.

[42] E. Jamelot. `https://github.com/cea-trust-platform/Stokes_NCFEM`.