T-coercivity for solving Stokes problem with nonconforming finite elements

Erell Jamelot

T-COERCIVITY FOR SOLVING STOKES PROBLEM WITH NONCONFORMING FINITE ELEMENTS

ERELL JAMELOT

Université Paris-Saclay, CEA, Service de Thermo-hydraulique et de Mécanique des Fluides, 91191 Gif-sur-Yvette, France

Abstract. We propose to analyse the discretization of the Stokes problem with nonconforming finite elements in light of the T-coercivity (cf. [1] for Helmholtz-like problems, see [2], [3] and [4] for the neutron diffusion equation). We propose explicit expressions of the stability constants. Finally, we give numerical results illustrating the importance of using divergence-free velocity reconstruction.

Keywords. Stokes problem, T-coercitivity, nonconforming finite elements, Fortin operator

2020 Mathematics Subject Classification. 65N30, 35J57, 76D07

Funding. CEA SIMU/SITHY project

1. Introduction

The Stokes problem describes the steady state of incompressible Newtonian flows. They are derived from the Navier–Stokes equations [5]. With regard to numerical analysis, the study of Stokes problem helps to build an appropriate approximation of the Navier–Stokes equations. We consider here a discretization with nonconforming finite elements [6, 7]. We propose to state the discrete inf-sup condition in light of the T-coercivity (cf. [1] for Helmholtz-like problems, see [2], [3] and [4] for the neutron diffusion equation), which allows to estimate the discrete error constant. In Section 2, we recall the T-coercivity theory as written in [1]. In Section 3 we apply it to the continuous Stokes Problem. We give details on the triangulation in Section 4, and we apply the T-coercivity to the discretization of Stokes problem with nonconforming mixed finite elements in Section 5. In Section 6 (resp. 7), we precise the proof of the well-posedness in the case of order 1 (resp. order 2) nonconforming mixed finite elements. Finally, we give numerical results illustrating the importance of using divergence-free velocity reconstruction.

2. T-coercivity

We recall here the T-coercivity theory as written in [1]. Consider first the variational problem, where V and W are two Hilbert spaces and $f \in V'$:

$$\text{Find } u \in V \text{ such that } \forall v \in W,\ a(u,v) = \langle f, v\rangle_V. \tag{2.1}$$

E-mail address: `erell.jamelot@cea.fr`.

Date: October 28, 2022.

Classically, we know that Problem (2.1) is well-posed if $a(\cdot,\cdot)$ satisfies the stability and the solvability conditions of the so-called Banach–Nečas–Babuška (BNB) Theorem (see a.e. [8, Thm. 25.9]). For some models, one can also prove the well-posedness using the T-coercivity theory (cf. [1] for Helmholtz-like problems, see [2], [3] and [4] for the neutron diffusion equation).

Definition 1. *Let V and W be two Hilbert spaces and $a(\cdot,\cdot)$ be a continuous and bilinear form over $V \times W$. It is T-coercive if*

$$\exists T \in \mathcal{L}(V,W), \text{ bijective}, \exists \alpha > 0, \forall v \in V, |a(v,Tv)| \geq \alpha \|v\|_V^2. \tag{2.2}$$

If in addition $a(\cdot,\cdot)$ is symmetric, it is T-coercive if

$$\exists T \in \mathcal{L}(V,V), \exists \alpha > 0, \forall v \in V, |a(v,Tv)| \geq \alpha \|v\|_V^2. \tag{2.3}$$

When the bilinear form $a(\cdot,\cdot)$ is symmetric, the requirement that the operator T is bijective can be dropped. It is proved in [1] that the T-coercivity condition is equivalent to the stability and solvability conditions of the BNB Theorem. Whereas the BNB theorem relies on an abstract inf-sup condition, T-coercivity uses explicit inf–sup operators, both at the continuous and discrete levels.

Theorem 1. *(well-posedness) Let $a(\cdot,\cdot)$ be a continuous and bilinear form. Suppose that the form $a(\cdot,\cdot)$ is T-coercive. Then Problem (2.1) is well-posed.*

3. Stokes problem

Let Ω be a connected bounded domain of $\mathbb{R}^d$, $d = 2, 3$, with a polygonal ($d = 2$) or Lipschitz polyhedral ($d = 3$) boundary $\partial\Omega$. We consider Stokes problem:

$$\text{Find } (\mathbf{u}, p) \text{ such that } \left\{ \begin{array}{rcl} -\nu\, \boldsymbol{\Delta}\, \mathbf{u} + \mathbf{grad}\, p & = & \mathbf{f}, \\ \operatorname{div} \mathbf{u} & = & 0. \end{array} \right. \tag{3.1}$$

with Dirichlet boundary conditions for the velocity $\mathbf{u}$ and a normalization condition for the pressure p:

$$\mathbf{u} = 0 \text{ on } \partial\Omega, \qquad \int_\Omega p = 0.$$

The vector field $\mathbf{u}$ represents the velocity of the fluid and the scalar field p represents its pressure divided by the fluid density which is supposed to be constant. Thus, the SI unit of the components of $\mathbf{u}$ is $m \cdot s^{-1}$ and the SI unit of p is $m^2 \cdot s^{-2}$). The first equation of (3.1) corresponds to the momentum balance equation and the second one corresponds to the conservation of the mass. The constant parameter $\nu > 0$ is the kinematic viscosity of the fluid, its SI unit is $m^2 \cdot s^{-1}$. The vector field $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ represents a body forces divided by the fluid density, its SI unit is $m \cdot s^{-2}$.

Before stating the variational formulation of Problem (3.1), we provide some definition and reminders. Let us set $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^1_0(\Omega) = (H^1_0(\Omega))^d$, $\mathbf{H}^{-1}(\Omega) = (H^{-1}(\Omega))^d$ its dual space and $L^2_{zmv}(\Omega) = \{q \in L^2(\Omega) \,|\, \int_\Omega q = 0\}$. We recall that $\mathbf{H}(\mathrm{div};\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,|\, \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$. Let us first recall Poincaré-Steklov inequality:

$$\exists C_{PS} > 0 \,|\, \forall v \in H^1_0(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C_{PS} \|\, \mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}. \tag{3.2}$$

The SI unit of C_{PS} is m.
Thanks to this result, in $H^1_0(\Omega)$, the semi-norm is equivalent to the natural norm, so that the scalar product reads $(v,w)_{H^1_0(\Omega)} = (\mathbf{grad}\, v, \mathbf{grad}\, w)_{\mathbf{L}^2(\Omega)}$ and the norm

is $\|v\|_{H^1_0(\Omega)} = \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}$. Let $\mathbf{v}$, $\mathbf{w} \in \mathbf{H}^1_0(\Omega)$. We denote by $(v_i)_{i=1}^d$ (resp. $(w_i)_{i=1}^d$) the components of $\mathbf{v}$ (resp. $\mathbf{w}$), and we set $\mathbf{Grad}\,\mathbf{v} = (\partial_j v_i)_{i,j=1}^d \in \mathbb{L}^2(\Omega)$, where $\mathbb{L}^2(\Omega) = [L^2(\Omega)]^{d\times d}$. We have:

$$(\mathbf{Grad}\,\mathbf{v}, \mathbf{Grad}\,\mathbf{w})_{\mathbb{L}^2(\Omega)} = (\mathbf{v}, \mathbf{w})_{\mathbf{H}^1_0(\Omega)} = \sum_{i=1}^d (v_i, w_i)_{H^1_0(\Omega)}$$

and:

$$\|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)} = \left(\sum_{j=1}^d \|v_j\|^2_{H^1_0(\Omega)}\right)^{1/2} = \|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}.$$

Let us set $\mathbf{V} = \left\{\mathbf{v} \in \mathbf{H}^1_0(\Omega) \mid \operatorname{div}\mathbf{v} = 0\right\}$. The space $\mathbf{V}$ is a closed subset of $\mathbf{H}^1_0(\Omega)$. We denote by $\mathbf{V}^\perp$ the orthogonal of $\mathbf{V}$ in $\mathbf{H}^1_0(\Omega)$. Let $\nu_p > 0$ be a kinematic viscosity. We recall that [5, cor. I.2.4]:

Proposition 1. *The operator* $\operatorname{div} : \mathbf{H}^1_0(\Omega) \to L^2(\Omega)$ *is an isomorphism of* $\mathbf{V}^\perp$ *onto* $L^2_{zmv}(\Omega)$. *Let* $\nu_p > 0$ *be a constant kinematic viscosity. We call* C_{div} *the constant such that:*

$$\forall p \in L^2_{zmv}(\Omega),\ \exists! \mathbf{v}_p \in \mathbf{V}^\perp \mid \operatorname{div}\mathbf{v}_p = \frac{1}{\nu_p} p \text{ and } \|\mathbf{v}_p\|_{\mathbf{H}^1_0(\Omega)} \le \frac{C_{\operatorname{div}}}{\nu_p}\|p\|_{L^2(\Omega)}. \tag{3.3}$$

Here, the constant C_{div} has no unit. It depends only on the domain Ω. Notice that we have: $C_{\operatorname{div}} = 1/\beta(\Omega)$ where $\beta(\Omega)$ is the inf-sup condition (or Ladyzhenskaya–Babuška–Brezzi condition):

$$\beta(\Omega) = \inf_{q\in L^2_{zmv}(\Omega)\setminus\{0\}} \sup_{\mathbf{v}\in\mathbf{H}^1_0(\Omega)\setminus\{0\}} \frac{(q, \operatorname{div}\mathbf{v})_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)}\,\|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)}}. \tag{3.4}$$

Generally, the value of $\beta(\Omega)$ is not known explicitly. In [9], Bernardi et al established results on the discrete approximation of $\beta(\Omega)$ using conforming finite elements. Recently, Gallistl proposed in [10] a numerical scheme with adaptive meshes for computing approximations to $\beta(\Omega)$. In the case of $d = 2$, Costabel and Dauge [11] established the following bound:

Theorem 2. *Let* $\Omega \subset \mathbb{R}^2$ *be a domain contained in a ball of radius* R*, star-shaped with respect to a concentric ball of radius* ρ*. Then*

$$\beta(\Omega) \ge \frac{\rho}{\sqrt{2}\,R}\left(1 + \sqrt{1 - \frac{\rho^2}{R^2}}\right)^{-1/2} \ge \frac{\rho}{2\,R}. \tag{3.5}$$

Let us detail the bound for some remarkable domains. If Ω is a ball, $\beta(\Omega) \ge \frac{1}{2}$ and if Ω is a square, $\beta(\Omega) \ge \frac{1}{2\sqrt{2}}$. Suppose now that Ω is stretched in some direction by a factor k, then $\beta(\Omega) \ge \frac{1}{2\,k}$. Finally, if Ω is L-shaped (resp. cross-shaped) such that $L = k\,l$, where L is the largest length and l is the smallest length of an edge, then $\beta(\Omega) \ge \frac{1}{2\sqrt{2}\,k}$ (resp. $\beta(\Omega) \ge \frac{1}{4\,k}$).

The variational formulation of Problem (3.1) reads:
Find $(\mathbf{u}, p) \in \mathbf{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$ such that

$$\left\{\begin{array}{rcll} \nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}^1_0(\Omega)} - (p, \operatorname{div}\mathbf{v})_{L^2(\Omega)} &=& \langle \mathbf{f}, \mathbf{v}\rangle_{\mathbf{H}^1_0(\Omega)} & \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega)\ ; \\ (q, \operatorname{div}\mathbf{u})_{L^2(\Omega)} &=& 0 & \forall q \in L^2_{zmv}(\Omega). \end{array}\right. \tag{3.6}$$

Classically, one proves that Problem (3.6) is well-posed using Poincaré-Steklov inequality (3.2) and Prop. 1. Check for instance the proof of [5, Thm. I.5.1].

Let us set $\mathcal{X} = \mathbf{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$ which is a Hilbert space which we endow with the following norm:

$$\|(\mathbf{v}, q)\|_{\mathcal{X}} = \|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)} + \nu^{-1}\, \|q\|_{L^2(\Omega)}. \tag{3.7}$$

We consider now the following bilinear symmetric and continuous form:

$$\left\{ \begin{array}{ccl} a_S : \mathcal{X} \times \mathcal{X} & \rightarrow & \mathbb{R} \\ (\mathbf{u}', p') \times (\mathbf{v}, q) & \mapsto & \nu(\mathbf{u}', \mathbf{v})_{\mathbf{H}^1_0(\Omega)} - (p', \operatorname{div} \mathbf{v})_{L^2(\Omega)} - (q, \operatorname{div} \mathbf{u}')_{L^2(\Omega)} \end{array} \right. . \tag{3.8}$$

We can write Problem (3.1) in an equivalent way as follows:

$$\text{Find } (\mathbf{u}, p) \in \mathcal{X} \text{ such that} \quad a_S\,((\mathbf{u}, p), (\mathbf{v}, q)) = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}^1_0(\Omega)} \quad \forall (\mathbf{v}, q) \in \mathcal{X}. \tag{3.9}$$

Let us prove that Problem (3.9) is well-posed using the T-coercivity theory.

Theorem 3. *Problem* (3.9) *is well-posed. It admits one and only one solution such that:*

$$\forall \mathbf{f} \in \mathbf{H}^{-1}(\Omega), \qquad \left\{ \begin{array}{l} \|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega)} \leq \nu^{-1}\, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}, \\ \|p\|_{L^2(\Omega)} \leq C_{\rm div}\, \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}. \end{array} \right. \tag{3.10}$$

Proof. We follow here the proof given in [12, 13]. Let us consider $(\mathbf{u}', p') \in \mathcal{X}$ and let us build $(\mathbf{v}^\star, q^\star) = T(\mathbf{u}', p') \in \mathcal{X}$ satisfying (2.3) (with $V = \mathcal{X}$). We need three main steps.

1. According to Prop. 1, there exists $\mathbf{v}_{p'} \in \mathbf{H}^1_0(\Omega)$ such that: $\operatorname{div} \mathbf{v}_{p'} = \nu^{-1}\, p'$ in Ω and

$$\|\mathbf{v}_{p'}\|^2_{\mathbf{H}^1_0(\Omega)} \leq \left(\frac{C_{\rm div}}{\nu}\right)^2 \|p'\|^2_{L^2(\Omega)}. \tag{3.11}$$

Let us set $(\mathbf{v}^\star, q^\star) := (\gamma\, \mathbf{u}' - \mathbf{v}_{p'}, -\gamma\, p')$, with $\gamma > 0$. We obtain:

$$a_S\,(\,(\mathbf{u}', p'),\, (\mathbf{v}^\star, q^\star)\,) = \nu\, \gamma\, \|\mathbf{u}'\|^2_{\mathbf{H}^1_0(\Omega)} + \nu^{-1}\, \|p'\|^2_{L^2(\Omega)} - \nu\, (\mathbf{u}', \mathbf{v}_{p'})_{\mathbf{H}^1_0(\Omega)}. \tag{3.12}$$

2. In order to bound the last term of (3.12), we use Young inequality and then inequality (3.11):

$$(\mathbf{u}', \mathbf{v}_{p'})_{\mathbf{H}^1_0(\Omega)} \leq \frac{\eta}{2} \|\mathbf{u}'\|^2_{\mathbf{H}^1_0(\Omega)} + \frac{\eta^{-1}}{2} \left(\frac{C_{\rm div}}{\nu}\right)^2 \|p'\|^2_{L^2(\Omega)}. \tag{3.13}$$

3. Using the bound (3.13) in (3.12) and choosing $\eta = \gamma$, we get:

$$a_S\,(\,(\mathbf{u}', p'),\, (\mathbf{v}^\star, q^\star)\,) \geq \frac{\gamma}{2}\, \nu\, \|\mathbf{u}'\|^2_{\mathbf{H}^1_0(\Omega)} + \nu^{-1} \left(1 + \frac{\gamma^{-1}}{2} (C_{\rm div})^2\right) \|p'\|^2_{L^2(\Omega)}$$

Consider now $\gamma = (C_{\rm div})^2$.
Noticing that $\nu \|\mathbf{u}\|^2_{\mathbf{H}^1_0(\Omega)} + \nu^{-1} \|p'\|^2_{L^2(\Omega)} \geq \dfrac{\nu}{2} \|(\mathbf{u}', p')\|^2_{\mathcal{X}}$, we obtain:

$$a_S\,(\,(\mathbf{u}', p'),\, (\mathbf{v}^\star, q^\star)\,) \geq \frac{\nu}{2}\, C_{\min}\, \|(\mathbf{u}', p')\|^2_{\mathcal{X}} \text{ where } C_{\min} = \frac{1}{2} \min(\,(C_{\rm div})^2, 1\,).$$

The operator T such that $T(\mathbf{u}',p') = (\mathbf{v}^\star, q^\star)$ is linear and continuous:

$$\begin{array}{rcl} \|T(\mathbf{u}',p')\|_{\mathcal{X}} & := & \|\mathbf{v}^\star\|_{\mathbf{H}^1_0(\Omega)} + \nu^{-1}\,\|q^\star\|_{L^2(\Omega)} \\ & \le & \gamma\,\|\mathbf{u}'\|_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{v}_{p'}\|_{\mathbf{H}^1_0(\Omega)} + \gamma\,\nu^{-1}\,\|p'\|_{L^2(\Omega)}, \\ & \le & \gamma\,\|\mathbf{u}'\|^2_{\mathbf{H}^1_0(\Omega)} + \,(C_{\mathrm{div}} + \gamma)\,\nu^{-1}\,\|p'\|^2_{L^2(\Omega)}, \\ & \le & C_{\max}\,\|(\mathbf{u}',p')\|_{\mathcal{X}}, \end{array}$$

where $C_{\max} = C_{\mathrm{div}}\,(1 + C_{\mathrm{div}})$.
[1] The symmetric and continuous bilinear form $a(\cdot,\cdot)$ is then T-coercive and according to Theorem 1, Problem (3.9) is well-posed. Let us prove (3.10). Consider $(\mathbf{u},p)$ the unique solution of Problem (3.9). Choosing $\mathbf{v} = 0$, we obtain that $\forall q \in L^2_{zmv}(\Omega)$, $(q, \operatorname{div}\mathbf{u})_{L^2(\Omega)} = 0$, so that $\mathbf{u} \in \mathbf{V}$. Now, choosing $\mathbf{v} = \mathbf{u}$ and using Cauchy-Schwarz inequality, we have: $\nu\,\|\mathbf{u}\|^2_{\mathbf{H}^1_0(\Omega} = \langle \mathbf{f}, \mathbf{u}\rangle_{\mathbf{H}^1_0(\Omega)} \le \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega)}$, so that: $\|\mathbf{u}\|_{\mathbf{H}^1_0(\Omega} \le \nu^{-1}\,\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$. Next, we choose in (3.9) $\mathbf{v} = \mathbf{v}_p \in \mathbf{V}^\perp$, where $\operatorname{div}\mathbf{v}_p = -\nu^{-1}\,p$ (see Prop. 1). Noticing that $\mathbf{u} \in \mathbf{V}$ and $\mathbf{v}_p \in \mathbf{V}^\perp$, it holds[2]: $(\mathbf{u}, \mathbf{v}_p)_{\mathbf{H}^1_0(\Omega)} = 0$. This gives:

$$\begin{array}{rcl} -(p, \operatorname{div}\mathbf{v}_p)_{L^2(\Omega)} & = & \nu^{-1}\,\|p\|^2_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{v}_p\rangle_{\mathbf{H}^1_0(\Omega)}, \\ & \le & \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|\mathbf{v}_p\|_{\mathbf{H}^1_0(\Omega)} \le C_{\mathrm{div}}\,\nu^{-1}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\,\|p\|_{L^2(\Omega)}, \end{array}$$

so that: $\|p\|_{L^2(\Omega)} \le C_{\mathrm{div}}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$. □

Remark 1. *We recover the first Banach–Nečas–Babuška condition* [8, Thm. 25.9, (BNB1)]*:*

$$a_S\,(\,(\mathbf{u}',p'),\,(\mathbf{v}^\star,q^\star)\,) \ge \frac{\nu}{2}\,C_{\min}\,(C_{\max})^{-1}\,\|(\mathbf{u}',p')\|_{\mathcal{X}}\,\|(\mathbf{v}^\star,q^\star)\|_{\mathcal{X}}.$$

Let us call $C_{\mathrm{stab}} = \frac{\nu}{2}\,C_{\min}\,(C_{\max})^{-1}$ the stability constant. With the choice of our parameters, C_{stab} is such that:

$$C_{\mathrm{stab}} = \begin{cases} \dfrac{\nu}{4}\,\dfrac{C_{\mathrm{div}}}{1 + C_{\mathrm{div}}} & \text{if } 0 < C_{\mathrm{div}} \le 1, \\[2ex] \dfrac{\nu}{4}\,\dfrac{(C_{\mathrm{div}})^{-1}}{1 + C_{\mathrm{div}}} & \text{if } 1 \le C_{\mathrm{div}}. \end{cases}$$

Thus, the T-coercivity approach allows to give an estimate of the stability constant. In our computations, it depends on the choice of the parameters η and γ, so that it could be optimized.
If we were using a conforming discretization to solve Problem (3.9) (a.e. Taylor-Hood finite elements [14]), we would use the bilinear form $a_S(\cdot,\cdot)$ to state the discrete variational formulation. Let us call the discrete spaces $\mathbf{X}_{c,h} \subset \mathbf{H}^1_0(\Omega)$ and $Q_{c,h} \subset L^2_{zmv}(\Omega)$. Then to prove the discrete T-coercivity, we would need to state

[1] Remark that $(\mathbf{v}^\star, q^\star) = (\mathbf{0}, 0) \Leftrightarrow (\mathbf{u}', p') = (\mathbf{0}, 0)$: the operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ is bijective.

[2] According to [5, Cor. I.2.3], since $\mathbf{v}_p \in \mathbf{V}^\perp$, $\exists p \in L^2_{zmv}(\Omega)\,|\,\Delta\mathbf{v}_p = \mathbf{grad}\,q$ in $\mathbf{H}^{-1}(\Omega)$. Integrating by parts twice, we have: $(\mathbf{u}, \mathbf{v}_p)_{\mathbf{H}^1_0(\Omega)} = -\langle \mathbf{grad}\,q, \mathbf{u}\rangle_{\mathbf{H}^1_0(\Omega)} = (q, \operatorname{div}\mathbf{u})_{L^2(\Omega)} = 0$.

the discrete counterpart to Proposition 1. To do so, we can build a linear operator $\Pi_c : \mathbf{X} \to \mathbf{X}_h$, known as Fortin operator, such that (see a.e. [15, §8.4.1]):

$$\exists C_c \,|\, \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \qquad \|\,\mathbf{Grad}\,\Pi_c \mathbf{v}\|_{\mathbb{L}^2(\Omega)} \le C_c \|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{3.14}$$

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\operatorname{div} \Pi_c \mathbf{v}, q_h)_{L^2(\Omega)} = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q_h \in Q_{c,h}. \tag{3.15}$$

Using a nonconforming discretization, we will not use the bilinear form $a_S(\cdot,\cdot)$ to exhibit the discrete variational formulation, but we will need a similar operator to (3.14)-(3.15) to prove the discrete T-coercivity, which is stated in Theorem 4.

4. Discretization

We call $(O, (x_{d'})_{d'=1}^d)$ the Cartesian coordinates system, of orthonormal basis $(\mathbf{e}_{d'})_{d'=1}^d$. Consider $(\mathcal{T}_h)_h$ a simplicial triangulation sequence of Ω. For a triangulation $\mathcal{T}_h$, we use the following index sets:

- $\mathcal{I}_K$ denotes the index set of the elements, such that $\mathcal{T}_h := \bigcup_{\ell \in \mathcal{I}_K} K_\ell$ is the set of elements.
- $\mathcal{I}_F$ denotes the index set of the facets[3], such that $\mathcal{F}_h := \bigcup_{f \in \mathcal{I}_F} F_f$ is the set of facets.
  Let $\mathcal{I}_F = \mathcal{I}_F^i \cup \mathcal{I}_F^b$, where $\forall f \in \mathcal{I}_F^i$, $F_f \in \Omega$ and $\forall f \in \mathcal{I}_F^b$, $F_f \in \partial\Omega$.
- $\mathcal{I}_S$ denotes the index set of the vertices, such that $(S_j)_{j \in \mathcal{I}_S}$ is the set of vertices.
  Let $\mathcal{I}_S = \mathcal{I}_S^i \cup \mathcal{I}_S^b$, where $\forall j \in \mathcal{I}_S^i$, $S_j \in \Omega$ and $\forall j \in \mathcal{I}_S^b$, $S_j \in \partial\Omega$.

We also define the following index subsets:

- $\forall \ell \in \mathcal{I}_K$, $\mathcal{I}_{F,\ell} = \{f \in \mathcal{I}_F \,|\, F_f \in K_\ell\}$, $\quad \mathcal{I}_{S,\ell} = \{j \in \mathcal{I}_S \,|\, S_j \in K_\ell\}$.
- $\forall j \in \mathcal{I}_S$, $\mathcal{I}_{K,j} = \{\ell \in \mathcal{I}_K \,|\, S_j \in K_\ell\}$, $\quad N_j := \operatorname{card}(\mathcal{I}_{K,j})$.

For all $\ell \in \mathcal{I}_K$, we call h_ℓ and ρ_ℓ the diameters of K_ℓ and its inscribed sphere respectively, and we let: $\sigma_\ell = \frac{h_\ell}{\rho_\ell}$. When the $(\mathcal{T}_h)_h$ is a shape-regular triangulation sequence (see a.e. [16, def. 11.2]), there exists a constant $\sigma > 1$ such that for all h, for all $\ell \in \mathcal{I}_K$, $\sigma_\ell \le \sigma$. For all $f \in \mathcal{I}_F$, M_f denotes the barycentre of F_f, and by $\mathbf{n}_f$ its unit normal (outward oriented if $F_f \in \partial\Omega$). For all $j \in \mathcal{I}_S$, for all $\ell \in \mathcal{I}_{K,j}$, $\lambda_{j,\ell}$ denotes the barycentric coordinate of S_j in K_ℓ; $F_{j,\ell}$ denotes the face opposite to vertex S_j in element K_ℓ, and $\mathbf{x}_{j,\ell}$ denotes its barycentre. We call $\mathcal{S}_{j,\ell}$ the outward normal vector of $F_{j,\ell}$ and of norm $|\mathcal{S}_{j,\ell}| = |F_{j,\ell}|$. We remind the expression of $\lambda_{j,\ell}$ and the integration formula (25.14) p. 187 of [17]:

$$\forall \mathbf{x} \in K_\ell,\ \lambda_{j,\ell}(\mathbf{x}) \quad = \quad (d\,|K_\ell|)^{-1}\,(\mathbf{x}_{j,\ell} - \mathbf{x}) \cdot \mathcal{S}_{j,\ell}\ ;$$

$$\int_{K_\ell} \prod_{i \in \mathcal{I}_{S,\ell}} \lambda_{i,\ell}^{\alpha_{i\ell}} \quad = \quad |K_\ell|\, \frac{d! \displaystyle\prod_{i \in \mathcal{I}_{S,\ell}} \alpha_{i,\ell}!}{\left(d + \displaystyle\sum_{i \in \mathcal{I}_{S,\ell}} \alpha_{i,\ell}\right)!}. \tag{4.1}$$

Let introduce spaces of piecewise regular elements:
We set $\mathcal{P}_h H^1 = \left\{v \in L^2(\Omega)\,;\quad \forall \ell \in \mathcal{I}_K,\ v_{|K_\ell} \in H^1(K_\ell)\right\}$, endowed with the scalar

[3]The term facet stands for face (resp. edge) when $d = 3$ (resp. $d = 2$).

product :

$$(v,w)_h := \sum_{\ell\in\mathcal{I}_K} (\mathbf{grad}\, v, \mathbf{grad}\, w)_{\mathbf{L}^2(K_\ell)} \quad \|v\|_h^2 = \sum_{\ell\in\mathcal{I}_K} \|\,\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)}.$$

We set $\mathcal{P}_h\mathbf{H}^1 = [\mathcal{P}_h H^1]^d$, endowed with the scalar product :

$$(\mathbf{v},\mathbf{w})_h := \sum_{\ell\in\mathcal{I}_K} (\mathbf{Grad}\,\mathbf{v}, \mathbf{Grad}\,\mathbf{w})_{\mathbb{L}^2(K_\ell)} \quad \|\mathbf{v}\|_h^2 = \sum_{\ell\in\mathcal{I}_K} \|\,\mathbf{Grad}\,\mathbf{v}\|^2_{\mathbb{L}^2(K_\ell)}.$$

Let $f\in\mathcal{I}_F^i$ such that $F_f = \partial K_L \cap \partial K_R$ and $\mathbf{n}_f$ is outward K_L oriented.
The jump (resp. average) of a function $v\in\mathcal{P}_h H^1$ across the facet F_f is defined as follows: $[v]_{F_f} := v_{|K_L} - v_{|K_R}$ (resp. $\{v\}_{F_f} := \frac{1}{2}(v_{|K_L} + v_{|K_R})$). For $f\in\mathcal{I}_F^b$, we set: $[v]_{F_f} := v_{|F_f}$ and $\{v\}_{F_f} := v_{|F_f}$.
We set $\mathcal{P}_h\mathbf{H}(\mathrm{div}) = \left\{\mathbf{v}\in\mathbf{L}^2(\Omega)\,;\quad \forall\ell\in\mathcal{I}_K,\, \mathbf{v}_{|K_\ell}\in\mathbf{H}(\mathrm{div};\,K_\ell)\right\}$, and we define the operator div_h such that:

$$\forall\mathbf{v}\in\mathcal{P}_h\mathbf{H}(\mathrm{div}),\, \forall q\in L^2(\Omega), \quad (\mathrm{div}_h\,\mathbf{v}, q) = \sum_{\ell\in\mathcal{I}_K} (\mathrm{div}\,\mathbf{v}, q)_{L^2(K_\ell)}.$$

We recall classical finite elements estimates [16]. Let $\hat{K}$ be the reference simplex and $\hat{F}$ be the reference facet. For $\ell\in\mathcal{I}_K$ (resp. $f\in\mathcal{I}_F$), we denote by $T_\ell : \hat{K}\to K_\ell$ (resp. $T_f : \hat{F}\to F_f$) the geometric mapping such that $\forall\hat{\mathbf{x}}\in\hat{K}$, $\mathbf{x}_{|K_\ell} = T_\ell(\hat{\mathbf{x}}) = \mathbb{B}_\ell\hat{\mathbf{x}} + \mathbf{b}_\ell$ (resp. $\mathbf{x}_{|F_f} = T_f(\hat{\mathbf{x}}) = \mathbb{B}_f\hat{\mathbf{x}} + \mathbf{b}_f$), and we set $J_\ell = \det(\mathbb{B}_\ell)$ (resp. $J_f = \det(\mathbb{B}_f)$). There holds:

$$|J_\ell| = d!\,|K_\ell|, \quad \|\mathbb{B}_\ell\| = \frac{h_\ell}{\rho_{\hat{K}}}, \quad \|\mathbb{B}_\ell{}^{-1}\| = \frac{h_{\hat{K}}}{\rho_\ell}, \quad |J_f| = (d-1)!\,|F_f|.$$

For $v\in L^2(K_\ell)$, we set $\hat{v}_\ell = v\circ T_\ell$. For $v\in v^2(F_f)$, we set: $\hat{v}_f = v\circ T_f$. Changing the variable, we get:

$$\|v\|^2_{L^2(K_\ell)} = |J_\ell|\,\|\hat{v}_\ell\|^2_{L^2(\hat{K})}, \quad \text{and} \quad \|v\|^2_{L^2(F_f)} = |J_f|\,\|\hat{v}_f\|^2_{L^2(\hat{F})}. \tag{4.2}$$

Let $v\in\mathcal{P}_h H^1$. By changing the variable, $\mathbf{grad}\, v_{|K_\ell} = (\mathbb{B}_\ell{}^{-1})^T\,\mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{v}_\ell$, and it holds:

$$\begin{aligned} \|\,\mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{v}_\ell\|^2_{\mathbf{L}^2(\hat{K})} &\le \|\mathbb{B}_\ell\|^2\;|J_\ell|^{-1}\,\|\,\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)}, \\ &\lesssim \sigma^2\,(\rho_\ell)^{-(d-2)}\,\|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)}. \end{aligned} \tag{4.3}$$

Let us recall some useful inequalities that we will need:

- The Poincaré-Steklov inequality in cells [16, Lemma 12.11]:
for all $\ell\in\mathcal{I}_K$ (K_ℓ is a convex set), $\forall v\in H^1(K_\ell)$:

$$\|\underline{v}_\ell\|_{L^2(K_\ell)} \le \pi^{-1}h_\ell\|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)}, \quad \text{where } \underline{v}_\ell = v_{|K_\ell} - \frac{\int_{K_\ell} v}{|K_\ell|}. \tag{4.4}$$

- The multiplicative trace inequality as written in the proof of [16, lemma 12.15] for $p=2$: for all $\ell\in\mathcal{I}_K$, for all $f\in\mathcal{I}_{F,\ell}$, $\forall v\in H^1(K_\ell)$:

$$\|v\|^2_{L^2(F_f)} \le \frac{|F_f|}{|K_\ell|}\|v\|_{L^2(K_\ell)}\left(\|v\|_{L^2(K_\ell)} + \frac{2}{d}l^{\perp}_{(\ell,f)}\|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)}\right), \tag{4.5}$$

where $l^{\perp}_{(\ell,f)}$ is the largest length of an edge in K_ℓ and not belonging to F_f.
- Combining (4.4) and (4.5), we get that $\forall v\in H^1(K_\ell)$:

$$\|\underline{v}_\ell\|^2_{L^2(F_f)} \le \frac{|F_f|}{|K_\ell|}\pi^{-1}h_\ell\left(\pi^{-1}h_\ell + \frac{2}{d}l^{\perp}_{(\ell,f)}\right)\|\,\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)}. \tag{4.6}$$

Notice that in the reference element, inequality (4.6) reads:

$$\forall \ell \in \mathcal{I}_K,\, \forall f \in \mathcal{I}_{F,\ell}, \quad \|\widehat{(\underline{v}_\ell)}_f\|^2_{L^2(\hat{F})} \lesssim \|\,\mathbf{grad}_{\hat{\mathbf{x}}}\, \hat{v}_\ell\|^2_{\mathbf{L}^2(\hat{K})}. \tag{4.7}$$

For all $D \subset \mathbb{R}^d$, we call $P^k(D)$ the set of order k polynomials on D, $\mathbf{P}^k(D) = (P^k(D))^d$, and we consider the broken polynomial space:

$$P^k_{disc}(\mathcal{T}_h) = \left\{q \in L^2(\Omega); \quad \forall \ell \in \mathcal{I}_K,\, q_{|K_\ell} \in P^k(K_\ell)\right\}.$$

5. The nonconforming mixed finite element method for Stokes

The nonconforming finite element method was introduced by Crouzeix and Raviart in [6] to solve Stokes Problem (3.1). We approximate the vector space $\mathbf{H}^1(\Omega)$ component by component by piecewise polynomials of order $k \in \mathbb{N}^\star$. Let us consider X_h (resp. $X_{0,h}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order k:

$$\begin{aligned} X_h &= \left\{ v_h \in P^k_{disc}(\mathcal{T}_h)\,; \quad \forall f \in \mathcal{I}^i_F,\, \forall q_h \in P^{k-1}(F_f),\, \int_{F_f} [v_h]\, q_h = 0 \right\}; \\ X_{0,h} &= \left\{ v_h \in X_h\,; \quad \forall f \in \mathcal{I}^b_F,\, \forall q_h \in P^{k-1}(F_f),\, \int_{F_f} v_h\, q_h = 0 \right\}. \end{aligned} \tag{5.1}$$

The condition on the jumps of v_h on the inner facets is often called the patch-test condition.

Proposition 2. *The broken norm $v_h \to \|v_h\|_h$ is a norm over $X_{0,h}$.*

Proof. Let $v_h \in X_{0,h}$ such that $\|v_h\|_h = 0$. Then for all $\ell \in \mathcal{I}_K$, $v_{h|K_\ell}$ is a constant. For all $f \in \mathcal{I}^i_F$ the jump $[v_h]_{F_f}$ vanishes, so that v_h is a constant over Ω. We deduce from the discrete boundary condition that $v_h = 0$. □

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order k is $\mathbf{X}_h = (X_h)^d$ (resp. $\mathbf{X}_{0,h} = (X_{0,h})^d$). We set $\mathcal{X}_h := \mathbf{X}_{0,h} \times Q_h$ where $Q_h = P^{k-1}_{disc}(\mathcal{T}_h) \cap L^2_{zmv}(\Omega)$. We deduce from Proposition 2 the

Proposition 3. *The broken norm defined below is a norm on $\mathcal{X}_h$:*

$$\|(\cdot,\cdot)\|_{\mathcal{X}_h} : \left\{ \begin{array}{ccc} \mathcal{X}_h & \mapsto & \mathbb{R} \\ (\mathbf{v}_h, q_h) & \to & \|\mathbf{v}_h\|_h + \nu^{-1}\, \|q_h\|_{L^2(\Omega)} \end{array} \right. . \tag{5.2}$$

Thus, the product space $\mathcal{X}_h$ endowed with the broken norm $\|\cdot\|_{\mathcal{X}_h}$ is a Hilbert space.

Proposition 4. *The following discrete Poincaré–Steklov inequality holds: there exists a constant C^{nc}_{PS} independent of $\mathcal{T}_h$ such that*

$$\forall \mathbf{v}_h \in \mathbf{X}_{0,h}, \quad \|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \le C^{nc}_{PS}\, \|\mathbf{v}_h\|_h, \tag{5.3}$$

where C^{nc}_{PS} is independent of $\mathcal{T}_h$ and is proportional to the diameter of Ω.

Proof. Inequality (5.3) is stated in [8, Lemma 36.6] for $k = 1$, but one can check that the proof holds true for higher orders, thanks to the patch-test condition. An alternative proof is given in [18, Theorem C.1]. □

We consider the discrete continuous bilinear form $a_{S,h}(\cdot,\cdot)$ such that :

$$\left\{\begin{array}{rcl} a_{S,h}:\mathcal{X}_h\times\mathcal{X}_h & \to & \mathbb{R} \\ (\mathbf{u}_h',p_h')\times(\mathbf{v}_h,q_h) & \mapsto & \nu(\mathbf{u}_h',\mathbf{v}_h)_h-(p_h',\operatorname{div}_h\mathbf{v}_h)-(q_h,\operatorname{div}_h\mathbf{u}_h') \end{array}\right. .$$

Let $\ell_{\mathbf{f}}\in\mathcal{L}(\mathcal{X}_h,\mathbb{R})$ be such that :

$$\forall(\mathbf{v}_h,q_h)\in\mathcal{X}_h,\quad \ell_{\mathbf{f}}\,(\,(\mathbf{v}_h,q_h)\,)=\left\{\begin{array}{rl} (\mathbf{f},\mathbf{v}_h)_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f}\in\mathbf{L}^2(\Omega) \\ \langle\mathbf{f},\mathcal{I}_h(\mathbf{v}_h)\rangle_{\mathbf{H}_0^1(\Omega)} & \text{if } \mathbf{f}\notin\mathbf{L}^2(\Omega) \end{array}\right. ,$$

where $\mathcal{I}_h:\mathbf{X}_{0,h}\to\mathbf{Y}_{0,h}$, with $\mathbf{Y}_{0,h}=\{\mathbf{v}_h\in\mathbf{H}_0^1(\Omega)\,;\quad\forall\ell\in\mathcal{I}_K,\,\mathbf{v}_{h|K_\ell}\in\mathbf{P}^k(K_\ell)\}$, is the averaging operator described in [16, §22.4.1]. There exists a constant $C_{\mathcal{I}_h}^{nc}>0$ independent of $\mathcal{T}_h$ such that :

$$\|\mathcal{I}_h\mathbf{v}_h\|_{\mathbf{H}_0^1(\Omega)}\le C_{\mathcal{I}_h}^{nc}\,\|\mathbf{v}_h\|_h,\quad\forall\mathbf{v}_h\in\mathbf{X}_{0,h}. \tag{5.4}$$

The nonconforming discretization of Problem (3.9) reads:
Find $(\mathbf{u}_h,p_h)\in\mathcal{X}_h$ such that

$$a_{S,h}\,((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h))=\ell_{\mathbf{f}}\,(\,(\mathbf{v}_h,q_h)\,)\quad\forall(\mathbf{v}_h,q_h)\in\mathcal{X}_h. \tag{5.5}$$

Let us prove that Problem (5.5) is well-posed using the T-coercivity theory.

Theorem 4. *Suppose that there exists a Fortin operator $\Pi_{nc}:\mathbf{H}^1(\Omega)\to\mathbf{X}_h$ such that*

$$\exists C_{nc}\,|\,\forall\mathbf{v}\in\mathbf{H}^1(\Omega)\quad\|\Pi_{nc}\mathbf{v}\|_h\le C_{nc}\|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{5.6}$$

$$\forall\mathbf{v}\in\mathbf{H}^1(\Omega)\quad(\operatorname{div}_h\Pi_{nc}\mathbf{v},q_h)=(\operatorname{div}\mathbf{v},q_h)_{L^2(\Omega)},\quad\forall q\in Q_h, \tag{5.7}$$

where the constant C_{nc} does not depend on h. Then Problem (5.5) *is well-posed. Moreover, it admits one and only one solution $(\mathbf{u}_h,p_h)$ such that:*

$$\begin{array}{l} \text{if } \mathbf{f}\in\mathbf{L}^2(\Omega): \left\{\begin{array}{rcl} \|\mathbf{u}_h\|_h & \le & C_{PS}^{nc}\,\nu^{-1}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \\ \|p_h\|_{L^2(\Omega)} & \le & 2\,C_{PS}^{nc}\,C_{\operatorname{div}}^{nc}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \end{array}\right. , \\ \text{if } \mathbf{f}\notin\mathbf{L}^2(\Omega): \left\{\begin{array}{rcl} \|\mathbf{u}_h\|_h & \le & C_{\mathcal{I}_h}^{nc}\,\nu^{-1}\,\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \\ \|p_h\|_{L^2(\Omega)} & \le & 2\,C_{\mathcal{I}_h}^{nc}\,C_{\operatorname{div}}^{nc}\,\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \end{array}\right. , \end{array} \tag{5.8}$$

where $C_{\operatorname{div}}^{nc}=C_{\operatorname{div}}\,C_{nc}$. Additionally, we can compute classical a priori error estimates (see [6, Theorems 3, 4 and 6]*). Suppose that $(\mathbf{u},p)\in\mathbf{H}^{1+k}(\Omega)\times H^k(\Omega)$, we then have the estimate:*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}\le C\sigma^\ell\,h^{k+1}\,\left(|\mathbf{u}|_{\mathbf{H}^{k+1}(\Omega)}+\nu^{-1}\,|p|_{H^k(\Omega)}\right), \tag{5.9}$$

where the constant $C>0$ is independent of h, σ is the shape regularity constant and the exponent $\ell\in\mathbb{N}^\star$ depends on k.

Proof. Let us consider $(\mathbf{u}_h',p_h')\in\mathcal{X}_h$ and let us build $(\mathbf{v}_h^\star,q_h^\star)\in\mathcal{X}_h$ satisfying (2.3) (with $V=\mathcal{X}_h$). We follow the three main steps of the proof of Theorem 1.

1. According to Proposition 1, there exists $\mathbf{v}_{p_h'}\in\mathbf{V}^\perp$ such that $\operatorname{div}\mathbf{v}_{p_h'}=\nu^{-1}\,p_h'$ in Ω and:

$$\|\mathbf{v}_{p_h'}\|_{\mathbf{H}_0^1(\Omega)}^2\le\left(\frac{C_{\operatorname{div}}}{\nu}\right)^2\,\|p_h'\|_{L^2(\Omega)}^2.$$

Consider $\mathbf{v}_{h,p'_h} = \Pi_{nc}\mathbf{v}_{p'_h}$, for all $q_h \in Q_h$, we have: $(\mathrm{div}_h\,\mathbf{v}_{h,p'_h}, q_h) = \nu^{-1}\,(p'_h, q_h)_{L^2(\Omega)}$ and

$$\|\mathbf{v}_{h,p'_h}\|_h^2 \leq \left(\frac{C^{nc}_{\mathrm{div}}}{\nu}\right)^2 \|p'_h\|^2_{L^2(\Omega)} \text{ where } C^{nc}_{\mathrm{div}} = C_{nc}\,C_{\mathrm{div}}. \tag{5.10}$$

Let us set $(\mathbf{v}^\star_h, q^\star_h) := (\gamma_{nc}\mathbf{u}'_h - \mathbf{v}_{h,p'_h}, -\gamma_{nc}\,p'_h)$, with $\gamma_{nc} > 0$. We obtain:

$$a_{S,h}\,(\,(\mathbf{u}'_h, p'_h),\,(\mathbf{v}^\star_h, q^\star_h)\,) = \nu\,\gamma_{nc}\|\mathbf{u}'_h\|_h^2 + \nu^{-1}\|p'_h\|^2_{L^2(\Omega)} - \nu(\mathbf{u}'_h, \mathbf{v}_{h,p'_h})_h. \tag{5.11}$$

2. In order to bound the last term of (5.11), we use Young inequality and then inequality (5.10):

$$(\mathbf{u}'_h, \mathbf{v}_{h,p'_h})_h \leq \frac{\eta_{nc}}{2}\|\mathbf{u}'_h\|_h^2 + \frac{\eta_{nc}^{-1}}{2}\left(\frac{C^{nc}_{\mathrm{div}}}{\nu}\right)^2 \|p'_h\|^2_{L^2(\Omega)}. \tag{5.12}$$

3. Using the bound (5.12) in (5.11) and choosing $\eta_{nc} = \gamma_{nc}$, we get:

$$a_{S,h}\,(\,(\mathbf{u}'_h, p'_h),\,(\mathbf{v}^\star_h, q^\star_h)\,) \geq \frac{\gamma_{nc}}{2}\nu\,\|\mathbf{u}'_h\|_h^2 + \nu^{-1}\left(1 + \frac{(\gamma_{nc})^{-1}}{2}\,(C^{nc}_{\mathrm{div}})^2\right)\|p'_h\|^2_{L^2(\Omega)}$$

Consider now $\gamma_{nc} = (C^{nc}_{\mathrm{div}})^2$.
Noticing that $\nu\|\mathbf{u}_h\|_h^2 + \nu^{-1}\|p'_h\|^2_{L^2(\Omega)} \geq \dfrac{\nu}{2}\|(\mathbf{u}'_h, p'_h)\|^2_{\mathcal{X}_h}$, we obtain:

$$a_{S,h}\,(\,(\mathbf{u}'_h, p'_h),\,(\mathbf{v}^\star_h, q^\star_h)\,) \geq \frac{\nu}{2}C^{nc}_{\min}\,\|(\mathbf{u}'_h, p'_h)\|^2_{\mathcal{X}_h},$$

where $C^{nc}_{\min} = \dfrac{1}{2}\,\min(\,(C_{\mathrm{div}})^2, 1\,)$.
The operator T_h such that $T_h(\mathbf{u}'_h, p'_h) = (\mathbf{v}^\star_h, p^\star_h)$ is linear and continuous:

$$\|T_h(\mathbf{u}'_h, p'_h)\|_{\mathcal{X}_h} = \|\mathbf{v}^\star_h\|_h + \nu^{-1}\,\|q^\star_h\|_{L^2(\Omega)} \leq C^{nc}_{\max}\,\|(\mathbf{u}'_h, p'_h)\|^2_{\mathcal{X}_h}$$

where $C^{nc}_{\max} = C^{nc}_{\mathrm{div}}\,(C^{nc}_{\mathrm{div}} + 1)$. [4] The discrete continuous bilinear form $a_{S,h}(\cdot,\cdot)$ is then T_h-coercive and according to Theorem 1, Problem (5.5) is well posed. Consider $(\mathbf{u}_h, p_h)$ the unique solution of Problem (5.5). Choosing $\mathbf{v}_h = 0$, we obtain that $\mathrm{div}_h\,\mathbf{u}_h = 0$. Now, choosing $\mathbf{v}_h = \mathbf{u}_h$ in (5.5) and using Cauchy-Schwarz inequality, we get that:

$$\left\{\begin{array}{lcll} \|\mathbf{u}_h\|_h & \leq & \nu^{-1}\,C^{nc}_{PS}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega),\ \text{using (5.3)}\ ; \\ \|\mathbf{u}_h\|_h & \leq & \nu^{-1}\,C^{nc}_{\mathcal{I}_h}\,\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} & \text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega),\ \text{using (5.4).} \end{array}\right. \tag{5.13}$$

Consider $(\mathbf{v}_h, q_h) = (\mathbf{v}_{h,p_h}, 0)$ in (5.5), where $\mathbf{v}_{h,p_h} = \Pi_{nc}\mathbf{v}_{p_h}$ is built as $\mathbf{v}_{h,p'_h}$ in point 1, setting $p'_h = p_h$. Suppose that $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Using the triangular inequality, Cauchy-Schwarz inequality, Poincaré-Steklov inequality (5.3), Theorem 4, and estimate (5.13), we have:

$$\begin{array}{rcl} \|p_h\|^2_{L^2(\Omega)} & = & \nu\,(\mathbf{u}_h, \mathbf{v}_{h,p_h})_h - (\mathbf{f}, \mathbf{v}_{h,p_h})_{\mathbf{L}^2(\Omega)}\,, \\ & \leq & \nu\,\|\mathbf{u}_h\|_h\,\|\mathbf{v}_{h,p_h}\|_h + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|\mathbf{v}_{h,p_h}\|_{\mathbf{L}^2(\Omega)} \\ & \leq & 2\,C^{nc}_{PS}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|\mathbf{v}_{h,p_h}\|_h \leq 2\,C^{nc}_{PS}\,C_{nc}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|\,\mathbf{Grad}\,\mathbf{v}_{p_h}\|_{\mathbb{L}^2(\Omega)}\,, \\ & \leq & 2\,C^{nc}_{PS}\,C^{nc}_{\mathrm{div}}\,\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\,\|p_h\|_{L^2(\Omega)}. \end{array}$$

[4] Note that $(\mathbf{v}^\star_h, q^\star_h) = (\mathbf{0}, 0) \Leftrightarrow (\mathbf{u}'_h, p'_h) = (\mathbf{0}, 0)$, so that the operator $T_h \in \mathcal{L}(\mathcal{X}_h, \mathcal{X}_h)$ is bijective.

Applying the same reasoning when $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, we get that:

$$(5.14)\qquad \begin{cases} \|p_h\|_{L^2(\Omega)} \leq 2\, C_{PS}^{nc}\, C_{\mathrm{div}}^{nc}\, \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega), \text{ using (5.3) ;} \\ \|p_h\|_{L^2(\Omega)} \leq 2\, C_{\mathcal{I}_h}^{nc}\, C_{\mathrm{div}}^{nc} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} & \text{if } \mathbf{f} \notin \mathbf{L}^2(\Omega), \text{ using (5.4).} \end{cases}$$

The a priori error estimate corresponds to [6, Theorem 4]. □

Remark 2. *Again, we recover the first Banach–Nečas–Babuška condition* [8, Thm. 25.9, (BNB1)]*:*

$$a_{S,h}\ (\,(\mathbf{u}'_h, p'_h),\, (\mathbf{v}^\star_h, q^\star_h)\,) \geq \frac{\nu}{2}\,(C_{\max}^{nc})^{-1}\, \|(\mathbf{u}'_h, p'_h)\|_{\mathcal{X}_h}\, \|(\mathbf{v}^\star_h, q^\star_h)\|_{\mathcal{X}_h}.$$

Let us call $C_{\mathrm{stab}}^{nc} = \frac{\nu}{2}\, C_{\min}^{nc}\, (C_{\max}^{nc})^{-1}$ the stability constant. With the choice of our parameters, C_{stab}^{nc} is such that:

$$C_{\mathrm{stab}}^{nc} = \begin{cases} \dfrac{\nu}{4}\, \dfrac{C_{\mathrm{div}}^{nc}}{1 + C_{\mathrm{div}}^{nc}} & \text{if } 0 < C_{\mathrm{div}}^{nc} \leq 1, \\ \dfrac{\nu}{4}\, \dfrac{(C_{\mathrm{div}}^{nc})^{-1}}{1 + C_{\mathrm{div}}^{nc}} & \text{if } 1 \leq C_{\mathrm{div}}^{nc}. \end{cases}$$

The main issue with nonconforming mixed finite elements is the construction the basis functions. In a recent paper, Sauter explains such a construction in two dimensions [18, Corollary 2.4], and gives a bound to the discrete counterpart $\beta_{\mathcal{T}}(\Omega)$ of $\beta(\Omega)$ defined in (3.4):

$$(5.15)\qquad \beta_{\mathcal{T}}(\Omega) = \inf_{q_h \in Q_h \setminus \{0\}}\ \sup_{\mathbf{v}_h \in \mathbf{X}_{0,h} \setminus \{0\}} \frac{(\mathrm{div}_h\, \mathbf{v}_h)}{\|q_h\|_{L^2(\Omega)}\, \|\mathbf{v}_h\|_h} \geq c_{\mathcal{T}}\, k^{-\alpha}.$$

This bound is in $c_{\mathcal{T}}\, k^{-\alpha}$, where the parameter α is explicit and depends on k and on the mesh topology; and the constant $c_{\mathcal{T}}$ depends only on the shape-regularity of the mesh.

6. Nonconforming Crouzeix-Raviart mixed finite elements

We study the lowest order nonconforming Crouzeix-Raviart mixed finite elements [6]. Let us consider X_{CR} (resp. $X_{0,CR}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order 1:

$$(6.1)\qquad \begin{aligned} X_{CR} &= \left\{ v_h \in P^1_{disc}(\mathcal{T}_h)\,;\quad \forall f \in \mathcal{I}_F^i,\ \int_{F_f} [v_h] = 0 \right\}; \\ X_{0,CR} &= \left\{ v_h \in X_{CR}\,;\quad \forall f \in \mathcal{I}_F^b,\ \int_{F_f} v_h = 0 \right\}. \end{aligned}$$

The space of nonconforming approximation of of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order 1 is $\mathbf{X}_{CR} = (X_{CR})^d$ (resp. $\mathbf{X}_{0,CR} = (X_{0,CR})^d$). We set $\mathcal{X}_{CR} := \mathbf{X}_{0,CR} \times Q_{CR}$ where $Q_{CR} = P^0_{disc}(\mathcal{T}_h) \cap L^2_{zmv}(\Omega)$.
We can endow X_{CR} with the basis $(\psi_f)_{f \in \mathcal{I}_F}$ such that: $\forall \ell \in \mathcal{I}_K$,

$$\psi_{f|K_\ell} = \begin{cases} 1 - d\lambda_{i,\ell} & \text{if } f \in \mathcal{I}_{F,\ell}, \\ 0 & \text{otherwise,} \end{cases}$$

where S_i is the vertex opposite to F_f in K_ℓ. We then have $\psi_{f|F_f} = 1$, so that $[\psi_f]_{F_f} = 0$ if $f \in \mathcal{I}_F^i$ (i.e. $F_f \in \overset{\circ}{\Omega}$), and $\forall f' \neq f$, $\int_{F_{f'}} \psi_f = 0$.

We have: $X_{CR} = \mathrm{vect}\left((\psi_f)_{f\in\mathcal{I}_F}\right)$ and $X_{0,CR} = \mathrm{vect}\left((\psi_f)_{f\in\mathcal{I}_F^i}\right)$.

The Crouzeix-Raviart interpolation operator π_{CR} for scalar functions is defined by:

$$\pi_{CR} : \left\{ \begin{array}{rcl} H^1(\Omega) & \to & X_{CR} \\ v & \mapsto & \displaystyle\sum_{f\in\mathcal{I}_F} \pi_f v\, \psi_f \end{array} \right. , \text{ where } \pi_f v = \frac{1}{|F_f|}\int_{F_f} v.$$

Notice that $\forall f \in \mathcal{I}_F$, $\int_{F_f} \pi_{CR} v = \int_{F_f} v$. Moreover, the Crouzeix-Raviart interpolation operator preserves the constants, so that $\pi_{CR}\underline{v}_\Omega = \underline{v}_\Omega$ where $\underline{v}_\Omega = \int_\Omega v/|\Omega|$. We recall the following result [19, Lemma 2]):

Lemma 1. *The Crouzeix-Raviart interpolation operator π_{CR} is such that:*

$$\forall v \in H^1(\Omega), \quad \|\pi_{CR} v\|_h \leq \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)}. \tag{6.2}$$

Proof. We have, integrating by parts twice and using Cauchy-Schwarz inequality:

$$\begin{array}{rcl} \mathbf{grad}\, \pi_{CR} v_{|K_\ell} & = & |K_\ell|^{-1} \displaystyle\int_{K_\ell} \mathbf{grad}\, \pi_{CR} v = |K_\ell|^{-1} \sum_{f\in\mathcal{I}_{F,\ell}} \int_{F_f} \pi_{CR} v\, \mathbf{n}_f, \\ & = & |K_\ell|^{-1} \displaystyle\sum_{f\in\mathcal{I}_{F,\ell}} \int_{F_f} v\, \mathbf{n}_f = |K_\ell|^{-1} \int_{K_\ell} \mathbf{grad}\, v, \\ |\,\mathbf{grad}\, \pi_{CR} v_{|K_\ell}| & \leq & |K_\ell|^{-1/2} \|\,\mathbf{grad}\, v\|_{\mathbf{L}^2(K_\ell)} \\ \Rightarrow \quad \|\,\mathbf{grad}\, \pi_{CR} v\|^2_{\mathbf{L}^2(K_\ell)} & \leq & \|\,\mathbf{grad}\, v\|^2_{\mathbf{L}^2(K_\ell)}. \end{array}$$

Summing these local estimates over $\ell \in \mathcal{I}_K$, we obtain (6.2). □

For a vector $\mathbf{v} \in \mathbf{H}^1(\Omega)$ of components $(v_{d'})_{d'=1}^d$, the Crouzeix-Raviart interpolation operator is such that: $\Pi_{CR}\mathbf{v} = (\pi_{CR} v_{d'})_{d'=1}^d$. Let us set $\Pi_f \mathbf{v} = (\pi_f v_{d'})_{d'=1}^d$.

Lemma 2. *The Crouzeix-Raviart interpolation operator Π_{CR} can play the role of the Fortin operator:*

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \|\Pi_{CR}\mathbf{v}\|_h \leq \|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}, \tag{6.3}$$

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\mathrm{div}_h\, \Pi_{CR}\mathbf{v}, q_h) = (\mathrm{div}\,\mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q \in Q_h, \tag{6.4}$$

Moreover, for all $\mathbf{v} \in \mathbf{P}^1(\Omega)$, $\Pi_{CR}\mathbf{v} = \mathbf{v}$.

Proof. We obtain (6.3) applying Lemma (1) component by component. By integrating by parts, we have $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\forall \ell \in \mathcal{I}_K$:

$$\begin{array}{rcl} \displaystyle\int_{K_\ell} \mathrm{div}\, \Pi_{CR}\mathbf{v} & = & \displaystyle\sum_{f\in\mathcal{I}_{F,\ell}} \int_{F_f} \Pi_{CR}\mathbf{v}\cdot\mathbf{n}_f = \sum_{f\in\mathcal{I}_{F,\ell}} \int_{F_f} \Pi_f \mathbf{v}\cdot\mathbf{n}_f, \\ & = & \displaystyle\sum_{f\in\mathcal{I}_{F,\ell}} \int_{F_f} \mathbf{v}\cdot\mathbf{n}_f = \int_{K_\ell} \mathrm{div}\,\mathbf{v}, \end{array}$$

so that (6.4) is satisfied. □

We can apply the T-coercivity theory to show the next following result:

Theorem 5. *Let $\mathcal{X}_h = \mathcal{X}_{CR}$. Then the continuous bilinear form $a_{S,h}(\cdot,\cdot)$ is T_h-coercive and Problem* (5.5) *is well-posed.*

Proof. Using estimates (6.3) and (5.3), we apply the proof of Theorem 4. □

Since the constant of the interpolation operator Π_{CR} is equal to 1, we have $C_{min}^{CR} = C_{min}$ and $C_{max}^{CR} = C_{max}$: the stability constant of the nonconforming Crouzeix-Raviart mixed finite elements is independent of the mesh. This is not the case for higher order (see [20, Theorem 2.2]).

7. Fortin-Soulie mixed finite elements

We consider here the case $d = 2$ and we study Fortin-Soulie mixed finite elements [7]. We consider a shape-regular triangulation sequence $(\mathcal{T}_h)_h$.
Let us consider X_{FS} (resp. $X_{0,FS}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H_0^1(\Omega)$) of order 2:

$$X_{FS} = \left\{ v_h \in P^2_{disc}(\mathcal{T}_h)\,; \quad \forall f \in \mathcal{I}_F^i,\, \forall q_h \in P^1(F_f),\, \int_{F_f} [v_h]\, q_h = 0 \right\};$$

$$X_{0,FS} = \left\{ v_h \in X_{FS}\,; \quad \forall f \in \mathcal{I}_F^b,\, \forall q_h \in P^1(F_f),\, \int_{F_f} v_h\, q_h = 0 \right\}. \tag{7.1}$$

The space of nonconforming approximation of $\mathbf{H}^1(\Omega)$ (resp. $\mathbf{H}^1_0(\Omega)$) of order 2 is $\mathbf{X}_{FS} = (X_{FS})^2$ (resp. $\mathbf{X}_{0,FS} = (X_{0,FS})^2$). We set $\mathcal{X}_{FS} = \mathbf{X}_{0,FS} \times Q_{FS}$ where $Q_{FS} := P^1_{disc}(\mathcal{T}_h) \cap L^2_{zmv}(\Omega)$.
The building of a basis for $X_{0,FS}$ is more involved than for $X_{0,CR}$ since we cannot use two points per facet as degrees of freedom. Indeed, for all $\ell \in K_\ell$, there exists a polynomial of order 2 vanishing on the Gauss-Legendre points of the facets of the boundary ∂K_ℓ. Let $f \in \mathcal{I}_F$. The barycentric coordinates of the two Gauss-Legendre points $(p_{+,f}, p_{-,f})$ on F_f are such that:

$$p_{+,f} = (c_+, c_-),\ p_{-,f} = (c_-, c_+),\ \text{where } c_\pm = (1 \pm 1/\sqrt{3})/2.$$

These points can be used to integrate exactly order three polynomials:

$$\forall g \in P^3(F_f),\ \int_{F_f} g = \frac{|F_f|}{2}\left(g(p_{+,f}) + g(p_{-,f})\right).$$

For all $\ell \in \mathcal{I}_K$, we define the quadratic function ϕ_{K_ℓ} that vanishes on the six Gauss-Legendre points of the facets of K_ℓ (see Fig. 1):

$$\phi_{K_\ell} := 2 - 3 \sum_{i \in \mathcal{I}_{S,\ell}} \lambda_{i,\ell}^2 \text{ such that } \quad \forall f \in \mathcal{I}_{F,\ell},\, \forall q \in P^1(F), \quad \int_{F_f} \phi_{K_\ell}\, q = 0. \tag{7.2}$$

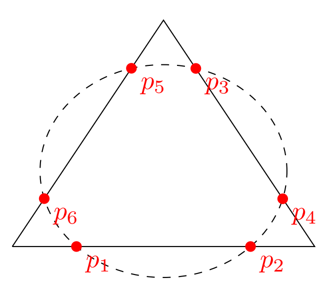

Figure 1. The six Gauss-Legendre points of an element K_ℓ and the elliptic function ϕ_{K_ℓ}.

We also define the spaces of P^2-Lagrange functions:

$$\begin{array}{rcl} X_{LG} & := & \left\{v_h \in H^1(\Omega); \quad \forall \ell \in \mathcal{I}_K,\, v_{h|K_\ell} \in P^2(K_\ell)\right\}, \\ X_{0,LG} & := & \left\{v_h \in X_{LG}; \quad v_{h|\partial\Omega} = 0\right\}. \end{array}$$

The Proposition below proved in [7, Prop. 1] allows to build a basis for $X_{0,FS}$:

Proposition 5. *We have the following decomposition:* $X_{FS} = X_{LG} + \Phi_h$ *with* $\dim(X_{LG} \cap \Phi_h) = 1$. *Any function of* X_{FS} *can be written as the sum of a function of* X_{LG} *and a function of* Φ_h. *This representation can be made unique by specifying one degree of freedom.*

Notice that $\Phi_h \cap X_{LG} = \mathrm{vect}(v_\Phi)$, where for all $\ell \in \mathcal{I}_K$, $v_{\Phi|K_\ell} = \phi_{K_\ell}$. Then, counting the degrees of freedom, one can show that $\dim(X_{FS}) = \dim(X_{LG}) + \dim(\Phi_h) + 1$. For problems involving Dirichlet boundary conditions we can prove thus that for $X_{0,FS}$ the representation is unique and $X_{0,FS} = X_{0,LG} \oplus \Phi_h$. We have $X_{LG} = \mathrm{vect}\left((\phi_{S_i})_{i\in\mathcal{I}_S}, (\phi_{F_f})_{f\in\mathcal{I}_F}\right)$ where the basis functions are such that: $\forall \ell \in \mathcal{I}_K$,

$$\begin{array}{rcl} \forall i \in \mathcal{I}_S, \quad \phi_{S_i|K_\ell} & = & \left\{\begin{array}{ll} \lambda_{i,\ell}\,(2\lambda_{j,\ell} - 1) & \text{if } i \in \mathcal{I}_{S,\ell} \\ 0 & \text{if } j \notin \mathcal{I}_{S,\ell} \end{array}\right. \\ \forall f \in \mathcal{I}_F, \quad \phi_{F_f|K_\ell} & = & \left\{\begin{array}{ll} 4\,\lambda_{i,\ell}\,\lambda_{j,\ell} & \text{if } f \in \mathcal{I}_{F,\ell}, \text{ and } F_f = S_iS_j \\ 0 & \text{if } f \notin \mathcal{I}_{F,\ell} \end{array}\right. \end{array}. \tag{7.3}$$

For all $\ell \in \mathcal{I}_K$, we will denote by $(\phi_{\ell,j})_{j=1}^6$ the local nodal basis such that:

$$(\phi_{\ell,j})_{j=1}^3 = (\phi_{S_i|K_\ell})_{i\in\mathcal{I}_{S,\ell}} \quad \text{and} \quad (\phi_{\ell,j})_{j=4}^6 = (\phi_{F_f|K_\ell})_{f\in\mathcal{I}_{F,\ell}}.$$

The spaces X_{FS} and $X_{0,FS}$ are such that:

$$\begin{array}{rcl} X_{FS} & = & \mathrm{vect}\left((\phi_{S_i})_{i\in\mathcal{I}_S}, (\phi_{F_f})_{f\in\mathcal{I}_F}, (\phi_{K_\ell})_{\ell\in\mathcal{I}_K}\right), \\ X_{0,FS} & = & \mathrm{vect}\left((\phi_{S_i})_{i\in\mathcal{I}_S^i}, (\phi_{F_f})_{f\in\mathcal{I}_F^i}, (\phi_{K_\ell})_{\ell\in\mathcal{I}_K}\right). \end{array} \tag{7.4}$$

We propose here an alternative definition of the Fortin interpolation operator proposed in [7]. Let us first recall the Scott-Zhang interpolation operator [21, 22]. For all $i \in \mathcal{I}_S$, we choose some $\ell_i \in \mathcal{I}_{K,i}$, and we build the $L^2(K_{\ell_i})$-dual basis $(\tilde\phi_{\ell_i,j})_{j=1}^6$ of the local nodal basis such that:

$$\forall j,\, j' \in \{1,\cdots,6\}, \quad \int_{K_{\ell_i}} \tilde\phi_{\ell_i,j}\,\phi_{\ell_i,j'} = \delta_{j,j'}.$$

Let us define the Fortin-Soulie interpolation operator for scalar functions by:

$$\begin{array}{c} \pi_{FS} : \left\{\begin{array}{rcl} \mathcal{P}_h H^1 & \to & X_{FS} \\ v & \mapsto & \tilde\pi v + \displaystyle\sum_{\ell\in\mathcal{I}_K} v_{K_\ell}\phi_{K_\ell} \end{array}\right., \\ \text{with} \quad \tilde\pi v = \displaystyle\sum_{i\in\mathcal{I}_S} v_{S_i}\phi_{S_i} + \sum_{f\in\mathcal{I}_F} \tilde v_{F_f}\phi_{F_f}. \end{array} \tag{7.5}$$

- The coefficients $(v_{S_i})_{i\in\mathcal{I}_S}$ are fixed so that: $\forall i \in \mathcal{I}_S$, $v_{S_i} = \displaystyle\int_{K_{\ell,i}} v\,\tilde{\phi}_{\ell_i,j_i}$, where j_i is the index such that $\displaystyle\int_{K_{\ell_i}} \tilde{\phi}_{\ell_i,j_i}\,\phi_{S_i|K_{\ell_i}} = 1$. Using Cauchy-Schwarz inequality and inequality (4.2), we have:

$$(7.6)\qquad |v_{S_i}| \le \left(\int_{K_{\ell_i}} \tilde{\phi}^2_{\ell_i,j_i}\right)^{1/2} \|v\|_{L^2(K_{\ell_i})} \lesssim |K_{\ell_i}|^{-1/2}\,\|v\|_{L^2(K_{\ell_i})} \lesssim \|\hat{v}_{\ell_i}\|_{L^2(\hat{K})}.$$

- The coefficients $\left(\tilde{v}_{F_f}\right)_{f\in\mathcal{I}_F}$ are fixed so that: $\forall f \in \mathcal{I}_F$, $\displaystyle\int_{F_f} \tilde{\pi}\tilde{v} = \int_{F_f} \{v\}$.

We then have:

$$(7.7)\qquad \tilde{v}_{F_f} = \frac{3}{2}\,v_{F_f} - \frac{1}{4}\sum_{i\in\mathcal{I}_{S,f}} v_{S_i},\ \text{ where } v_{F_f} := \frac{1}{|F_f|}\int_{F_f}\{v\}$$

For all $\ell \in \mathcal{I}_K$, the coefficient v_{K_ℓ} is fixed so that: $\displaystyle\int_{K_\ell} \pi_{FS} v = \int_{K_\ell} v$.
The definition (7.5) is more general than the one given in [7], which holds for $v \in H^2(\Omega)$.
We set $\mathbf{v}_{S_i} := (\,\tilde{\pi}v_1(S_i), \tilde{\pi}v_2(S_i))^T$ and $\tilde{\mathbf{v}}_{F_f} := (\,\tilde{\pi}v_1(F_f), \tilde{\pi}v_2(F_f)\,)^T$.
We now define the Fortin-Soulie interpolation operator for vector functions by:

$$\Pi_{FS} : \left\{ \begin{array}{rcl} \mathbf{H}^1(\Omega) & \to & \mathbf{X}_{FS} \\ \mathbf{v} & \mapsto & \displaystyle\sum_{i\in\mathcal{I}_S} \mathbf{v}_{S_i}\phi_{S_i} + \sum_{f\in\mathcal{I}_F} \tilde{\mathbf{v}}_{F_f}\phi_{F_f} + \sum_{\ell\in\mathcal{I}_K} \mathbf{v}_{K_\ell}\phi_{K_\ell}. \end{array} \right.$$

For all $\ell \in \mathcal{I}_K$, the vector coefficient $\mathbf{v}_{K_\ell} \in \mathbb{R}^2$ is now fixed so that condition (5.7) is satisfied. We can impose for example that the projection $\Pi_{FS}\mathbf{v}$ satisfies:

$$(7.8)\qquad \int_{K_\ell} T_\ell^{-1}(\mathbf{x})\ \mathrm{div}\,\Pi_{FS}\mathbf{v} = \int_{K_\ell} T_\ell^{-1}(\mathbf{x})\ \mathrm{div}\,\mathbf{v}.$$

Notice that due to (7.2), the patch-test condition is still satisfied. Moreover, one can check that for all $\mathbf{v} \in \mathbf{P}^2(\Omega)$, $\Pi_{FS}\mathbf{v} = \mathbf{v}$. In particular, if $\mathbf{v} \in \mathbf{P}^1(\Omega)$, we obtain that for all $\ell \in \mathcal{I}_K$, $\mathbf{v}_{K_\ell} = 0$. Using definitions (7.3) and (7.7), we obtain for all $\ell \in \mathcal{I}_K$:

$$(7.9)\qquad (\Pi_{FS}\mathbf{v})_{|K_\ell} = \sum_{i\in\mathcal{I}_{S,\ell}} \mathbf{v}_{S_i}\,\psi_{S_i} + \frac{3}{2}\sum_{f\in\mathcal{I}_{F,\ell}} \mathbf{v}_{F_f}\phi_{F_f} + \mathbf{v}_{K_\ell}\phi_{K_\ell},$$

where $\psi_{S_i|K_\ell} = 3\,\lambda^2_{i,\ell} - 2\,\lambda_{i,\ell}$.
Let us estimate $\mathbf{v}_{K_\ell}$. By changing the variable, setting $\hat{\mathbf{v}}_\ell(\hat{\mathbf{x}}) = \mathbf{v}\circ T_\ell(\hat{\mathbf{x}})$, the linear system (7.8) is written as follows, for $d' \in \{1,2\}$:

$$\begin{array}{rcl} (\mathbb{B}_\ell{}^{-1}\,\mathbf{v}_{K_\ell})\cdot \displaystyle\int_{\hat{K}} \hat{x}_{d'}\ \mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{\phi}_{K_\ell} & = & \displaystyle\int_{\hat{K}} \hat{x}_{d'}\ \mathrm{div}_{\hat{\mathbf{x}}}(\mathbb{B}_\ell{}^{-1}\hat{\mathbf{v}}_\ell) \\ & & \displaystyle -\sum_{i\in\mathcal{I}_{S,\ell}} (\mathbb{B}_\ell{}^{-1}\,\mathbf{v}_{S_i})\cdot\int_{\hat{K}} \hat{x}_{d'}\ \mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{\psi}_{S_i} \\ & & \displaystyle -\frac{3}{2}\sum_{f\in\mathcal{I}_{F,\ell}} (\mathbb{B}_\ell{}^{-1}\,\mathbf{v}_{F_f})\cdot\int_{\hat{K}} \hat{x}_{d'}\ \mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{\phi}_{F_f}. \end{array}$$

Noticing that $\int_{\hat{K}} \hat{x}_{d'}\ \mathbf{grad}_{\hat{\mathbf{x}}}\, \hat{\phi}_{K_\ell} = -\frac{1}{4}\mathbf{e}_{d'}$, we have:

$$\begin{aligned}\frac{1}{4}\mathbb{B}_\ell^{-1}\,\mathbf{v}_{K_\ell} \quad &= \quad \sum_{i\in\mathcal{I}_{S,\ell}} \int_{\hat{K}} \hat{\mathbf{x}}\,(\mathbb{B}_\ell^{-1}\,\mathbf{v}_{S_i})\cdot \mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{\psi}_{S_i} \\ &\quad + \frac{3}{2}\sum_{f\in\mathcal{I}_{F,\ell}} \int_{\hat{K}} \hat{\mathbf{x}}\,(\mathbb{B}_\ell^{-1}\,\mathbf{v}_{F_f})\cdot \mathbf{grad}_{\hat{\mathbf{x}}}\,\hat{\phi}_{F_f} \\ &\quad - \int_{\hat{K}} \hat{\mathbf{x}}\ \mathrm{div}_{\hat{\mathbf{x}}}(\mathbb{B}_\ell^{-1}\hat{\mathbf{v}}_\ell).\end{aligned} \tag{7.10}$$

Using integration formula (4.1), and Cauchy-Schwarz inequality to bound the last term of (7.10), we have:

$$|\mathbf{v}_{K_\ell}|^2 \lesssim \sigma^2 \left(\sum_{i\in\mathcal{I}_{S,\ell}} |\mathbf{v}_{S_i}|^2 + \sum_{f\in\mathcal{I}_{F,\ell}} |\mathbf{v}_{F_f}|^2 + \|\mathbf{Grad}_{\hat{\mathbf{x}}}\,\hat{\mathbf{v}}_\ell\|^2_{\mathbb{L}^2(\hat{K})} \right). \tag{7.11}$$

Proposition 6. *The Fortin-Soulie interpolation operator Π_{FS} is such that:*

$$\exists C_{FS}>0,\ \forall \mathbf{v}\in \mathbf{H}^1(\Omega), \quad \|\Pi_{FS}\mathbf{v}\|_h \le C_{FS}\|\,\mathbf{Grad}\,\mathbf{v}\|_{\mathbb{L}^2(\Omega)}. \tag{7.12}$$

Proof. Let $\mathbf{v}\in\mathbf{H}^1(\Omega)$. Let us set $\underline{\mathbf{v}}\in\mathcal{P}_h\mathbf{H}^1$ such that $\forall\ell\in\mathcal{I}_K$, $\underline{\mathbf{v}}_{|K_\ell} := \mathbf{v} - \int_{K_\ell}\mathbf{v}/|K_\ell|$. We have:

$$\|\Pi_{FS}\mathbf{v}\|_h^2 = \sum_{\ell\in\mathcal{I}_K} \|\mathbf{Grad}\,\Pi_{FS}\mathbf{v}\|^2_{\mathbb{L}^2(K_\ell)} = \sum_{\ell\in\mathcal{I}_K} \|\mathbf{Grad}\,\Pi_{FS}\underline{\mathbf{v}}\|^2_{\mathbb{L}^2(K_\ell)}.$$

For $\mathbf{a}$, $\mathbf{b}\in\mathbb{R}^2$, we set: $\mathbf{a}\otimes\mathbf{b} := (a_i\,b_j)_{i,j=1}^2 \in \mathbb{R}^{2\times 2}$. According to equation (7.9), we have:

$$\mathbf{Grad}\,\Pi_{FS}\underline{\mathbf{v}} = \sum_{i\in\mathcal{I}_{S,\ell}} \underline{\mathbf{v}}_{S_i}\otimes\mathbf{grad}\,\psi_{S_i} + \frac{3}{2}\sum_{f\in\mathcal{I}_{F,\ell}} \underline{\mathbf{v}}_{F_f}\otimes\mathbf{grad}\,\phi_{F_f} + \underline{\mathbf{v}}_{K_\ell}\otimes\mathbf{grad}\,\phi_{K_\ell}.$$

We can then make the estimate:

$$\begin{aligned}\|\,\mathbf{Grad}\,\Pi_{FS}\underline{\mathbf{v}}\|^2_{\mathbb{L}^2(K_\ell)} \quad &\lesssim \quad \sum_{i\in\mathcal{I}_{S,\ell}} |\underline{\mathbf{v}}_{S_i}|^2\,\|\,\mathbf{grad}\,\psi_{S_i}\|^2_{\mathbb{L}^2(K_\ell)} \\ &\quad + \sum_{f\in\mathcal{I}_{F,\ell}} |\tilde{\underline{\mathbf{v}}}_{F_f}|^2\,\|\,\mathbf{grad}\,\phi_{F_f}\|^2_{\mathbb{L}^2(K_\ell)} \\ &\quad + |\underline{\mathbf{v}}_{K_\ell}|^2\,\|\,\mathbf{grad}\,\phi_{K_\ell}\|^2_{\mathbb{L}^2(K_\ell)}, \\ &\lesssim \quad \left\|\mathbb{B}_\ell^{-1}\right\|^2\,|J_\ell| \left(\sum_{i\in\mathcal{I}_{S,\ell}} |\underline{\mathbf{v}}_{S_i}|^2 + \sum_{f\in\mathcal{I}_{F,\ell}} |\underline{\mathbf{v}}_{F_f}|^2 + |\underline{\mathbf{v}}_{K_\ell}|^2 \right).\end{aligned}$$

Thus, using estimate (7.11), and noticing that $\left\|\mathbb{B}_\ell^{-1}\right\|^2\,|J_\ell| \lesssim \sigma^2$, we have:

$$\begin{aligned}&\|\,\mathbf{Grad}\,\Pi_{FS}\underline{\mathbf{v}}\|^2_{\mathbb{L}^2(K_\ell)} \\ &\lesssim \sigma^4 \left(\sum_{i\in\mathcal{I}_{S,\ell}} |\underline{\mathbf{v}}_{S_i}|^2 + \sum_{f\in\mathcal{I}_{F,\ell}} |\underline{\mathbf{v}}_{F_f}|^2 + \|\,\mathbf{Grad}_{\hat{\mathbf{x}}}\,\hat{\mathbf{v}}_\ell\|^2_{\mathbb{L}^2(\hat{K})} \right).\end{aligned} \tag{7.13}$$

Using inequality (7.6) and Poincaré-Steklov inequality (4.4) in cell, component by component, we have:

$$|\underline{\mathbf{v}}_{S_i}|^2 \lesssim \| \mathbf{Grad}_{\hat{\mathbf{x}}}\, \hat{\mathbf{v}}_{\ell_i} \|^2_{\mathbb{L}^2(\hat{K})}. \tag{7.14}$$

Since the triangulation $\mathcal{T}_h$ is suppose to be shape-regular, there exists a constant $N_\theta \lesssim \sigma^2$ such that for all $i \in \mathcal{I}_{K,i}$, $N_i \le N_\theta$ [16, Rmk. 11.5]. We then obtain that:

$$\sum_{\ell \in \mathcal{I}_K} \sum_{i \in \mathcal{I}_{S,\ell}} \| \mathbf{Grad}_{\hat{\mathbf{x}}}\, \hat{\mathbf{v}}_{\ell_i} \|^2_{\mathbb{L}^2(\hat{K})} \lesssim N_\theta \sum_{\ell \in \mathcal{I}_K} \| \mathbf{Grad}_{\hat{\mathbf{x}}}\, \hat{\mathbf{v}}_{\ell} \|^2_{\mathbb{L}^2(\hat{K})}.$$

Thus, summing (7.14), we get:

$$\sum_{\ell \in \mathcal{I}_K} \sum_{i \in \mathcal{I}_{S,\ell}} |\underline{\mathbf{v}}_{S_i}|^2 \lesssim N_\theta \sum_{\ell \in \mathcal{I}_K} \| \mathbf{Grad}_{\hat{\mathbf{x}}}\, \hat{\mathbf{v}}_{\ell} \|^2_{\mathbb{L}^2(\hat{K})}. \tag{7.15}$$

Using the triangular inequality, equality (4.2), and inequality (4.7), we obtain:

$$|\underline{\mathbf{v}}_{F_f}|^2 \lesssim |F_f|^{-1} \sum_{\ell' \in \mathcal{I}_{K,f}} \|\underline{\mathbf{v}}_{\ell'|F_f}\|^2_{\mathbf{L}^2(F_f)} \lesssim \sum_{\ell' \in \mathcal{I}_{K,f}} \| \mathbf{Grad}_{\hat{\mathbf{x}}}\, \hat{\mathbf{v}}_{\ell'} \|^2_{\mathbb{L}^2(\hat{K})}.$$

Notice that: $\displaystyle\sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} \sum_{\ell' \in \mathcal{I}_{K,f}} \| \mathbf{Grad}_{\hat{\mathbf{x}}}\, \hat{\mathbf{v}}_{\ell'} \|^2_{\mathbb{L}^2(\hat{K})} \le 6 \sum_{\ell \in \mathcal{I}_K} \| \mathbf{Grad}_{\hat{\mathbf{x}}}\, \hat{\mathbf{v}}_{\ell} \|^2_{\mathbb{L}^2(\hat{K})}$, so that:

$$\sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} |F_f|^{-1} \|\underline{\mathbf{v}}_{\ell'|F_f}\|^2_{\mathbf{L}^2(F_f)} \lesssim \sum_{\ell \in \mathcal{I}_K} \| \mathbf{Grad}_{\hat{\mathbf{x}}}\, \hat{\mathbf{v}}_{\ell} \|^2_{\mathbb{L}^2(\hat{K})}. \tag{7.16}$$

Summing (7.13) over $\ell \in \mathcal{I}_K$, using (7.15) and (7.16), we get that:

$$\|\Pi_{FS}\mathbf{v}\|_h^2 \lesssim \sigma^2\, N_\theta \sum_{\ell \in \mathcal{I}_K} \| \mathbf{Grad}_{\hat{\mathbf{x}}}\, \hat{\mathbf{v}}_{\ell} \|^2_{\mathbb{L}^2(\hat{K})}, \tag{7.17}$$

Considering (4.3) for each component, and noticing that $\|\mathbb{B}_\ell\|^2\, |J_\ell|^{-1} \lesssim \sigma^2$, we obtain that:

$$\|\Pi_{FS}\mathbf{v}\|_h^2 \lesssim \sigma^4\, N_\theta \sum_{\ell \in \mathcal{I}_K} \|\mathbf{Grad}\, \mathbf{v}_\ell\|^2_{\mathbb{L}^2(K_\ell)}\,. \tag{7.18}$$

We obtain then (7.12) with $C_{FS} \approx \sigma^2\,(N_\theta)^{1/2}$. □

We recall that the discrete Poincaré–Steklov inequality (5.3) holds.

Theorem 6. *Let $\mathcal{X}_h = \mathcal{X}_{FS}$. Then the continuous bilinear form $a_{S,h}(\cdot,\cdot)$ is T_h-coercive and Problem* (5.5) *is well-posed.*

Proof. Apply the proof of Theorem 4. □

Notice that in the recent paper [23], the inf-sup condition of the mixed Fortin-Soulie finite element is proven directly on a triangle and then using the macro-element technique [24], but it seems difficult to use this technique to build a Fortin operator, which is needed to compute error estimates.
The study can be extended to higher orders for $d = 2$ using the following papers: [25] for $k \ge 4$, k even, [26] for $k = 3$ and [20] for $k \ge 5$, k odd. In [27], the authors propose a local Fortin operator for the lowest order Taylor-Hood finite element [14] for $d = 3$ which could be used to prove the T-coercivity.

8. Numerical results

Consider Problem (3.1) with data $\mathbf{f} = -\,\mathbf{grad}\,\phi$, where $\phi \in H^1(\Omega) \cap L^2_{zmv}(\Omega)$. The unique solution is then $(\mathbf{u}, p) := (0, \phi)$. By integrating by parts, the source term in (3.6) reads:

$$\forall \mathbf{v} \in \mathbf{H}^1_0(\Omega), \quad \int_\Omega \mathbf{f} \cdot \mathbf{v} = \int_\Omega \phi \,\mathrm{div}\,\mathbf{v}. \tag{8.1}$$

Recall that the nonconforming space $\mathbf{X}_h$ defined in (5.1) is a subset of $\mathcal{P}_h\mathbf{H}^1$: using a nonconforming finite element method, the integration by parts must be done on each element of the triangulation, and we have:

$$\forall \mathbf{v} \in \mathcal{P}_h\mathbf{H}^1, \quad \int_\Omega \mathbf{f} \cdot \mathbf{v} = (\mathrm{div}_h\,\mathbf{v}, \phi) + \sum_{f \in \mathcal{I}_F} \int_{F_f} [\mathbf{v} \cdot \mathbf{n}_f]\,\phi. \tag{8.2}$$

When we apply this result to the right-hand-side of (5.5), we notice that the term with the jumps acts as a numerical source, which numerical influence is proportional to $1/\nu$. Thus, we cannot obtain exactly $\mathbf{u}_h = 0$ (see also (5.9)). Linke proposed in [28] to project the test function $\mathbf{v}_h \in \mathbf{X}_h$ on a discrete subspace of $\mathbf{H}(\mathrm{div};\Omega)$, like Raviart-Thomas or Brezzi-Douglas-Marini finite elements (see [29, 30], or the monograph [15]). Let $\Pi_{\mathrm{div}} : \mathbf{X}_{0,h} \to P^k_{disc}(\mathcal{T}_h) \cap \mathbf{H}_0(\mathrm{div};\Omega)$ be some interpolation operator built so that for all $\mathbf{v}_h \in \mathbf{X}_{0,h}$, for all $\ell \in \mathcal{I}_K$, $(\mathrm{div}\,\Pi_{\mathrm{div}}\mathbf{v}_h)_{|K_\ell} = \mathrm{div}\,\mathbf{v}_{h|K_\ell}$. Integrating by parts, we have for all $\mathbf{v}_h \in \mathbf{X}_{0,h}$:

$$\begin{aligned}
\int_\Omega \mathbf{f} \cdot \Pi_{\mathrm{div}}\mathbf{v}_h &= \int_\Omega \phi \,\mathrm{div}\,\Pi_{\mathrm{div}}\mathbf{v}_h = \sum_{\ell \in K_\ell} \int_{K_\ell} \phi \,\mathrm{div}\,\Pi_{\mathrm{div}}\mathbf{v}_h, \\
&= \sum_{\ell \in K_\ell} \int_{K_\ell} \phi \,\mathrm{div}\,\mathbf{v}_h = (\mathrm{div}_h\,\mathbf{v}_h, \phi).
\end{aligned}$$

The projection Π_{div} allows to eliminate the terms of the integrals of the jumps in (8.2).
Let us write Problem (5.5) as:
Find $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$ such that

$$a_{S,h}\,((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \ell_{\mathbf{f}}\,(\,(\Pi_{\mathrm{div}}\mathbf{v}_h, q_h)\,) \quad \forall (\mathbf{v}_h, q_h) \in \mathcal{X}_h. \tag{8.3}$$

In the case of $\mathcal{X}_h = \mathcal{X}_{CR}$ and a projection on Brezzi-Douglas-Marini finite elements, the following error estimate holds if $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$:

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \le \widetilde{C}\, h^2\, |\mathbf{u}|_{\mathbf{H}^2(\Omega)}, \tag{8.4}$$

where the constant $\widetilde{C}$ if independent of h. The proof is detailed in [31] for shape-regular meshes and [32] for anisotropic meshes. We remark that the error doesn't depend on the norm of the pressure nor on the ν parameter. We will provide some numerical results to illustrate the effectiveness of this formulation, even with a projection on the Raviart-Thomas finite elements, which, for a fixed polynomial order, are less precise than the Brezzi-Douglas-Marini finite elements.
For all $\ell \in \mathcal{I}_K$, we let $P^k_H(K_\ell)$ be the set of homogeneous polynomials of order k on K_ℓ.
For $k \in \mathbb{N}^\star$, the space of Raviart-Thomas finite elements can be defined as:

$$\mathbf{X}_{RT_k} := \left\{ \mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega);\, \forall \ell \in \mathcal{I}_k,\, \mathbf{v}_{|K_\ell} = \mathbf{a}_\ell + b_\ell \mathbf{x} \,|\, (\mathbf{a}_\ell, b_\ell) \in P^k(K_\ell)^d \times P^k_H(K_\ell) \right\}.$$

Let $k \leq 1$.
The Raviart–Thomas interpolation operator $\Pi_{RT_k} : \mathbf{H}^1(\Omega) \cup \mathbf{X}_h \to \mathbf{X}_{RT_k}$ is defined by: $\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \cup \mathbf{X}_h$,

$$(8.5)\qquad \left\{ \begin{array}{ll} \forall f \in \mathcal{I}_F, & \displaystyle\int_{F_f} \Pi_{RT_k}\mathbf{v} \cdot \mathbf{n}_f\, q = \int_{F_f} \mathbf{v} \cdot \mathbf{n}_f\, q, \quad \forall q \in P^k(F_f) \\ \text{for } k=1,\ \forall \ell \in \mathcal{I}_K, & \displaystyle\int_{K_\ell} \Pi_{RT_1}\mathbf{v} = \int_{K_\ell} \mathbf{v} \end{array} \right. .$$

Note that the Raviart–Thomas interpolation operator preserves the constants. Let $\mathbf{v}_h \in \mathbf{X}_h$. In order to compute the left-hand-side of (8.2), we must evaluate $(\Pi_{RT_k}\mathbf{v}_h)_{|K_\ell}$ for all $\ell \in \mathcal{I}_K$. Calculations are performed using the proposition below, which corresponds to [33, Lemma 3.11]:

Proposition 7. *Let $k \leq 1$. Let $\hat{\Pi}_{RT_k} : \mathbf{H}^1(\hat{K}) \to \mathbf{P}^k(\hat{K})$ be the Raviart–Thomas interpolation operator restricted to the reference element, so that: $\forall \hat{\mathbf{v}} \in \mathbf{H}^1(\hat{K})$,*

$$(8.6)\qquad \left\{ \begin{array}{ll} \forall \hat{F} \in \partial\hat{K}, & \displaystyle\int_{\hat{F}} \hat{\Pi}_{RT_k}\hat{\mathbf{v}} \cdot \mathbf{n}_{\hat{F}}\, \hat{q} = \int_{\hat{F}} \hat{\mathbf{v}} \cdot \mathbf{n}_{\hat{F}}\, \hat{q}, \quad \forall \hat{q} \in P^k(\hat{F}) \\ \text{for } k=1, & \displaystyle\int_{\hat{K}} \hat{\Pi}_{RT_k}\hat{\mathbf{v}} = \int_{\hat{K}} \hat{\mathbf{v}} \end{array} \right. .$$

We then have: $\forall \ell \in \mathcal{I}_K$,

$$(8.7)\qquad (\Pi_{RT_k}\mathbf{v})_{|K_\ell}(\mathbf{x}) = \mathbb{B}_\ell \left(\hat{\Pi}_{RT_k} \mathbb{B}_\ell^{-1} \hat{\mathbf{v}}_\ell \right) \circ T_\ell^{-1}(\mathbf{x}) \;\; \textit{where} \;\; \hat{\mathbf{v}}_\ell = \mathbf{v} \circ T_\ell(\hat{\mathbf{x}}).$$

The proof is based on the equality of the $\hat{F}$ and $\hat{K}$-moments of $(\Pi_{RT_k}\mathbf{v})_{|K_\ell} \circ T_\ell(\hat{\mathbf{x}})$ and $\mathbb{B}_\ell \left(\hat{\Pi}_{RT_k} \mathbb{B}_\ell^{-1} \hat{\mathbf{v}}_\ell \right)(\hat{\mathbf{x}})$. For $k=0$, setting for $d' \in \{1, \cdots, d\}$: $\boldsymbol{\psi}_{f,d'} := \psi_f\, \mathbf{e}_{d'}$, we obtain that:

$$(8.8)\quad \forall \ell \in \mathcal{I}_K,\ \forall f \in \mathcal{I}_{F,\ell}, \quad (\Pi_{RT_0}\boldsymbol{\psi}_{f,d'})_{|K_\ell} = (d\,|K_\ell|)^{-1} \left(\mathbf{x} - \vec{OS}_{f,\ell} \right) \mathcal{S}_{f,\ell} \cdot \mathbf{e}_{d'}\,,$$

where $S_{f,\ell}$ is the vertex opposite to F_f in K_ℓ.
For $k=1$, the vector $(\Pi_{RT_1}\mathbf{v}_h)_{|K_\ell}$ is described by eight unknowns:

$$(\Pi_{RT_1}\mathbf{v}_h)_{|K_\ell} = \mathbb{A}_\ell\, \mathbf{x} + (\mathbf{b}_\ell \cdot \mathbf{x})\, \mathbf{x} + \mathbf{d}_\ell, \text{ where } \mathbb{A}_\ell \in \mathbb{R}^{2\times 2},\ \mathbf{b}_\ell \in \mathbb{R}^2,\ \mathbf{d}_\ell \in \mathbb{R}^2.$$

We compute only once the inverse of the matrix of the linear system (8.6), in $\mathbb{R}^{8\times 8}$.
In the Table 1 (resp. Tables 2 and 3), we call $\varepsilon_0(\mathbf{u}) = \|\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}$ (resp. $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} / \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}$) the velocity error in $\mathbf{L}^2(\Omega)$-norm, where $\mathbf{u}_h$ is the solution to Problem (5.5) (columns $\mathbf{X}_{CR}$ and $\mathbf{X}_{FS}$) or (8.3) (columns $\mathbf{X}_{CR} + \Pi_{RT_0}$ and $\mathbf{X}_{FS} + \Pi_{RT_1}$) and h is the mesh step.
We first consider Stokes Problem (3.1) in $\Omega = (0,1)^2$ with $\mathbf{u} = 0$, $p = (x_1)^3 + (x_2)^3 - 0.5$, $\mathbf{f} = \mathbf{grad}\, p = 3\, \left((x_1)^2, (x_2)^2 \right)^T$. We report in Table 1 $\varepsilon_0(\mathbf{u})$ for $h = 5.00\,e-2$ and for different values of ν.

ν	$\mathbf{X}_{CR}$	$\mathbf{X}_{CR} + \Pi_{RT_0}$	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS} + \Pi_{RT_1}$
$1.00\,e+0$	$3.19\,e-4$	$1.34\,e-18$	$3.53\,e-7$	$9.09\,e-19$
$1.00\,e-3$	$3.19\,e-1$	$1.34\,e-15$	$3.53\,e-4$	$9.09\,e-16$
$1.00\,e-4$	$3.19\,e+0$	$1.34\,e-14$	$3.53\,e-3$	$9.09\,e-15$

TABLE 1. Values of $\varepsilon_0(\mathbf{u})$ for $h = 5.00\,e-2$

When there is no projection, the error is inversely proportional to the ν parameter,

whereas using the projection, we obtain $\mathbf{u}_h = 0$ up to machine precision.
We now consider Stokes Problem (3.1) in $\Omega = (0,1)^2$ with:

$$\mathbf{u} = \begin{pmatrix} (1-\cos(2\,\pi\,x_1))\,\sin(2\,\pi\,x_2) \\ (\cos(2\,\pi\,x_2)-1)\,\sin(2\,\pi\,x_1) \end{pmatrix}, \quad \left\{ \begin{array}{rcl} p & = & \sin(2\,\pi\,x_1)\,\sin(2\,\pi\,x_2), \\ \mathbf{f} & = & -\nu\,\Delta\mathbf{u} + \mathbf{grad}\,p. \end{array} \right.$$

We report in Table 2 (resp. 3) the values of $\varepsilon_0(\mathbf{u})$ in the case $\nu = 1.00\,e-3$ (resp. $\nu = 1.00\,e-4$) for different level of mesh refinement. When there is no projection, $\varepsilon_0(\mathbf{u})$ is inversely proportional to ν, whereas using the projection, $\varepsilon_0(\mathbf{u})$ is independent of ν.

h	$\mathbf{X}_{CR}$	$\mathbf{X}_{CR} + \Pi_{RT_0}$	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS} + \Pi_{RT_1}$
$5.00\,e-2$	$5.66\,e-1$	$1.13\,e-2$	$2.35\,e-3$	$2.06\,e-4$
$2.50\,e-2$	$1.33\,e-1$	$2.89\,e-3$	$3.21\,e-4$	$2.59\,e-5$
$1.25\,e-2$	$3.88\,e-2$	$5.40\,e-4$	$4.20\,e-5$	$3.40\,e-6$
$6.25\,e-3$	$8.40\,e-3$	$1.79\,e-4$	$5.04\,e-6$	$4.15\,e-7$
Rate	$h^{2.05}$	$h^{2.07}$	$h^{2.96}$	$h^{2.98}$

TABLE 2. Values of $\varepsilon_0(\mathbf{u})$ for $\nu = 1.00\,e-3$

h	$\mathbf{X}_{CR}$	$\mathbf{X}_{CR} + \Pi_{RT_0}$	$\mathbf{X}_{FS}$	$\mathbf{X}_{FS} + \Pi_{RT_1}$
$5.00\,e-2$	$5.66\,e-0$	$1.13\,e-2$	$2.35\,e-2$	$2.06\,e-4$
$2.50\,e-2$	$1.33\,e-0$	$2.89\,e-3$	$3.20\,e-3$	$2.59\,e-5$
$1.25\,e-2$	$3.38\,e-1$	$5.40\,e-4$	$4.20\,e-4$	$3.40\,e-6$
$6.25\,e-3$	$8.40\,e-2$	$1.79\,e-4$	$5.04\,e-5$	$4.15\,e-7$
Rate	$h^{2.05}$	$h^{2.07}$	$h^{2.96}$	$h^{2.98}$

TABLE 3. Values of $\varepsilon_0(\mathbf{u})$ for $\nu = 1.00\,e-4$

Let $\mathbf{u}_{FS}$ (resp. $\mathbf{u}_{FS+RT_1}$) the solution to Problem (5.5) (resp. (8.3)) computed with Fortin-Soulie finite elements. We represent on Figure 2 the values of the Lagrange projection of $(\mathbf{u}_{FS} - \mathbf{u}_{FS+RT_1})$ in the case where $h = 2.50\,e-2$ and $\nu = 1.00\,e-4$. We observe local oscillations, of order the mesh step, which are caused by the numerical source exhibited in (8.2).
In order to enhance the numerical results, one can also use a posteriori error estimators to adapt the mesh (see [34, 35] for order 1 and [36] for order 2).
Alternatively, using the nonconforming Crouzeix-Raviart mixed finite element method, one can build a divergence-free basis, as described in [37]. Notice that using conforming finite elements, the Scott-Vogelius finite elements [38, 39] produce velocity approximations that are exactly divergence free.
The code used to get the numerical results can be downloaded on GitHub [40].

9. Conclusion

We analysed the discretization of Stokes problem with nonconforming finite elements in light of the T-coercivity theory, we computed stability coefficients for $k = 1$, $d = 2$ or 3 without regularity assumption; and for $k = 2$, $d = 2$ in the case of a shape-regular simplicial triangulation sequence. For $k = 2$, we used an alternative definition of the Fortin-Soulie interpolation operator. We then provided numerical results to illustrate the importance of using $\mathbf{H}(\mathrm{div})$-conforming projection. Further, we intend to extend the study to other mixed finite element methods.

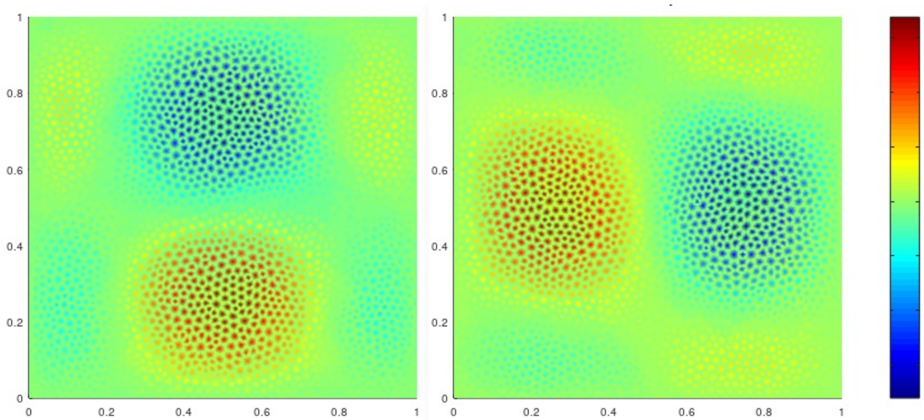

FIGURE 2. Values of $(\mathbf{u}_{FS} - \mathbf{u}_{FS+RT_1})$. Left: x_1-component, right: x_2-component.

Acknowledgements

The author acknowledges Mahran Rihani and Albéric Lefort.

References

[1] P. Ciarlet Jr. T-coercivity: Application to the discretization of Helmhotz-like problems. *Computers & Mathematics with Applications*, 64(1):22–24, 2012.

[2] E. Jamelot and P. Ciarlet, Jr. Fast non-overlapping Schwarz domain decomposition methods for solving the neutron diffusion equation. *Journal of Computational Physics*, 241:445–463, 2013.

[3] P. Ciarlet Jr., E. Jamelot, and F. D. Kpadonou. Domain decomposition methods for the diffusion equation with low-regularity solution. *Computers & Mathematics with Applications*, 74(10):2369–2384, 2017.

[4] L. Giret. *Non-Conforming Domain Decomposition for the Multigroup Neutron SPN Equation*. PhD thesis, Université Paris-Saclay, 2018.

[5] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*. Springer-Verlag, 1986.

[6] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *RAIRO, Sér. Anal. Numér.*, 7(3):33–75, 1973.

[7] M. Fortin and M. Soulie. A non-conforming piecewise quadratic finite element on triangles. *International Journal for Numerical Methods in Engineering*, 19(4):505–520, 1983.

[8] A. Ern and J.-L. Guermond. *Finite elements II*, volume 73 of *Texts in Applied Mathematics*. Springer, 2021.

[9] C. Bernardi, M. Costabel, M. Dauge, and V. Girault. Continuity properties of the inf-sup constant for the divergence. *SIAM J. Math. Anal.*, 48(2):1250–1271, 2016.

[10] D. Gallistl. Rayleigh-Ritz approximation of the inf-sup constant for the divergence. *Mathematics of Computation*, 88(315):73–89, 2019.

[11] M. Costabel and M. Dauge. On the inequalities of Babuška-Aziz, Friedrichs and Horgan-Payne. *Arch. Rational Mech. Anal.*, 217:873–898, 2015.

[12] M. Barré and P. Ciarlet Jr. T-coercivité et problèmes mixtes. working paper or preprint, October 2022.

[13] P. Ciarlet Jr. Méthodes variationnelles pour l'analyse de problèmes non coercifs, 2021. M.Sc. AMS lecture notes (ENSTA-IPP).

[14] C. Taylor and T. Hood. Numerical solution of the Navier-Stokes equations using the finite element technique. *Computers & Fluids*, 1:73–100, 1973.

[15] D. Boffi, F. Brezzi, and M. Fortin. *Mixed and hybrid finite element methods and applications*. Springer-Verlag, 2013.

[16] A. Ern and J.-L. Guermond. *Finite elements I*, volume 72 of *Texts in Applied Mathematics*. Springer, 2021.
[17] P. G. Ciarlet. The Effect of Numerical integration for Second-Order Problems. In *Finite Element Methods (Part 1)*, volume II of *Handbook of Numerical Analysis*. Elsevier, 1991.
[18] S. Sauter. The inf-sup constant for Crouzeix-Raviart triangular elements of any polynomial order, 2022.
[19] T. Apel, S. Nicaise, and J. Schöberl. Crouzeix-Raviart type finite elements on anisotropic meshes. *Numerische Mathematik*, 89(2):193–223, 2001.
[20] C. Carstensen and S. Sauter. Critical functions and inf-sup stability of Crouzeix-Raviart elements. *Computers and Mathematics with Applications*, 108:12–23, 2022.
[21] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54:483–493, 1990.
[22] P. Ciarlet Jr. Analysis of the Scott–Zhang interpolation in the fractional order Sobolev spaces. *J. Numer. Math.*, 21(3):173—-180, 2013.
[23] S. Sauter and C. Torres. On the Inf-Sup Stabillity of Crouzeix-Raviart Stokes Elements in 3D, 2022.
[24] R. Stenberg. Error analysis of some finite element methods for the stokes problem. *Math. Comp.*, 54:495–508, 1990.
[25] À. Baran and G. Stoyan. Gauss-Legendre elements: a stable, higher order non-conforming finite element family. *Computing*, 79(1):1–21, 2007.
[26] C. Carstensen and S. Sauter. Crouzeix-Raviart triangular elements are inf-sup stable, 2021. Preprint.
[27] L. Diening, J. Storn, and T. Tscherpel. Fortin operator for Taylor-Hood element. *Numerische Mathematik*, 150(2):671–689, 2022.
[28] A. Linke. On the Role of the Helmholtz-Decomposition in Mixed Methods for Incompressible Flows and a New Variational Crime. *Comput. Methods Appl. Mech. Engrg.*, 268:782–800, 2014.
[29] P.-A. Raviart and J.-M. Thomas. A mixed finite element method for second order elliptic problems. In *Mathematical aspects of finite element methods*, volume 606 of *Lecture Notes in Mathematics*, pages 292–315. Springer, 1977.
[30] F. Brezzi, J. Douglas, and L. D. Marini. Two families of mixed finite elements for second order elliptic problems. *Numerische Mathematik*, 47(2):217–235, 1985.
[31] C. Brennecke, A. Linke, C. Merdon, and J. Schöberl. Optimal and pressure independent L^2 velocity error estimates for a modified Crouzeix-Raviart Stokes element with BDM reconstructions. *Journal of Computational Mathematics*, 33(2):191–208, 2015.
[32] T. Apel, V. Kempf, A. Linke, and C. Merdon. A nonconforming pressure-robust finite element method for the Stokes equations on anisotropic meshes. *IMA Journal of Numerical Analysis*, 42(1):392–416, 2022.
[33] G. N. Gatica. *A Simple Introduction to the Mixed Finite Element Method: Theory and Applications*. SpringerBriefs in Mathematics. Springer, 2014.
[34] E. Dari, R. Durán, and C. Padra. Error estimators for nonconforming finite element approximations of the Stokes problem. *Mathematics of Computation*, 64(211):1017–1033, 1995.
[35] W. Dörfler and M. Ainsworth. Reliable a posteriori error control for nonconforming finite element approximation of Stokes flow. *Mathematics of Computation*, 74(252):1599–1619, 2005.
[36] M. Ainsworth, A. Allendes, G. R. Barrenechea, and R. Rankin. Computable error bounds for nonconforming Fortin–Soulie finite element approximation of the Stokes problem. *IMA Journal of Numerical Analysis*, 32(2):417–447, 2011.
[37] F. Hecht. Construction d'une base de fonctions P_1 non conforme à divergence nulle dans $\mathbb{R}^3$. *RAIRO, Sér. Anal. Numér.*, 15(2):119–150, 1981.
[38] L. R. Scott and M. Vogelius. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *RAIRO, Sér. Anal. Numér.*, 19(1):111–143, 1985.
[39] S. Zhang. A new family of stable mixed finite elements for the 3D Stokes equations. *Mathematics of Computation*, 74:543–554, 2005.
[40] E. Jamelot. `https://github.com/cea-trust-platform/Stokes_NCFEM`.