IMPROVED STABILITY ESTIMATES FOR SOLVING STOKES PROBLEM WITH FORTIN-SOULIE FINITE ELEMENTS

ERELL JAMELOT

Université Paris-Saclay, CEA, Service de Thermo-hydraulique et de Mécanique des Fluides, 91191 Gif-sur-Yvette, France

Abstract. We propose to analyse the discretization of the Stokes problem with nonconforming finite elements in light of the T-coercivity. First we explicit the stability constants with respect to the shape regularity parameter for order 1 in 2 or 3 dimension, and order 2 in 2 dimension. In this last case, we improve the result of the original Crouzeix-Raviart paper. Second, we illustrate the importance of using a divergence-free velocity reconstruction on some numerical experiments.

Keywords. Stokes problem, T-coercitivity, Fortin-Soulie finite elements, Fortin operator

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1. Introduction

The Stokes problem describes the steady state of incompressible Newtonian flows. They are derived from the Navier–Stokes equations [1]. With regard to numerical analysis, the study of Stokes problem helps to build an appropriate approximation of the Navier–Stokes equations. We consider here a discretization with nonconforming finite elements [2, 3]. We propose to state the discrete inf-sup condition in light of the T-coercivity (cf. [4] for Helmholtz-like problems, see [5], [6] and [7] for the neutron diffusion equation), which allows to estimate the discrete error constant. In Section 2, we recall the T-coercivity theory [4], which is known to be an equivalent reformulation of the Banach–Nečas–Babuška Theorem. In Section 3 we apply it to the continuous Stokes Problem. We give details on the triangulation in Section 4, and we apply the T-coercivity to the discretization of Stokes problem with nonconforming mixed finite elements in Section 5. For the Stokes problem, in the discrete case, this amounts to finding a Fortin operator. In Section 6 (resp. 7), we precise the proof of the well-posedness in the case of order 1 (resp. order 2) nonconforming mixed finite elements. In Section 8, we illustrate the importance of using a divergence-free velocity on some numerical experiments.

E-mail address: `erell.jamelot@cea.fr`.

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2. T-coercivity

We recall here the T-coercivity theory as written in [4]. Consider first the variational problem, where V and W are two Hilbert spaces and $f \in V'$:

$$\text{Find } u \in V \text{ such that } \forall v \in W,\ a(u,v) = \langle f, v\rangle_V. \tag{2.1}$$

Classically, we know that Problem (2.1) is well-posed if $a(\cdot,\cdot)$ satisfies the stability and the solvability conditions of the so-called Banach–Nečas–Babuška (BNB) Theorem (see a.e. [8, Thm. 25.9]). For some models, one can also prove the well-posedness using the T-coercivity theory (cf. [4] for Helmholtz-like problems, see [5], [6] and [7] for the neutron diffusion equation).

Definition 1. *Let V and W be two Hilbert spaces and $a(\cdot,\cdot)$ be a continuous and bilinear form over $V \times W$. It is T-coercive if*

$$\exists T \in \mathcal{L}(V,W),\ \textit{bijective},\ \exists \alpha > 0,\ \forall v \in V,\ |a(v,Tv)| \geq \alpha \|v\|_V^2. \tag{2.2}$$

If in addition $a(\cdot,\cdot)$ is symmetric, it is T-coercive if

$$\exists T \in \mathcal{L}(V,V),\ \exists \alpha > 0,\ \forall v \in V,\ |a(v,Tv)| \geq \alpha \|v\|_V^2. \tag{2.3}$$

When the bilinear form $a(\cdot,\cdot)$ is symmetric, the requirement that the operator T is bijective can be dropped. It is proved in [4] that the T-coercivity condition is equivalent to the stability and solvability conditions of the BNB Theorem. Whereas the BNB theorem relies on an abstract inf–sup condition, T-coercivity uses explicit inf–sup operators, both at the continuous and discrete levels.

Theorem 1. *(well-posedness) Let $a(\cdot,\cdot)$ be a continuous and bilinear form. Suppose that the form $a(\cdot,\cdot)$ is T-coercive. Then Problem* (2.1) *is well-posed.*

3. Stokes problem

Let Ω be a connected bounded domain of $\mathbb{R}^d$, $d = 2, 3$, with a polygonal ($d = 2$) or Lipschitz polyhedral ($d = 3$) boundary $\partial\Omega$. We consider Stokes problem:

$$\text{Find } (\mathbf{u}, p) \text{ such that } \left\{ \begin{array}{rcl} -\nu\, \boldsymbol{\Delta}\, \mathbf{u} + \mathbf{grad}\, p &=& \mathbf{f}, \\ \operatorname{div} \mathbf{u} &=& 0. \end{array} \right. \tag{3.1}$$

with Dirichlet boundary conditions for the velocity $\mathbf{u}$ and a normalization condition for the pressure p:

$$\mathbf{u} = 0 \text{ on } \partial\Omega, \qquad \int_\Omega p = 0.$$

The vector field $\mathbf{u}$ represents the velocity of the fluid and the scalar field p represents its pressure divided by the fluid density which is supposed to be constant. Thus, the SI unit of the components of $\mathbf{u}$ is $m \cdot s^{-1}$ and the SI unit of p is $m^2 \cdot s^{-2}$). The first equation of (3.1) corresponds to the momentum balance equation and the second one corresponds to the conservation of the mass. The constant parameter $\nu > 0$ is the kinematic viscosity of the fluid, its SI unit is $m^2 \cdot s^{-1}$. The vector field $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ represents a body forces divided by the fluid density, its SI unit is $m \cdot s^{-2}$.

Before stating the variational formulation of Problem (3.1), we provide some definition and reminders. Let us set $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^d$, $\mathbf{H}^{-1}(\Omega) = (H^{-1}(\Omega))^d$ its dual space and $L^2_{zmv}(\Omega) = \{q \in L^2(\Omega) \,|\, \int_\Omega q = 0\}$.

We recall that $\mathbf{H}(\mathrm{div};\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,|\, \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}$. Let us first recall Poincaré-Steklov inequality:

$$\exists C_{PS} > 0 \,|\, \forall v \in H_0^1(\Omega), \quad \|v\|_{L^2(\Omega)} \le C_{PS} \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}. \tag{3.2}$$

The SI unit of C_{PS} is m.
Thanks to this result, in $H_0^1(\Omega)$, the semi-norm is equivalent to the natural norm, so that the scalar product reads $(v,w)_{H_0^1(\Omega)} = (\mathbf{grad}\, v, \mathbf{grad}\, w)_{\mathbf{L}^2(\Omega)}$ and the norm is $\|v\|_{H_0^1(\Omega)} = \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}$. Let $\mathbf{v}$, $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$. We denote by $(v_i)_{i=1}^d$ (resp. $(w_i)_{i=1}^d$) the components of $\mathbf{v}$ (resp. $\mathbf{w}$), and we set $\mathbf{Grad}\,\mathbf{v} = (\partial_j v_i)_{i,j=1}^d \in \mathbb{L}^2(\Omega)$, where $\mathbb{L}^2(\Omega) = [L^2(\Omega)]^{d\times d}$. We have:

$$(\mathbf{Grad}\,\mathbf{v}, \mathbf{Grad}\,\mathbf{w})_{\mathbb{L}^2(\Omega)} = (\mathbf{v},\mathbf{w})_{\mathbf{H}_0^1(\Omega)} = \sum_{i=1}^d (v_i, w_i)_{H_0^1(\Omega)}$$

and:

$$\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} = \left(\sum_{j=1}^d \|v_j\|^2_{H_0^1(\Omega)} \right)^{1/2} = \|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}.$$

Let us set $\mathbf{V} = \big\{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \mid \mathrm{div}\,\mathbf{v} = 0\big\}$. The space $\mathbf{V}$ is a closed subset of $\mathbf{H}_0^1(\Omega)$. We denote by $\mathbf{V}^\perp$ the orthogonal of $\mathbf{V}$ in $\mathbf{H}_0^1(\Omega)$. Let $\nu_p > 0$ be a kinematic viscosity. We recall that [1, cor. I.2.4]:

Proposition 1. *The operator* $\mathrm{div} : \mathbf{H}_0^1(\Omega) \to L^2(\Omega)$ *is an isomorphism of* $\mathbf{V}^\perp$ *onto* $L^2_{zmv}(\Omega)$*. We call* C_{div} *the constant such that:*

$$\forall p \in L^2_{zmv}(\Omega),\ \exists!\mathbf{v} \in \mathbf{V}^\perp \mid \mathrm{div}\,\mathbf{v} = p \text{ and } \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} \le C_{\mathrm{div}} \|p\|_{L^2(\Omega)}. \tag{3.3}$$

The constant C_{div} depends only on the domain Ω. Notice that we have: $C_{\mathrm{div}} = 1/\beta(\Omega)$ where $\beta(\Omega)$ is the inf-sup condition (or Ladyzhenskaya–Babuška–Brezzi condition):

$$\beta(\Omega) = \inf_{q \in L^2_{zmv}(\Omega)\setminus\{0\}} \ \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)\setminus\{0\}} \frac{(q, \mathrm{div}\,\mathbf{v})_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)} \, \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)}}. \tag{3.4}$$

Generally, the value of $\beta(\Omega)$ is not known explicitly. In [9], Bernardi et al established results on the discrete approximation of $\beta(\Omega)$ using conforming finite elements. Recently, Gallistl proposed in [10] a numerical scheme with adaptive meshes for computing approximations to $\beta(\Omega)$. In the case of $d = 2$, Costabel and Dauge [11] established the following bound:

Theorem 2. *Let* $\Omega \subset \mathbb{R}^2$ *be a domain contained in a ball of radius* R*, star-shaped with respect to a concentric ball of radius* ρ*. Then*

$$\beta(\Omega) \ge \frac{\rho}{\sqrt{2}\,R} \left(1 + \sqrt{1 - \frac{\rho^2}{R^2}} \right)^{-1/2} \ge \frac{\rho}{2\,R}. \tag{3.5}$$

Let us detail the bound for some remarkable domains. If Ω is a ball, $\beta(\Omega) \ge \frac{1}{2}$ and if Ω is a square, $\beta(\Omega) \ge \frac{1}{2\sqrt{2}}$. Suppose now that Ω is stretched in some direction by a factor k, then $\beta(\Omega) \ge \frac{1}{2\,k}$. Finally, if Ω is L-shaped (resp. cross-shaped) such that $L = k\,l$, where L is the largest length and l is the smallest length of an edge, then $\beta(\Omega) \ge \frac{1}{2\sqrt{2}\,k}$ (resp. $\beta(\Omega) \ge \frac{1}{4\,k}$).

The variational formulation of Problem (3.1) reads:
Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$ such that

$$\left\{ \begin{array}{rcll} \nu(\mathbf{u},\mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} & = & \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \ ; \\ (q, \operatorname{div} \mathbf{u})_{L^2(\Omega)} & = & 0 & \forall q \in L^2_{zmv}(\Omega). \end{array} \right. \tag{3.6}$$

Classically, one proves that Problem (3.6) is well-posed using Poincaré-Steklov inequality (3.2) and Prop. 1. Check for instance the proof of [1, Thm. I.5.1].

Let us set $\mathcal{X} = \mathbf{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$ which is a Hilbert space which we endow with the following norm:

$$\|(\mathbf{v}, q)\|_{\mathcal{X}} = \left(\|\mathbf{v}\|^2_{\mathbf{H}_0^1(\Omega)} + \nu^{-2} \, \|q\|^2_{L^2(\Omega)} \right)^{1/2}. \tag{3.7}$$

We consider now the following bilinear symmetric and continuous form:

$$\left\{ \begin{array}{rcl} a_S : \mathcal{X} \times \mathcal{X} & \to & \mathbb{R} \\ (\mathbf{u}', p') \times (\mathbf{v}, q) & \mapsto & \nu(\mathbf{u}', \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p', \operatorname{div} \mathbf{v})_{L^2(\Omega)} - (q, \operatorname{div} \mathbf{u}')_{L^2(\Omega)} \end{array} \right. . \tag{3.8}$$

We can write Problem (3.1) in an equivalent way as follows:

$$\text{Find } (\mathbf{u}, p) \in \mathcal{X} \text{ such that } \quad a_S\left((\mathbf{u}, p), (\mathbf{v}, q) \right) = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} \quad \forall (\mathbf{v}, q) \in \mathcal{X}. \tag{3.9}$$

Let us prove that Problem (3.9) is well-posed using the T-coercivity theory.

Theorem 3. *Problem* (3.9) *is well-posed. It admits one and only one solution such that:*

$$\forall \mathbf{f} \in \mathbf{H}^{-1}(\Omega), \quad \left\{ \begin{array}{l} \|\mathbf{u}\|_{\mathbf{H}_0^1(\Omega)} \le \nu^{-1} \, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}, \\ \|p\|_{L^2(\Omega)} \le C_{\mathrm{div}} \, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}. \end{array} \right. \tag{3.10}$$

Proof. We follow here the proof given in [12, 13]. Let us consider $(\mathbf{u}', p') \in \mathcal{X}$ and let us build $(\mathbf{v}^\star, q^\star) = T(\mathbf{u}', p') \in \mathcal{X}$ satisfying (2.3) (with $V = \mathcal{X}$). We need three main steps.

1. According to Prop. 1, there exists $\tilde{\mathbf{v}}_{p'} \in \mathbf{H}_0^1(\Omega)$ such that: $\operatorname{div} \tilde{\mathbf{v}}_{p'} = p'$ in Ω and $\|\tilde{\mathbf{v}}_{p'}\|_{\mathbf{H}_0^1(\Omega)} \le C_{\mathrm{div}} \, \|p'\|_{L^2(\Omega)}$. Let us set $\mathbf{v}_{p'} = \nu^{-1} \tilde{\mathbf{v}}_{p'}$ so that $\operatorname{div} \mathbf{v}_{p'} = \nu^{-1} \, p'$ and

$$\|\mathbf{v}_{p'}\|_{\mathbf{H}_0^1(\Omega)} \le \nu^{-1} \, C_{\mathrm{div}} \, \|p'\|_{L^2(\Omega)}. \tag{3.11}$$

Let us set $(\mathbf{v}^\star, q^\star) := (\gamma \, \mathbf{u}' - \mathbf{v}_{p'}, -\gamma \, p')$, with $\gamma > 0$. We obtain:

$$a_S \left(\, (\mathbf{u}', p'), \, (\mathbf{v}^\star, q^\star) \, \right) = \nu \, \gamma \, \|\mathbf{u}'\|^2_{\mathbf{H}_0^1(\Omega)} + \nu^{-1} \, \|p'\|^2_{L^2(\Omega)} - \nu \, (\mathbf{u}', \mathbf{v}_{p'})_{\mathbf{H}_0^1(\Omega)}. \tag{3.12}$$

2. In order to bound the last term of (3.12), we use Young inequality and then inequality (3.11) so that for all $\eta > 0$:

$$(\mathbf{u}', \mathbf{v}_{p'})_{\mathbf{H}_0^1(\Omega)} \le \frac{\eta}{2} \|\mathbf{u}'\|^2_{\mathbf{H}_0^1(\Omega)} + \frac{\eta^{-1}}{2} \left(\frac{C_{\mathrm{div}}}{\nu} \right)^2 \|p'\|^2_{L^2(\Omega)}. \tag{3.13}$$

3. Using the bound (3.13) in (3.12) and choosing $\eta = \gamma$, we get:

$$a_S \left(\, (\mathbf{u}', p'), \, (\mathbf{v}^\star, q^\star) \, \right) \ge \nu \left(\frac{\gamma}{2} \, \|\mathbf{u}'\|^2_{\mathbf{H}_0^1(\Omega)} + \nu^{-2} \left(1 + \frac{\gamma^{-1}}{2} (C_{\mathrm{div}})^2 \right) \|p'\|^2_{L^2(\Omega)} \right).$$

Consider now $\gamma = (C_{\mathrm{div}})^2$. We obtain:

$$a_S \left(\, (\mathbf{u}', p'), \, (\mathbf{v}^\star, q^\star) \, \right) \ge \nu \, C_{\min} \, \|(\mathbf{u}', p')\|^2_{\mathcal{X}} \text{ where } C_{\min} = \frac{1}{2} \min(\, (C_{\mathrm{div}})^2, 1 \,).$$

The operator T such that $T(\mathbf{u}', p') = (\mathbf{v}^\star, q^\star)$ is linear and continuous:

$$\begin{aligned}\|T(\mathbf{u}', p')\|_{\mathcal{X}}^2 &:= \|\mathbf{v}^\star\|^2_{\mathbf{H}^1_0(\Omega)} + \nu^{-2}\,\|q^\star\|^2_{L^2(\Omega)}\\ &\le \gamma^2\,\|\mathbf{u}'\|^2_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{v}_{p'}\|^2_{\mathbf{H}^1_0(\Omega)} + \gamma^2\,\nu^{-2}\,\|p'\|^2_{L^2(\Omega)},\\ &\le \gamma^2\,\|\mathbf{u}'\|^2_{\mathbf{H}^1_0(\Omega)} + (\,(C_{\mathrm{div}})^2 + \gamma)\,\nu^{-2}\,\|p'\|^2_{L^2(\Omega)},\\ &\le (C_{\max})^2\,\|(\mathbf{u}', p')\|^2_{\mathcal{X}},\end{aligned}$$

where $C_{\max} = C_{\mathrm{div}}\,(1 + (C_{\mathrm{div}})^2)^{1/2}$.

[1] The symmetric and continuous bilinear form $a(\cdot,\cdot)$ is then T-coercive and according to Theorem 1, Problem (3.9) is well-posed. Let us prove (3.10). Consider $(\mathbf{u}, p)$ the unique solution of Problem (3.9). Choosing $\mathbf{v} = 0$, we obtain that $\forall q \in L^2_{zmv}(\Omega)$, $(q, \operatorname{div}\mathbf{u})_{L^2(\Omega)} = 0$, so that $\mathbf{u} \in \mathbf{V}$. Now, choosing $\mathbf{v} = \mathbf{u}$ and using Cauchy-Schwarz inequality, we have: $\nu\,\|\mathbf{u}\|^2_{\mathbf{H}^1_0(\Omega} = \langle \mathbf{f}, \mathbf{u}\rangle_{\mathbf{H}^1_0(\Omega)} \le \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega)}$, so that: $\|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega} \le \nu^{-1}\,\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$. Next, we choose in (3.9) $\mathbf{v} = \mathbf{v}_p \in \mathbf{V}^\perp$, where $\operatorname{div}\mathbf{v}_p = -\nu^{-1}\,p$ (see Prop. 1). Noticing that $\mathbf{u} \in \mathbf{V}$ and $\mathbf{v}_p \in \mathbf{V}^\perp$, it holds[2]: $(\mathbf{u}, \mathbf{v}_p)_{\mathbf{H}^1_0(\Omega)} = 0$. This gives:

$$\begin{aligned}-(p, \operatorname{div}\mathbf{v}_p)_{L^2(\Omega)} &= \nu^{-1}\,\|p\|^2_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{v}_p\rangle_{\mathbf{H}^1_0(\Omega)},\\ &\le \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|\mathbf{v}_p\|_{\mathbf{H}^1_0(\Omega)} \le C_{\mathrm{div}}\,\nu^{-1}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|p\|_{L^2(\Omega)},\end{aligned}$$

so that: $\|p\|_{L^2(\Omega)} \le C_{\mathrm{div}}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$. □

Remark 1. *We recover the first Banach–Nečas–Babuška condition* [8, Thm. 25.9, (BNB1)]*:*

$$a_S\,(\,(\mathbf{u}', p'), (\mathbf{v}^\star, q^\star)\,) \ge \nu\,C_{\min}\,(C_{\max})^{-1}\,\|(\mathbf{u}', p')\|_{\mathcal{X}}\,\|(\mathbf{v}^\star, q^\star)\|_{\mathcal{X}}.$$

Let us call $C_{\mathrm{stab}} = \nu\,C_{\min}\,(C_{\max})^{-1}$ the stability constant. With the choice of our parameters, C_{stab} is such that:

$$C_{\mathrm{stab}} = \begin{cases} \dfrac{\nu}{2}\,\dfrac{C_{\mathrm{div}}}{(1 + (C_{\mathrm{div}})^2))^{1/2}} & \text{if } 0 < C_{\mathrm{div}} \le 1,\\[2ex] \dfrac{\nu}{2}\,\dfrac{(C_{\mathrm{div}})^{-1}}{(1 + (C_{\mathrm{div}})^2))^{1/2}} & \text{if } 1 \le C_{\mathrm{div}}.\end{cases}$$

Thus, the T-coercivity approach allows to give an estimate of the stability constant. In our computations, it depends on the choice of the parameters η and γ, so that it could be optimized.

If we were using a conforming discretization to solve Problem (3.9) (a.e. Taylor-Hood finite elements [14]), we would use the bilinear form $a_S(\cdot,\cdot)$ to state the discrete variational formulation. Let us call the discrete spaces $\mathbf{X}_{c,h} \subset \mathbf{H}^1_0(\Omega)$ and $Q_{c,h} \subset L^2_{zmv}(\Omega)$. Then to prove the discrete T-coercivity, we would need to state

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[1] Remark that $(\mathbf{v}^\star, q^\star) = (\mathbf{0}, 0) \Leftrightarrow (\mathbf{u}', p') = (\mathbf{0}, 0)$: the operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ is bijective.

[2] According to [1, Cor. I.2.3], since $\mathbf{v}_p \in \mathbf{V}^\perp$, $\exists p \in L^2_{zmv}(\Omega)\,|\,\Delta\mathbf{v}_p = \mathbf{grad}\,q$ in $\mathbf{H}^{-1}(\Omega)$. Integrating by parts twice, we have: $(\mathbf{u}, \mathbf{v}_p)_{\mathbf{H}^1_0(\Omega)} = -\langle \mathbf{grad}\,q, \mathbf{u}\rangle_{\mathbf{H}^1_0(\Omega)} = (q, \operatorname{div}\mathbf{u})_{L^2(\Omega)} = 0$.

the discrete counterpart to Proposition 1. To do so, we can build a linear operator $\Pi_c : \mathbf{X} \to \mathbf{X}_h$, known as Fortin operator, such that (see a.e. [15, §8.4.1]):

$$\exists C_c \,|\, \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \qquad \| \,\mathbf{Grad}\, \Pi_c \mathbf{v}\|_{\mathbb{L}^2(\Omega)} \le C_c \| \,\mathbf{Grad}\, \mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{3.14}$$

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\operatorname{div} \Pi_c \mathbf{v}, q_h)_{L^2(\Omega)} = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q_h \in Q_{c,h}. \tag{3.15}$$

Using a nonconforming discretization, we will not use the bilinear form $a_S(\cdot,\cdot)$ to exhibit the discrete variational formulation, but we will need a similar operator to (3.14)-(3.15) to prove the discrete T-coercivity, which is stated in Theorem 4.

4. Discretization

We call $(O, (x_{d'})_{d'=1}^d)$ the Cartesian coordinates system, of orthonormal basis $(\mathbf{e}_{d'})_{d'=1}^d$. Consider $(\mathcal{T}_h)_h$ a simplicial triangulation sequence of Ω. For a triangulation $\mathcal{T}_h$, we use the following index sets:

- $\mathcal{I}_K$ denotes the index set of the elements, such that $\mathcal{T}_h := \bigcup_{\ell \in \mathcal{I}_K} K_\ell$ is the set of elements.
- $\mathcal{I}_F$ denotes the index set of the facets[3], such that $\mathcal{F}_h := \bigcup_{f \in \mathcal{I}_F} F_f$ is the set of facets.
  Let $\mathcal{I}_F = \mathcal{I}_F^i \cup \mathcal{I}_F^b$, where $\forall f \in \mathcal{I}_F^i$, $F_f \in \Omega$ and $\forall f \in \mathcal{I}_F^b$, $F_f \in \partial\Omega$.
- $\mathcal{I}_S$ denotes the index set of the vertices, such that $(S_j)_{j \in \mathcal{I}_S}$ is the set of vertices.
  Let $\mathcal{I}_S = \mathcal{I}_S^i \cup \mathcal{I}_S^b$, where $\forall j \in \mathcal{I}_S^i$, $S_j \in \Omega$ and $\forall j \in \mathcal{I}_S^b$, $S_j \in \partial\Omega$.

We also define the following index subsets:

- $\forall \ell \in \mathcal{I}_K$, $\mathcal{I}_{F,\ell} = \{f \in \mathcal{I}_F \,|\, F_f \in K_\ell\}$, $\quad \mathcal{I}_{S,\ell} = \{j \in \mathcal{I}_S \,|\, S_j \in K_\ell\}$.
- $\forall j \in \mathcal{I}_S$, $\mathcal{I}_{K,j} = \{\ell \in \mathcal{I}_K \,|\, S_j \in K_\ell\}$, $\quad N_j := \operatorname{card}(\mathcal{I}_{K,j})$.

For all $\ell \in \mathcal{I}_K$, we call h_ℓ and ρ_ℓ the diameters of K_ℓ and its inscribed sphere respectively, and we let: $\sigma_\ell = \frac{h_\ell}{\rho_\ell}$. When the $(\mathcal{T}_h)_h$ is a shape-regular triangulation sequence (see a.e. [16, def. 11.2]), there exists a constant $\sigma > 1$, called the shape regularity parameter, such that for all h, for all $\ell \in \mathcal{I}_K$, $\sigma_\ell \le \sigma$. For all $f \in \mathcal{I}_F$, M_f denotes the barycentre of F_f, and by $\mathbf{n}_f$ its unit normal (outward oriented if $F_f \in \partial\Omega$). For all $j \in \mathcal{I}_S$, for all $\ell \in \mathcal{I}_{K,j}$, $\lambda_{j,\ell}$ denotes the barycentric coordinate of S_j in K_ℓ; $F_{j,\ell}$ denotes the face opposite to vertex S_j in element K_ℓ, and $\mathbf{x}_{j,\ell}$ denotes its barycentre. We call $\mathcal{S}_{j,\ell}$ the outward normal vector of $F_{j,\ell}$ and of norm $|\mathcal{S}_{j,\ell}| = |F_{j,\ell}|$.

Let introduce spaces of piecewise regular elements:
We set $\mathcal{P}_h H^1 = \left\{ v \in L^2(\Omega)\,; \quad \forall \ell \in \mathcal{I}_K,\, v_{|K_\ell} \in H^1(K_\ell) \right\}$, endowed with the scalar product :

$$(v, w)_h := \sum_{\ell \in \mathcal{I}_K} (\mathbf{grad}\, v, \mathbf{grad}\, w)_{\mathbf{L}^2(K_\ell)} \quad \|v\|_h^2 = \sum_{\ell \in \mathcal{I}_K} \| \,\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)}^2.$$

We set $\mathcal{P}_h \mathbf{H}^1 = [\mathcal{P}_h H^1]^d$, endowed with the scalar product :

$$(\mathbf{v}, \mathbf{w})_h := \sum_{\ell \in \mathcal{I}_K} (\mathbf{Grad}\, \mathbf{v}, \mathbf{Grad}\, \mathbf{w})_{\mathbb{L}^2(K_\ell)} \quad \|\mathbf{v}\|_h^2 = \sum_{\ell \in \mathcal{I}_K} \| \,\mathbf{Grad}\, \mathbf{v}\|_{\mathbb{L}^2(K_\ell)}^2.$$

[3] The term facet stands for face (resp. edge) when $d = 3$ (resp. $d = 2$).

Let $f \in \mathcal{I}_F^i$ such that $F_f = \partial K_L \cap \partial K_R$ and $\mathbf{n}_f$ is outward K_L oriented.
The jump (resp. average) of a function $v \in \mathcal{P}_h H^1$ across the facet F_f is defined as follows: $[v]_{F_f} := v_{|K_L} - v_{|K_R}$ (resp. $\{v\}_{F_f} := \frac{1}{2}(v_{|K_L} + v_{|K_R})$). For $f \in \mathcal{I}_F^b$, we set: $[v]_{F_f} := v_{|F_f}$ and $\{v\}_{F_f} := v_{|F_f}$.
We set $\mathcal{P}_h\mathbf{H}(\mathrm{div}) = \left\{\mathbf{v} \in \mathbf{L}^2(\Omega)\,; \quad \forall \ell \in \mathcal{I}_K,\, \mathbf{v}_{|K_\ell} \in \mathbf{H}(\mathrm{div};\, K_\ell)\right\}$, and we define the operator div_h such that:

$$\forall \mathbf{v} \in \mathcal{P}_h\mathbf{H}(\mathrm{div}), \forall q \in L^2(\Omega), \quad (\mathrm{div}_h\, \mathbf{v}, q) = \sum_{\ell \in \mathcal{I}_K} (\mathrm{div}\, \mathbf{v}, q)_{L^2(K_\ell)}.$$

We recall classical finite elements estimates [16]. Let $\hat{K}$ be the reference simplex and $\hat{F}$ be the reference facet. For $\ell \in \mathcal{I}_K$ (resp. $f \in \mathcal{I}_F$), we denote by $T_\ell : \hat{K} \to K_\ell$ (resp. $T_f : \hat{F} \to F_f$) the geometric mapping such that $\forall \hat{\mathbf{x}} \in \hat{K}$, $\mathbf{x}_{|K_\ell} = T_\ell(\hat{\mathbf{x}}) = \mathbb{B}_\ell \hat{\mathbf{x}} + \mathbf{b}_\ell$ (resp. $\mathbf{x}_{|F_f} = T_f(\hat{\mathbf{x}}) = \mathbb{B}_f \hat{\mathbf{x}} + \mathbf{b}_f$), and we set $J_\ell = \det(\mathbb{B}_\ell)$ (resp. $J_f = \det(\mathbb{B}_f)$). There holds:

$$|J_\ell| = d!\,|K_\ell|, \quad \|\mathbb{B}_\ell\| = \frac{h_\ell}{\rho_{\hat{K}}}, \quad \|\mathbb{B}_\ell{}^{-1}\| = \frac{h_{\hat{K}}}{\rho_\ell}, \quad |J_f| = (d-1)!\,|F_f|. \tag{4.1}$$

For $v \in L^2(K_\ell)$, we set $\hat{v}_\ell = v \circ T_\ell$. For $v \in v^2(F_f)$, we set: $\hat{v}_f = v \circ T_f$. Changing the variable, we get:

$$\|v\|^2_{L^2(K_\ell)} = |J_\ell|\,\|\hat{v}_\ell\|^2_{L^2(\hat{K})}, \quad \text{and} \quad \|v\|^2_{L^2(F_f)} = |J_f|\,\|\hat{v}_f\|^2_{L^2(\hat{F})}. \tag{4.2}$$

Let $v \in \mathcal{P}_h H^1$. By changing the variable, $\mathbf{grad}\, v_{|K_\ell} = (\mathbb{B}_\ell{}^{-1})^T\, \mathbf{grad}_{\hat{\mathbf{x}}}\, \hat{v}_\ell$, and it holds:

$$\begin{array}{llll} (i) & \|\,\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)} & \leq & \|\mathbb{B}_\ell{}^{-1}\|^2\,|K_\ell|\,\|\,\mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{v}_\ell\|^2_{\mathbf{L}^2(\hat{K})}, \\ (ii) & \|\,\mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{v}_\ell\|^2_{\mathbf{L}^2(\hat{K})} & \leq & \|\mathbb{B}_\ell\|^2\,|K_\ell|^{-1}\,\|\,\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)}. \end{array} \tag{4.3}$$

We recall the Poincaré-Steklov inequality in cells [16, Lemma 12.11]:
for all $\ell \in \mathcal{I}_K$ (K_ℓ is a convex set), $\forall v \in H^1(K_\ell)$:

$$\|\underline{v}_\ell\|_{L^2(K_\ell)} \leq \pi^{-1} h_\ell \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)}, \quad \text{where } \underline{v}_\ell = v_{|K_\ell} - \frac{\int_{K_\ell} v}{|K_\ell|}. \tag{4.4}$$

For all $D \subset \mathbb{R}^d$, and $k \in \mathbb{N}^*$, we call $P^k(D)$ the set of order k polynomials on D, $\mathbf{P}^k(D) = (P^k(D))^d$, and we consider the broken polynomial space:

$$P^k_{disc}(\mathcal{T}_h) = \left\{q \in L^2(\Omega); \quad \forall \ell \in \mathcal{I}_K,\, q_{|K_\ell} \in P^k(K_\ell)\right\}, \quad \mathbf{P}^k_{disc}(\mathcal{T}_h) := (P^k_{disc}(\mathcal{T}_h))^d.$$

We let $P^0(\mathcal{T}_h)$ be the space of piecewise constant functions on $\mathcal{T}_h$.

5. The nonconforming mixed finite element method for Stokes

The nonconforming finite element method was introduced by Crouzeix and Raviart in [2] to solve Stokes Problem (3.1). We approximate the vector space $\mathbf{H}^1(\Omega)$ component by component by piecewise polynomials of order $k \in \mathbb{N}^\star$. Let us consider X_h (resp. $X_{0,h}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$)

of order k:

$$X_h = \left\{ v_h \in P^k_{disc}(\mathcal{T}_h)\,; \quad \forall f \in \mathcal{I}^i_F,\, \forall q_h \in P^{k-1}(F_f),\, \int_{F_f} [v_h]\, q_h = 0 \right\}\,;$$

$$X_{0,h} = \left\{ v_h \in X_h\,; \quad \forall f \in \mathcal{I}^b_F,\, \forall q_h \in P^{k-1}(F_f),\, \int_{F_f} v_h\, q_h = 0 \right\}. \tag{5.1}$$

The condition on the jumps of v_h on the inner facets is often called the patch-test condition.

Proposition 2. *The broken norm $v_h \to \|v_h\|_h$ is a norm over $X_{0,h}$.*

Proof. Let $v_h \in X_{0,h}$ such that $\|v_h\|_h = 0$. Then for all $\ell \in \mathcal{I}_K$, $v_{h|K_\ell}$ is a constant. For all $f \in \mathcal{I}^i_F$ the jump $[v_h]_{F_f}$ vanishes, so that v_h is a constant over Ω. We deduce from the discrete boundary condition that $v_h = 0$. □

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order k is $\mathbf{X}_h = (X_h)^d$ (resp. $\mathbf{X}_{0,h} = (X_{0,h})^d$). We set $\mathcal{X}_h := \mathbf{X}_{0,h} \times Q_h$ where $Q_h = P^{k-1}_{disc}(\mathcal{T}_h) \cap L^2_{zmv}(\Omega)$. We deduce from Proposition 2 the

Proposition 3. *The broken norm defined below is a norm on $\mathcal{X}_h$:*

$$\|(\cdot,\cdot)\|_{\mathcal{X}_h} : \left\{ \begin{array}{ccc} \mathcal{X}_h & \mapsto & \mathbb{R} \\ (\mathbf{v}_h, q_h) & \to & \left(\|\mathbf{v}_h\|^2_h + \nu^{-2}\, \|q_h\|^2_{L^2(\Omega)} \right)^{1/2} \end{array} \right. . \tag{5.2}$$

Thus, the product space $\mathcal{X}_h$ endowed with the broken norm $\|\cdot\|_{\mathcal{X}_h}$ is a Hilbert space.

Proposition 4. *The following discrete Poincaré–Steklov inequality holds: there exists a constant C^{nc}_{PS} independent of $\mathcal{T}_h$ such that*

$$\forall \mathbf{v}_h \in \mathbf{X}_{0,h}, \quad \|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \le C^{nc}_{PS}\, \|\mathbf{v}_h\|_h, \tag{5.3}$$

where C^{nc}_{PS} is independent of $\mathcal{T}_h$ and is proportional to the diameter of Ω.

Proof. Inequality (5.3) is stated in [8, Lemma 36.6] for $k = 1$, but one can check that the proof holds true for higher-order, thanks to the patch-test condition. An alternative proof is given in [17, Theorem C.1]. □

We consider the discrete continuous bilinear form $a_{S,h}(\cdot,\cdot)$ such that :

$$\left\{ \begin{array}{ccc} a_{S,h} : \mathcal{X}_h \times \mathcal{X}_h & \to & \mathbb{R} \\ (\mathbf{u}'_h, p'_h) \times (\mathbf{v}_h, q_h) & \mapsto & \nu(\mathbf{u}'_h, \mathbf{v}_h)_h - (\mathrm{div}_h\, \mathbf{v}_h, p'_h) - (\mathrm{div}_h\, \mathbf{u}'_h, q_h) \end{array} \right. .$$

Let $\ell_{\mathbf{f}} \in \mathcal{L}(\mathcal{X}_h, \mathbb{R})$ be such that :

$$\forall (\mathbf{v}_h, q_h) \in \mathcal{X}_h, \quad \ell_{\mathbf{f}}\left((\mathbf{v}_h, q_h) \right) = \left\{ \begin{array}{ll} (\mathbf{f}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega) \\ \langle \mathbf{f}, \mathcal{I}_h(\mathbf{v}_h) \rangle_{\mathbf{H}^1_0(\Omega)} & \text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega) \end{array} \right. ,$$

where $\mathcal{I}_h : \mathbf{X}_{0,h} \to \mathbf{Y}_{0,h}$, with $\mathbf{Y}_{0,h} = \{\mathbf{v}_h \in \mathbf{H}^1_0(\Omega)\,; \quad \forall \ell \in \mathcal{I}_K,\, \mathbf{v}_{h|K_\ell} \in \mathbf{P}^k(K_\ell)\}$, is the averaging operator described in [16, §22.4.1]. There exists a constant $C^{nc}_{\mathcal{I}_h} > 0$ independent of $\mathcal{T}_h$ such that :

$$\|\mathcal{I}_h \mathbf{v}_h\|_{\mathbf{H}^1_0(\Omega)} \le C^{nc}_{\mathcal{I}_h}\, \|\mathbf{v}_h\|_h, \quad \forall \mathbf{v}_h \in \mathbf{X}_{0,h}. \tag{5.4}$$

The nonconforming discretization of Problem (3.9) reads:
Find $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$ such that

$$a_{S,h}\ ((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \ell_{\mathbf{f}}\ (\ (\mathbf{v}_h, q_h)\ ) \quad \forall (\mathbf{v}_h, q_h) \in \mathcal{X}_h. \tag{5.5}$$

Let us prove that Problem (5.5) is well-posed using the T-coercivity theory.

Theorem 4. *Suppose that there exists a Fortin operator $\Pi_{nc} : \mathbf{H}^1(\Omega) \to \mathbf{X}_h$ such that*

$$\exists C_{nc} \,|\, \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\Pi_{nc}\mathbf{v}\|_h \le C_{nc} \|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{5.6}$$

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\mathrm{div}_h\, \Pi_{nc}\mathbf{v}, q_h) = (\mathrm{div}\,\mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q \in Q_h, \tag{5.7}$$

where the constant C_{nc} does not depend on h. Then Problem (5.5) *is well-posed. Moreover, it admits one and only one solution $(\mathbf{u}_h, p_h)$ such that:*

$$\begin{aligned} &\text{if } \mathbf{f} \in \mathbf{L}^2(\Omega) : \left\{ \begin{array}{rcl} \|\mathbf{u}_h\|_h & \le & C_{PS}^{nc}\, \nu^{-1}\, \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \\ \|p_h\|_{L^2(\Omega)} & \le & 2\, C_{PS}^{nc}\, C_{\mathrm{div}}^{nc}\, \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \end{array} \right. , \\ &\text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega) : \left\{ \begin{array}{rcl} \|\mathbf{u}_h\|_h & \le & C_{\mathcal{I}_h}^{nc}\, \nu^{-1}\, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \\ \|p_h\|_{L^2(\Omega)} & \le & 2\, C_{\mathcal{I}_h}^{nc}\, C_{\mathrm{div}}^{nc}\, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \end{array} \right. , \end{aligned} \tag{5.8}$$

where $C_{\mathrm{div}}^{nc} = C_{\mathrm{div}}\, C_{nc}$.

Proof. Let us consider $(\mathbf{u}'_h, p'_h) \in \mathcal{X}_h$ and let us build $(\mathbf{v}^\star_h, q^\star_h) \in \mathcal{X}_h$ satisfying (2.3) (with $V = \mathcal{X}_h$). We follow the three main steps of the proof of Theorem 1.

1. According to Proposition 1, there exists $\tilde{\mathbf{v}}_{p'_h} \in \mathbf{V}^\perp$ such that $\mathrm{div}\, \tilde{\mathbf{v}}_{p'_h} = p'_h$ in Ω and $\|\tilde{\mathbf{v}}_{p'_h}\|_{\mathbf{H}^1_0(\Omega)} \le C_{\mathrm{div}} \|p'_h\|_{L^2(\Omega)}$. Let us set $\mathbf{v}_{p'_h} = \nu^{-1}\tilde{\mathbf{v}}_{p'_h}$ so that $\mathrm{div}\, \mathbf{v}_{p'_h} = \nu^{-1} p'_h$ and $\|\mathbf{v}_{p'_h}\|_{\mathbf{H}^1_0(\Omega)} \le \nu^{-1}\, C_{\mathrm{div}}\, \|p'_h\|^2_{L^2(\Omega)}$. Consider $\mathbf{v}_{h,p'_h} = \Pi_{nc}\mathbf{v}_{p'_h}$, for all $q_h \in Q_h$, we have: $(\mathrm{div}_h\, \mathbf{v}_{h,p'_h}, q_h) = \nu^{-1}\, (p'_h, q_h)_{L^2(\Omega)}$ and

$$\|\mathbf{v}_{h,p'_h}\|_h \le \nu^{-1}\, C_{\mathrm{div}}^{nc} \nu\, \|p'_h\|_{L^2(\Omega)} \text{ where } C_{\mathrm{div}}^{nc} = C_{nc}\, C_{\mathrm{div}}. \tag{5.9}$$

   Let us set $(\mathbf{v}^\star_h, q^\star_h) := (\gamma_{nc}\mathbf{u}'_h - \mathbf{v}_{h,p'_h}, -\gamma_{nc}\, p'_h)$, with $\gamma_{nc} > 0$. We obtain:

$$a_{S,h}\ (\ (\mathbf{u}'_h, p'_h),\ (\mathbf{v}^\star_h, q^\star_h)\ ) = \nu\, \gamma_{nc} \|\mathbf{u}'_h\|_h^2 + \nu^{-1} \|p'_h\|^2_{L^2(\Omega)} - \nu(\mathbf{u}'_h, \mathbf{v}_{h,p'_h})_h. \tag{5.10}$$

2. In order to bound the last term of (5.10), we use Young inequality and then inequality (5.9) so that for all $\eta_{nc} > 0$:

$$(\mathbf{u}'_h, \mathbf{v}_{h,p'_h})_h \le \frac{\eta_{nc}}{2} \|\mathbf{u}'_h\|_h^2 + \frac{\eta_{nc}^{-1}}{2} \left(\frac{C_{\mathrm{div}}^{nc}}{\nu} \right)^2 \|p'_h\|^2_{L^2(\Omega)}. \tag{5.11}$$

3. Using the bound (5.11) in (5.10) and choosing $\eta_{nc} = \gamma_{nc}$, we get:

$$a_{S,h}\ (\ (\mathbf{u}'_h, p'_h),\ (\mathbf{v}^\star_h, q^\star_h)\ ) \ge \nu \left(\frac{\gamma_{nc}}{2} \nu\, \|\mathbf{u}'_h\|_h^2 + \nu^{-2} \left(1 + \frac{(\gamma_{nc})^{-1}}{2}\, (C_{\mathrm{div}}^{nc})^2 \right) \|p'_h\|^2_{L^2(\Omega)} \right).$$

   Consider now $\gamma_{nc} = (C_{\mathrm{div}}^{nc})^2$. We obtain:

$$a_{S,h}\ (\ (\mathbf{u}'_h, p'_h),\ (\mathbf{v}^\star_h, q^\star_h)\ ) \ge \frac{\nu}{2} C_{\min}^{nc}\, \|(\mathbf{u}'_h, p'_h)\|^2_{\mathcal{X}_h},$$

where $C^{nc}_{\min} = \dfrac{1}{2}\,\min(\,(C_{\rm div})^2, 1\,)$.
The operator T_h such that $T_h(\mathbf{u}'_h, p'_h) = (\mathbf{v}^\star_h, p^\star_h)$ is linear and continuous:

$$\|T_h(\mathbf{u}'_h, p'_h)\|^2_{\mathcal{X}_h} = \|\mathbf{v}^\star_h\|^2_h + \nu^{-2}\,\|q^\star_h\|_{L^2(\Omega)} \le (C^{nc}_{\max})^2\,\|(\mathbf{u}'_h, p'_h)\|^2_{\mathcal{X}_h}$$

where $C^{nc}_{\max} = C^{nc}_{\rm div}\,(1+(C^{nc}_{\rm div})^2)^{1/2}$. [4] The discrete continuous bilinear form $a_{S,h}(\cdot,\cdot)$ is then T_h-coercive and according to Theorem 1, Problem (5.5) is well posed. Consider $(\mathbf{u}_h, p_h)$ the unique solution of Problem (5.5). Choosing $\mathbf{v}_h = 0$, we obtain that $\operatorname{div}_h \mathbf{u}_h = 0$. Now, choosing $\mathbf{v}_h = \mathbf{u}_h$ in (5.5) and using Cauchy-Schwarz inequality, we get that:

$$\left\{\begin{array}{lll} \|\mathbf{u}_h\|_h & \le & \nu^{-1}\, C^{nc}_{PS}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \quad \text{if } \mathbf{f}\in\mathbf{L}^2(\Omega),\ \text{using (5.3)}\ ; \\ \|\mathbf{u}_h\|_h & \le & \nu^{-1}\, C^{nc}_{\mathcal{I}_h}\,\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \quad \text{if } \mathbf{f}\notin\mathbf{L}^2(\Omega),\ \text{using (5.4)}. \end{array}\right. \tag{5.12}$$

Consider $(\mathbf{v}_h, q_h) = (\mathbf{v}_{h,p_h}, 0)$ in (5.5), where $\mathbf{v}_{h,p_h} = \Pi_{nc}\mathbf{v}_{p_h}$ is built as $\mathbf{v}_{h,p'_h}$ in point 1, setting $p'_h = p_h$. Suppose that $\mathbf{f}\in\mathbf{L}^2(\Omega)$. Using the triangular inequality, Cauchy-Schwarz inequality, Poincaré-Steklov inequality (5.3), Theorem 4, and estimate (5.12), we have:

$$\begin{array}{rcl} \|p_h\|^2_{L^2(\Omega)} & = & \nu\,(\mathbf{u}_h, \mathbf{v}_{h,p_h})_h - (\mathbf{f}, \mathbf{v}_{h,p_h})_{\mathbf{L}^2(\Omega)}\,, \\ & \le & \nu\,\|\mathbf{u}_h\|_h\,\|\mathbf{v}_{h,p_h}\|_h + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|\mathbf{v}_{h,p_h}\|_{\mathbf{L}^2(\Omega)} \\ & \le & 2\,C^{nc}_{PS}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|\mathbf{v}_{h,p_h}\|_h \le 2\,C^{nc}_{PS}\,C_{nc}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|\,\mathbf{Grad}\,\mathbf{v}_{p_h}\|_{\mathbb{L}^2(\Omega)}\,, \\ & \le & 2\,C^{nc}_{PS}\,C^{nc}_{\rm div}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|p_h\|_{L^2(\Omega)}. \end{array}$$

Applying the same reasoning when $\mathbf{f}\in\mathbf{H}^{-1}(\Omega)$, we get that:

$$\left\{\begin{array}{lll} \|p_h\|_{L^2(\Omega)} & \le & 2\,C^{nc}_{PS}\,C^{nc}_{\rm div}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \quad \text{if } \mathbf{f}\in\mathbf{L}^2(\Omega),\ \text{using (5.3)}\ ; \\ \|p_h\|_{L^2(\Omega)} & \le & 2\,C^{nc}_{\mathcal{I}_h}\,C^{nc}_{\rm div}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \quad \text{if } \mathbf{f}\notin\mathbf{L}^2(\Omega),\ \text{using (5.4)}. \end{array}\right. \tag{5.13}$$

□

Corollary 1. *Under the assumption of Theorem 4, suppose that* $(\mathbf{u}, p)\in\mathbf{H}^{1+k}(\Omega)\times H^k(\Omega)$*, we then have the estimate:*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \le C\sigma^\ell\, h^{k+1}\,\left(|\mathbf{u}|_{\mathbf{H}^{k+1}(\Omega)} + \nu^{-1}\,|p|_{H^k(\Omega)}\right), \tag{5.14}$$

where the constant $C>0$ is independent of h, σ is the shape regularity parameter and the exponent $\ell\in\mathbb{N}^\star$ depends on k.

Proof. The a priori error estimate corresponds to [2, Theorem 4]. □

Remark 2. *Again, we recover the first Banach–Nečas–Babuška condition* [8, Thm. 25.9, (BNB1)]*:*

$$a_{S,h}\,(\,(\mathbf{u}'_h, p'_h),\,(\mathbf{v}^\star_h, q^\star_h)\,) \ge \nu\,C^{nc}_{\min}\,(C^{nc}_{\max})^{-1}\,\|(\mathbf{u}'_h, p'_h)\|_{\mathcal{X}_h}\,\|(\mathbf{v}^\star_h, q^\star_h)\|_{\mathcal{X}_h}.$$

[4] Note that $(\mathbf{v}^\star_h, q^\star_h) = (\mathbf{0}, 0) \Leftrightarrow (\mathbf{u}'_h, p'_h) = (\mathbf{0}, 0)$, so that the operator $T_h\in\mathcal{L}(\mathcal{X}_h, \mathcal{X}_h)$ is bijective.

Let us call $C^{nc}_{\rm stab} = \nu\, C^{nc}_{\min}\, (C^{nc}_{\max})^{-1}$ the stability constant. With the choice of our parameters, $C^{nc}_{\rm stab}$ is such that:

$$C^{nc}_{\rm stab} = \begin{cases} \dfrac{\nu}{2}\, \dfrac{C^{nc}_{\rm div}}{(1+(C^{nc}_{\rm div})^2)^{1/2}} & \text{if } 0 < C^{nc}_{\rm div} \leq 1, \\[2ex] \dfrac{\nu}{2}\, \dfrac{(C^{nc}_{\rm div})^{-1}}{(1+(C^{nc}_{\rm div})^2)^{1/2}} & \text{if } 1 \leq C^{nc}_{\rm div}. \end{cases}$$

The main issue with nonconforming mixed finite elements is the construction the basis functions. In a recent paper, Sauter explains such a construction in two dimensions [17, Corollary 2.4], and gives a bound to the discrete counterpart $\beta_{\mathcal{T}}(\Omega)$ of $\beta(\Omega)$ defined in (3.4):

$$\beta_{\mathcal{T}}(\Omega) = \inf_{q_h \in Q_h\setminus\{0\}}\ \sup_{\mathbf{v}_h \in \mathbf{X}_{0,h}\setminus\{0\}} \frac{(\mathrm{div}_h\, \mathbf{v}_h, q_h)}{\|q_h\|_{L^2(\Omega)}\, \|\mathbf{v}_h\|_h} \geq c_{\mathcal{T}}\, k^{-\alpha}. \tag{5.15}$$

This bound is in $c_{\mathcal{T}}\, k^{-\alpha}$, where the parameter α is explicit and depends on k and on the mesh topology; and the constant $c_{\mathcal{T}}$ depends only on the shape-regularity of the mesh.

6. Nonconforming Crouzeix-Raviart mixed finite elements

We study the lowest order nonconforming Crouzeix-Raviart mixed finite elements [2]. Let us consider X_{CR} (resp. $X_{0,CR}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order 1:

$$\begin{aligned} X_{CR} &= \left\{ v_h \in P^1_{disc}(\mathcal{T}_h)\,;\quad \forall f \in \mathcal{I}^i_F,\ \int_{F_f} [v_h] = 0 \right\}; \\ X_{0,CR} &= \left\{ v_h \in X_{CR}\,;\quad \forall f \in \mathcal{I}^b_F,\ \int_{F_f} v_h = 0 \right\}. \end{aligned} \tag{6.1}$$

The space of nonconforming approximation of of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order 1 is $\mathbf{X}_{CR} = (X_{CR})^d$ (resp. $\mathbf{X}_{0,CR} = (X_{0,CR})^d$). We set $\mathcal{X}_{CR} := \mathbf{X}_{0,CR} \times Q_{CR}$ where $Q_{CR} = P^0(\mathcal{T}_h) \cap L^2_{zmv}(\Omega)$.

We can endow X_{CR} with the basis $(\psi_f)_{f\in\mathcal{I}_F}$ such that: $\forall \ell \in \mathcal{I}_K$,

$$\psi_{f|K_\ell} = \begin{cases} 1 - d\lambda_{i,\ell} & \text{if } f \in \mathcal{I}_{F,\ell}, \\ 0 & \text{otherwise}, \end{cases}$$

where S_i is the vertex opposite to F_f in K_ℓ. We then have $\psi_{f|F_f} = 1$, so that $[\psi_f]_{F_f} = 0$ if $f \in \mathcal{I}^i_F$ (i.e. $F_f \in \overset{\circ}{\Omega}$), and $\forall f' \neq f$, $\int_{F_{f'}} \psi_f = 0$.

We have: $X_{CR} = \mathrm{vect}\left((\psi_f)_{f\in\mathcal{I}_F}\right)$ and $X_{0,CR} = \mathrm{vect}\left((\psi_f)_{f\in\mathcal{I}^i_F}\right)$.

The Crouzeix-Raviart interpolation operator π_{CR} for scalar functions is defined by:

$$\pi_{CR} : \begin{cases} H^1(\Omega) & \to & X_{CR} \\ v & \mapsto & \displaystyle\sum_{f\in\mathcal{I}_F} \pi_f v\, \psi_f \end{cases}, \text{ where } \pi_f v = \frac{1}{|F_f|}\int_{F_f} v.$$

Notice that $\forall f \in \mathcal{I}_F$, $\int_{F_f} \pi_{CR} v = \int_{F_f} v$. Moreover, the Crouzeix-Raviart interpolation operator preserves the constants, so that $\pi_{CR}\underline{v}_\Omega = \underline{v}_\Omega$ where $\underline{v}_\Omega = \int_\Omega v/|\Omega|$.

We recall the following result [18, Lemma 2]):

Lemma 1. *The Crouzeix-Raviart interpolation operator π_{CR} is such that:*

$$\forall v \in H^1(\Omega), \quad \|\pi_{CR} v\|_h \leq \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}. \tag{6.2}$$

Proof. We have, integrating by parts twice and using Cauchy-Schwarz inequality:

$$\begin{array}{rcl} \mathbf{grad}\, \pi_{CR} v_{|K_\ell} & = & |K_\ell|^{-1} \displaystyle\int_{K_\ell} \mathbf{grad}\, \pi_{CR} v = |K_\ell|^{-1} \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} \pi_{CR} v\, \mathbf{n}_f, \\ & = & |K_\ell|^{-1} \displaystyle\sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} v\, \mathbf{n}_f = |K_\ell|^{-1} \int_{K_\ell} \mathbf{grad}\, v, \\ |\,\mathbf{grad}\, \pi_{CR} v_{|K_\ell}| & \leq & |K_\ell|^{-1/2} \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)} \\ \Rightarrow \quad \|\,\mathbf{grad}\, \pi_{CR} v\|^2_{\mathbf{L}^2(K_\ell)} & \leq & \|\,\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)}. \end{array}$$

Summing these local estimates over $\ell \in \mathcal{I}_K$, we obtain (6.2). □

For a vector $\mathbf{v} \in \mathbf{H}^1(\Omega)$ of components $(v_{d'})_{d'=1}^d$, the Crouzeix-Raviart interpolation operator is such that: $\Pi_{CR}\mathbf{v} = (\pi_{CR} v_{d'})_{d'=1}^d$. Let us set $\Pi_f \mathbf{v} = (\pi_f v_{d'})_{d'=1}^d$.

Lemma 2. *The Crouzeix-Raviart interpolation operator Π_{CR} can play the role of the Fortin operator:*

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\Pi_{CR}\mathbf{v}\|_h \leq \|\,\mathbf{Grad}\, \mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{6.3}$$

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\operatorname{div}_h \Pi_{CR}\mathbf{v}, q_h) = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q \in Q_h, \tag{6.4}$$

Moreover, for all $\mathbf{v} \in \mathbf{P}^1(\Omega)$, $\Pi_{CR}\mathbf{v} = \mathbf{v}$.

Proof. We obtain (6.3) applying Lemma (1) component by component. By integrating by parts, we have $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\forall \ell \in \mathcal{I}_K$:

$$\begin{array}{rcl} \displaystyle\int_{K_\ell} \operatorname{div} \Pi_{CR}\mathbf{v} & = & \displaystyle\sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} \Pi_{CR}\mathbf{v} \cdot \mathbf{n}_f = \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} \Pi_f \mathbf{v} \cdot \mathbf{n}_f, \\ & = & \displaystyle\sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} \mathbf{v} \cdot \mathbf{n}_f = \int_{K_\ell} \operatorname{div} \mathbf{v}, \end{array}$$

so that (6.4) is satisfied. □

We can apply the T-coercivity theory to show the next following result:

Theorem 5. *Let $\mathcal{X}_h = \mathcal{X}_{CR}$. Then the continuous bilinear form $a_{S,h}(\cdot,\cdot)$ is T_h-coercive and Problem* (5.5) *is well-posed.*

Proof. Using estimates (6.3) and (5.3), we apply the proof of Theorem 4. □

Since the constant of the interpolation operator Π_{CR} is equal to 1, we have $C_{min}^{CR} = C_{min}$ and $C_{max}^{CR} = C_{max}$: the stability constant of the nonconforming Crouzeix-Raviart mixed finite elements is independent of the mesh. This is not the case for higher-order (see [19, Theorem 2.2]).
For higher-order, we cannot built the interpolation operator component by component, since higher-order divergence moments must be preserved. Thus, for $k > 1$, we must build Π_{nc} so that for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $\ell \in \mathcal{I}_K$, for all $q \in P^{k-1}(K_\ell)$:

$$\int_{K_\ell} q \operatorname{div} \Pi_{nc}\mathbf{v} = \int_{K_\ell} q \operatorname{div} \mathbf{v}.$$

We recall that by integration by parts, we have:

$$\int_{K_\ell} q \,\operatorname{div} \Pi_{nc}\mathbf{v} + \int_{K_\ell} \mathbf{grad}\, q \cdot \Pi_{nc}\mathbf{v} = \int_{\partial K_\ell} q\, \Pi_{nc}\mathbf{v} \cdot \mathbf{n}_{|\partial K_\ell}. \tag{6.5}$$

Hence, to obtain a local estimate of $\|\,\mathbf{Grad}\, \Pi_{nc}\mathbf{v}\|_{\mathbb{L}(K_\ell)}$, we will need the following Lemma:

Lemma 3. *Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and $q \in P^{k-1}(K_\ell)$. We set $\underline{\mathbf{v}}_\ell := \mathbf{v}_\ell - \dfrac{\int_{K_\ell} \mathbf{v}}{|K_\ell|}$, where $\mathbf{v}_\ell = \mathbf{v}_{|K_\ell}$. We have:*

$$\left| \int_{\partial K_\ell} q\, \underline{\mathbf{v}}_\ell \cdot \mathbf{n}_{|\partial K_\ell} \right| \le |K_\ell|^{k/2}\, \|\,\mathbf{Grad}\,\mathbf{v}_\ell\|_{\mathbb{L}^2(K_\ell)} \tag{6.6}$$

Proof. We have by integration by parts, and then using Cauchy-Schwarz inequality:

$$\begin{aligned}
\left| \int_{\partial K_\ell} q\, \underline{\mathbf{v}}_\ell \cdot \mathbf{n}_f \right| &\le \left| \int_{K_\ell} q \,\operatorname{div} \underline{\mathbf{v}}_\ell \right| + \left| \int_{K_\ell} \mathbf{grad}\, q \cdot \underline{\mathbf{v}}_\ell \right|, \\
&\le \|q\|_{L^2(K_\ell)}\, \|\,\mathbf{Grad}\, \underline{\mathbf{v}}_\ell\|_{\mathbb{L}^2(K_\ell)} + \|\,\mathbf{grad}\, q\|_{\mathbf{L}^2(K_\ell)}\, \|\underline{\mathbf{v}}_\ell\|_{\mathbf{L}^2(K_\ell)}, \\
&\le |K_\ell|^{k/2} \|\,\mathbf{Grad}\, \underline{\mathbf{v}}_\ell\|_{\mathbb{L}^2(K_\ell)} + |K_\ell|^{(k-1)/2}\, \|\underline{\mathbf{v}}_\ell\|_{\mathbf{L}^2(K_\ell)}, \\
&\lesssim |K_\ell|^{k/2} \|\,\mathbf{Grad}\, \mathbf{v}_\ell\|_{\mathbb{L}^2(K_\ell)} \text{ using (4.4).}
\end{aligned}$$

□

7. Fortin-Soulie mixed finite elements

We consider here the case $d = 2$ and we study Fortin-Soulie mixed finite elements [3]. We consider a shape-regular triangulation sequence $(\mathcal{T}_h)_h$.
Let us consider X_{FS} (resp. $X_{0,FS}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order 2:

$$\begin{aligned}
X_{FS} &= \left\{ v_h \in P^2_{disc}(\mathcal{T}_h)\,; \quad \forall f \in \mathcal{I}^i_F,\, \forall q_h \in P^1(F_f),\, \int_{F_f} [v_h]\, q_h = 0 \right\}; \\
X_{0,FS} &= \left\{ v_h \in X_{FS}\,; \quad \forall f \in \mathcal{I}^b_F,\, \forall q_h \in P^1(F_f),\, \int_{F_f} v_h\, q_h = 0 \right\}.
\end{aligned} \tag{7.1}$$

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order 2 is $\mathbf{X}_{FS} = (X_{FS})^2$ (resp. $\mathbf{X}_{0,FS} = (X_{0,FS})^2$). We set $\mathcal{X}_{FS} = \mathbf{X}_{0,FS} \times Q_{FS}$ where $Q_{FS} := P^1_{disc}(\mathcal{T}_h) \cap L^2_{zmv}(\Omega)$.
The building of a basis for $X_{0,FS}$ is more involved than for $X_{0,CR}$ since we cannot use two points per facet as degrees of freedom. Indeed, for all $\ell \in K_\ell$, there exists a polynomial of order 2 vanishing on the Gauss-Legendre points of the facets of the boundary ∂K_ℓ. Let $f \in \mathcal{I}_F$. The barycentric coordinates of the two Gauss-Legendre points $(p_{+,f}, p_{-,f})$ on F_f are such that:

$$p_{+,f} = (c_+, c_-),\; p_{-,f} = (c_-, c_+), \text{ where } c_\pm = (1 \pm 1/\sqrt{3})/2.$$

These points can be used to integrate exactly order three polynomials:

$$\forall g \in P^3(F_f),\ \int_{F_f} g = \frac{|F_f|}{2}\left(g(p_{+,f}) + g(p_{-,f})\right).$$

For all $\ell \in \mathcal{I}_K$, we define the quadratic function ϕ_{K_ℓ} that vanishes on the six Gauss-Legendre points of the facets of K_ℓ (see Fig. 1):

$$\phi_{K_\ell} := 2 - 3\sum_{i\in\mathcal{I}_{S,\ell}} \lambda_{i,\ell}^2 \text{ such that } \quad \forall f \in \mathcal{I}_{F,\ell},\ \forall q \in P^1(F), \quad \int_{F_f} \phi_{K_\ell}\, q = 0. \tag{7.2}$$

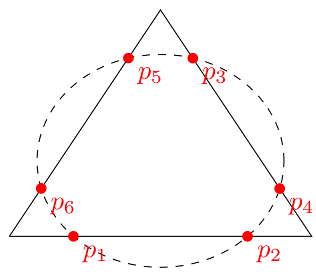

FIGURE 1. The six Gauss-Legendre points of an element K_ℓ and the elliptic function ϕ_{K_ℓ}.

We also define the spaces of P^2-Lagrange functions:

$$\begin{aligned} X_{LG} &:= \left\{v_h \in H^1(\Omega); \quad \forall \ell \in \mathcal{I}_K,\ v_{h|K_\ell} \in P^2(K_\ell)\right\}, \\ X_{0,LG} &:= \left\{v_h \in X_{LG}; \quad v_{h|\partial\Omega} = 0\right\}. \end{aligned}$$

The Proposition below proved in [3, Prop. 1] allows to build a basis for $X_{0,FS}$:

Proposition 5. *We have the following decomposition:* $X_{FS} = X_{LG} + \Phi_h$ *with* $\dim(X_{LG} \cap \Phi_h) = 1$. *Any function of* X_{FS} *can be written as the sum of a function of* X_{LG} *and a function of* Φ_h. *This representation can be made unique by specifying one degree of freedom.*

Notice that $\Phi_h \cap X_{LG} = \mathrm{vect}(v_\Phi)$, where for all $\ell \in \mathcal{I}_K$, $v_{\Phi|K_\ell} = \phi_{K_\ell}$. Then, counting the degrees of freedom, one can show that $\dim(X_{FS}) = \dim(X_{LG}) + \dim(\Phi_h) + 1$. For problems involving Dirichlet boundary conditions we can prove thus that for $X_{0,FS}$ the representation is unique and $X_{0,FS} = X_{0,LG} \oplus \Phi_h$. We have $X_{LG} = \mathrm{vect}\left((\phi_{S_i})_{i\in\mathcal{I}_S}, (\phi_{F_f})_{f\in\mathcal{I}_F}\right)$ where the basis functions are such that:

$$\begin{aligned} &\forall i, j \in \mathcal{I}_S,\ \forall f, g \in \mathcal{I}_F : \\ &\quad \phi_{S_i}(S_j) = \delta_{ij},\ \phi_{S_i}(M_f) = 0, \quad \phi_{M_f}(M_g) = \delta_{fg},\ \phi_{M_f}(S_i) = 0. \end{aligned}$$

For all $\ell \in \mathcal{I}_K$, we will denote by $(\phi_{\ell,j})_{j=1}^6$ the local nodal basis such that:

$$(\phi_{\ell,j})_{j=1}^3 = (\phi_{S_i|K_\ell})_{i\in\mathcal{I}_{S,\ell}} \quad \text{and} \quad (\phi_{\ell,j})_{j=4}^6 = (\phi_{F_f|K_\ell})_{f\in\mathcal{I}_{F,\ell}}.$$

The spaces X_{FS} and $X_{0,FS}$ are such that:

$$\begin{aligned} X_{FS} &= \mathrm{vect}\left((\phi_{S_i})_{i\in\mathcal{I}_S}, (\phi_{F_f})_{f\in\mathcal{I}_F}, (\phi_{K_\ell})_{\ell\in\mathcal{I}_K}\right), \\ X_{0,FS} &= \mathrm{vect}\left((\phi_{S_i})_{i\in\mathcal{I}_S^i}, (\phi_{F_f})_{f\in\mathcal{I}_F^i}, (\phi_{K_\ell})_{\ell\in\mathcal{I}_K}\right). \end{aligned} \tag{7.3}$$

We propose here an alternative definition of the Fortin interpolation operator proposed in [3]. Let us first recall the Scott-Zhang interpolation operator [20, 21]. For all $i \in \mathcal{I}_S$, we choose some $\ell_i \in \mathcal{I}_{K,i}$, and we build the $L^2(K_{\ell_i})$-dual basis $(\tilde{\phi}_{\ell_i,j})_{j=1}^6$ of the local nodal basis such that:

$$\forall j,\, j' \in \{1, \cdots, 6\}, \quad \int_{K_{\ell_i}} \tilde{\phi}_{\ell_i,j}\, \phi_{\ell_i,j'} = \delta_{j,j'}.$$

Let us define the Fortin-Soulie interpolation operator for scalar functions by:

$$\pi_{FS} : \left\{ \begin{array}{ccl} H^1(\Omega) & \to & X_{FS} \\ v & \mapsto & \tilde{\pi} v + \displaystyle\sum_{\ell \in \mathcal{I}_K} v_{K_\ell} \phi_{K_\ell} \end{array} \right. , \tag{7.4}$$
$$\text{with} \quad \tilde{\pi} v = \sum_{i \in \mathcal{I}_S} v_{S_i} \phi_{S_i} + \sum_{f \in \mathcal{I}_F} \tilde{v}_{F_f} \phi_{F_f}.$$

- The coefficients $(v_{S_i})_{i \in \mathcal{I}_S}$ are fixed so that: $\forall i \in \mathcal{I}_S,\ v_{S_i} = \displaystyle\int_{K_{\ell,i}} v\, \tilde{\phi}_{\ell_i,j_i}$, where j_i is the index such that $\displaystyle\int_{K_{\ell_i}} \tilde{\phi}_{\ell_i,j_i}\, \phi_{S_i|K_{\ell_i}} = 1$.
- The coefficients $\left(\tilde{v}_{F_f}\right)_{f \in \mathcal{I}_F}$ are fixed so that: $\forall f \in \mathcal{I}_F,\ \displaystyle\int_{F_f} \tilde{\pi} \tilde{v} = \int_{F_f} v$.

For all $\ell \in \mathcal{I}_K$, the coefficient v_{K_ℓ} is fixed so that: $\displaystyle\int_{K_\ell} \pi_{FS} v = \int_{K_\ell} v$.
The definition (7.4) is more general than the one given in [3], which holds for $v \in H^2(\Omega)$.
We set $\mathbf{v}_{S_i} := (\,\tilde{\pi} v_1(S_i), \tilde{\pi} v_2(S_i))^T$ and $\tilde{\mathbf{v}}_{F_f} := (\,\tilde{\pi} v_1(F_f), \tilde{\pi} v_2(F_f)\,)^T$.
We can define two different Fortin-Soulie interpolation operators for vector functions. First, let

$$\tilde{\Pi}_{FS} : \left\{ \begin{array}{ccl} \mathbf{H}^1(\Omega) & \to & \mathbf{X}_{FS} \\ \mathbf{v} & \mapsto & (\pi_{FS}(\mathbf{v})_1, \pi_{FS}(\mathbf{v})_2)^T. \end{array} \right.$$

We remind that the coefficients $(\tilde{\mathbf{v}}_{K_\ell})_{\ell \in \mathcal{I}_K}$ are such that:

$$\forall \ell \in \mathcal{I}_K, \quad \int_{K_\ell} \tilde{\Pi}_{FS} \mathbf{v} = \int_{K_\ell} \mathbf{v}. \tag{7.5}$$

The interpolation operator $\tilde{\Pi}_{FS}$ preserves the local averages, but it doesn't preserve the divergence. We then define a second interpolation operator which preserves the divergence in a weak sense:

$$\Pi_{FS} : \left\{ \begin{array}{ccl} \mathbf{H}^1(\Omega) & \to & \mathbf{X}_{FS} \\ \mathbf{v} & \mapsto & \displaystyle\sum_{i \in \mathcal{I}_S} \mathbf{v}_{S_i} \phi_{S_i} + \sum_{f \in \mathcal{I}_F} \tilde{\mathbf{v}}_{F_f} \phi_{F_f} + \sum_{\ell \in \mathcal{I}_K} \mathbf{v}_{K_\ell} \phi_{K_\ell} \end{array} \right. .$$

For all $\ell \in \mathcal{I}_K$, the vector coefficient $\mathbf{v}_{K_\ell} \in \mathbb{R}^2$ is now fixed so that condition (5.7) is satisfied. We can impose for example that the projection $\Pi_{FS} \mathbf{v}$ satisfies:

$$\int_{K_\ell} T_\ell^{-1}(\mathbf{x})\ \mathrm{div}\, \Pi_{FS} \mathbf{v} = \int_{K_\ell} T_\ell^{-1}(\mathbf{x})\ \mathrm{div}\, \mathbf{v}. \tag{7.6}$$

Notice that due to (7.2), the patch-test condition is still satisfied.

Proposition 6. *Let $\sigma_D > 0$. The Fortin-Soulie interpolation operator Π_{FS} is such for all $\mathbf{v} \in \bigcap_{0<s<\sigma_D} \mathbf{H}^{1+s}(\Omega)$ we have:*

$$\forall\, s \in]0, \sigma_D[,\ \forall \ell \in \mathcal{I}_K,\ \|\,\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim (\sigma_\ell)^2\,(h_\ell)^s\,|\mathbf{v}|_{1+s,K_\ell}, \tag{7.7}$$

$$\forall\, s \in]0, \sigma_D[,\ \exists C_{FS} = \mathcal{O}(\sigma^2), \quad \|\Pi_{FS}\mathbf{v} - \mathbf{v}\|_h \le C_{FS}\, h^s\, |\mathbf{v}|_{1+s,\Omega}. \tag{7.8}$$

Remark 3. *Albeit we are inspired by the proof of* [2, Lemma 4]*, we changed the transition from equation (4.27) to (4.29) there by using only the properties related to the normal component of the velocity, cf* (6.6)*. As a matter of fact, in the original proof, one ends up with either $C_{FS} = \mathcal{O}(\sigma^3)$ with the help of the multiple trace inequality or with $C_{FS} = \mathcal{O}(\sigma^2)$ at the cost of imposing a stronger assumption on the regularity of $\mathbf{v}$ (namely, $\sigma_D > 1/2$). Finally, because we do not split the integral over the boundaries of elements into the sum of $d+1$ integrals over the facets, we obtain purely local estimates, which appear to be new for the Fortin-Soulie element in the case of low-regularity fields $\mathbf{v}$.*

Proof. Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$. By construction, we have:

$$\int_{K_\ell} (\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v}) = 0 \text{ and for all } f \in \mathcal{I}_{F,\ell},\ \int_{F_f} (\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})_{|K_\ell} = 0. \tag{7.9}$$

We have:

$$\begin{aligned} \|\,\mathbf{Grad}\,(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \le \quad & \|\,\mathbf{Grad}\,(\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \\ & + \|\,\mathbf{Grad}\,(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}. \end{aligned} \tag{7.10}$$

Notice that for all $\ell \in \mathcal{I}_K$, $(\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v})_{|K_\ell} = (\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell})\,\phi_{K_\ell}$.
Using (4.3)-(i), we obtain that:

$$\begin{aligned} \|\,\mathbf{Grad}\,(\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \quad & \lesssim \quad |\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|\,\|\,\mathbf{grad}\ \phi_{K_\ell}\|_{\mathbf{L}^2(K_\ell)}, \\ & \lesssim \quad \|\mathbb{B}_\ell{}^{-1}\|\,|K_\ell|^{1/2}\,|\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|, \\ & \lesssim \quad \sigma_\ell\,|\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|. \end{aligned} \tag{7.11}$$

Let us estimate $|\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}|$. On the one hand, we have:

$$\begin{aligned} \int_{K_\ell} (\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v}) \quad &= \quad \int_{K_\ell} (\Pi_{FS}\mathbf{v} - \mathbf{v}) \text{ from (7.5)}, \\ &= \quad \int_{\partial K_\ell} \mathbf{x}\,(\Pi_{FS}\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{|\partial K_\ell} \text{ by IBP}, \\ &= \quad \int_{\partial K_\ell} \mathbf{x}\,(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{|\partial K_\ell} \text{ from (7.2)}. \end{aligned}$$

Hence, using (7.9), we obtain:

$$\left| \int_{K_\ell} (\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v}) \right| \le |K_\ell|\,\|\,\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}. \tag{7.12}$$

On the other hand, it holds:

$$\int_{K_\ell} (\Pi_{FS}\mathbf{v} - \tilde{\Pi}_{FS}\mathbf{v}) = (\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}) \int_{K_\ell} \phi_{K_\ell} = \frac{|K_\ell|}{4}\,(\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}). \tag{7.13}$$

Hence, combining (7.12) and (7.13), we have:

$$|\mathbf{v}_{K_\ell} - \tilde{\mathbf{v}}_{K_\ell}| \le 4\,\|\,\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}$$

We deduce from (7.10) that for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, for all $\ell \in \mathcal{I}_K$ we have:

$$\|\,\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{K_\ell} \lesssim \sigma_\ell \,\|\,\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)}. \tag{7.14}$$

For all $\mathbf{v} \in \mathbf{P}^2(K_\ell)$ we have $\tilde{\Pi}_{FS}(\mathbf{v}) = \mathbf{v}$ and $(\widehat{\tilde{\Pi}_{FS}\mathbf{v}})_\ell = \hat{\tilde{\Pi}}_{FS}\hat{\mathbf{v}}_\ell$. Hence, using Bramble-Hilbert/Deny-Lions Lemma [16, Lemma 11.9], we have:

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\,\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim \sigma_\ell\, |\mathbf{v}|_{1,K_\ell},$$
$$\forall \mathbf{v} \in \mathbf{H}^2(\Omega) \quad \|\,\mathbf{Grad}(\tilde{\Pi}_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim \sigma_\ell\, h_\ell\, |\mathbf{v}|_{2,K_\ell}.$$

We deduce that:

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\,\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim (\sigma_\ell)^2\, |\mathbf{v}|_{1,K_\ell}, \tag{7.15}$$
$$\forall \mathbf{v} \in \mathbf{H}^2(\Omega) \quad \|\,\mathbf{Grad}(\Pi_{FS}\mathbf{v} - \mathbf{v})\|_{\mathbb{L}^2(K_\ell)} \lesssim (\sigma_\ell)^2\, h_\ell\, |\mathbf{v}|_{2,K_\ell}. \tag{7.16}$$

Using interpolation property [22, Lemma 22.2], we obtain (7.8). □

We recall that the discrete Poincaré–Steklov inequality (5.3) holds.

Theorem 6. *Let $\mathcal{X}_h = \mathcal{X}_{FS}$. Then the continuous bilinear form $a_{S,h}(\cdot,\cdot)$ is T_h-coercive and Problem* (5.5) *is well-posed.*

Proof. According to Proposition 6, the Fortin-Soulie interpolation operator Π_{FS} satisfies (5.6)-(5.7), so that we can apply the proof of Theorem 4. □

Notice that in the recent paper [23], the inf-sup condition of the mixed Fortin-Soulie finite element is proven directly on a triangle and then using the macro-element technique [24], but it seems difficult to use this technique to build a Fortin operator, which is needed to compute error estimates.
The study can be extended to higher orders for $d = 2$ using the following papers: [25] for $k \geq 4$, k even, [26] for $k = 3$ and [19] for $k \geq 5$, k odd. In [27], the authors propose a local Fortin operator for the lowest order Taylor-Hood finite element [14] for $d = 3$ which could be used to prove the T-coercivity.

8. Numerical results

Consider Problem (3.1) with data $\mathbf{f} = -\,\mathbf{grad}\,\phi$, where $\phi \in H^1(\Omega) \cap L^2_{zmv}(\Omega)$. The unique solution is then $(\mathbf{u}, p) := (0, \phi)$. By integrating by parts, the source term in (3.6) reads:

$$\forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \quad \int_\Omega \mathbf{f} \cdot \mathbf{v} = \int_\Omega \phi \,\mathrm{div}\,\mathbf{v}. \tag{8.1}$$

Recall that the nonconforming space $\mathbf{X}_h$ defined in (5.1) is a subset of $\mathcal{P}_h\mathbf{H}^1$: using a nonconforming finite element method, the integration by parts must be done on each element of the triangulation, and we have:

$$\forall \mathbf{v} \in \mathcal{P}_h\mathbf{H}^1, \quad \int_\Omega \mathbf{f} \cdot \mathbf{v} = (\mathrm{div}_h\,\mathbf{v}, \phi) + \sum_{f \in \mathcal{I}_F} \int_{F_f} [\mathbf{v} \cdot \mathbf{n}_f]\, \phi. \tag{8.2}$$

When we apply this result to the right-hand-side of (5.5), we notice that the term with the jumps acts as a numerical source, which numerical influence is proportional to $1/\nu$. Thus, we cannot obtain exactly $\mathbf{u}_h = 0$ (see also (5.14)). Linke proposed in [28] to project the test function $\mathbf{v}_h \in \mathbf{X}_h$ on a discrete subspace of $\mathbf{H}(\mathrm{div};\Omega)$, like Raviart-Thomas or Brezzi-Douglas-Marini finite elements (see [29, 30], or the monograph [15]). Let $\Pi_{\mathrm{div}} : \mathbf{X}_{0,h} \to P^k_{disc}(\mathcal{T}_h) \cap \mathbf{H}_0(\mathrm{div};\Omega)$ be some interpolation operator built so that for all $\mathbf{v}_h \in \mathbf{X}_{0,h}$, for all $\ell \in \mathcal{I}_K$, $(\operatorname{div}\Pi_{\operatorname{div}}\mathbf{v}_h)_{|K_\ell} = \operatorname{div}\mathbf{v}_{h|K_\ell}$. Integrating by parts, we have for all $\mathbf{v}_h \in \mathbf{X}_{0,h}$:

$$\begin{aligned}\int_\Omega \mathbf{f}\cdot\Pi_{\operatorname{div}}\mathbf{v}_h &= \int_\Omega \phi\,\operatorname{div}\Pi_{\operatorname{div}}\mathbf{v}_h = \sum_{\ell\in K_\ell}\int_{K_\ell}\phi\,\operatorname{div}\Pi_{\operatorname{div}}\mathbf{v}_h,\\ &= \sum_{\ell\in K_\ell}\int_{K_\ell}\phi\,\operatorname{div}\mathbf{v}_h = (\operatorname{div}_h\mathbf{v}_h,\phi).\end{aligned}$$

The projection Π_{div} allows to eliminate the terms of the integrals of the jumps in (8.2).
Let us write Problem (5.5) as:
Find $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$ such that

$$a_{S,h}\ ((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)) = \ell_{\mathbf{f}}\ ((\Pi_{\operatorname{div}}\mathbf{v}_h,q_h))\quad \forall(\mathbf{v}_h,q_h)\in\mathcal{X}_h. \tag{8.3}$$

In the case of $\mathcal{X}_h = \mathcal{X}_{CR}$ and a projection on Brezzi-Douglas-Marini finite elements, the following error estimate holds if $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega)\times H^1(\Omega)$:

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \le \widetilde{C}\,h^2\,|\mathbf{u}|_{\mathbf{H}^2(\Omega)}, \tag{8.4}$$

where the constant $\widetilde{C}$ if independent of h. The proof is detailed in [31] for shape-regular meshes and [32] for anisotropic meshes. We remark that the error doesn't depend on the norm of the pressure nor on the ν parameter. We will provide some numerical results to illustrate the effectiveness of this formulation, even with a projection on the Raviart-Thomas finite elements, which, for a fixed polynomial order, are less precise than the Brezzi-Douglas-Marini finite elements.
For all $\ell \in \mathcal{I}_K$, we let $P_H^k(K_\ell)$ be the set of homogeneous polynomials of order k on K_ℓ.
For $k \in \mathbb{N}^\star$, the space of Raviart-Thomas finite elements can be defined as:

$$\mathbf{X}_{RT_k} := \left\{\mathbf{v}\in\mathbf{H}(\operatorname{div};\Omega);\,\forall\ell\in\mathcal{I}_k,\,\mathbf{v}_{|K_\ell} = \mathbf{a}_\ell + b_\ell\mathbf{x}\,|\,(\mathbf{a}_\ell,b_\ell)\in P^k(K_\ell)^d\times P_H^k(K_\ell)\right\}.$$

Let $k \le 1$.
The Raviart–Thomas interpolation operator $\Pi_{RT_k} : \mathbf{H}^1(\Omega)\cup\mathbf{X}_h \to \mathbf{X}_{RT_k}$ is defined by: $\forall\mathbf{v}\in\mathbf{H}^1(\Omega)\cup\mathbf{X}_h$,

$$\left\{\begin{array}{rl}\forall f\in\mathcal{I}_F, & \displaystyle\int_{F_f}\Pi_{RT_k}\mathbf{v}\cdot\mathbf{n}_f\,q = \int_{F_f}\mathbf{v}\cdot\mathbf{n}_f\,q,\quad \forall q\in P^k(F_f)\\ \text{for } k=1,\,\forall\ell\in\mathcal{I}_K, & \displaystyle\int_{K_\ell}\Pi_{RT_1}\mathbf{v} = \int_{K_\ell}\mathbf{v}\end{array}\right.. \tag{8.5}$$

Note that the Raviart–Thomas interpolation operator preserves the constants. Let $\mathbf{v}_h \in \mathbf{X}_h$. In order to compute the left-hand-side of (8.2), we must evaluate $(\Pi_{RT_k}\mathbf{v}_h)_{|K_\ell}$ for all $\ell \in \mathcal{I}_K$. Calculations are performed using the proposition below, which corresponds to [33, Lemma 3.11]:

Proposition 7. *Let $k \le 1$. Let $\hat{\Pi}_{RT_k} : \mathbf{H}^1(\hat{K}) \to \mathbf{P}^k(\hat{K})$ be the Raviart–Thomas interpolation operator restricted to the reference element, so that: $\forall\hat{\mathbf{v}}\in\mathbf{H}^1(\hat{K})$,*

$$\left\{\begin{array}{ll}\forall\hat{F}\in\partial\hat{K}, & \displaystyle\int_{\hat{F}}\hat{\Pi}_{RT_k}\hat{\mathbf{v}}\cdot\mathbf{n}_{\hat{F}}\,\hat{q} = \int_{\hat{F}}\hat{\mathbf{v}}\cdot\mathbf{n}_{\hat{F}}\,\hat{q},\quad\forall\hat{q}\in P^k(\hat{F})\\ \text{for } k=1, & \displaystyle\int_{\hat{K}}\hat{\Pi}_{RT_k}\hat{\mathbf{v}} = \int_{\hat{K}}\hat{\mathbf{v}}\end{array}\right.. \tag{8.6}$$

We then have: $\forall \ell \in \mathcal{I}_K$,

$$(\Pi_{RT_k}\mathbf{v})_{|K_\ell}(\mathbf{x}) = \mathbb{B}_\ell \left(\hat{\Pi}_{RT_k} \mathbb{B}_\ell{}^{-1} \hat{\mathbf{v}}_\ell \right) \circ T_\ell{}^{-1}(\mathbf{x}) \;\; \textit{where} \;\; \hat{\mathbf{v}}_\ell = \mathbf{v} \circ T_\ell(\hat{\mathbf{x}}). \tag{8.7}$$

The proof is based on the equality of the $\hat{F}$ and $\hat{K}$-moments of $(\Pi_{RT_k}\mathbf{v})_{|K_\ell} \circ T_\ell(\hat{\mathbf{x}})$ and $\mathbb{B}_\ell \left(\hat{\Pi}_{RT_k} \mathbb{B}_\ell{}^{-1} \hat{\mathbf{v}}_\ell \right)(\hat{\mathbf{x}})$. For $k = 0$, setting for $d' \in \{1, \cdots, d\}$: $\boldsymbol{\psi}_{f,d'} := \psi_f \, \mathbf{e}_{d'}$, we obtain that:

$$\forall \ell \in \mathcal{I}_K \,,\; \forall f \in \mathcal{I}_{F,\ell} \,, \quad (\Pi_{RT_0} \boldsymbol{\psi}_{f,d'})_{|K_\ell} = (d\,|K_\ell|)^{-1} \left(\mathbf{x} - \vec{OS}_{f,\ell} \right) \mathcal{S}_{f,\ell} \cdot \mathbf{e}_{d'} \,, \tag{8.8}$$

where $S_{f,\ell}$ is the vertex opposite to F_f in K_ℓ.
For $k = 1$, the vector $(\Pi_{RT_1}\mathbf{v}_h)_{|K_\ell}$ is described by eight unknowns:

$$(\Pi_{RT_1}\mathbf{v}_h)_{|K_\ell} = \mathbb{A}_\ell \, \mathbf{x} + (\mathbf{b}_\ell \cdot \mathbf{x}) \, \mathbf{x} + \mathbf{d}_\ell, \text{ where } \mathbb{A}_\ell \in \mathbb{R}^{2\times 2},\, \mathbf{b}_\ell \in \mathbb{R}^2,\, \mathbf{d}_\ell \in \mathbb{R}^2.$$

We compute only once the inverse of the matrix of the linear system (8.6), in $\mathbb{R}^{8\times 8}$. In the Table 1 (resp. Tables 2 and 3), we call $\varepsilon_0(\mathbf{u}) = \|\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}$ (resp. $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}/\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}$) the velocity error in $\mathbf{L}^2(\Omega)$-norm, where $\mathbf{u}_h$ is the solution to Problem (5.5) (columns $\mathbf{X}_{CR}$ and $\mathbf{X}_{FS}$) or (8.3) (columns $\mathbf{X}_{CR} + \Pi_{RT_0}$ and $\mathbf{X}_{FS} + \Pi_{RT_1}$) and h is the mesh step.
We first consider Stokes Problem (3.1) in $\Omega = (0,1)^2$ with $\mathbf{u} = 0$, $p = (x_1)^3 + (x_2)^3 - 0.5$, $\mathbf{f} = \mathbf{grad}\, p = 3\, \left((x_1)^2, (x_2)^2 \right)^T$. We report in Table 1 $\varepsilon_0(\mathbf{u})$ for $h = 5.00\,e-2$ and for different values of ν.

ν	$\mathbf{X}_{CR}$	$\mathbf{X}_{CR} + \Pi_{RT_0}$	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS} + \Pi_{RT_1}$
$1.00\,e+0$	$3.19\,e-4$	$1.34\,e-18$	$3.53\,e-7$	$9.09\,e-19$
$1.00\,e-3$	$3.19\,e-1$	$1.34\,e-15$	$3.53\,e-4$	$9.09\,e-16$
$1.00\,e-4$	$3.19\,e+0$	$1.34\,e-14$	$3.53\,e-3$	$9.09\,e-15$

TABLE 1. Values of $\varepsilon_0(\mathbf{u})$ for $h = 5.00\,e-2$

When there is no projection, the error is inversely proportional to the ν parameter, whereas using the projection, we obtain $\mathbf{u}_h = 0$ up to machine precision.
We now consider Stokes Problem (3.1) in $\Omega = (0,1)^2$ with:

$$\mathbf{u} = \begin{pmatrix} (1 - \cos(2\,\pi\,x_1)) \, \sin(2\,\pi\,x_2) \\ (\cos(2\,\pi\,x_2) - 1) \, \sin(2\,\pi\,x_1) \end{pmatrix}, \quad \left\{ \begin{array}{lcl} p & = & \sin(2\,\pi\,x_1)\, \sin(2\,\pi\,x_2), \\ \mathbf{f} & = & -\nu\, \Delta \mathbf{u} + \mathbf{grad}\, p. \end{array} \right.$$

We report in Table 2 (resp. 3) the values of $\varepsilon_0(\mathbf{u})$ in the case $\nu = 1.00\,e-3$ (resp. $\nu = 1.00\,e-4$) for different level of mesh refinement. When there is no projection, $\varepsilon_0(\mathbf{u})$ is inversely proportional to ν, whereas using the projection, $\varepsilon_0(\mathbf{u})$ is independent of ν.

h	$\mathbf{X}_{CR}$	$\mathbf{X}_{CR} + \Pi_{RT_0}$	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS} + \Pi_{RT_1}$
$5.00\,e-2$	$5.66\,e-1$	$1.13\,e-2$	$2.35\,e-3$	$2.06\,e-4$
$2.50\,e-2$	$1.33\,e-1$	$2.89\,e-3$	$3.21\,e-4$	$2.59\,e-5$
$1.25\,e-2$	$3.88\,e-2$	$5.40\,e-4$	$4.20\,e-5$	$3.40\,e-6$
$6.25\,e-3$	$8.40\,e-3$	$1.79\,e-4$	$5.04\,e-6$	$4.15\,e-7$
Rate	$h^{2.05}$	$h^{2.07}$	$h^{2.96}$	$h^{2.98}$

TABLE 2. Values of $\varepsilon_0(\mathbf{u})$ for $\nu = 1.00\,e-3$

h	$\mathbf{X}_{CR}$	$\mathbf{X}_{CR} + \Pi_{RT_0}$	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS} + \Pi_{RT_1}$
$5.00\,e-2$	$5.66\,e-0$	$1.13\,e-2$	$2.35\,e-2$	$2.06\,e-4$
$2.50\,e-2$	$1.33\,e-0$	$2.89\,e-3$	$3.20\,e-3$	$2.59\,e-5$
$1.25\,e-2$	$3.38\,e-1$	$5.40\,e-4$	$4.20\,e-4$	$3.40\,e-6$
$6.25\,e-3$	$8.40\,e-2$	$1.79\,e-4$	$5.04\,e-5$	$4.15\,e-7$
Rate	$h^{2.05}$	$h^{2.07}$	$h^{2.96}$	$h^{2.98}$

TABLE 3. Values of $\varepsilon_0(\mathbf{u})$ for $\nu = 1.00\,e-4$

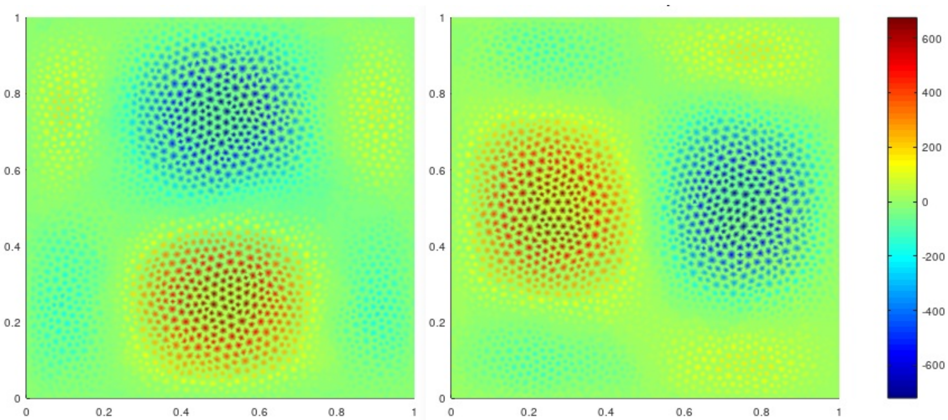

FIGURE 2. Values of $(\mathbf{u}_{FS} - \mathbf{u}_{FS+RT_1})$. Left: x_1-component, right: x_2-component.

Let $\mathbf{u}_{FS}$ (resp. $\mathbf{u}_{FS+RT_1}$) the solution to Problem (5.5) (resp. (8.3)) computed with Fortin-Soulie finite elements. We represent on Figure 2 the values of the Lagrange projection of $(\mathbf{u}_{FS} - \mathbf{u}_{FS+RT_1})$ in the case where $h = 2.50\,e-2$ and $\nu = 1.00\,e-4$. We observe local oscillations, of order the mesh step, which are caused by the numerical source exhibited in (8.2).
In order to enhance the numerical results, one can also use a posteriori error estimators to adapt the mesh (see [34, 35] for order 1 and [36] for order 2).
Alternatively, using the nonconforming Crouzeix-Raviart mixed finite element method, one can build a divergence-free basis, as described in [37]. Notice that using conforming finite elements, the Scott-Vogelius finite elements [38, 39] produce velocity approximations that are exactly divergence free.
The code used to get the numerical results can be downloaded on GitHub [40].
In principle, one can also obtain results with low-regularity velocity field.

9. Conclusion

We analysed the discretization of Stokes problem with nonconforming finite elements in light of the T-coercivity theory, we computed stability coefficients for order 1 in 2 or 3 dimension without mesh regularity assumption; and for order 1 in 2 dimension in the case of a shape-regular triangulation sequence. We then provided numerical results to illustrate the importance of using $\mathbf{H}(\mathrm{div})$-conforming projection. Further, we intend to extend the study to other mixed finite element methods.

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