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IMPROVED STABILITY ESTIMATES FOR SOLVING STOKES PROBLEM WITH FORTIN-SOULIE FINITE ELEMENTS

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ABSTRACT. We propose to analyse the discretization of the Stokes problem with nonconforming finite elements in light of the T-coercivity. First we explicit the stability constants with respect to the shape regularity parameter for order 1 in 2 or 3 dimension, and order 2 in 2 dimension. [In this last case, we improve the result of the original Crouzeix-Raviart paper.](#) Second, we illustrate the importance of using a divergence-free velocity reconstruction on some numerical experiments.

Keywords. Stokes problem, T-coercitivity, Fortin-Soulie finite elements, Fortin operator

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1. INTRODUCTION

The Stokes problem describes the steady state of incompressible Newtonian flows. They are derived from the Navier–Stokes equations [1]. With regard to numerical analysis, the study of Stokes problem helps to build an appropriate approximation of the Navier–Stokes equations. We consider here a discretization with nonconforming finite elements [2, 3]. We propose to state the discrete inf-sup condition in light of the T-coercivity (cf. [4] for Helmholtz-like problems, see [5], [6] and [7] for the neutron diffusion equation), which allows to estimate the discrete error constant. In Section 2, we recall the T-coercivity theory [4], [which is known to be an equivalent reformulation of the Banach–Nečas–Babuška Theorem.](#) In Section 3 we apply it to the continuous Stokes Problem. We give details on the triangulation in Section 4, and we apply the T-coercivity to the discretization of Stokes problem with nonconforming mixed finite elements in Section 5. [For the Stokes problem, in the discrete case, this amounts to finding a Fortin operator.](#) In Section 6 (resp. 7), we precise the proof of the well-posedness in the case of order 1 (resp. order 2) nonconforming mixed finite elements. In Section 8, we illustrate the importance of using a divergence-free velocity on some numerical experiments.

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2. T-COERCIVITY

We recall here the T-coercivity theory as written in [4]. Consider first the variational problem, where V and W are two Hilbert spaces and $f \in V'$:

$$(2.1) \quad \text{Find } u \in V \text{ such that } \forall v \in W, a(u, v) = \langle f, v \rangle_V.$$

Classically, we know that Problem (2.1) is well-posed if $a(\cdot, \cdot)$ satisfies the stability and the solvability conditions of the so-called Banach–Nečas–Babuška (BNB) Theorem (see a.e. [8, Thm. 25.9]). For some models, one can also prove the well-posedness using the T-coercivity theory (cf. [4] for Helmholtz-like problems, see [5], [6] and [7] for the neutron diffusion equation).

Definition 1. *Let V and W be two Hilbert spaces and $a(\cdot, \cdot)$ be a continuous and bilinear form over $V \times W$. It is T-coercive if*

$$(2.2) \quad \exists T \in \mathcal{L}(V, W), \text{ bijective}, \exists \alpha > 0, \forall v \in V, |a(v, Tv)| \geq \alpha \|v\|_V^2.$$

If in addition $a(\cdot, \cdot)$ is symmetric, it is T-coercive if

$$(2.3) \quad \exists T \in \mathcal{L}(V, V), \exists \alpha > 0, \forall v \in V, |a(v, Tv)| \geq \alpha \|v\|_V^2.$$

When the bilinear form $a(\cdot, \cdot)$ is symmetric, the requirement that the operator T is bijective can be dropped. It is proved in [4] that the T-coercivity condition is equivalent to the stability and solvability conditions of the BNB Theorem. Whereas the BNB theorem relies on an abstract inf-sup condition, T-coercivity uses explicit inf-sup operators, both at the continuous and discrete levels.

Theorem 1. *(well-posedness) Let $a(\cdot, \cdot)$ be a continuous and bilinear form. Suppose that the form $a(\cdot, \cdot)$ is T-coercive. Then Problem (2.1) is well-posed.*

3. STOKES PROBLEM

Let Ω be a connected bounded domain of \mathbb{R}^d , $d = 2, 3$, with a polygonal ($d = 2$) or Lipschitz polyhedral ($d = 3$) boundary $\partial\Omega$. We consider Stokes problem:

$$(3.1) \quad \text{Find } (\mathbf{u}, p) \text{ such that } \begin{cases} -\nu \Delta \mathbf{u} + \mathbf{grad} p &= \mathbf{f}, \\ \operatorname{div} \mathbf{u} &= 0. \end{cases}$$

with Dirichlet boundary conditions for the velocity \mathbf{u} and a normalization condition for the pressure p :

$$\mathbf{u} = 0 \text{ on } \partial\Omega, \quad \int_{\Omega} p = 0.$$

The vector field \mathbf{u} represents the velocity of the fluid and the scalar field p represents its pressure divided by the fluid density which is supposed to be constant. Thus, the SI unit of the components of \mathbf{u} is $m \cdot s^{-1}$ and the SI unit of p is $m^2 \cdot s^{-2}$. The first equation of (3.1) corresponds to the momentum balance equation and the second one corresponds to the conservation of the mass. The constant parameter $\nu > 0$ is the kinematic viscosity of the fluid, its SI unit is $m^2 \cdot s^{-1}$. The vector field $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ represents a body forces divided by the fluid density, its SI unit is $m \cdot s^{-2}$.

Before stating the variational formulation of Problem (3.1), we provide some definition and reminders. Let us set $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^d$, $\mathbf{H}^{-1}(\Omega) = (H^{-1}(\Omega))^d$ its dual space and $L_{zmv}^2(\Omega) = \{q \in L^2(\Omega) \mid \int_{\Omega} q = 0\}$.

We recall that $\mathbf{H}(\text{div}; \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid \text{div } \mathbf{v} \in L^2(\Omega)\}$. Let us first recall Poincaré-Steklov inequality:

$$(3.2) \quad \exists C_{PS} > 0 \mid \forall v \in H_0^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C_{PS} \|\mathbf{grad } v\|_{\mathbf{L}^2(\Omega)}.$$

The SI unit of C_{PS} is m .

Thanks to this result, in $H_0^1(\Omega)$, the semi-norm is equivalent to the natural norm, so that the scalar product reads $(v, w)_{H_0^1(\Omega)} = (\mathbf{grad } v, \mathbf{grad } w)_{\mathbf{L}^2(\Omega)}$ and the norm is $\|v\|_{H_0^1(\Omega)} = \|\mathbf{grad } v\|_{\mathbf{L}^2(\Omega)}$. Let $\mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega)$. We denote by $(v_i)_{i=1}^d$ (resp. $(w_i)_{i=1}^d$) the components of \mathbf{v} (resp. \mathbf{w}), and we set $\mathbf{Grad } \mathbf{v} = (\partial_j v_i)_{i,j=1}^d \in \mathbb{L}^2(\Omega)$, where $\mathbb{L}^2(\Omega) = [L^2(\Omega)]^{d \times d}$. We have:

$$(\mathbf{Grad } \mathbf{v}, \mathbf{Grad } \mathbf{w})_{\mathbb{L}^2(\Omega)} = (\mathbf{v}, \mathbf{w})_{\mathbf{H}_0^1(\Omega)} = \sum_{i=1}^d (v_i, w_i)_{H_0^1(\Omega)}$$

and:

$$\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} = \left(\sum_{j=1}^d \|v_j\|_{H_0^1(\Omega)}^2 \right)^{1/2} = \|\mathbf{Grad } \mathbf{v}\|_{\mathbb{L}^2(\Omega)}.$$

Let us set $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \mid \text{div } \mathbf{v} = 0\}$. The space \mathbf{V} is a closed subset of $\mathbf{H}_0^1(\Omega)$. We denote by \mathbf{V}^\perp the orthogonal of \mathbf{V} in $\mathbf{H}_0^1(\Omega)$. Let $\nu_p > 0$ be a kinematic viscosity. We recall that [1, cor. I.2.4]:

Proposition 1. *The operator $\text{div} : \mathbf{H}_0^1(\Omega) \rightarrow L^2(\Omega)$ is an isomorphism of \mathbf{V}^\perp onto $L_{zmv}^2(\Omega)$. We call C_{div} the constant such that:*

$$(3.3) \quad \forall p \in L_{zmv}^2(\Omega), \exists! \mathbf{v} \in \mathbf{V}^\perp \mid \text{div } \mathbf{v} = p \text{ and } \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\text{div}} \|p\|_{L^2(\Omega)}.$$

The constant C_{div} depends only on the domain Ω . Notice that we have: $C_{\text{div}} = 1/\beta(\Omega)$ where $\beta(\Omega)$ is the inf-sup condition (or Ladyzhenskaya–Babuška–Brezzi condition):

$$(3.4) \quad \beta(\Omega) = \inf_{q \in L_{zmv}^2(\Omega) \setminus \{0\}} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \setminus \{0\}} \frac{(q, \text{div } \mathbf{v})_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)} \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)}}.$$

Generally, the value of $\beta(\Omega)$ is not known explicitly. In [9], Bernardi et al established results on the discrete approximation of $\beta(\Omega)$ using conforming finite elements. Recently, Gallistl proposed in [10] a numerical scheme with adaptive meshes for computing approximations to $\beta(\Omega)$. In the case of $d = 2$, Costabel and Dauge [11] established the following bound:

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be a domain contained in a ball of radius R , star-shaped with respect to a concentric ball of radius ρ . Then*

$$(3.5) \quad \beta(\Omega) \geq \frac{\rho}{\sqrt{2}R} \left(1 + \sqrt{1 - \frac{\rho^2}{R^2}} \right)^{-1/2} \geq \frac{\rho}{2R}.$$

Let us detail the bound for some remarkable domains. If Ω is a ball, $\beta(\Omega) \geq \frac{1}{2}$ and if Ω is a square, $\beta(\Omega) \geq \frac{1}{2\sqrt{2}}$. Suppose now that Ω is stretched in some direction by a factor k , then $\beta(\Omega) \geq \frac{1}{2k}$. Finally, if Ω is L-shaped (resp. cross-shaped) such that $L = kl$, where L is the largest length and l is the smallest length of an edge, then $\beta(\Omega) \geq \frac{1}{2\sqrt{2}k}$ (resp. $\beta(\Omega) \geq \frac{1}{4k}$).

The variational formulation of Problem (3.1) reads:

Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_{zmv}^2(\Omega)$ such that

$$(3.6) \quad \begin{cases} \nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} &= \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega); \\ (q, \operatorname{div} \mathbf{u})_{L^2(\Omega)} &= 0 & \forall q \in L_{zmv}^2(\Omega). \end{cases}$$

Classically, one proves that Problem (3.6) is well-posed using Poincaré-Steklov inequality (3.2) and Prop. 1. Check for instance the proof of [1, Thm. I.5.1].

Let us set $\mathcal{X} = \mathbf{H}_0^1(\Omega) \times L_{zmv}^2(\Omega)$ which is a Hilbert space which we endow with the following norm:

$$(3.7) \quad \|(\mathbf{v}, q)\|_{\mathcal{X}} = \left(\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)}^2 + \nu^{-2} \|q\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

We consider now the following bilinear symmetric and continuous form:

$$(3.8) \quad \begin{cases} a_S : \mathcal{X} \times \mathcal{X} &\rightarrow \mathbb{R} \\ (\mathbf{u}', p') \times (\mathbf{v}, q) &\mapsto \nu(\mathbf{u}', \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p', \operatorname{div} \mathbf{v})_{L^2(\Omega)} - (q, \operatorname{div} \mathbf{u}')_{L^2(\Omega)}. \end{cases}$$

We can write Problem (3.1) in an equivalent way as follows:

$$(3.9) \quad \text{Find } (\mathbf{u}, p) \in \mathcal{X} \text{ such that } a_S((\mathbf{u}, p), (\mathbf{v}, q)) = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} \quad \forall (\mathbf{v}, q) \in \mathcal{X}.$$

Let us prove that Problem (3.9) is well-posed using the T-coercivity theory.

Theorem 3. *Problem (3.9) is well-posed. It admits one and only one solution such that:*

$$(3.10) \quad \forall \mathbf{f} \in \mathbf{H}^{-1}(\Omega), \quad \begin{cases} \|\mathbf{u}\|_{\mathbf{H}_0^1(\Omega)} \leq \nu^{-1} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}, \\ \|p\|_{L^2(\Omega)} \leq C_{\operatorname{div}} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}. \end{cases}$$

Proof. We follow here the proof given in [12, 13]. Let us consider $(\mathbf{u}', p') \in \mathcal{X}$ and let us build $(\mathbf{v}^*, q^*) = T(\mathbf{u}', p') \in \mathcal{X}$ satisfying (2.3) (with $V = \mathcal{X}$). We need three main steps.

1. According to Prop. 1, there exists $\tilde{\mathbf{v}}_{p'} \in \mathbf{H}_0^1(\Omega)$ such that: $\operatorname{div} \tilde{\mathbf{v}}_{p'} = p'$ in Ω and $\|\tilde{\mathbf{v}}_{p'}\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|p'\|_{L^2(\Omega)}$. Let us set $\mathbf{v}_{p'} = \nu^{-1} \tilde{\mathbf{v}}_{p'}$ so that $\operatorname{div} \mathbf{v}_{p'} = \nu^{-1} p'$ and

$$(3.11) \quad \|\mathbf{v}_{p'}\|_{\mathbf{H}_0^1(\Omega)} \leq \nu^{-1} C_{\operatorname{div}} \|p'\|_{L^2(\Omega)}.$$

Let us set $(\mathbf{v}^*, q^*) := (\gamma \mathbf{u}' - \mathbf{v}_{p'}, -\gamma p')$, with $\gamma > 0$. We obtain:

$$(3.12) \quad a_S((\mathbf{u}', p'), (\mathbf{v}^*, q^*)) = \nu \gamma \|\mathbf{u}'\|_{\mathbf{H}_0^1(\Omega)}^2 + \nu^{-1} \|p'\|_{L^2(\Omega)}^2 - \nu(\mathbf{u}', \mathbf{v}_{p'})_{\mathbf{H}_0^1(\Omega)}.$$

2. In order to bound the last term of (3.12), we use Young inequality and then inequality (3.11) so that for all $\eta > 0$:

$$(3.13) \quad (\mathbf{u}', \mathbf{v}_{p'})_{\mathbf{H}_0^1(\Omega)} \leq \frac{\eta}{2} \|\mathbf{u}'\|_{\mathbf{H}_0^1(\Omega)}^2 + \frac{\eta^{-1}}{2} \left(\frac{C_{\operatorname{div}}}{\nu} \right)^2 \|p'\|_{L^2(\Omega)}^2.$$

3. Using the bound (3.13) in (3.12) and choosing $\eta = \gamma$, we get:

$$a_S((\mathbf{u}', p'), (\mathbf{v}^*, q^*)) \geq \nu \left(\frac{\gamma}{2} \|\mathbf{u}'\|_{\mathbf{H}_0^1(\Omega)}^2 + \nu^{-2} \left(1 + \frac{\gamma^{-1}}{2} (C_{\operatorname{div}})^2 \right) \|p'\|_{L^2(\Omega)}^2 \right).$$

Consider now $\gamma = (C_{\operatorname{div}})^2$. We obtain:

$$a_S((\mathbf{u}', p'), (\mathbf{v}^*, q^*)) \geq \nu C_{\min} \|(\mathbf{u}', p')\|_{\mathcal{X}}^2 \quad \text{where } C_{\min} = \frac{1}{2} \min((C_{\operatorname{div}})^2, 1).$$

The operator T such that $T(\mathbf{u}', p') = (\mathbf{v}^*, q^*)$ is linear and continuous:

$$\begin{aligned} \|T(\mathbf{u}', p')\|_{\mathcal{X}}^2 &:= \|\mathbf{v}^*\|_{\mathbf{H}_0^1(\Omega)}^2 + \nu^{-2} \|q^*\|_{L^2(\Omega)}^2 \\ &\leq \gamma^2 \|\mathbf{u}'\|_{\mathbf{H}_0^1(\Omega)}^2 + \|\mathbf{v}_{p'}\|_{\mathbf{H}_0^1(\Omega)}^2 + \gamma^2 \nu^{-2} \|p'\|_{L^2(\Omega)}^2, \\ &\leq \gamma^2 \|\mathbf{u}'\|_{\mathbf{H}_0^1(\Omega)}^2 + ((C_{\text{div}})^2 + \gamma) \nu^{-2} \|p'\|_{L^2(\Omega)}^2, \\ &\leq (C_{\text{max}})^2 \|(\mathbf{u}', p')\|_{\mathcal{X}}^2, \end{aligned}$$

where $C_{\text{max}} = C_{\text{div}} (1 + (C_{\text{div}})^2)^{1/2}$.

¹ The symmetric and continuous bilinear form $a(\cdot, \cdot)$ is then T -coercive and according to Theorem 1, Problem (3.9) is well-posed. Let us prove (3.10). Consider (\mathbf{u}, p) the unique solution of Problem (3.9). Choosing $\mathbf{v} = 0$, we obtain that $\forall q \in L_{zmv}^2(\Omega)$, $(q, \text{div } \mathbf{u})_{L^2(\Omega)} = 0$, so that $\mathbf{u} \in \mathbf{V}$. Now, choosing $\mathbf{v} = \mathbf{u}$ and using Cauchy-Schwarz inequality, we have: $\nu \|\mathbf{u}\|_{\mathbf{H}_0^1(\Omega)}^2 = \langle \mathbf{f}, \mathbf{u} \rangle_{\mathbf{H}_0^1(\Omega)} \leq \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{u}\|_{\mathbf{H}_0^1(\Omega)}$, so that: $\|\mathbf{u}\|_{\mathbf{H}_0^1(\Omega)} \leq \nu^{-1} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$. Next, we choose in (3.9) $\mathbf{v} = \mathbf{v}_p \in \mathbf{V}^\perp$, where $\text{div } \mathbf{v}_p = -\nu^{-1} p$ (see Prop. 1). Noticing that $\mathbf{u} \in \mathbf{V}$ and $\mathbf{v}_p \in \mathbf{V}^\perp$, it holds²: $(\mathbf{u}, \mathbf{v}_p)_{\mathbf{H}_0^1(\Omega)} = 0$. This gives:

$$\begin{aligned} -(p, \text{div } \mathbf{v}_p)_{L^2(\Omega)} &= \nu^{-1} \|p\|_{L^2(\Omega)}^2 = \langle \mathbf{f}, \mathbf{v}_p \rangle_{\mathbf{H}_0^1(\Omega)}, \\ &\leq \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{v}_p\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\text{div}} \nu^{-1} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \|p\|_{L^2(\Omega)}, \end{aligned}$$

so that: $\|p\|_{L^2(\Omega)} \leq C_{\text{div}} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$. \square

Remark 1. We recover the first Banach–Nečas–Babuška condition [8, Thm. 25.9, (BNB1)]:

$$a_S((\mathbf{u}', p'), (\mathbf{v}^*, q^*)) \geq \nu C_{\text{min}} (C_{\text{max}})^{-1} \|(\mathbf{u}', p')\|_{\mathcal{X}} \|(\mathbf{v}^*, q^*)\|_{\mathcal{X}}.$$

Let us call $C_{\text{stab}} = \nu C_{\text{min}} (C_{\text{max}})^{-1}$ the stability constant. With the choice of our parameters, C_{stab} is such that:

$$C_{\text{stab}} = \begin{cases} \frac{\nu}{2} \frac{C_{\text{div}}}{(1 + (C_{\text{div}})^2)^{1/2}} & \text{if } 0 < C_{\text{div}} \leq 1, \\ \frac{\nu}{2} \frac{(C_{\text{div}})^{-1}}{(1 + (C_{\text{div}})^2)^{1/2}} & \text{if } 1 \leq C_{\text{div}}. \end{cases}$$

Thus, the T-coercivity approach allows to give an estimate of the stability constant. In our computations, it depends on the choice of the parameters η and γ , so that it could be optimized.

If we were using a conforming discretization to solve Problem (3.9) (a.e. Taylor-Hood finite elements [14]), we would use the bilinear form $a_S(\cdot, \cdot)$ to state the discrete variational formulation. Let us call the discrete spaces $\mathbf{X}_{c,h} \subset \mathbf{H}_0^1(\Omega)$ and $Q_{c,h} \subset L_{zmv}^2(\Omega)$. Then to prove the discrete T-coercivity, we would need to state

¹Remark that $(\mathbf{v}^*, q^*) = (\mathbf{0}, 0) \Leftrightarrow (\mathbf{u}', p') = (\mathbf{0}, 0)$: the operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ is bijective.

²According to [1, Cor. I.2.3], since $\mathbf{v}_p \in \mathbf{V}^\perp$, $\exists p \in L_{zmv}^2(\Omega) \mid \Delta \mathbf{v}_p = \mathbf{grad } q$ in $\mathbf{H}^{-1}(\Omega)$. Integrating by parts twice, we have: $(\mathbf{u}, \mathbf{v}_p)_{\mathbf{H}_0^1(\Omega)} = -\langle \mathbf{grad } q, \mathbf{u} \rangle_{\mathbf{H}_0^1(\Omega)} = (q, \text{div } \mathbf{u})_{L^2(\Omega)} = 0$.

the discrete counterpart to Proposition 1. To do so, we can build a linear operator $\Pi_c : \mathbf{X} \rightarrow \mathbf{X}_h$, known as Fortin operator, such that (see a.e. [15, §8.4.1]):

$$(3.14) \quad \exists C_c \mid \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\mathbf{Grad} \Pi_c \mathbf{v}\|_{\mathbb{L}^2(\Omega)} \leq C_c \|\mathbf{Grad} \mathbf{v}\|_{\mathbb{L}^2(\Omega)},$$

$$(3.15) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\operatorname{div} \Pi_c \mathbf{v}, q_h)_{L^2(\Omega)} = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q_h \in Q_{c,h}.$$

Using a nonconforming discretization, we will not use the bilinear form $a_S(\cdot, \cdot)$ to exhibit the discrete variational formulation, but we will need a similar operator to (3.14)-(3.15) to prove the discrete T-coercivity, which is stated in Theorem 4.

4. DISCRETIZATION

We call $(O, (x_{d'}^d)_{d'=1}^d)$ the Cartesian coordinates system, of orthonormal basis $(\mathbf{e}_{d'}^d)_{d'=1}^d$. Consider $(\mathcal{T}_h)_h$ a simplicial triangulation sequence of Ω . For a triangulation \mathcal{T}_h , we use the following index sets:

- \mathcal{I}_K denotes the index set of the elements, such that $\mathcal{T}_h := \bigcup_{\ell \in \mathcal{I}_K} K_\ell$ is the set of elements.
- \mathcal{I}_F denotes the index set of the facets³, such that $\mathcal{F}_h := \bigcup_{f \in \mathcal{I}_F} F_f$ is the set of facets.
Let $\mathcal{I}_F = \mathcal{I}_F^i \cup \mathcal{I}_F^b$, where $\forall f \in \mathcal{I}_F^i, F_f \in \Omega$ and $\forall f \in \mathcal{I}_F^b, F_f \in \partial\Omega$.
- \mathcal{I}_S denotes the index set of the vertices, such that $(S_j)_{j \in \mathcal{I}_S}$ is the set of vertices.
Let $\mathcal{I}_S = \mathcal{I}_S^i \cup \mathcal{I}_S^b$, where $\forall j \in \mathcal{I}_S^i, S_j \in \Omega$ and $\forall j \in \mathcal{I}_S^b, S_j \in \partial\Omega$.

We also define the following index subsets:

- $\forall \ell \in \mathcal{I}_K, \mathcal{I}_{F,\ell} = \{f \in \mathcal{I}_F \mid F_f \in K_\ell\}, \quad \mathcal{I}_{S,\ell} = \{j \in \mathcal{I}_S \mid S_j \in K_\ell\}.$
- $\forall j \in \mathcal{I}_S, \mathcal{I}_{K,j} = \{\ell \in \mathcal{I}_K \mid S_j \in K_\ell\}, \quad N_j := \operatorname{card}(\mathcal{I}_{K,j}).$

For all $\ell \in \mathcal{I}_K$, we call h_ℓ and ρ_ℓ the diameters of K_ℓ and its inscribed sphere respectively, and we let: $\sigma_\ell = \frac{h_\ell}{\rho_\ell}$. When the $(\mathcal{T}_h)_h$ is a shape-regular triangulation sequence (see a.e. [16, def. 11.2]), there exists a constant $\sigma > 1$, called the [shape regularity parameter](#), such that for all h , for all $\ell \in \mathcal{I}_K$, $\sigma_\ell \leq \sigma$. For all $f \in \mathcal{I}_F$, M_f denotes the barycentre of F_f , and by \mathbf{n}_f its unit normal (outward oriented if $F_f \in \partial\Omega$). For all $j \in \mathcal{I}_S$, for all $\ell \in \mathcal{I}_{K,j}$, $\lambda_{j,\ell}$ denotes the barycentric coordinate of S_j in K_ℓ ; $F_{j,\ell}$ denotes the face opposite to vertex S_j in element K_ℓ , and $\mathbf{x}_{j,\ell}$ denotes its barycentre. We call $\mathcal{S}_{j,\ell}$ the outward normal vector of $F_{j,\ell}$ and of norm $|\mathcal{S}_{j,\ell}| = |F_{j,\ell}|$.

Let introduce spaces of piecewise regular elements:

We set $\mathcal{P}_h H^1 = \{v \in L^2(\Omega); \quad \forall \ell \in \mathcal{I}_K, v|_{K_\ell} \in H^1(K_\ell)\}$, endowed with the scalar product :

$$(v, w)_h := \sum_{\ell \in \mathcal{I}_K} (\mathbf{grad} v, \mathbf{grad} w)_{L^2(K_\ell)} \quad \|v\|_h^2 = \sum_{\ell \in \mathcal{I}_K} \|\mathbf{grad} v\|_{L^2(K_\ell)}^2.$$

We set $\mathcal{P}_h \mathbf{H}^1 = [\mathcal{P}_h H^1]^d$, endowed with the scalar product :

$$(\mathbf{v}, \mathbf{w})_h := \sum_{\ell \in \mathcal{I}_K} (\mathbf{Grad} \mathbf{v}, \mathbf{Grad} \mathbf{w})_{L^2(K_\ell)} \quad \|\mathbf{v}\|_h^2 = \sum_{\ell \in \mathcal{I}_K} \|\mathbf{Grad} \mathbf{v}\|_{L^2(K_\ell)}^2.$$

³The term facet stands for face (resp. edge) when $d = 3$ (resp. $d = 2$).

Let $f \in \mathcal{I}_F^i$ such that $F_f = \partial K_L \cap \partial K_R$ and \mathbf{n}_f is outward K_L oriented.

The jump (resp. average) of a function $v \in \mathcal{P}_h H^1$ across the facet F_f is defined as follows: $[v]_{F_f} := v|_{K_L} - v|_{K_R}$ (resp. $\{v\}_{F_f} := \frac{1}{2}(v|_{K_L} + v|_{K_R})$). For $f \in \mathcal{I}_F^b$, we set: $[v]_{F_f} := v|_{F_f}$ and $\{v\}_{F_f} := v|_{F_f}$.

We set $\mathcal{P}_h \mathbf{H}(\text{div}) = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \quad \forall \ell \in \mathcal{I}_K, \mathbf{v}|_{K_\ell} \in \mathbf{H}(\text{div}; K_\ell)\}$, and we define the operator div_h such that:

$$\forall \mathbf{v} \in \mathcal{P}_h \mathbf{H}(\text{div}), \forall q \in L^2(\Omega), \quad (\text{div}_h \mathbf{v}, q) = \sum_{\ell \in \mathcal{I}_K} (\text{div} \mathbf{v}, q)_{L^2(K_\ell)}.$$

We recall classical finite elements estimates [16]. Let \hat{K} be the reference simplex and \hat{F} be the reference facet. For $\ell \in \mathcal{I}_K$ (resp. $f \in \mathcal{I}_F$), we denote by $T_\ell : \hat{K} \rightarrow K_\ell$ (resp. $T_f : \hat{F} \rightarrow F_f$) the geometric mapping such that $\forall \hat{\mathbf{x}} \in \hat{K}$, $\mathbf{x}|_{K_\ell} = T_\ell(\hat{\mathbf{x}}) = \mathbb{B}_\ell \hat{\mathbf{x}} + \mathbf{b}_\ell$ (resp. $\mathbf{x}|_{F_f} = T_f(\hat{\mathbf{x}}) = \mathbb{B}_f \hat{\mathbf{x}} + \mathbf{b}_f$), and we set $J_\ell = \det(\mathbb{B}_\ell)$ (resp. $J_f = \det(\mathbb{B}_f)$). There holds:

$$(4.1) \quad |J_\ell| = d! |K_\ell|, \quad \|\mathbb{B}_\ell\| = \frac{h_\ell}{\rho_{\hat{K}}}, \quad \|\mathbb{B}_\ell^{-1}\| = \frac{h_{\hat{K}}}{\rho_\ell}, \quad |J_f| = (d-1)! |F_f|.$$

For $v \in L^2(K_\ell)$, we set $\hat{v}_\ell = v \circ T_\ell$. For $v \in v^2(F_f)$, we set: $\hat{v}_f = v \circ T_f$. Changing the variable, we get:

$$(4.2) \quad \|v\|_{L^2(K_\ell)}^2 = |J_\ell| \|\hat{v}_\ell\|_{L^2(\hat{K})}^2, \quad \text{and} \quad \|v\|_{L^2(F_f)}^2 = |J_f| \|\hat{v}_f\|_{L^2(\hat{F})}^2.$$

Let $v \in \mathcal{P}_h H^1$. By changing the variable, $\mathbf{grad} v|_{K_\ell} = (\mathbb{B}_\ell^{-1})^T \mathbf{grad}_{\hat{\mathbf{x}}} \hat{v}_\ell$, and it holds:

$$(4.3) \quad \begin{aligned} (i) \quad & \|\mathbf{grad} v\|_{\mathbf{L}^2(K_\ell)}^2 \leq \|\mathbb{B}_\ell^{-1}\|^2 |K_\ell| \|\mathbf{grad}_{\hat{\mathbf{x}}} \hat{v}_\ell\|_{\mathbf{L}^2(\hat{K})}^2, \\ (ii) \quad & \|\mathbf{grad}_{\hat{\mathbf{x}}} \hat{v}_\ell\|_{\mathbf{L}^2(\hat{K})}^2 \leq \|\mathbb{B}_\ell\|^2 |K_\ell|^{-1} \|\mathbf{grad} v\|_{\mathbf{L}^2(K_\ell)}^2. \end{aligned}$$

We recall the Poincaré-Steklov inequality in cells [16, Lemma 12.11]:

for all $\ell \in \mathcal{I}_K$ (K_ℓ is a convex set), $\forall v \in H^1(K_\ell)$:

$$(4.4) \quad \|\underline{v}_\ell\|_{L^2(K_\ell)} \leq \pi^{-1} h_\ell \|\mathbf{grad} v\|_{\mathbf{L}^2(K_\ell)}, \quad \text{where } \underline{v}_\ell = v|_{K_\ell} - \frac{\int_{K_\ell} v}{|K_\ell|}.$$

For all $D \subset \mathbb{R}^d$, and $k \in \mathbb{N}^*$, we call $P^k(D)$ the set of order k polynomials on D , $\mathbf{P}^k(D) = (P^k(D))^d$, and we consider the broken polynomial space:

$$P_{disc}^k(\mathcal{T}_h) = \{q \in L^2(\Omega); \quad \forall \ell \in \mathcal{I}_K, q|_{K_\ell} \in P^k(K_\ell)\}, \quad \mathbf{P}_{disc}^k(\mathcal{T}_h) := (P_{disc}^k(\mathcal{T}_h))^d.$$

We let $P^0(\mathcal{T}_h)$ be the space of piecewise constant functions on \mathcal{T}_h .

5. THE NONCONFORMING MIXED FINITE ELEMENT METHOD FOR STOKES

The nonconforming finite element method was introduced by Crouzeix and Raviart in [2] to solve Stokes Problem (3.1). We approximate the vector space $\mathbf{H}^1(\Omega)$ component by component by piecewise polynomials of order $k \in \mathbb{N}^*$. Let us consider X_h (resp. $X_{0,h}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H_0^1(\Omega)$)

of order k :

$$(5.1) \quad \begin{aligned} X_h &= \left\{ v_h \in P_{disc}^k(\mathcal{T}_h); \quad \forall f \in \mathcal{I}_F^i, \forall q_h \in P^{k-1}(F_f), \int_{F_f} [v_h] q_h = 0 \right\}; \\ X_{0,h} &= \left\{ v_h \in X_h; \quad \forall f \in \mathcal{I}_F^b, \forall q_h \in P^{k-1}(F_f), \int_{F_f} v_h q_h = 0 \right\}. \end{aligned}$$

The condition on the jumps of v_h on the inner facets is often called the patch-test condition.

Proposition 2. *The broken norm $v_h \rightarrow \|v_h\|_h$ is a norm over $X_{0,h}$.*

Proof. Let $v_h \in X_{0,h}$ such that $\|v_h\|_h = 0$. Then for all $\ell \in \mathcal{I}_K$, $v_h|_{K_\ell}$ is a constant. For all $f \in \mathcal{I}_F^i$ the jump $[v_h]_{F_f}$ vanishes, so that v_h is a constant over Ω . We deduce from the discrete boundary condition that $v_h = 0$. \square

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}_0^1(\Omega)$) of order k is $\mathbf{X}_h = (X_h)^d$ (resp. $\mathbf{X}_{0,h} = (X_{0,h})^d$). We set $\mathcal{X}_h := \mathbf{X}_{0,h} \times Q_h$ where $Q_h = P_{disc}^{k-1}(\mathcal{T}_h) \cap L_{zmv}^2(\Omega)$. We deduce from Proposition 2 the

Proposition 3. *The broken norm defined below is a norm on \mathcal{X}_h :*

$$(5.2) \quad \|(\cdot, \cdot)\|_{\mathcal{X}_h} : \begin{cases} \mathcal{X}_h & \mapsto \mathbb{R} \\ (\mathbf{v}_h, q_h) & \rightarrow \left(\|\mathbf{v}_h\|_h^2 + \nu^{-2} \|q_h\|_{L^2(\Omega)}^2 \right)^{1/2}. \end{cases}$$

Thus, the product space \mathcal{X}_h endowed with the broken norm $\|\cdot\|_{\mathcal{X}_h}$ is a Hilbert space.

Proposition 4. *The following discrete Poincaré–Steklov inequality holds: there exists a constant C_{PS}^{nc} independent of \mathcal{T}_h such that*

$$(5.3) \quad \forall \mathbf{v}_h \in \mathbf{X}_{0,h}, \quad \|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \leq C_{PS}^{nc} \|\mathbf{v}_h\|_h,$$

where C_{PS}^{nc} is independent of \mathcal{T}_h and is proportional to the diameter of Ω .

Proof. Inequality (5.3) is stated in [8, Lemma 36.6] for $k = 1$, but one can check that the proof holds true for higher-order, thanks to the patch-test condition. An alternative proof is given in [17, Theorem C.1]. \square

We consider the discrete continuous bilinear form $a_{S,h}(\cdot, \cdot)$ such that :

$$\begin{cases} a_{S,h} : \mathcal{X}_h \times \mathcal{X}_h & \rightarrow \mathbb{R} \\ (\mathbf{u}'_h, p'_h) \times (\mathbf{v}_h, q_h) & \mapsto \nu(\mathbf{u}'_h, \mathbf{v}_h)_h - (\operatorname{div}_h \mathbf{v}_h, p'_h) - (\operatorname{div}_h \mathbf{u}'_h, q_h) \end{cases}.$$

Let $\ell_{\mathbf{f}} \in \mathcal{L}(\mathcal{X}_h, \mathbb{R})$ be such that :

$$\forall (\mathbf{v}_h, q_h) \in \mathcal{X}_h, \quad \ell_{\mathbf{f}}((\mathbf{v}_h, q_h)) = \begin{cases} (\mathbf{f}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega) \\ \langle \mathbf{f}, \mathcal{I}_h(\mathbf{v}_h) \rangle_{\mathbf{H}_0^1(\Omega)} & \text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega) \end{cases},$$

where $\mathcal{I}_h : \mathbf{X}_{0,h} \rightarrow \mathbf{Y}_{0,h}$, with $\mathbf{Y}_{0,h} = \{\mathbf{v}_h \in \mathbf{H}_0^1(\Omega); \quad \forall \ell \in \mathcal{I}_K, \mathbf{v}_h|_{K_\ell} \in \mathbf{P}^k(K_\ell)\}$, is the averaging operator described in [16, §22.4.1]. There exists a constant $C_{\mathcal{I}_h}^{nc} > 0$ independent of \mathcal{T}_h such that :

$$(5.4) \quad \|\mathcal{I}_h \mathbf{v}_h\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\mathcal{I}_h}^{nc} \|\mathbf{v}_h\|_h, \quad \forall \mathbf{v}_h \in \mathbf{X}_{0,h}.$$

The nonconforming discretization of Problem (3.9) reads:

Find $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$ such that

$$(5.5) \quad a_{S,h}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \ell_{\mathbf{f}}((\mathbf{v}_h, q_h)) \quad \forall (\mathbf{v}_h, q_h) \in \mathcal{X}_h.$$

Let us prove that Problem (5.5) is well-posed using the T-coercivity theory.

Theorem 4. *Suppose that there exists a Fortin operator $\Pi_{nc} : \mathbf{H}^1(\Omega) \rightarrow \mathbf{X}_h$ such that*

$$(5.6) \quad \exists C_{nc} \mid \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\Pi_{nc} \mathbf{v}\|_h \leq C_{nc} \|\mathbf{Grad} \mathbf{v}\|_{\mathbb{L}^2(\Omega)},$$

$$(5.7) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\operatorname{div}_h \Pi_{nc} \mathbf{v}, q_h) = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q_h \in Q_h,$$

where the constant C_{nc} does not depend on h . Then Problem (5.5) is well-posed. Moreover, it admits one and only one solution (\mathbf{u}_h, p_h) such that:

$$(5.8) \quad \begin{aligned} \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega) : & \quad \begin{cases} \|\mathbf{u}_h\|_h \leq C_{PS}^{nc} \nu^{-1} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \\ \|p_h\|_{L^2(\Omega)} \leq 2 C_{PS}^{nc} C_{\operatorname{div}}^{nc} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \end{cases}, \\ \text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega) : & \quad \begin{cases} \|\mathbf{u}_h\|_h \leq C_{T_h}^{nc} \nu^{-1} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \\ \|p_h\|_{L^2(\Omega)} \leq 2 C_{T_h}^{nc} C_{\operatorname{div}}^{nc} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \end{cases}, \end{aligned}$$

where $C_{\operatorname{div}}^{nc} = C_{\operatorname{div}} C_{nc}$.

Proof. Let us consider $(\mathbf{u}'_h, p'_h) \in \mathcal{X}_h$ and let us build $(\mathbf{v}^*_h, q^*_h) \in \mathcal{X}_h$ satisfying (2.3) (with $V = \mathcal{X}_h$). We follow the three main steps of the proof of Theorem 1.

1. According to Proposition 1, there exists $\tilde{\mathbf{v}}_{p'_h} \in \mathbf{V}^\perp$ such that $\operatorname{div} \tilde{\mathbf{v}}_{p'_h} = p'_h$ in Ω and $\|\tilde{\mathbf{v}}_{p'_h}\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|p'_h\|_{L^2(\Omega)}$. Let us set $\mathbf{v}_{p'_h} = \nu^{-1} \tilde{\mathbf{v}}_{p'_h}$ so that $\operatorname{div} \mathbf{v}_{p'_h} = \nu^{-1} p'_h$ and $\|\mathbf{v}_{p'_h}\|_{\mathbf{H}_0^1(\Omega)} \leq \nu^{-1} C_{\operatorname{div}} \|p'_h\|_{L^2(\Omega)}$. Consider $\mathbf{v}_{h,p'_h} = \Pi_{nc} \mathbf{v}_{p'_h}$, for all $q_h \in Q_h$, we have: $(\operatorname{div}_h \mathbf{v}_{h,p'_h}, q_h) = \nu^{-1} (p'_h, q_h)_{L^2(\Omega)}$ and

$$(5.9) \quad \|\mathbf{v}_{h,p'_h}\|_h \leq \nu^{-1} C_{\operatorname{div}}^{nc} \nu \|p'_h\|_{L^2(\Omega)} \text{ where } C_{\operatorname{div}}^{nc} = C_{nc} C_{\operatorname{div}}.$$

Let us set $(\mathbf{v}^*_h, q^*_h) := (\gamma_{nc} \mathbf{u}'_h - \mathbf{v}_{h,p'_h}, -\gamma_{nc} p'_h)$, with $\gamma_{nc} > 0$. We obtain:

$$(5.10) \quad a_{S,h}((\mathbf{u}'_h, p'_h), (\mathbf{v}^*_h, q^*_h)) = \nu \gamma_{nc} \|\mathbf{u}'_h\|_h^2 + \nu^{-1} \|p'_h\|_{L^2(\Omega)}^2 - \nu (\mathbf{u}'_h, \mathbf{v}_{h,p'_h})_h.$$

2. In order to bound the last term of (5.10), we use Young inequality and then inequality (5.9) so that for all $\eta_{nc} > 0$:

$$(5.11) \quad (\mathbf{u}'_h, \mathbf{v}_{h,p'_h})_h \leq \frac{\eta_{nc}}{2} \|\mathbf{u}'_h\|_h^2 + \frac{\eta_{nc}^{-1}}{2} \left(\frac{C_{\operatorname{div}}^{nc}}{\nu} \right)^2 \|p'_h\|_{L^2(\Omega)}^2.$$

3. Using the bound (5.11) in (5.10) and choosing $\eta_{nc} = \gamma_{nc}$, we get:

$$a_{S,h}((\mathbf{u}'_h, p'_h), (\mathbf{v}^*_h, q^*_h)) \geq \nu \left(\frac{\gamma_{nc}}{2} \nu \|\mathbf{u}'_h\|_h^2 + \nu^{-2} \left(1 + \frac{(\gamma_{nc})^{-1}}{2} (C_{\operatorname{div}}^{nc})^2 \right) \|p'_h\|_{L^2(\Omega)}^2 \right).$$

Consider now $\gamma_{nc} = (C_{\operatorname{div}}^{nc})^2$. We obtain:

$$a_{S,h}((\mathbf{u}'_h, p'_h), (\mathbf{v}^*_h, q^*_h)) \geq \frac{\nu}{2} C_{\min}^{nc} \|(\mathbf{u}'_h, p'_h)\|_{\mathcal{X}_h}^2,$$

where $C_{\min}^{nc} = \frac{1}{2} \min((C_{\text{div}})^2, 1)$.

The operator T_h such that $T_h(\mathbf{u}'_h, p'_h) = (\mathbf{v}_h^*, p_h^*)$ is linear and continuous:

$$\|T_h(\mathbf{u}'_h, p'_h)\|_{\mathcal{X}_h}^2 = \|\mathbf{v}_h^*\|_h^2 + \nu^{-2} \|q_h^*\|_{L^2(\Omega)}^2 \leq (C_{\max}^{nc})^2 \|(\mathbf{u}'_h, p'_h)\|_{\mathcal{X}_h}^2$$

where $C_{\max}^{nc} = C_{\text{div}}^{nc} (1 + (C_{\text{div}}^{nc})^2)^{1/2}$.⁴ The discrete continuous bilinear form $a_{S,h}(\cdot, \cdot)$ is then T_h -coercive and according to Theorem 1, Problem (5.5) is well posed. Consider (\mathbf{u}_h, p_h) the unique solution of Problem (5.5). Choosing $\mathbf{v}_h = 0$, we obtain that $\text{div}_h \mathbf{u}_h = 0$. Now, choosing $\mathbf{v}_h = \mathbf{u}_h$ in (5.5) and using Cauchy-Schwarz inequality, we get that:

$$(5.12) \quad \begin{cases} \|\mathbf{u}_h\|_h \leq \nu^{-1} C_{PS}^{nc} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega), \text{ using (5.3) ;} \\ \|\mathbf{u}_h\|_h \leq \nu^{-1} C_{T_h}^{nc} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} & \text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega), \text{ using (5.4).} \end{cases}$$

Consider $(\mathbf{v}_h, q_h) = (\mathbf{v}_{h,p_h}, 0)$ in (5.5), where $\mathbf{v}_{h,p_h} = \Pi_{nc} \mathbf{v}_{p_h}$ is built as \mathbf{v}_{h,p'_h} in point 1, setting $p'_h = p_h$. Suppose that $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Using the triangular inequality, Cauchy-Schwarz inequality, Poincaré-Steklov inequality (5.3), Theorem 4, and estimate (5.12), we have:

$$\begin{aligned} \|p_h\|_{L^2(\Omega)}^2 &= \nu (\mathbf{u}_h, \mathbf{v}_{h,p_h})_h - (\mathbf{f}, \mathbf{v}_{h,p_h})_{\mathbf{L}^2(\Omega)}, \\ &\leq \nu \|\mathbf{u}_h\|_h \|\mathbf{v}_{h,p_h}\|_h + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}_{h,p_h}\|_{\mathbf{L}^2(\Omega)} \\ &\leq 2 C_{PS}^{nc} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}_{h,p_h}\|_h \leq 2 C_{PS}^{nc} C_{nc} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{Grad} \mathbf{v}_{p_h}\|_{\mathbf{L}^2(\Omega)}, \\ &\leq 2 C_{PS}^{nc} C_{\text{div}}^{nc} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|p_h\|_{L^2(\Omega)}. \end{aligned}$$

Applying the same reasoning when $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, we get that:

$$(5.13) \quad \begin{cases} \|p_h\|_{L^2(\Omega)} \leq 2 C_{PS}^{nc} C_{\text{div}}^{nc} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega), \text{ using (5.3) ;} \\ \|p_h\|_{L^2(\Omega)} \leq 2 C_{T_h}^{nc} C_{\text{div}}^{nc} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} & \text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega), \text{ using (5.4).} \end{cases}$$

□

Corollary 1. *Under the assumption of Theorem 4, suppose that $(\mathbf{u}, p) \in \mathbf{H}^{1+k}(\Omega) \times H^k(\Omega)$, we then have the estimate:*

$$(5.14) \quad \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \leq C \sigma^\ell h^{k+1} (\|\mathbf{u}\|_{\mathbf{H}^{k+1}(\Omega)} + \nu^{-1} \|p\|_{H^k(\Omega)}),$$

where the constant $C > 0$ is independent of h , σ is the shape regularity parameter and the exponent $\ell \in \mathbb{N}^*$ depends on k .

Proof. The a priori error estimate corresponds to [2, Theorem 4].

□

Remark 2. *Again, we recover the first Banach-Nečas-Babuška condition [8, Thm. 25.9, (BNB1)]:*

$$a_{S,h}((\mathbf{u}'_h, p'_h), (\mathbf{v}_h^*, q_h^*)) \geq \nu C_{\min}^{nc} (C_{\max}^{nc})^{-1} \|(\mathbf{u}'_h, p'_h)\|_{\mathcal{X}_h} \|(\mathbf{v}_h^*, q_h^*)\|_{\mathcal{X}_h}.$$

⁴Note that $(\mathbf{v}_h^*, q_h^*) = (\mathbf{0}, 0) \Leftrightarrow (\mathbf{u}'_h, p'_h) = (\mathbf{0}, 0)$, so that the operator $T_h \in \mathcal{L}(\mathcal{X}_h, \mathcal{X}_h)$ is bijective.

Let us call $C_{\text{stab}}^{nc} = \nu C_{\text{min}}^{nc} (C_{\text{max}}^{nc})^{-1}$ the stability constant. With the choice of our parameters, C_{stab}^{nc} is such that:

$$C_{\text{stab}}^{nc} = \begin{cases} \frac{\nu}{2} \frac{C_{\text{div}}^{nc}}{(1 + (C_{\text{div}}^{nc})^2)^{1/2}} & \text{if } 0 < C_{\text{div}}^{nc} \leq 1, \\ \frac{\nu}{2} \frac{(C_{\text{div}}^{nc})^{-1}}{(1 + (C_{\text{div}}^{nc})^2)^{1/2}} & \text{if } 1 \leq C_{\text{div}}^{nc}. \end{cases}$$

The main issue with nonconforming mixed finite elements is the construction the basis functions. In a recent paper, Sauter explains such a construction in two dimensions [17, Corollary 2.4], and gives a bound to the discrete counterpart $\beta_{\mathcal{T}}(\Omega)$ of $\beta(\Omega)$ defined in (3.4):

$$(5.15) \quad \beta_{\mathcal{T}}(\Omega) = \inf_{q_h \in Q_h \setminus \{0\}} \sup_{\mathbf{v}_h \in \mathbf{X}_{0,h} \setminus \{0\}} \frac{(\text{div}_h \mathbf{v}_h, q_h)}{\|q_h\|_{L^2(\Omega)} \|\mathbf{v}_h\|_h} \geq c_{\mathcal{T}} k^{-\alpha}.$$

This bound is in $c_{\mathcal{T}} k^{-\alpha}$, where the parameter α is explicit and depends on k and on the mesh topology; and the constant $c_{\mathcal{T}}$ depends only on the shape-regularity of the mesh.

6. NONCONFORMING CROUZEIX-RAVIART MIXED FINITE ELEMENTS

We study the lowest order nonconforming Crouzeix-Raviart mixed finite elements [2]. Let us consider X_{CR} (resp. $X_{0,CR}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H_0^1(\Omega)$) of order 1:

$$(6.1) \quad \begin{aligned} X_{CR} &= \left\{ v_h \in P_{\text{disc}}^1(\mathcal{T}_h); \quad \forall f \in \mathcal{I}_F^i, \int_{F_f} [v_h] = 0 \right\}; \\ X_{0,CR} &= \left\{ v_h \in X_{CR}; \quad \forall f \in \mathcal{I}_F^b, \int_{F_f} v_h = 0 \right\}. \end{aligned}$$

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}_0^1(\Omega)$) of order 1 is $\mathbf{X}_{CR} = (X_{CR})^d$ (resp. $\mathbf{X}_{0,CR} = (X_{0,CR})^d$). We set $\mathcal{X}_{CR} := \mathbf{X}_{0,CR} \times Q_{CR}$ where $Q_{CR} = P^0(\mathcal{T}_h) \cap L_{zmv}^2(\Omega)$.

We can endow X_{CR} with the basis $(\psi_f)_{f \in \mathcal{I}_F}$ such that: $\forall \ell \in \mathcal{I}_K$,

$$\psi_{f|K_\ell} = \begin{cases} 1 - d\lambda_{i,\ell} & \text{if } f \in \mathcal{I}_{F,\ell}, \\ 0 & \text{otherwise,} \end{cases}$$

where S_i is the vertex opposite to F_f in K_ℓ . We then have $\psi_{f|F_f} = 1$, so that $[\psi_f]_{F_f} = 0$ if $f \in \mathcal{I}_F^i$ (i.e. $F_f \in \overset{\circ}{\Omega}$), and $\forall f' \neq f$, $\int_{F_{f'}} \psi_f = 0$.

We have: $X_{CR} = \text{vect}((\psi_f)_{f \in \mathcal{I}_F})$ and $X_{0,CR} = \text{vect}((\psi_f)_{f \in \mathcal{I}_F^i})$.

The Crouzeix-Raviart interpolation operator π_{CR} for scalar functions is defined by:

$$\pi_{CR} : \begin{cases} H^1(\Omega) & \rightarrow X_{CR} \\ v & \mapsto \sum_{f \in \mathcal{I}_F} \pi_f v \psi_f \quad , \quad \text{where } \pi_f v = \frac{1}{|F_f|} \int_{F_f} v. \end{cases}$$

Notice that $\forall f \in \mathcal{I}_F$, $\int_{F_f} \pi_{CR} v = \int_{F_f} v$. Moreover, the Crouzeix-Raviart interpolation operator preserves the constants, so that $\pi_{CR} \underline{v}_\Omega = \underline{v}_\Omega$ where $\underline{v}_\Omega = \int_\Omega v / |\Omega|$. We recall the following result [18, Lemma 2]):

Lemma 1. *The Crouzeix-Raviart interpolation operator π_{CR} is such that:*

$$(6.2) \quad \forall v \in H^1(\Omega), \quad \|\pi_{CR}v\|_h \leq \|\mathbf{grad} v\|_{\mathbf{L}^2(\Omega)}.$$

Proof. We have, integrating by parts twice and using Cauchy-Schwarz inequality:

$$\begin{aligned} \mathbf{grad} \pi_{CR}v|_{K_\ell} &= |K_\ell|^{-1} \int_{K_\ell} \mathbf{grad} \pi_{CR}v = |K_\ell|^{-1} \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} \pi_{CR}v \mathbf{n}_f, \\ &= |K_\ell|^{-1} \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} v \mathbf{n}_f = |K_\ell|^{-1} \int_{K_\ell} \mathbf{grad} v, \\ |\mathbf{grad} \pi_{CR}v|_{K_\ell} &\leq |K_\ell|^{-1/2} \|\mathbf{grad} v\|_{\mathbf{L}^2(K_\ell)} \\ \Rightarrow \|\mathbf{grad} \pi_{CR}v\|_{\mathbf{L}^2(K_\ell)}^2 &\leq \|\mathbf{grad} v\|_{\mathbf{L}^2(K_\ell)}^2. \end{aligned}$$

Summing these local estimates over $\ell \in \mathcal{I}_K$, we obtain (6.2). \square

For a vector $\mathbf{v} \in \mathbf{H}^1(\Omega)$ of components $(v_{d'})_{d'=1}^d$, the Crouzeix-Raviart interpolation operator is such that: $\Pi_{CR}\mathbf{v} = (\pi_{CR}v_{d'})_{d'=1}^d$. Let us set $\Pi_f\mathbf{v} = (\pi_f v_{d'})_{d'=1}^d$.

Lemma 2. *The Crouzeix-Raviart interpolation operator Π_{CR} can play the role of the Fortin operator:*

$$(6.3) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\Pi_{CR}\mathbf{v}\|_h \leq \|\mathbf{Grad} \mathbf{v}\|_{\mathbf{L}^2(\Omega)},$$

$$(6.4) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\operatorname{div}_h \Pi_{CR}\mathbf{v}, q_h) = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q_h \in Q_h,$$

Moreover, for all $\mathbf{v} \in \mathbf{P}^1(\Omega)$, $\Pi_{CR}\mathbf{v} = \mathbf{v}$.

Proof. We obtain (6.3) applying Lemma (1) component by component. By integrating by parts, we have $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\forall \ell \in \mathcal{I}_K$:

$$\begin{aligned} \int_{K_\ell} \operatorname{div} \Pi_{CR}\mathbf{v} &= \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} \Pi_{CR}\mathbf{v} \cdot \mathbf{n}_f = \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} \Pi_f\mathbf{v} \cdot \mathbf{n}_f, \\ &= \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} \mathbf{v} \cdot \mathbf{n}_f = \int_{K_\ell} \operatorname{div} \mathbf{v}, \end{aligned}$$

so that (6.4) is satisfied. \square

We can apply the T-coercivity theory to show the next following result:

Theorem 5. *Let $\mathcal{X}_h = \mathcal{X}_{CR}$. Then the continuous bilinear form $a_{S,h}(\cdot, \cdot)$ is T_h -coercive and Problem (5.5) is well-posed.*

Proof. Using estimates (6.3) and (5.3), we apply the proof of Theorem 4. \square

Since the constant of the interpolation operator Π_{CR} is equal to 1, we have $C_{min}^{CR} = C_{min}$ and $C_{max}^{CR} = C_{max}$: the stability constant of the nonconforming Crouzeix-Raviart mixed finite elements is independent of the mesh. This is not the case for higher-order (see [19, Theorem 2.2]).

For higher-order, we cannot build the interpolation operator component by component, since higher-order divergence moments must be preserved. Thus, for $k > 1$, we must build Π_{nc} so that for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $\ell \in \mathcal{I}_K$, for all $q \in P^{k-1}(K_\ell)$:

$$\int_{K_\ell} q \operatorname{div} \Pi_{nc}\mathbf{v} = \int_{K_\ell} q \operatorname{div} \mathbf{v}.$$

We recall that by integration by parts, we have:

$$(6.5) \quad \int_{K_\ell} q \operatorname{div} \Pi_{nc} \mathbf{v} + \int_{K_\ell} \mathbf{grad} q \cdot \Pi_{nc} \mathbf{v} = \int_{\partial K_\ell} q \Pi_{nc} \mathbf{v} \cdot \mathbf{n}|_{\partial K_\ell}.$$

Hence, to obtain a local estimate of $\|\mathbf{Grad} \Pi_{nc} \mathbf{v}\|_{\mathbb{L}(K_\ell)}$, we will need the following Lemma:

Lemma 3. *Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and $q \in P^{k-1}(K_\ell)$. We set $\underline{\mathbf{v}}_\ell := \mathbf{v}_\ell - \frac{\int_{K_\ell} \mathbf{v}}{|K_\ell|}$, where $\mathbf{v}_\ell = \mathbf{v}|_{K_\ell}$. We have:*

$$(6.6) \quad \left| \int_{\partial K_\ell} q \underline{\mathbf{v}}_\ell \cdot \mathbf{n}|_{\partial K_\ell} \right| \leq |K_\ell|^{k/2} \|\mathbf{Grad} \underline{\mathbf{v}}_\ell\|_{\mathbb{L}^2(K_\ell)}$$

Proof. We have by integration by parts, and then using Cauchy-Schwarz inequality:

$$\begin{aligned} \left| \int_{\partial K_\ell} q \underline{\mathbf{v}}_\ell \cdot \mathbf{n}_f \right| &\leq \left| \int_{K_\ell} q \operatorname{div} \underline{\mathbf{v}}_\ell \right| + \left| \int_{K_\ell} \mathbf{grad} q \cdot \underline{\mathbf{v}}_\ell \right|, \\ &\leq \|q\|_{L^2(K_\ell)} \|\mathbf{Grad} \underline{\mathbf{v}}_\ell\|_{\mathbb{L}^2(K_\ell)} + \|\mathbf{grad} q\|_{\mathbb{L}^2(K_\ell)} \|\underline{\mathbf{v}}_\ell\|_{\mathbb{L}^2(K_\ell)}, \\ &\leq |K_\ell|^{k/2} \|\mathbf{Grad} \underline{\mathbf{v}}_\ell\|_{\mathbb{L}^2(K_\ell)} + |K_\ell|^{(k-1)/2} \|\underline{\mathbf{v}}_\ell\|_{\mathbb{L}^2(K_\ell)}, \\ &\lesssim |K_\ell|^{k/2} \|\mathbf{Grad} \underline{\mathbf{v}}_\ell\|_{\mathbb{L}^2(K_\ell)} \text{ using (4.4)}. \end{aligned}$$

□

7. FORTIN-SOULIE MIXED FINITE ELEMENTS

We consider here the case $d = 2$ and we study Fortin-Soulie mixed finite elements [3]. We consider a shape-regular triangulation sequence $(\mathcal{T}_h)_h$. Let us consider X_{FS} (resp. $X_{0,FS}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H_0^1(\Omega)$) of order 2:

$$(7.1) \quad \begin{aligned} X_{FS} &= \left\{ v_h \in P_{disc}^2(\mathcal{T}_h); \quad \forall f \in \mathcal{I}_F^i, \forall q_h \in P^1(F_f), \int_{F_f} [v_h] q_h = 0 \right\}; \\ X_{0,FS} &= \left\{ v_h \in X_{FS}; \quad \forall f \in \mathcal{I}_F^b, \forall q_h \in P^1(F_f), \int_{F_f} v_h q_h = 0 \right\}. \end{aligned}$$

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}_0^1(\Omega)$) of order 2 is $\mathbf{X}_{FS} = (X_{FS})^2$ (resp. $\mathbf{X}_{0,FS} = (X_{0,FS})^2$). We set $\mathcal{X}_{FS} = \mathbf{X}_{0,FS} \times Q_{FS}$ where $Q_{FS} := P_{disc}^1(\mathcal{T}_h) \cap L_{zmv}^2(\Omega)$.

The building of a basis for $X_{0,FS}$ is more involved than for $X_{0,CR}$ since we cannot use two points per facet as degrees of freedom. Indeed, for all $\ell \in K_\ell$, there exists a polynomial of order 2 vanishing on the Gauss-Legendre points of the facets of the boundary ∂K_ℓ . Let $f \in \mathcal{I}_F$. The barycentric coordinates of the two Gauss-Legendre points $(p_{+,f}, p_{-,f})$ on F_f are such that:

$$p_{+,f} = (c_+, c_-), p_{-,f} = (c_-, c_+), \text{ where } c_\pm = (1 \pm 1/\sqrt{3})/2.$$

These points can be used to integrate exactly order three polynomials:

$$\forall g \in P^3(F_f), \int_{F_f} g = \frac{|F_f|}{2} (g(p_{+,f}) + g(p_{-,f})).$$

For all $\ell \in \mathcal{I}_K$, we define the quadratic function ϕ_{K_ℓ} that vanishes on the six Gauss-Legendre points of the facets of K_ℓ (see Fig. 1):

$$(7.2) \quad \phi_{K_\ell} := 2 - 3 \sum_{i \in \mathcal{I}_{S,\ell}} \lambda_{i,\ell}^2 \text{ such that } \forall f \in \mathcal{I}_{F,\ell}, \forall q \in P^1(F), \int_{F_f} \phi_{K_\ell} q = 0.$$

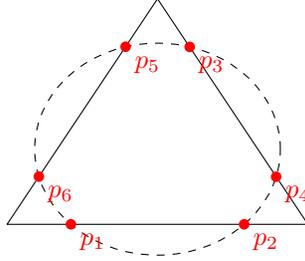


FIGURE 1. The six Gauss-Legendre points of an element K_ℓ and the elliptic function ϕ_{K_ℓ} .

We also define the spaces of P^2 -Lagrange functions:

$$\begin{aligned} X_{LG} &:= \{v_h \in H^1(\Omega); \forall \ell \in \mathcal{I}_K, v_h|_{K_\ell} \in P^2(K_\ell)\}, \\ X_{0,LG} &:= \{v_h \in X_{LG}; v_h|_{\partial\Omega} = 0\}. \end{aligned}$$

The Proposition below proved in [3, Prop. 1] allows to build a basis for $X_{0,FS}$:

Proposition 5. *We have the following decomposition: $X_{FS} = X_{LG} + \Phi_h$ with $\dim(X_{LG} \cap \Phi_h) = 1$. Any function of X_{FS} can be written as the sum of a function of X_{LG} and a function of Φ_h . This representation can be made unique by specifying one degree of freedom.*

Notice that $\Phi_h \cap X_{LG} = \text{vect}(v_\Phi)$, where for all $\ell \in \mathcal{I}_K$, $v_\Phi|_{K_\ell} = \phi_{K_\ell}$. Then, counting the degrees of freedom, one can show that $\dim(X_{FS}) = \dim(X_{LG}) + \dim(\Phi_h) + 1$. For problems involving Dirichlet boundary conditions we can prove thus that for $X_{0,FS}$ the representation is unique and $X_{0,FS} = X_{0,LG} \oplus \Phi_h$. We have $X_{LG} = \text{vect}((\phi_{S_i})_{i \in \mathcal{I}_S}, (\phi_{F_f})_{f \in \mathcal{I}_F})$ where the basis functions are such that:

$$\begin{aligned} \forall i, j \in \mathcal{I}_S, \forall f, g \in \mathcal{I}_F : \\ \phi_{S_i}(S_j) = \delta_{ij}, \phi_{S_i}(M_f) = 0, \quad \phi_{M_f}(M_g) = \delta_{fg}, \phi_{M_f}(S_i) = 0. \end{aligned}$$

For all $\ell \in \mathcal{I}_K$, we will denote by $(\phi_{\ell,j})_{j=1}^6$ the local nodal basis such that:

$$(\phi_{\ell,j})_{j=1}^3 = (\phi_{S_i|K_\ell})_{i \in \mathcal{I}_{S,\ell}} \quad \text{and} \quad (\phi_{\ell,j})_{j=4}^6 = (\phi_{F_f|K_\ell})_{f \in \mathcal{I}_{F,\ell}}.$$

The spaces X_{FS} and $X_{0,FS}$ are such that:

$$(7.3) \quad \begin{aligned} X_{FS} &= \text{vect} \left((\phi_{S_i})_{i \in \mathcal{I}_S}, (\phi_{F_f})_{f \in \mathcal{I}_F}, (\phi_{K_\ell})_{\ell \in \mathcal{I}_K} \right), \\ X_{0,FS} &= \text{vect} \left((\phi_{S_i})_{i \in \mathcal{I}_S^i}, (\phi_{F_f})_{f \in \mathcal{I}_F^i}, (\phi_{K_\ell})_{\ell \in \mathcal{I}_K} \right). \end{aligned}$$

We propose here an alternative definition of the Fortin interpolation operator proposed in [3]. Let us first recall the Scott-Zhang interpolation operator [20, 21]. For all $i \in \mathcal{I}_S$, we choose some $\ell_i \in \mathcal{I}_{K,i}$, and we build the $L^2(K_{\ell_i})$ -dual basis $(\tilde{\phi}_{\ell_i,j})_{j=1}^6$ of the local nodal basis such that:

$$\forall j, j' \in \{1, \dots, 6\}, \quad \int_{K_{\ell_i}} \tilde{\phi}_{\ell_i,j} \phi_{\ell_i,j'} = \delta_{j,j'}.$$

Let us define the Fortin-Soulie interpolation operator for scalar functions by:

$$(7.4) \quad \pi_{FS} : \begin{cases} H^1(\Omega) & \rightarrow X_{FS} \\ v & \mapsto \tilde{\pi}v + \sum_{\ell \in \mathcal{I}_K} v_{K_\ell} \phi_{K_\ell} \end{cases}, \\ \text{with } \tilde{\pi}v = \sum_{i \in \mathcal{I}_S} v_{S_i} \phi_{S_i} + \sum_{f \in \mathcal{I}_F} \tilde{v}_{F_f} \phi_{F_f}.$$

- The coefficients $(v_{S_i})_{i \in \mathcal{I}_S}$ are fixed so that: $\forall i \in \mathcal{I}_S, v_{S_i} = \int_{K_{\ell_i}} v \tilde{\phi}_{\ell_i,j_i}$, where j_i is the index such that $\int_{K_{\ell_i}} \tilde{\phi}_{\ell_i,j_i} \phi_{S_i|K_{\ell_i}} = 1$.
- The coefficients $(\tilde{v}_{F_f})_{f \in \mathcal{I}_F}$ are fixed so that: $\forall f \in \mathcal{I}_F, \int_{F_f} \tilde{\pi}v = \int_{F_f} v$.

For all $\ell \in \mathcal{I}_K$, the coefficient v_{K_ℓ} is fixed so that: $\int_{K_\ell} \pi_{FS}v = \int_{K_\ell} v$.

The definition (7.4) is more general than the one given in [3], which holds for $v \in H^2(\Omega)$.

We set $\mathbf{v}_{S_i} := (\tilde{\pi}v_1(S_i), \tilde{\pi}v_2(S_i))^T$ and $\tilde{\mathbf{v}}_{F_f} := (\tilde{\pi}v_1(F_f), \tilde{\pi}v_2(F_f))^T$.

We can define two different Fortin-Soulie interpolation operators for vector functions. First, let

$$\tilde{\Pi}_{FS} : \begin{cases} \mathbf{H}^1(\Omega) & \rightarrow \mathbf{X}_{FS} \\ \mathbf{v} & \mapsto (\pi_{FS}(\mathbf{v})_1, \pi_{FS}(\mathbf{v})_2)^T. \end{cases}$$

We remind that the coefficients $(\tilde{\mathbf{v}}_{K_\ell})_{\ell \in \mathcal{I}_K}$ are such that:

$$(7.5) \quad \forall \ell \in \mathcal{I}_K, \quad \int_{K_\ell} \tilde{\Pi}_{FS} \mathbf{v} = \int_{K_\ell} \mathbf{v}.$$

The interpolation operator $\tilde{\Pi}_{FS}$ preserves the local averages, but it doesn't preserve the divergence. We then define a second interpolation operator which preserves the divergence in a weak sense:

$$\Pi_{FS} : \begin{cases} \mathbf{H}^1(\Omega) & \rightarrow \mathbf{X}_{FS} \\ \mathbf{v} & \mapsto \sum_{i \in \mathcal{I}_S} \mathbf{v}_{S_i} \phi_{S_i} + \sum_{f \in \mathcal{I}_F} \tilde{\mathbf{v}}_{F_f} \phi_{F_f} + \sum_{\ell \in \mathcal{I}_K} \mathbf{v}_{K_\ell} \phi_{K_\ell} \end{cases}.$$

For all $\ell \in \mathcal{I}_K$, the vector coefficient $\mathbf{v}_{K_\ell} \in \mathbb{R}^2$ is now fixed so that [condition \(5.7\) is satisfied](#). We can impose for example that the projection $\Pi_{FS} \mathbf{v}$ satisfies:

$$(7.6) \quad \int_{K_\ell} T_\ell^{-1}(\mathbf{x}) \operatorname{div} \Pi_{FS} \mathbf{v} = \int_{K_\ell} T_\ell^{-1}(\mathbf{x}) \operatorname{div} \mathbf{v}.$$

Notice that due to (7.2), the patch-test condition is still satisfied.

Proposition 6. *Let $\sigma_D > 0$. The Fortin-Soulie interpolation operator Π_{FS} is such for all $\mathbf{v} \in \bigcap_{0 < s < \sigma_D} \mathbf{H}^{1+s}(\Omega)$ we have:*

$$(7.7) \quad \forall s \in]0, \sigma_D[, \forall \ell \in \mathcal{I}_K, \|\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim (\sigma_\ell)^2 (h_\ell)^s |\mathbf{v}|_{1+s, K_\ell},$$

$$(7.8) \quad \forall s \in]0, \sigma_D[, \exists C_{FS} = \mathcal{O}(\sigma^2), \quad \|\Pi_{FS}\mathbf{v} - \mathbf{v}\|_h \leq C_{FS} h^s |\mathbf{v}|_{1+s, \Omega}.$$

Remark 3. *Albeit we are inspired by the proof of [2, Lemma 4], we changed the transition from equation (4.27) to (4.29) there by using only the properties related to the normal component of the velocity, cf (6.6). As a matter of fact, in the original proof, one ends up with either $C_{FS} = \mathcal{O}(\sigma^3)$ with the help of the multiple trace inequality or with $C_{FS} = \mathcal{O}(\sigma^2)$ at the cost of imposing a stronger assumption on the regularity of \mathbf{v} (namely, $\sigma_D > 1/2$). Finally, because we do not split the integral over the boundaries of elements into the sum of $d + 1$ integrals over the facets, we obtain purely local estimates, which appear to be new for the Fortin-Soulie element in the case of low-regularity fields \mathbf{v} .*

Proof. Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$. By construction, we have:

$$(7.9) \quad \int_{K_\ell} (\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v}) = 0 \text{ and for all } f \in \mathcal{I}_{F, \ell}, \int_{F_f} (\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})|_{K_\ell} = 0.$$

We have:

$$(7.10) \quad \begin{aligned} \|\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} &\leq \|\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \\ &\quad + \|\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}. \end{aligned}$$

Notice that for all $\ell \in \mathcal{I}_K$, $(\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v})|_{K_\ell} = (\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}) \phi_{K_\ell}$.

Using (4.3)-(i), we obtain that:

$$(7.11) \quad \begin{aligned} \|\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v})\|_{\mathbb{L}^2(K_\ell)} &\lesssim |\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}| \|\mathbf{grad} \phi_{K_\ell}\|_{\mathbb{L}^2(K_\ell)}, \\ &\lesssim \|\mathbb{B}_\ell^{-1}\| |K_\ell|^{1/2} |\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|, \\ &\lesssim \sigma_\ell |\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|. \end{aligned}$$

Let us estimate $|\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|$. On the one hand, we have:

$$\begin{aligned} \int_{K_\ell} (\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v}) &= \int_{K_\ell} (\Pi_{FS}\mathbf{v} - \mathbf{v}) \text{ from (7.5),} \\ &= \int_{\partial K_\ell} \mathbf{x} (\Pi_{FS}\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}|_{\partial K_\ell} \text{ by IBP,} \\ &= \int_{\partial K_\ell} \mathbf{x} (\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}|_{\partial K_\ell} \text{ from (7.2).} \end{aligned}$$

Hence, using (7.9), we obtain:

$$(7.12) \quad \left| \int_{K_\ell} (\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v}) \right| \leq |K_\ell| \|\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}.$$

On the other hand, it holds:

$$(7.13) \quad \int_{K_\ell} (\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v}) = (\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}) \int_{K_\ell} \phi_{K_\ell} = \frac{|K_\ell|}{4} (\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}).$$

Hence, combining (7.12) and (7.13), we have:

$$|\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}| \leq 4 \|\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}$$

We deduce from (7.10) that for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $\ell \in \mathcal{I}_K$ we have:

$$(7.14) \quad \|\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{K_\ell} \lesssim \sigma_\ell \|\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}.$$

For all $\mathbf{v} \in \mathbf{P}^2(K_\ell)$ we have $\tilde{\Pi}_{FS}(\mathbf{v}) = \mathbf{v}$ and $(\widehat{\tilde{\Pi}_{FS}\mathbf{v}})_\ell = \widehat{\tilde{\Pi}_{FS}\mathbf{v}}_\ell$. Hence, using Bramble-Hilbert/Deny-Lions Lemma [16, Lemma 11.9], we have:

$$\begin{aligned} \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad & \|\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim \sigma_\ell |\mathbf{v}|_{1, K_\ell}, \\ \forall \mathbf{v} \in \mathbf{H}^2(\Omega) \quad & \|\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim \sigma_\ell h_\ell |\mathbf{v}|_{2, K_\ell}. \end{aligned}$$

We deduce that:

$$(7.15) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim (\sigma_\ell)^2 |\mathbf{v}|_{1, K_\ell},$$

$$(7.16) \quad \forall \mathbf{v} \in \mathbf{H}^2(\Omega) \quad \|\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim (\sigma_\ell)^2 h_\ell |\mathbf{v}|_{2, K_\ell}.$$

Using interpolation property [22, Lemma 22.2], we obtain (7.8). \square

We recall that the discrete Poincaré–Steklov inequality (5.3) holds.

Theorem 6. *Let $\mathcal{X}_h = \mathcal{X}_{FS}$. Then the continuous bilinear form $a_{S,h}(\cdot, \cdot)$ is T_h -coercive and Problem (5.5) is well-posed.*

Proof. According to Proposition 6, the Fortin-Soulie interpolation operator Π_{FS} satisfies (5.6)-(5.7), so that we can apply the proof of Theorem 4. \square

Notice that in the recent paper [23], the inf-sup condition of the mixed Fortin-Soulie finite element is proven directly on a triangle and then using the macro-element technique [24], but it seems difficult to use this technique to build a Fortin operator, which is needed to compute error estimates.

The study can be extended to higher orders for $d = 2$ using the following papers: [25] for $k \geq 4$, k even, [26] for $k = 3$ and [19] for $k \geq 5$, k odd. In [27], the authors propose a local Fortin operator for the lowest order Taylor-Hood finite element [14] for $d = 3$ which could be used to prove the T-coercivity.

8. NUMERICAL RESULTS

Consider Problem (3.1) with data $\mathbf{f} = -\mathbf{grad} \phi$, where $\phi \in H^1(\Omega) \cap L^2_{zmv}(\Omega)$. The unique solution is then $(\mathbf{u}, p) := (0, \phi)$. By integrating by parts, the source term in (3.6) reads:

$$(8.1) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad \int_{\Omega} \mathbf{f} \cdot \mathbf{v} = \int_{\Omega} \phi \operatorname{div} \mathbf{v}.$$

Recall that the nonconforming space \mathbf{X}_h defined in (5.1) is a subset of $\mathcal{P}_h \mathbf{H}^1$: using a nonconforming finite element method, the integration by parts must be done on each element of the triangulation, and we have:

$$(8.2) \quad \forall \mathbf{v} \in \mathcal{P}_h \mathbf{H}^1, \quad \int_{\Omega} \mathbf{f} \cdot \mathbf{v} = (\operatorname{div}_h \mathbf{v}, \phi) + \sum_{f \in \mathcal{I}_F} \int_{F_f} [\mathbf{v} \cdot \mathbf{n}_f] \phi.$$

When we apply this result to the right-hand-side of (5.5), we notice that the term with the jumps acts as a numerical source, which numerical influence is proportional to $1/\nu$. Thus, we cannot obtain exactly $\mathbf{u}_h = 0$ (see also (5.14)). Linke proposed in [28] to project the test function $\mathbf{v}_h \in \mathbf{X}_h$ on a discrete subspace of $\mathbf{H}(\operatorname{div}; \Omega)$, like Raviart-Thomas or Brezzi-Douglas-Marini finite elements (see [29, 30], or the monograph [15]). Let $\Pi_{\operatorname{div}} : \mathbf{X}_{0,h} \rightarrow P_{disc}^k(\mathcal{T}_h) \cap \mathbf{H}_0(\operatorname{div}; \Omega)$ be some interpolation

operator built so that for all $\mathbf{v}_h \in \mathbf{X}_{0,h}$, for all $\ell \in \mathcal{I}_K$, $(\operatorname{div} \Pi_{\operatorname{div}} \mathbf{v}_h)|_{K_\ell} = \operatorname{div} \mathbf{v}_h|_{K_\ell}$. Integrating by parts, we have for all $\mathbf{v}_h \in \mathbf{X}_{0,h}$:

$$\begin{aligned} \int_{\Omega} \mathbf{f} \cdot \Pi_{\operatorname{div}} \mathbf{v}_h &= \int_{\Omega} \phi \operatorname{div} \Pi_{\operatorname{div}} \mathbf{v}_h = \sum_{\ell \in K_\ell} \int_{K_\ell} \phi \operatorname{div} \Pi_{\operatorname{div}} \mathbf{v}_h, \\ &= \sum_{\ell \in K_\ell} \int_{K_\ell} \phi \operatorname{div} \mathbf{v}_h = (\operatorname{div}_h \mathbf{v}_h, \phi). \end{aligned}$$

The projection Π_{div} allows to eliminate the terms of the integrals of the jumps in (8.2).

Let us write Problem (5.5) as:

Find $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$ such that

$$(8.3) \quad a_{S,h}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \ell_{\mathbf{f}}((\Pi_{\operatorname{div}} \mathbf{v}_h, q_h)) \quad \forall (\mathbf{v}_h, q_h) \in \mathcal{X}_h.$$

In the case of $\mathcal{X}_h = \mathcal{X}_{CR}$ and a projection on Brezzi-Douglas-Marini finite elements, the following error estimate holds if $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$:

$$(8.4) \quad \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \leq \tilde{C} h^2 |\mathbf{u}|_{\mathbf{H}^2(\Omega)},$$

where the constant \tilde{C} is independent of h . The proof is detailed in [31] for shape-regular meshes and [32] for anisotropic meshes. We remark that the error doesn't depend on the norm of the pressure nor on the ν parameter. We will provide some numerical results to illustrate the effectiveness of this formulation, even with a projection on the Raviart-Thomas finite elements, which, for a fixed polynomial order, are less precise than the Brezzi-Douglas-Marini finite elements.

For all $\ell \in \mathcal{I}_K$, we let $P_H^k(K_\ell)$ be the set of homogeneous polynomials of order k on K_ℓ .

For $k \in \mathbb{N}^*$, the space of Raviart-Thomas finite elements can be defined as:

$$\mathbf{X}_{RT_k} := \{ \mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega); \forall \ell \in \mathcal{I}_k, \mathbf{v}|_{K_\ell} = \mathbf{a}_\ell + b_\ell \mathbf{x} \mid (\mathbf{a}_\ell, b_\ell) \in P^k(K_\ell)^d \times P_H^k(K_\ell) \}.$$

Let $k \leq 1$.

The Raviart-Thomas interpolation operator $\Pi_{RT_k} : \mathbf{H}^1(\Omega) \cup \mathbf{X}_h \rightarrow \mathbf{X}_{RT_k}$ is defined by: $\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \cup \mathbf{X}_h$,

$$(8.5) \quad \left\{ \begin{array}{l} \forall f \in \mathcal{I}_F, \quad \int_{F_f} \Pi_{RT_k} \mathbf{v} \cdot \mathbf{n}_f q = \int_{F_f} \mathbf{v} \cdot \mathbf{n}_f q, \quad \forall q \in P^k(F_f) \\ \text{for } k = 1, \forall \ell \in \mathcal{I}_K, \quad \int_{K_\ell} \Pi_{RT_1} \mathbf{v} = \int_{K_\ell} \mathbf{v} \end{array} \right.$$

Note that the Raviart-Thomas interpolation operator preserves the constants. Let $\mathbf{v}_h \in \mathbf{X}_h$. In order to compute the left-hand-side of (8.2), we must evaluate $(\Pi_{RT_k} \mathbf{v}_h)|_{K_\ell}$ for all $\ell \in \mathcal{I}_K$. Calculations are performed using the proposition below, which corresponds to [33, Lemma 3.11]:

Proposition 7. *Let $k \leq 1$. Let $\hat{\Pi}_{RT_k} : \mathbf{H}^1(\hat{K}) \rightarrow \mathbf{P}^k(\hat{K})$ be the Raviart-Thomas interpolation operator restricted to the reference element, so that: $\forall \hat{\mathbf{v}} \in \mathbf{H}^1(\hat{K})$,*

$$(8.6) \quad \left\{ \begin{array}{l} \forall \hat{F} \in \partial \hat{K}, \quad \int_{\hat{F}} \hat{\Pi}_{RT_k} \hat{\mathbf{v}} \cdot \mathbf{n}_{\hat{F}} \hat{q} = \int_{\hat{F}} \hat{\mathbf{v}} \cdot \mathbf{n}_{\hat{F}} \hat{q}, \quad \forall \hat{q} \in P^k(\hat{F}) \\ \text{for } k = 1, \quad \int_{\hat{K}} \hat{\Pi}_{RT_k} \hat{\mathbf{v}} = \int_{\hat{K}} \hat{\mathbf{v}} \end{array} \right.$$

We then have: $\forall \ell \in \mathcal{I}_K$,

$$(8.7) \quad (\Pi_{RT_k} \mathbf{v})|_{K_\ell}(\mathbf{x}) = \mathbb{B}_\ell \left(\hat{\Pi}_{RT_k} \mathbb{B}_\ell^{-1} \hat{\mathbf{v}}_\ell \right) \circ T_\ell^{-1}(\mathbf{x}) \quad \text{where } \hat{\mathbf{v}}_\ell = \mathbf{v} \circ T_\ell(\hat{\mathbf{x}}).$$

The proof is based on the equality of the \hat{F} and \hat{K} -moments of $(\Pi_{RT_k} \mathbf{v})|_{K_\ell} \circ T_\ell(\hat{\mathbf{x}})$ and $\mathbb{B}_\ell \left(\hat{\Pi}_{RT_k} \mathbb{B}_\ell^{-1} \hat{\mathbf{v}}_\ell \right) (\hat{\mathbf{x}})$. For $k = 0$, setting for $d' \in \{1, \dots, d\}$: $\boldsymbol{\psi}_{f,d'} := \psi_f \mathbf{e}_{d'}$, we obtain that:

$$(8.8) \quad \forall \ell \in \mathcal{I}_K, \forall f \in \mathcal{I}_{F,\ell}, \quad (\Pi_{RT_0} \boldsymbol{\psi}_{f,d'})|_{K_\ell} = (d|K_\ell|)^{-1} \left(\mathbf{x} - \vec{O}S_{f,\ell} \right) \mathcal{S}_{f,\ell} \cdot \mathbf{e}_{d'},$$

where $S_{f,\ell}$ is the vertex opposite to F_f in K_ℓ .

For $k = 1$, the vector $(\Pi_{RT_1} \mathbf{v}_h)|_{K_\ell}$ is described by eight unknowns:

$$(\Pi_{RT_1} \mathbf{v}_h)|_{K_\ell} = \mathbb{A}_\ell \mathbf{x} + (\mathbf{b}_\ell \cdot \mathbf{x}) \mathbf{x} + \mathbf{d}_\ell, \quad \text{where } \mathbb{A}_\ell \in \mathbb{R}^{2 \times 2}, \mathbf{b}_\ell \in \mathbb{R}^2, \mathbf{d}_\ell \in \mathbb{R}^2.$$

We compute only once the inverse of the matrix of the linear system (8.6), in $\mathbb{R}^{8 \times 8}$. In the Table 1 (resp. Tables 2 and 3), we call $\varepsilon_0(\mathbf{u}) = \|\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}$ (resp. $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} / \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}$) the velocity error in $\mathbf{L}^2(\Omega)$ -norm, where \mathbf{u}_h is the solution to Problem (5.5) (columns \mathbf{X}_{CR} and \mathbf{X}_{FS}) or (8.3) (columns $\mathbf{X}_{CR} + \Pi_{RT_0}$ and $\mathbf{X}_{FS} + \Pi_{RT_1}$) and h is the mesh step.

We first consider Stokes Problem (3.1) in $\Omega = (0, 1)^2$ with $\mathbf{u} = 0$, $p = (x_1)^3 + (x_2)^3 - 0.5$, $\mathbf{f} = \mathbf{grad} p = 3 \left((x_1)^2, (x_2)^2 \right)^T$. We report in Table 1 $\varepsilon_0(\mathbf{u})$ for $h = 5.00 e - 2$ and for different values of ν .

ν	\mathbf{X}_{CR}	$\mathbf{X}_{CR} + \Pi_{RT_0}$	\mathbf{X}_{FS}	$\mathbf{X}_{FS} + \Pi_{RT_1}$
$1.00 e + 0$	$3.19 e - 4$	$1.34 e - 18$	$3.53 e - 7$	$9.09 e - 19$
$1.00 e - 3$	$3.19 e - 1$	$1.34 e - 15$	$3.53 e - 4$	$9.09 e - 16$
$1.00 e - 4$	$3.19 e + 0$	$1.34 e - 14$	$3.53 e - 3$	$9.09 e - 15$

TABLE 1. Values of $\varepsilon_0(\mathbf{u})$ for $h = 5.00 e - 2$

When there is no projection, the error is inversely proportional to the ν parameter, whereas using the projection, we obtain $\mathbf{u}_h = 0$ up to machine precision.

We now consider Stokes Problem (3.1) in $\Omega = (0, 1)^2$ with:

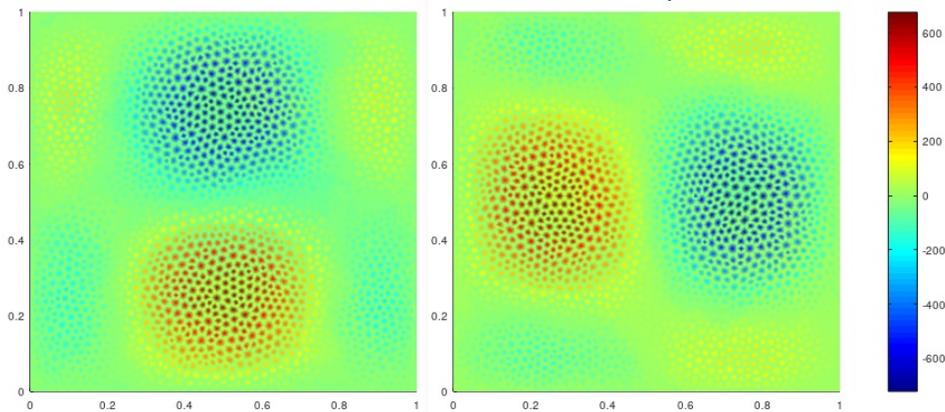
$$\mathbf{u} = \begin{pmatrix} (1 - \cos(2\pi x_1)) \sin(2\pi x_2) \\ (\cos(2\pi x_2) - 1) \sin(2\pi x_1) \end{pmatrix}, \quad \begin{cases} p = \sin(2\pi x_1) \sin(2\pi x_2), \\ \mathbf{f} = -\nu \Delta \mathbf{u} + \mathbf{grad} p. \end{cases}$$

We report in Table 2 (resp. 3) the values of $\varepsilon_0(\mathbf{u})$ in the case $\nu = 1.00 e - 3$ (resp. $\nu = 1.00 e - 4$) for different level of mesh refinement. When there is no projection, $\varepsilon_0(\mathbf{u})$ is inversely proportional to ν , whereas using the projection, $\varepsilon_0(\mathbf{u})$ is independent of ν .

h	\mathbf{X}_{CR}	$\mathbf{X}_{CR} + \Pi_{RT_0}$	\mathbf{X}_{FS}	$\mathbf{X}_{FS} + \Pi_{RT_1}$
$5.00 e - 2$	$5.66 e - 1$	$1.13 e - 2$	$2.35 e - 3$	$2.06 e - 4$
$2.50 e - 2$	$1.33 e - 1$	$2.89 e - 3$	$3.21 e - 4$	$2.59 e - 5$
$1.25 e - 2$	$3.88 e - 2$	$5.40 e - 4$	$4.20 e - 5$	$3.40 e - 6$
$6.25 e - 3$	$8.40 e - 3$	$1.79 e - 4$	$5.04 e - 6$	$4.15 e - 7$
Rate	$h^{2.05}$	$h^{2.07}$	$h^{2.96}$	$h^{2.98}$

TABLE 2. Values of $\varepsilon_0(\mathbf{u})$ for $\nu = 1.00 e - 3$

h	\mathbf{X}_{CR}	$\mathbf{X}_{CR} + \Pi_{RT_0}$	\mathbf{X}_{FS}	$\mathbf{X}_{FS} + \Pi_{RT_1}$
$5.00 e - 2$	$5.66 e - 0$	$1.13 e - 2$	$2.35 e - 2$	$2.06 e - 4$
$2.50 e - 2$	$1.33 e - 0$	$2.89 e - 3$	$3.20 e - 3$	$2.59 e - 5$
$1.25 e - 2$	$3.38 e - 1$	$5.40 e - 4$	$4.20 e - 4$	$3.40 e - 6$
$6.25 e - 3$	$8.40 e - 2$	$1.79 e - 4$	$5.04 e - 5$	$4.15 e - 7$
Rate	$h^{2.05}$	$h^{2.07}$	$h^{2.96}$	$h^{2.98}$

TABLE 3. Values of $\varepsilon_0(\mathbf{u})$ for $\nu = 1.00 e - 4$ FIGURE 2. Values of $(\mathbf{u}_{FS} - \mathbf{u}_{FS+RT_1})$. Left: x_1 -component, right: x_2 -component.

Let \mathbf{u}_{FS} (resp. \mathbf{u}_{FS+RT_1}) the solution to Problem (5.5) (resp. (8.3)) computed with Fortin-Soulie finite elements. We represent on Figure 2 the values of the Lagrange projection of $(\mathbf{u}_{FS} - \mathbf{u}_{FS+RT_1})$ in the case where $h = 2.50 e - 2$ and $\nu = 1.00 e - 4$. We observe local oscillations, of order the mesh step, which are caused by the numerical source exhibited in (8.2).

In order to enhance the numerical results, one can also use a posteriori error estimators to adapt the mesh (see [34, 35] for order 1 and [36] for order 2).

Alternatively, using the nonconforming Crouzeix-Raviart mixed finite element method, one can build a divergence-free basis, as described in [37]. Notice that using conforming finite elements, the Scott-Vogelius finite elements [38, 39] produce velocity approximations that are exactly divergence free.

The code used to get the numerical results can be downloaded on GitHub [40].

In principle, one can also obtain results with low-regularity velocity field.

9. CONCLUSION

We analysed the discretization of Stokes problem with nonconforming finite elements in light of the T-coercivity theory, we computed stability coefficients for order 1 in 2 or 3 dimension without mesh regularity assumption; and for order 1 in 2 dimension in the case of a shape-regular triangulation sequence. We then provided numerical results to illustrate the importance of using $\mathbf{H}(\text{div})$ -conforming projection. Further, we intend to extend the study to other mixed finite element methods.

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