

From asymptotic to closed forms for the Keiper/Li approach to the Riemann Hypothesis

André Voros

▶ **To cite this version:**

André Voros. From asymptotic to closed forms for the Keiper/Li approach to the Riemann Hypothesis. 2022. cea-03673957

HAL Id: cea-03673957
https://cea.hal.science/cea-03673957

Preprint submitted on 20 May 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

From asymptotic to closed forms for the Keiper/Li approach to the Riemann Hypothesis

André Voros
Université Paris–Saclay, CNRS, CEA, Institut de Physique Théorique
91191 Gif-sur-Yvette, France
E-mail: andre.voros@ipht.fr
dedicated to Yoshitsugu TAKEI for his 60^{th} birthday

May 9, 2022

Abstract

The Riemann Hypothesis (RH) - that all nonreal zeros of Riemann's zeta function shall have real part 1/2 - remains a major open problem. Its most concrete equivalent is that an infinite sequence of real numbers, the Keiper–Li constants, shall be everywhere positive (Li's criterion). But those numbers are analytically elusive and strenuous to compute, hence we seek simpler variants. The essential sensitivity to RH of that sequence lies in its asymptotic tail; then, retaining this feature, we can modify the Keiper–Li scheme to obtain a new sequence in elementary closed form. This makes for a more explicit analysis, with easier and faster computations. We can moreover show how the new sequence will signal RH-violating zeros if any, by observing its analogs for the Davenport–Heilbronn counterexamples to RH.

It is a great honor and pleasure to dedicate this talk to Professor Yoshitsugu Takei, for his major contributions to exact asymptotic analysis throughout his career since the early 90's, [16] and surely for many more years to come. I am most grateful to the RIMS[1] and the Organizers for their invitation - and many past ones.

After a digression on why and how (precisely 40 years ago!) we met exact asymptotic analysis, which was to be the source of our durable link and friendship with Y. Takei (§ 1), we will mainly survey our recent work on the Keiper–Li approach to the Riemann Hypothesis - referring to [40] for any further detail. In § 2 we review the original but elusive Keiper–Li sequence, and then (§ 3) a discretization step (from derivatives to finite differences) which leads to a modified sequence *in elementary closed form*; its $n \to \infty$ behavior provides a new, very

[1]This workshop was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

concrete, asymptotic criterion for the Riemann Hypothesis. Finally, in § 4 we test that criterion in generalized form, transposed to the Davenport–Heilbronn functions which are counterexamples to the Riemann Hypothesis (with some new material here: more data and discussions).

Riemann's zeta function has the equivalent definitions, for Re $x > 1$: [30]

$$\zeta(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k^{-x} \equiv \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{1}{\mathrm{e}^u - 1}\, u^{x-1}\, \mathrm{d}u \tag{1}$$

$$\equiv \prod_{\{p\}} (1 - p^{-x})^{-1} \text{ over all the prime numbers } p; \tag{2}$$

the last form (Euler product) is just quoted to recall why ζ is a crucial function in number theory ($\log \zeta$ encodes the primes).

Standard actions upon the integral (Mellin) representation in (1) yield:
- that ζ is meromorphic in all of $\mathbb{C}$, with the only pole $\zeta(x) = \dfrac{1}{x-1} + \cdots$;
- the explicit values

$$\zeta(-n) = (-1)^n B_{n+1}/(n+1),\ n = 0, 1, 2, \ldots \quad (B_n : \text{ Bernoulli numbers}); \tag{3}$$

- and Riemann's Functional Equation, best written as

$$2\xi(x) \equiv 2\xi(1-x), \quad \text{where } 2\xi(x) \stackrel{\text{def}}{=} x(x-1)\pi^{-x/2}\Gamma(x/2)\,\zeta(x); \tag{4}$$

2ξ is dubbed "completed zeta function" (ξ is the classic choice, but 2ξ is better normalized for us, with $2\xi(0) = 2\xi(1) = 1$). The Functional Equation and (3) imply the further explicit values

$$\zeta(2m) = \frac{|B_{2m}|}{2(2m)!}\,(2\pi)^{2m} \iff 2\xi(2m) = \frac{|B_{2m}|}{(2m-3)!!}\,(2\pi)^m \quad (m = 1, 2, \ldots) \tag{5}$$

where $k!! = 2^{(k+1)/2}\,\Gamma(\frac{1}{2}k + 1)/\sqrt{\pi}$ for k odd (the usual double factorial).

1 Aside: our first use of an exact WKB method

Our strong link with Y. Takei and the Japanese school of complex analysis stems from the growth of an exact form of asymptotics around 1980, itself much inspired and encouraged by M. Sato, T. Kawai, M. Kashiwara. [27] (Other precursors included Leray, Boutet de Monvel–Krée, Bender–Wu, Dingle, Balian–Bloch, Sibuya, Zinn-Justin, as detailed in [37, § 1.2].) But since our main topic will only touch standard (as opposed to exact) asymptotics, it may be timely here to share our personal recollection of why we met exact asymptotics at all, as this was by quite an accidental circumstance, not much told, and moreover sharing the preliminaries (1)–(5) of our later main topic: we wanted to generalize an asymptotic form of Riemann's Functional Equation (4), and that needed an exact complex-WKB treatment!

A pending problem in the 70's was the 1D quantum *quartic* oscillator (the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}q^2 + q^4$, $q \in \mathbb{R}$) and specially its spectrum, only known to be discrete ($\{E_\ell\}_{\ell=0,1,\ldots}$; $E_\ell > 0$, $E_\ell \uparrow +\infty$) and to solve an asymptotic condition of the form

$$\sum_{n=0}^{\infty} b_n E_\ell^{\,\frac{3}{4}(1-2n)} \sim 2\pi(\ell + \tfrac{1}{2}) \quad \text{for integer } \ell \to +\infty \tag{6}$$

(Bohr–Sommerfeld), with $b_0 = \oint_{\{p^2+q^4=1\}} p\,\mathrm{d}q$ (> 0) : a classical-action period.

In 1979, following [22, § 7][25], we considered the *spectral zeta function*

$$Z(x) \stackrel{\text{def}}{=} \sum_\ell E_\ell^{-x} \qquad (\text{Re } x > \tfrac{3}{4}) : \tag{7}$$

a Dirichlet series like (1) for $\zeta(x)$, but with the *thoroughly unknown* eigenvalues E_ℓ in place of the integers k. Remarkably though, $Z(x)$ kept many (of the non-arithmetic) *explicit* properties present in Riemann's $\zeta(x)$: [31][32]
- a Mellin representation, implying that $Z(x)$ is meromorphic in all of $\mathbb{C}$ and all its polar singularities can be written out;
- explicit finite values: all $Z(-n)$, $(n = 0, 1, 2, \ldots)$, [25] plus $Z'(0)$ and $Z(1)$.

There is just *no functional equation* for $Z(x)$ to generalize Riemann's eq. (4) for $\zeta(x)$ (which links to the *harmonic* (q^2) oscillator, of spectral zeta function $(1 - 2^{-x})\zeta(x)$). But if we only watch $x \to \pm\infty$ asymptotics, then $\zeta(x) \sim 1$ for $x \to +\infty$ reducing (4) to: $\zeta(x) \sim \Gamma(1-x)(2\pi)^x \sin(\frac{1}{2}\pi x)/\pi$ for $x \to -\infty$. Now this remnant of (4), just an *explicit* $(x \to -\infty)$ asymptotic formula, may generalize to other Mellin transforms: such a function $\int \Theta(u) u^{x-1} \mathrm{d}u$ *potentially* has its $x \to -\infty$ behavior dictated, and thus described, by the *nearest singularities of* $\Theta(u)$ in $\mathbb{C}^*$ - i.e., *provided the latter are isolated and computable.* And all that worked for $Z(x)$, under the specific Mellin representation

$$Z\big(\tfrac{3}{4}x\big) = \frac{1}{\Gamma(x)} \int_0^\infty \Theta_{3/4}(u)\, u^{x-1} \mathrm{d}u, \quad \Theta_{3/4}(u) \stackrel{\text{def}}{=} \sum_\ell \exp(-E_\ell^{\,3/4} u) \ (\text{Re } u > 0), \tag{8}$$

because *this* function $\Theta_{3/4}$ had a curious (and novel, at the time) analytic structure depicted in [5, § 4], and here in Fig. 1, using a rescaled variable s: $\Theta_{3/4}$ was a *ramified* function, with branch points all on a square lattice, and up to the rescaling, its discontinuity functions were: at 0, a Borel transform of the Bohr–Sommerfeld series (6), and at other lattice points, Borel transforms of various *exponentials of that same Bohr–Sommerfeld series* (essentially). In particular that gave the nearest discontinuity functions, at $s = \frac{1}{2}(1 \pm \mathrm{i})$, to all orders, [5, eq.(4.12)] implying this asymptotic expansion for $Z(x)$, [31, § V][32, § 7]

$$Z(\tfrac{3}{4}x) \sim \frac{\sin \frac{3}{4}\pi x}{\cos \frac{1}{2}\pi x} \left(\frac{b_0}{\sqrt{2}}\right)^x \frac{x}{\pi} \left[\sum_{j=0}^{\infty} \alpha_j \Gamma(-x-j)\right] \qquad (x \to -\infty), \tag{9}$$

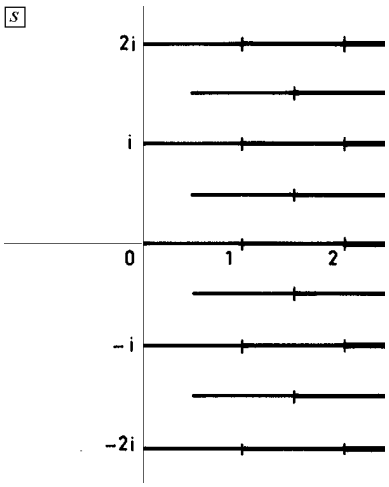


Figure 1: Cut plane (principal branch) for the multivalued function $\Theta_{3/4}(u)$, in the rescaled and negated variable $s = -b_0^{-1}u$. (Based on [5, Fig. 11].)

with $\{\alpha_j\}$ "bootstrapped" *in terms of* $\{b_n\}$ through the generating function

$$\sum_{j=0}^{\infty} \alpha_j \Big(\frac{v}{b_0}\Big)^j \equiv \exp\Big[\Big(\frac{b_1}{2}v + \frac{b_2}{2^2}v^3\Big) - \Big(\frac{b_3}{2^3}v^5 + \frac{b_4}{2^4}v^7\Big) + \cdots\Big] \quad (e.g.,\ \alpha_0 = 1). \tag{10}$$

Yet we noted that our crucial description of the function $\Theta_{3/4}$ (later understood as *resurgent* [13]) stayed somewhat empirical and incomplete. And however fast eq. (9) grew for $x \to -\infty$, it still eluded a standard asymptotic approach. So, to confirm (9)–(10) we needed tools able to *fully* describe $\Theta_{3/4}$. For that (encouraged also by Balian, Malgrange) we had to do WKB calculations with complex Planck's constant as in [4], using microfunction techniques [27] and a whole convolution algebra of Borel transforms; thus, from 1981 onwards [33, § 5.3][34][35][36] we ended up with *exact WKB results*... initially all for the sake of that $x \to -\infty$ behavior in the spectral zeta function $Z(x)$. (For which [36, p.281–286] fully analyzed the function $\Theta_{3/4}$ - named Z therein.)

As we also use (5) later, we mention that even without a functional equation for $Z(x)$, our further exact-WKB study of the potential q^4 extended the formulae (5) for $\zeta(2m)$, to explicit identities for $Z(3m)$, $m = 1, 2, \ldots$; e.g., $Z(3) = \frac{1}{6}Z(1)^3 - \frac{1}{2}Z(1)Z(2)$. [34] And similarly for higher-degree potentials q^{2M} $(M > 2)$, and for parity-twisted zeta functions $\sum_\ell (-1)^\ell E_\ell^{-x}$ as well; then, *one* such zeta-value exceptionally reduces almost as far as (5):

$$\sum_\ell (-1)^\ell E_\ell^{-2} = \tfrac{1}{128}\,[\pi\,\Gamma(\tfrac{1}{4})]^2/\Gamma(\tfrac{7}{8})^4 \quad \text{for the } \textit{sextic} \text{ potential } q^6 \tag{11}$$

by merging [34, eq. (16)] (exact-WKB) and [31, eq. (12)][32] (Weber–Schafheitlin) at $\mu \stackrel{\text{def}}{=} (2M+2)^{-1} = 1/8$.

Still, in our own work, for us several gaps *remain to be filled*: e.g., proof of full resurgence for the WKB solutions [18], validation of a very gentle behavior at infinity seen on Borel transforms [35, p.102–103], regularity and contractivity of an exact-quantization map beyond the case of homogeneous potentials which was settled by Avila [2] ([37, § 4.2][38, end §]), broader inclusion of nonpolynomial potentials [37, § 5.3]...

We conclude this digression by a salute to the impressive developments further carried out on exact WKB analysis in Japan, encompassing higher-order Ordinary Differential Equations, singular potentials, infinite-dimensional problems, ODE/IM correspondence, nonlinear problems (Painlevé), cluster algebras...

2 The Keiper–Li sequence

2.1 The Riemann zeros (basics) [30]

2.1.1 Known facts

By (4), ξ is a real entire function, with the two symmetry axes $\mathbb{R}$, and $L \stackrel{\text{def}}{=} \{\text{Re } x = \frac{1}{2}\}$ called the *critical line*.

The zeros of ξ or *Riemann zeros*, classically denoted ρ (and counted with multiplicities if any), all lie within the open strip $\{0 < \text{Re } x < 1\}$. They are infinitely many and their counting function $N(T)$, defined as the number of ρ in the rectangle $(0,1) \times (0, \mathrm{i}T)$, obeys the *Riemann–von Mangoldt* asymptotic law

$$N(T) = \frac{T}{2\pi}\Big(\log\frac{T}{2\pi} - 1\Big) + O(\log T) \qquad (T \to +\infty). \tag{12}$$

2.1.2 The Riemann Hypothesis (RH) (1859) [26]

All the zeros ρ of $\xi(x)$ lie on the critical line $\{\text{Re } x = \frac{1}{2}\}$.

This conjecture, most important for number theory (to understand the primes) has been neither proved nor disproved yet. On the other hand:
- Re $\rho \equiv \frac{1}{2}$ has been seen, and verified by computer, up to increasing ordinates $T = \text{Im } \rho$: since 2004, up to the 10^{13}-th zero ρ; [14] that sets the largest ordinate T_0 up to which RH is verified to a current value $\approx 2.4 \cdot 10^{12}$.
- numerous statements equivalent to RH, or criteria for RH, have been issued; many are highly abstract, but our focus will be on a specially concrete and simple-looking one.

2.2 The Keiper–Li tool to test RH

2.2.1 The Keiper vs Li sequences: generalities

Those sequences are defined: by the generating function (Keiper [17])

$$\sum_{n=1}^{\infty} \lambda_n^{\mathrm{K}} z^n \;\equiv\; \Phi(z) \stackrel{\text{def}}{=} \log 2\xi\Big(x = \frac{1}{1-z}\Big) \tag{13}$$

$$\iff \lambda_n^{\mathrm{K}} \;=\; \frac{1}{2\pi\mathrm{i}} \oint_C \frac{\mathrm{d}z}{z^{n+1}}\, \Phi(z), \quad C = \{|z| = \varepsilon \ll 1\} \text{ positively oriented;} \tag{14}$$

resp. by sums over all Riemann zeros grouped symmetrically (Li [20])

$$\lambda_n^{\mathrm{L}} \;\stackrel{\text{def}}{=}\; \sum_{\rho} [1 - (1 - 1/\rho)^n], \qquad n = 1, 2, \ldots \tag{15}$$

$$\equiv \;\sum_{\rho} [1 - \cos n\theta_\rho], \qquad \theta_\rho \stackrel{\text{def}}{=} -\mathrm{i} \log(1 - 1/\rho). \tag{16}$$

Both are denoted λ_n in the literature, but beware: $\lambda_n^{\mathrm{L}} \equiv n\lambda_n^{\mathrm{K}}$; so in way of a pun, Keiper's λ_n and Li's λ_n *differ by their common notation*. Neither normalization is nicer on all counts, so we rather keep both and use disambiguation superscripts K, L when the factor n matters.

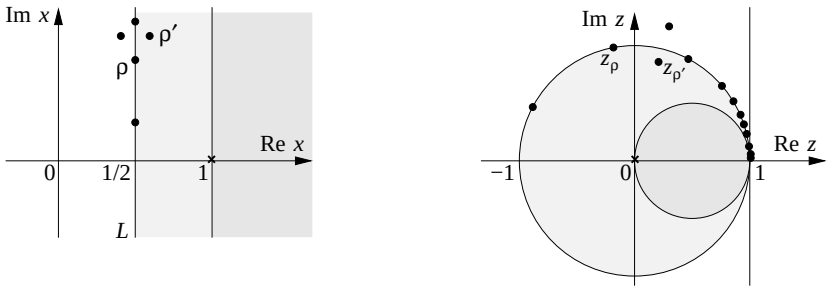


Figure 2: Riemann zeros (•) depicted in the x (left) and z (right) upper half-planes *schematically* (at mock locations, including a putative pair off the critical line L). The symmetrical zeros in the lower half-planes are not plotted. Domains are shaded only to mark which map to which.

A key element is the conformal mapping $x = 1/(1-z)$ in (13) which pulls back the half-plane $\{\mathrm{Re}\, x > \frac{1}{2}\}$ to the unit disk $\{|z| < 1\}$ (Fig. 2). This makes RH equivalent to: $\Phi(z)$ is analytic in *all of* that disk - that is why its Taylor coefficients λ_n are RH-sensitive. In quantitative terms, $\mathrm{d}\Phi/\mathrm{d}z$ is meromorphic with a simple pole of residue 1 at every preimage z_ρ of a zero ρ, and by (14),

$$\lambda_n^{\mathrm{L}} = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{\mathrm{d}z}{z^n} \frac{\mathrm{d}\Phi}{\mathrm{d}z}. \tag{17}$$

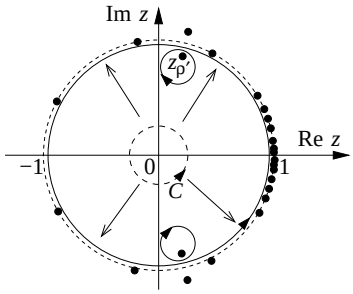


Figure 3: Contour deformation in Darboux's method for the asymptotics of (17).

Applying Darboux's method (i.e., the method of steepest descent in the variable $\log z$ [12, § 7.2]) we inflate C to $\{|z| = r\}$ with $r \uparrow 1^-$ (Fig. 3) and use the residue theorem to get the contributions from the poles $z_{\rho'}$ as

$$\lambda_n^{\mathrm{L}} = -\sum_{\{|z_{\rho'}|<1\}} z_{\rho'}^{-n} \quad + o(r^{-n})_{n\to\infty} \quad (\forall r < 1) \tag{18}$$

where we assign the notation ρ' to zeros (if any) having $\mathrm{Re}\ \rho' > \frac{1}{2}$ (in violation of RH, and amounting to $|z_{\rho'}| < 1$). If and only if RH is false, the sum in (18) is nonempty and then, ordered according to nondecreasing $|z_{\rho'}|$ it forms an asymptotic expansion in *exponentially growing* oscillations *about* 0.

2.2.2 Li's criterion for the Riemann Hypothesis

- If RH is false, the last sentence about (18) implies that $\lambda_n < 0$ *will* occur in the asymptotic regime $n \to \infty$.
- If RH is true, this amounts to all θ_ρ being real in the sums (16), which are therefore termwise *positive*, for all n. [17]

That pair of statements entails *Li's criterion*: [20]

$$\textbf{RH true} \iff \lambda_n > 0 \text{ for all } n.$$

However: [24]

$$\mathrm{Re}\ \rho \equiv \tfrac{1}{2} \text{ holds up to a height } T_0 \implies \lambda_n > 0 \text{ as long as } n < T_0^2.$$

This means that low values of n are actually inessential for Li's criterion: we may focus on the asymptotic $n \to \infty$ behavior of λ_n instead.

2.3 Asymptotic alternative for RH

The $n \to \infty$ form of λ_n is already fixed by the sum in (18) for RH false, but not so for RH true when that sum is empty. Instead, in the RH true case the sum (16) defining λ_n^{L} identifies with the Stieltjes integral $2\int_0^\infty (1 - \cos n\theta)\,\mathrm{d}N(T)$ where $T \equiv \frac{1}{2}\cot\frac{1}{2}\theta$; then, integration by parts gives

$$\lambda_n^{\mathrm{K}} = 2\int_0^\pi \sin n\theta\, N(\tfrac{1}{2}\cot\tfrac{1}{2}\theta)\,\mathrm{d}\theta. \tag{19}$$

The large-T law (12) now gives the $\theta \to 0$ form of the integrand in (19), which in turn converts to the large-n behavior of λ_n^{K} for RH true, as [24]

$$\lambda_n^{\mathrm{K}} = \tfrac{1}{2}[\log n + (\gamma - \log 2\pi - 1)] + o(1) \qquad (\gamma:\ \text{Euler's constant}), \tag{20}$$

giving a *tempered growth to* $+\infty$ (see [19][1] for stronger remainder estimates).

The asymptotic forms (18), (20) combine to give the ($n \to \infty$) *alternative* [39]

$$\lambda_n^{\mathrm{L}} \ \sim\ -\sum_{\{|z_{\rho'}|<1\}} z_{\rho'}^{-n} \qquad \textbf{if RH false} \tag{21}$$

$$\textbf{vs} \qquad \lambda_n^{\mathrm{L}} \ \sim\ \tfrac{1}{2} n\,[\log n + (\gamma - \log 2\pi - 1)] \qquad \textbf{if RH true.} \tag{22}$$

In practice, a term $z_{\rho'}^{-n}$ from (21) will compete in size with (22) if

$$n \gtrsim T^2/t \qquad (\text{for } \rho' = \tfrac{1}{2} + t \pm \mathrm{i}T, \quad t > 0). \tag{23}$$

This inequality (in order of magnitude) is also the *uncertainty principle* for the Fourier-conjugate variables θ and n in (19), which proves it a strict necessary condition as well. With $t < \frac{1}{2}$ and $T > T_0 \approx 2.4 \cdot 10^{12}$, (23) gives as concrete threshold: $n \gtrsim 10^{25}$, for λ_n to possibly sense violations of RH if any.

Unfortunately, the Keiper–Li numbers seem analytically quite challenging [7][10] and the complexity of their numerical evaluation steeply grows with n, mainly because it needs derivatives $(\log \xi)^{(n)}$ which are intricate to handle [17][21][9][15] ([15] reached $n = 10^5$). Currently, only the behavior (22) will show over the accessed ranges (Fig. 4 below), whereas values $n \gtrsim 10^{25}$ needed for new tests of RH appear way out of reach.

3 A closed-form variant of Keiper–Li [40]

The construction of the Keiper–Li sequence is not as inflexible as it may seem. Already, the unit disk of Fig. 2 can be remapped to itself by a (conformal) Möbius transformation, $z \mapsto z' = \frac{z - \tilde{z}}{1 - \tilde{z}^* z} \overset{\text{def}}{=} H_{\tilde{z}}(z)$ (for $|\tilde{z}| < 1$). Under the resulting composed map $x \mapsto z \mapsto z'$, $z' = 0$ can now correspond to an arbitrary point x_0 in the half-plane $\{\mathrm{Re}\ x > \frac{1}{2}\}$, and the transposition of (13) will define generalized Keiper–Li numbers in terms of $(\log \xi)$-derivatives now at $x = x_0$ instead of $x = 1$ [28] (which improves convergence if $\mathrm{Re}\ x_0 > 1$). But $\log \xi$ stays differentiated all the same, only elsewhere. To progress further, the deformation can be made more general.

3.1 Construction of an explicit sequence

The unwelcome differentiations on $\log \xi$ relate to the multiplicity of the pole $1/z^{n+1}$ of the integrand in (14). So we propose to split this pole into $n+1$ simple poles $1/z,\ 1/(z - z_1), \ldots 1/(z - z_n)$ by displacing each factor of z^{n+1} in (14)

differently. As previously we use hyperbolic displacements, i.e., Möbius transformations $H_{\tilde{z}}$, again to keep the unit disk invariant (thus preserving asymptotic sensitivity to RH). The integral form (14) for *Keiper's* λ_n^{K} thus gives rise to

$$\frac{1}{2\pi\mathrm{i}}\oint_C \frac{\mathrm{d}z}{z\,H_{z_1}(z)\cdots H_{z_n}(z)}\,\Phi(z) \equiv \sum_{m=1}^{n} \frac{1}{z_m[H_{z_1}(z)\cdots H_{z_n}(z)]'(z_m)}\,\Phi_m$$

by the elementary residue calculus for simple poles, with $\Phi_m \stackrel{\mathrm{def}}{=} \Phi(z_m)$: i.e., a *finite-difference* formula replaces a differential one. Finally specializing to $z_m = 1-(2m)^{-1}$, the above reduces to

$$\sum_{m=1}^{n}(-1)^m A_{nm}\Phi_m \stackrel{\mathrm{def}}{=} \Lambda_n, \qquad \text{with} \tag{24}$$

$$A_{nm} = \frac{2^{m-n}\,\big(2(n+m)-1\big)!!}{(2m-1)\,(n-m)!\,(2m)!}, \tag{25}$$

$$\text{and} \quad \Phi_m = \log 2\xi(2m) \equiv \log\left[\frac{|B_{2m}|}{(2m-3)!!}\,(2\pi)^m\right] \tag{26}$$

thanks to (5); thus, (24) specifies a deformed Keiper's sequence $\{\Lambda_n\}$ in elementary closed form. E.g., $\Lambda_1 = \frac{3}{2}\log\pi/3 \approx 0.0691764$, $\Lambda_2 = \frac{5}{24}\log[(2/5)^7 3^{11}/\pi^4] \approx 0.2274543$.

In summary: the original λ_n are elusive objects partly because the functions ζ hence ξ, $\log\xi$, have unwieldy derivatives. By discretizing the latter to finite differences, we inversely use the best in those same functions: their special values - countably many - upon which explicit finite differences can be built - to all orders, just as needed here.

3.2 Asymptotic alternative for RH with the sequence $\{\Lambda_n\}$

We only state our main result, in parallel to § 2.3: defining

$$F_n(x) \stackrel{\mathrm{def}}{=} (-1)^n\left[-\frac{1}{A_{n0}}\log(x-1) + \sum_{m=0}^{n}(-1)^m A_{nm}\log(x-2m)\right] \tag{27}$$

as single-valued in the cut plane $\mathbb{C}\setminus[0,2n]$, then for $n\to\infty$,

$$\Lambda_n \sim \sum_{\{\mathrm{Re}\ \rho'>1/2\}} F_n(\rho') \qquad \textbf{if RH false} \tag{28}$$

where for *each* $\rho' = \frac{1}{2}+t+\mathrm{i}T,\ t>0$ (with $\phi(\rho')$: a known phase function),

$$F_n(\rho') \sim \mathrm{e}^{\mathrm{i}\phi(\rho')}\,\frac{(-1)^n(2n)^{t+\mathrm{i}T}}{|T|^{2+t}\log n} \qquad \text{for } n \gg |T| \tag{29}$$

(*giving an oscillation of amplitude* $O(n^t/\log n)$ *about* 0);

$$\textbf{vs} \qquad \Lambda_n \sim \log n + \tfrac{1}{2}(\gamma - \log\pi - 1) \pmod{o(1)} \qquad \textbf{if RH true} \tag{30}$$

(*tempered growth to* $+\infty$).

(The derivations are similar to those sketched above for (21)–(22) but more elaborate; the explicit form (29) doesn't hold uniformly in $T = \mathrm{Im}\,\rho'$ hence cannot be substituted all at once into the full sum (28). [40, § 3])

In practice, a term from ρ' as in (29) will compete in size with (30) if

$$n \gtrsim T^{1+2/t} \qquad (\gtrsim T_0^5 \approx 10^{60} \text{ currently}): \text{ sufficient condition;} \tag{31}$$

now the uncertainty principle is much more favorable than either (31) or (23):

$$n \gtrsim \tfrac{1}{2} T\, \mathrm{e}^{1/t} \qquad (\gtrsim \tfrac{1}{2} T_0\, \mathrm{e}^2 \approx 10^{13} \text{ currently}): \text{ necessary condition.} \tag{32}$$

Still, current data (Figs. 4–5) will then only show the behavior (30).

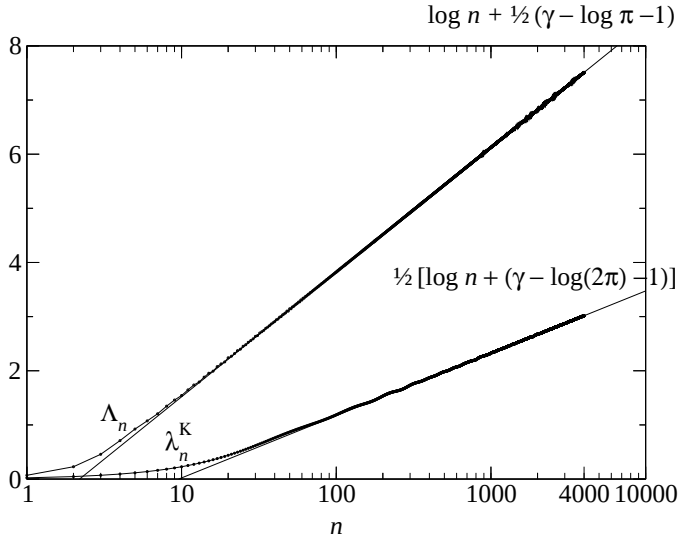


Figure 4: Upper plot: the sequence $\{\Lambda_n\}$ given by (24) displayed up to $n = 4000$ on a logarithmic n-scale (line segments connect data points only to aid the eye); straight line: the RH-true asymptotic form (30). Lower plot: the same for the original Keiper sequence (14) in comparison (data by courtesy of K. Maślanka [21]).

The two behaviors (28), (30) are mutually exclusive asymptotically ("alternative"), but numerically they superpose (they add): the form (30) sums the bulk effect of the zeros on the critical line, but the remainder therein, $\delta\Lambda_n = \Lambda_n - \log n - \frac{1}{2}(\gamma - \log\pi - 1)$ retains oscillations probably due to those zeros taken individually: e.g., the main oscillation is clearly synchronous with $(-1)^n n^\rho$, the same form as in (29) but for the first Riemann zero ($\rho = \frac{1}{2} + 14.134725\ldots\,\mathrm{i}$) (Fig. 5). Inversely, any growing oscillation (29) from an RH-violating zero ρ' will rise on top, not in place, of the smooth trend (30), see later counterexample to RH (Fig. 6).

3.3 Computational aspects

Calculations on Λ_n appear much simpler than for λ_n. A handful of command lines suffice in Mathematica (for instance) [41] to readily obtain values up to

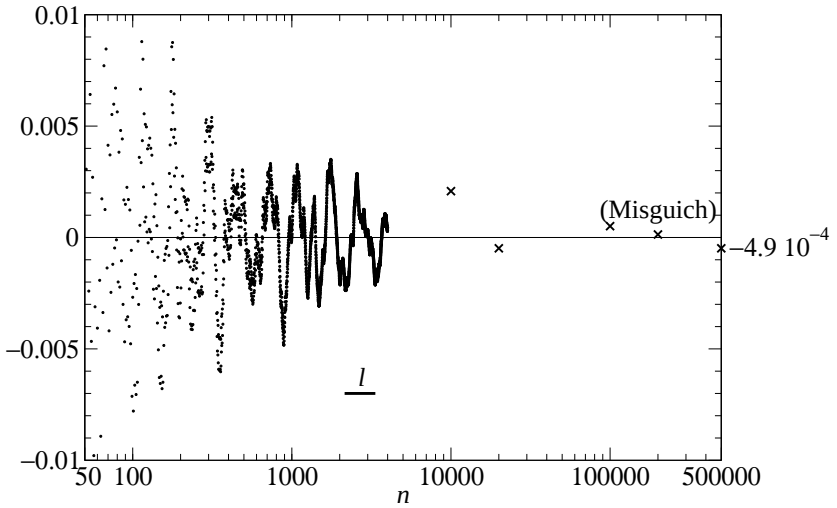


Figure 5: $(-1)^n$ times the remainder $\delta\Lambda_n$ in (30) ($= o(1)$ iff RH is true). This form oscillates with roughly the $(\log n)$-wavelength of n^ρ for the first Riemann zero $\rho = \frac{1}{2} + 14.134725\ldots$ i; this wavelength, $l = 2\pi/\operatorname{Im}\rho \approx 0.44452$, is depicted by a horizontal segment. Rightmost data: courtesy of G. Misguich (see § 3.3).

$n = 20,000$. G. Misguich has written a much faster parallel code for a 20-core machine, reaching $\Lambda_{500,000} \approx 12.33812102688$ in about 22 days. [23]

As a bonus, individual Λ_n can be accessed directly without a recursive build-up from $n = 1$ every time as with λ_n (evaluations of the Bernoulli numbers B_{2m} use recursion though, causing the major part of the workload).

An important fact is that the observed small Λ_n follow from huge cancellations between positive and negative terms in (24). Here, this too can be described explicitly: for $n \gg 1$ the Stirling formula shows $\max_m |A_{nm}|$ to grow like $\mathrm{e}^{(3+2\sqrt{2})n}$ (at $m \approx n/\sqrt{2}$). This means that $\approx 0.76555\,n$ decimal, or $2.5431\,n$ binary, leading significant digits have to cancel in the summation (24) to yield the final Λ_n. Arbitrary-precision computing is thus mandatory. This high instability in the specification of Λ_n (and already of λ_n, to a lesser degree) may be a price to pay for real-axis data that will signal phenomena located at very high imaginary parts.

3.4 Analytical questions

Fine-tuned as it is, the construction (24) still has residual flexibility: e.g., in [40, App.] we exhibit a more symmetrical - algebraically less elementary - variant (hence the plural in our main title). Inversely then, we may hope for simpler or better conditioned variants of (24) to emerge in some future.

The right-hand side of (28) only shows the start of a double expansion in integer powers of $1/\log n$ and complex powers of n, thus constituting a *transseries* in the variable $1/\log n$ - whose analytical properties wholly remain to be inves-

tigated.

The asymptotic alternative (28)–(30) entails an asymptotic Li-like criterion for the sequence $\{\Lambda_n\}$: RH true $\iff$ for some n_0, $\Lambda_n > 0$ $(\forall n > n_0)$. We cannot rule out $n_0 = 0$ (full Li's criterion), but we have no proof of this either.

4 The Davenport–Heilbronn counterexamples

They are "twisted zeta functions" defined by the Dirichlet series [11][6]

$$f_\pm(x) \stackrel{\text{def}}{=} \frac{1}{1^x} + \frac{\tau_\pm}{2^x} - \frac{\tau_\pm}{3^x} - \frac{1}{4^x} + \frac{0}{5^x} + \cdots \tag{33}$$

where $\tau_\pm = -\phi \pm \sqrt{1+\phi^2}$ with $\phi = \frac{1}{2}(1+\sqrt{5})$ (the golden ratio), and the numerators are repeated periodically mod 5.

For those specific $\tau_\pm$-values, $f_\pm(x)$ retain some properties like those of Riemann's $\zeta(x)$, e.g., countably many explicit values, and functional equations similar to (4). However, they lose the arithmetical properties of $\zeta(x)$ such as (2), and part of their (completed functions') zeros lie *off* the critical line L.

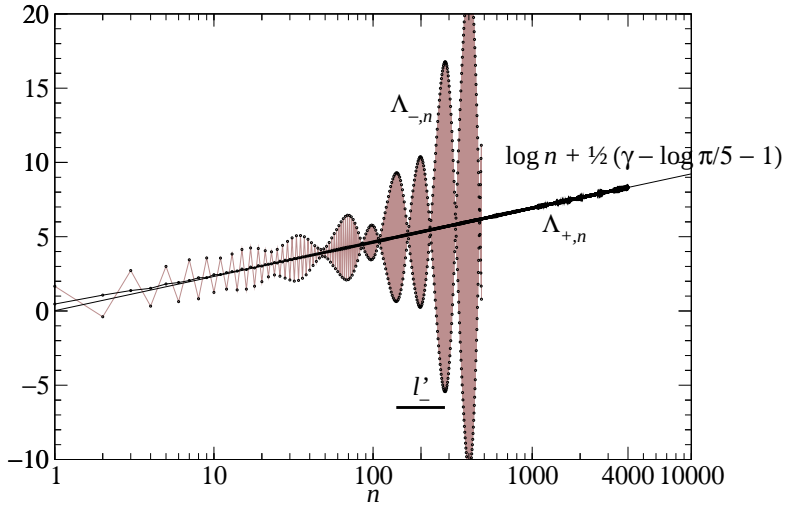


Figure 6: As Fig. 4, but for the two sequences $\{\Lambda_{\pm,n}\}$ associated with the Davenport–Heilbronn functions (33), stopping $\Lambda_{-,n}$ at $n = 480$ before severe overflow occurs; straight line: the generalized-RH-true asymptotic form (34). The horizontal segment depicts the $(\log n)$-wavelength l'_- of the leading oscillatory contribution $\propto n^{\rho'_- - 1/2}$ to $\Lambda_{-,n}$ (ignoring the factor $(-1)^n$ in (29)): $l'_- \approx 2\pi/8.91836 \approx 0.70452$.

As with $\zeta(x)$, Keiper–Li sequences $\{\lambda_{\pm,n}\}$ can be defined for $f_\pm$, and then, variants $\{\Lambda_{\pm,n}\}$ in elementary closed form as well (now using Bernoulli-polynomial values at 1/5 and 2/5). These sequences $\{\Lambda_{\pm,n}\}$ can then serve to numerically probe the RH-false branch (28) of our asymptotic alternative. [40, § 4.4]

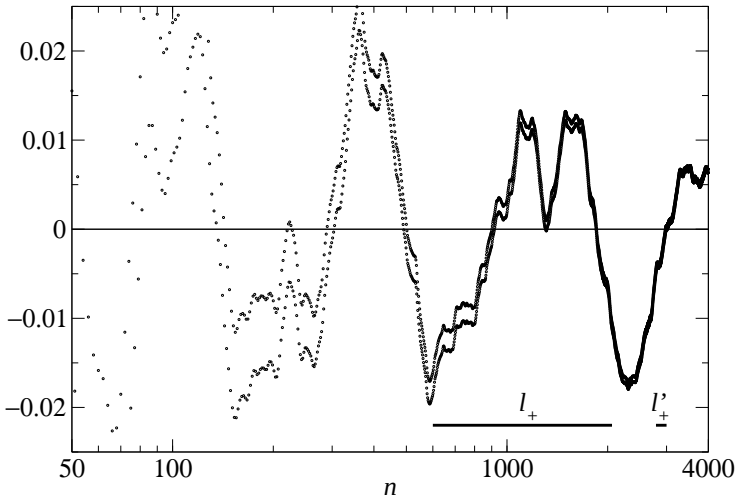


Figure 7: As Fig. 5, but for the remainder $\delta\Lambda_{+,n}$. The horizontal segments depict the $(\log n)$-wavelengths $l_+,\ l'_+$ of the expressions n^ρ for $\rho = \frac{1}{2} + 5.09416\,\mathrm{i}$ (the first zero of f_+, on L), and $\rho = \rho'_+$ (its first zero off L) respectively: $l_+ \approx 1.23341,\ l'_+ \approx 0.0733$.

The two functions $f_\pm$ yield contrasting numerical results (Fig. 6).

For (the completed function of) f_+, its lowest-T zero off the line L is $\rho'_+ \approx 0.808517 + 85.699348\,\mathrm{i}$ [29]. To detect it through the sequence $\{\Lambda_{+,n}\}$, (31) gives a threshold $n \approx T^{1+2/t} \approx (85.7)^{7.48} \approx 3 \cdot 10^{14}$: accordingly, in our computed range $\Lambda_{+,n}$ sticks to the RH-true prediction as generalized to this case,

$$\Lambda_{\pm,n} \approx \log n + \tfrac{1}{2}(\gamma - \log \pi/5 - 1). \tag{34}$$

Whereas for the case of f_-, the lowest-T zero off L is $\rho'_- \approx 2.30862 + 8.91836\,\mathrm{i}$ ([3], where our f_- is denoted f_2). To detect it through $\{\Lambda_{-,n}\}$, (31) now gives a threshold $n \approx T^{1+2/t} \approx (8.92)^{2.11} \approx 100$. Indeed, $\Lambda_{-,n}$ briefly starts along (34) on average, but the oscillating contribution like (29) from the zero ρ'_- quickly turns dominant. This actually models what one should see at much higher n for $\{\Lambda_{+,n}\}$, and at some still higher n for $\{\Lambda_n\}$ itself (the Riemann case) if RH is ultimately false.

On the other hand, f_+ is more suitable for practising to spot an early, hence weak, signal of RH-violation from ρ'_+ within the remainder $\delta\Lambda_{+,n}$ in (34). (With f_-, the signal from ρ'_- takes over too soon to allow that.) On f_+, the uncertainty-principle bound (32) for the first zero $\rho'_+ \notin L$ gives $n \gtrsim 1100$: that leaves a large n-interval $\approx [10^3, 3 \cdot 10^{14}]$ as training ground, to scan $(-1)^n \delta\Lambda_{+,n}$ for an oscillation $\propto n^{\rho'_+ - 1/2}/\log n$ (of $(\log n)$-wavelength: $l'_+ = 2\pi/\operatorname{Im} \rho'_+ \approx 0.0733$) (Fig. 7). At our highest data point $n = 4000$ its amplitude is $\approx 6 \cdot 10^{-5}$ by (29), still tiny, but it will grow like $n^{0.3085}/\log n$. Then, the faster the background part of $\delta\Lambda_{+,n}$ ($= o(1)$, due to zeros of f_+ on L) would decrease, the sooner that signal from ρ'_+ might stand out in the above interval. Such wishful

thinking suffices to suggest what to seek next: higher-n data for sure, but also stronger bounds on the remainder $\delta\Lambda_{+,n}$ ($\delta\Lambda_n$ for the Riemann case, just as [19] did with λ_n), and refined signal processings - the end goal being to most efficiently use the sequence $\{\Lambda_n\}$ itself for tests of RH.

References

[1] J. Arias de Reyna, *Asymptotics of Keiper–Li coefficients*, Funct. Approx. Comment. Math. **45** (2011) 7–21.

[2] A. Avila, *Convergence of an exact quantization scheme*, Commun. Math. Phys. **249** (2004) 305–318.

[3] E.P. Balanzario and J. Sánchez-Ortiz, *Zeros of the Davenport–Heilbronn counterexample*, Math. Comput. **76** (2007) 2045–2049.

[4] R. Balian and C. Bloch, *Solutions of the Schrödinger equation in terms of classical paths*, Ann. Phys. (NY) **85** (1974) 514–545.

[5] R. Balian, G. Parisi, A. Voros, *Quartic oscillator*, in: S. Albeverio et al. (eds.), *Feynman Path Integrals* [(Proceedings, Marseille 1978), Lecture Notes in Physics **106**, Springer, Berlin (1979) 337–360 (errata: [32, footnote p. 209] or [8, p. 203–204]).

[6] E. Bombieri and A. Ghosh, *Around the Davenport–Heilbronn function*, Uspekhi Mat. Nauk **66** (2011) 15–66, Russian Math. Surveys **66** (2011) 221–270.

[7] E. Bombieri and J.C. Lagarias, *Complements to Li's criterion for the Riemann Hypothesis*, J. Number Theory **77** (1999) 274–287.

[8] D. Chudnovsky and G. Chudnovsky (eds.), *The Riemann Problem, Complete Integrability and Arithmetic Applications* (Proceedings, IHÉS and Columbia University, 1979–1980), Lecture Notes in Mathematics **925**, Springer, Berlin (1982).

[9] M.W. Coffey, *Toward verification of the Riemann Hypothesis: application of the Li criterion*, Math. Phys. Anal. Geom. **8** (2005) 211–255.

[10] M.W. Coffey, *New results concerning power series expansions of the Riemann xi function and the Li/Keiper constants*, Proc. R. Soc. Lond. **A 464** (2008) 711–731.

[11] H. Davenport and H. Heilbronn, *On the zeros of certain Dirichlet series I, II*, J. London Math. Soc. **11** (1936) 181–185, 307–312.

[12] R.B. Dingle, *Asymptotic Expansions: their Derivation and Interpretation*, Academic Press, London (1973).

[13] J. Écalle, *Les fonctions résurgentes I*, Publications Mathématiques d'Orsay **81-05** (`http://sites.mathdoc.fr/PMO/PDF/E_ECALLE_81_05.pdf`) (and *II*, PMO **81-06** (1981); *III*, PMO **85-05** (1985)).

[14] X. Gourdon, *The 10^{13} first zeros of the Riemann Zeta function, and zeros computation at very large height*, preprint (Oct. 2004), `http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf`

[15] F. Johansson, *Rigorous high-precision computation of the Hurwitz zeta function and its derivatives*, Numer. Algor. **69** (2015) 253–270.

[16] T. Kawai and Y. Takei, *Algebraic Analysis of Singular Perturbation Theory*, Translations of Mathematical Monographs **227**, Amer. Math. Soc. (2005) [Japanese: Iwanami shoten (1998)], and refs. therein.

[17] J.B. Keiper, *Power series expansions of Riemann's ξ function*, Math. Comput. **58** (1992) 765–773.

[18] T. Koike and R. Schäfke, in preparation.

[19] J.C. Lagarias, *Li coefficients for automorphic L-functions*, Ann. Inst. Fourier, Grenoble **57** (2007) 1689–1740.

[20] X.-J. Li, *The positivity of a sequence of numbers and the Riemann Hypothesis*, J. Number Theory **65** (1997) 325–333.

[21] K. Maślanka, *Li's criterion for the Riemann hypothesis – numerical approach*, Opuscula Math. **24** (2004) 103–114.

[22] S. Minakshisundaram, Å. Pleijel, *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds*, Can. J. Math. **1** (1949) 242–256.

[23] G. Misguich, calculations for $n > 20000$, using `http://www.mpfr.org/` (private communications, 2017).

[24] J. Oesterlé, *Régions sans zéros de la fonction zêta de Riemann*, typescript (2000, revised 2001, uncirculated).

[25] G. Parisi, *Trace identities for the Schrödinger operator and the WKB method*, Preprint LPTENS 78/9 (École Normale Supérieure, Paris, March 1978), in [8, p.178–183].

[26] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsb. Preuss. Akad. Wiss. (Nov. 1859) 671–680; English translation, by R. Baker, Ch. Christenson and H. Orde: *Bernhard Riemann: Collected Papers*, paper VII, Kendrick Press, Heber City, UT (2004) 135–143.

[27] M. Sato, T. Kawai and M. Kashiwara, *Microfunctions and pseudo-differential equations*, in: H. Komatsu (ed.), *Hyperfunctions and pseudo-differential equations* (Proceedings, Katata 1971), Lecture Notes in Mathematics **287**, Springer, Berlin (1973) 265–529.

[28] S.K. Sekatskii, *Generalized Bombieri–Lagarias' theorem and generalized Li's criterion with its arithmetic interpretation*, Ukr. Mat. Zh. **66** (2014) 371–383, Ukr. Math. J. **66** (2014) 415–431.

[29] R. Spira, *Some zeros of the Titchmarsh counterexample*, Math. Comput. **63** (1994) 747–748.

[30] E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed. revised by D.R. Heath-Brown, Oxford Univ. Press, Oxford (1986).

[31] A. Voros, *Oscillateur quartique et méthodes semi-classiques*, in: *Séminaire Goulaouic–Schwartz 1979–1980*, § VI (nov. 1979) (`https://eudml.org/doc/111764` with errata in § "Notes").

[32] A. Voros, *The zeta function of the quartic oscillator*, Nucl. Phys. **B165** (1980) 209–236 (errata : `www.ipht.fr/Docspht//articles/t79/046/public/erratum.pdf`), augmented version: *Zeta functions of the quartic (and homogeneous anharmonic) oscillators*, in [8] p.184–208.

[33] A. Voros, *Spectre de l'équation de Schrödinger et méthode BKW*, Publications Mathématiques d'Orsay **81-09** (1981) (corrected copy: `https://www.ipht.fr/Docspht//articles/t81/120/public/81_120.pdf`).

[34] A. Voros, *Correspondance semi-classique et résultats exacts : cas des spectres d'opérateurs de Schrödinger homogènes*, C.R. Acad. Sci., Paris, Sér. I **293** (1981) 709–712, English translation: *Semiclassical correspondence and exact results: the case of the spectra of homogeneous Schrödinger operators*, J. Physique Lettres **43** (1982) L-1–L-4 (erratum: ibid. p.159).

[35] A. Voros, *Le problème spectral de Sturm–Liouville : le cas de l'oscillateur quartique*, (exposé no.602, Nov. 1982), in: Séminaire Bourbaki **25** (1982–1983), Astérisque **105–106**, Soc. Math. France (1983) 95–104 (`eudml.org/doc/110019` with reference updates in § "Notes").

[36] A. Voros, *The return of the quartic oscillator. The complex WKB method*, Ann. Inst. H. Poincaré **A39** (1983) 211–338 (`eudml.org/doc/76217` with errata in § "Notes").

[37] A. Voros, *The general 1D Schrödinger equation as an exactly solvable problem*, in: Y. Takei (ed.), *Recent trends in exponential asymptotics* (Proceedings, Kyoto 2004), RIMS Kôkyûroku **1424** (2005) 214–231.

[38] A. Voros, *From exact-WKB toward singular quantum perturbation theory II*, in: T. Aoki *et al.* (eds.), *Algebraic analysis of differential equations* (Festschrift in honor of T. Kawai, Kyoto 2005), Springer, Tokyo (2008) 321–334.

[39] A. Voros, *Sharpenings of Li's criterion for the Riemann Hypothesis*, Math. Phys. Anal. Geom. **9** (2006) 53–63 (erratum: our asymptotic forms for λ_n in the RH false case have wrong sign).

[40] A. Voros, *Discretized Keiper/Li approach to the Riemann Hypothesis*, Exp. Math. **29** (2020) 452–469 (publ. online Jul. 17, 2018).

[41] S. Wolfram, Mathematica, 3rd ed., Wolfram Media/Cambridge University Press, New York (1996).