Convergence analysis of the $Q1$ -finite element method for elliptic problems with non-boundary-fitted meshes

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SUMMARY

The aim of this paper is to derive *a priori* error estimates when the mesh does not fit the original domain's boundary. This problematic of the last century (e.g. the finite difference methodology) returns to topical studies with the huge development of domain embedding, fictitious domain or Cartesian-grid methods. These methods use regular structured meshes (most often Cartesian) for non-aligned domains. Although non-boundary fitted approaches become more and more applied, very few studies are devoted to theoretical error estimates.
In this paper, the convergence of a Q_1-nonconforming finite element method is analyzed for second-order elliptic problems with Dirichlet, Robin or Neumann boundary conditions. The finite element method uses standard Q_1 rectangular finite elements. As the finite element approximate space is not contained in the original solution space, this method is referred to as nonconforming. A stair-step boundary defined from the Cartesian mesh approximates the original domain's boundary. The convergence analysis of the finite element method for such a kind of non-boundary fitted stair-stepped approximation is no treated in the literature. The study of Dirichlet problems is based on similar techniques as those classically used with boundary-fitted linear triangular finite elements. The estimates obtained for Robin problems are novel and use some more technical arguments.
The rate of convergence is proved to be in $\mathcal{O}(h^{1/2})$ for the H^1 norm for all general boundary conditions, and classical duality arguments allow to obtain an $\mathcal{O}(h)$ error estimate in the L^2 norm for Dirichlet problems. Numerical results obtained with fictitious domain techniques, that impose original boundary conditions on a non-boundary fitted approximate immersed interface, are presented. These results confirm the theoretical rates of convergence. Copyright © 2007 John Wiley & Sons, Ltd.

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1. Introduction

We are interesting in solving second-order elliptic problems with general boundary conditions (Dirichlet, Robin or Neumann) using a structured Cartesian mesh. Due to the arbitrary shape of the original domain Ω, the mesh generally does not fit the original problem's boundary. The mesh is then *not boundary-fitted*. If the mesh step is h, an approximate domain Ω_h is thus defined such that $\text{meas}((\Omega \cup \Omega_h) \setminus (\Omega \cap \Omega_h)) = \mathcal{O}(h)$ (see Fig. 1). The boundary conditions are then imposed on the approximate stair-stepped boundary defined from the mesh.

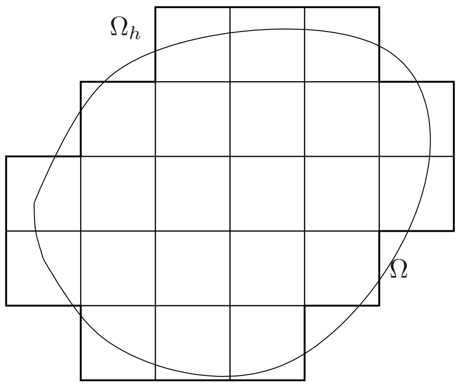


Figure 1. A non-boundary fitted mesh.

Although it may be surprising, to the best of our knowledges, error estimates for such a kind of problems have never been precisely studied in the literature. The existing results concern cases where the original boundary is at least piecewise linear approximated with P_1 finite elements (e.g. [1]). The stair-step case involves additional technical difficulties which are the purpose of the present paper. Moreover, homogeneous Dirichlet or Neumann boundary conditions are usually considered. Here, we generalize these boundary conditions with the innovative study of a Robin problem. In this case, we have to overcome the difficulties induced by the boundary integrals which are now performed on the stair-stepped boundary.

In the actual framework, this study is issued from fictitious domain methods where an approximate immersed interface is derived from the non-boundary fitted Cartesian fictitious domain mesh (see Ramière *et al.* [2, 3, 4, 5]). Indeed, in the fictitious domain context (see [6, 7]), the original domain Ω is embedded in a geometrically simpler domain (most often rectangular in 2D), called fictitious domain. The main interest of such a kind of methods is to use Cartesian meshes to facilitate the implementation of fast solvers. However, the original boundary conditions have to be enforced on an immersed interface which is non-aligned with the mesh. An approximate immersed interface has then to be defined. The analysis carried out in this paper can be considered as a subproblem of the convergence study of fictitious domain methods, where the approximate immersed interface does not fit the original immersed interface. In the present study, the immersed boundary conditions are supposed to be perfectly imposed on the approximate interface or, at least, the modelling error introduced by imposing the immersed boundary conditions is assumed to be negligible compared to the approximation error of the immersed interface.

The fictitious domain method was introduced in the sixties [6]. For several years now, fictitious domain methods have become increasingly popular. They have been applied in different fields such as computational fluid dynamics [8, 9] with the resolution of Navier-Stokes equations for viscous flows with rigid obstacles or particulate flows, medical simulation and biophysics [10] with problems of interaction fluid/structure specially for the study of the blood circulation, nuclear engineering [11] with the simulation of multi-phase flows, electromagnetic and geophysics [12] with problems of scattering by an obstacle, etc. Among the methods which deal with a fictitious domain approach, few studies are devoted to error estimates, especially for Neumann or Robin boundary conditions. Moreover, most of these works are carried out for a boundary-fitted approximation of the immersed interface.

One of the most popular approach to deal with embedded boundary conditions is to use boundary or distributed Lagrange multipliers (e.g. Glowinski *et al.* [13, 9]). For both approaches, to conserve the Cartesian fictitious domain mesh while correctly imposing the boundary conditions on the immersed interface, an independent additional mesh is introduced. In the boundary Lagrange multiplier approach a surface mesh of the immersed boundary is used whereas in the distributed approach a volume mesh is required on the complementary set of the original domain. Then the boundary conditions are weakly imposed. It involves the solution of a saddle-point problem which requires the inf-sup condition to be satisfied for the discrete subspaces. Some error analysis of the Lagrange multiplier/fictitious domain method can be found for Dirichlet boundary conditions in [14, 15, 16] for second-order elliptic problems. Using P_1 finite elements on a mesh independent of the original domain, Girault and Glowinski [14] prove that the boundary Lagrange multiplier approach converges in the H^1 norm with $h^{1/2-\epsilon}$, for all ϵ positive, in the fictitious domain. This non-optimal rate of convergence is due to the fictitious solution that may not belong to H^2 but only to $H^{3/2-\epsilon}$. Looking to the numerical error restricted to the original domain, an optimal estimate in $\mathcal{O}(h)$ is obtained. The estimates in the original domain seem not to depend on the regularity of the fictitious problem solution over the whole fictitious domain. The main point of the analysis led in [14] is the proof of the inf-sup condition that holds under a compatibility condition between the surface mesh and the fictitious domain mesh: the step size of the surface mesh has to be sufficiently large compared to the Cartesian grid step. For the fictitious domain method with volume or distributed Lagrange multipliers, Tomas [15] proves an error estimate in $\mathcal{O}(h^{1/2-\epsilon})$ for the H^1 norm when the mesh does not match the boundary of Ω. Maitre and Tomas [16] use Raviart-Thomas mixed finite elements and obtain an optimal convergence in $\mathcal{O}(h)$ for the $H(div)$ norm on a locally adapted mesh.

Another analysis of a Dirichlet fictitious problem with a non-smooth right-hand side can be found in [17]. The fictitious domain is discretized triangular finite elements that do not fit the immersed interface. It is proved that one may expect at least $\mathcal{O}(h^{1/2})$ precision in the L^2 norm. In the same meaning, with regard to a fluid-rigid system (Navier-Stokes equations for the fluid motion and Newton's laws for the rigid motion), San Martin *et al.* [18] obtains $h^{1/2}$-estimates in the H^1 norm and by applying the usual Aubin-Nitsche duality argument, h-estimates in the L^2 norm.

Another main class of fictitious domain methods derives from the immersed boundary method (IBM) of Peskin [19] (see an overview in [10]). The main idea of the IBM is to use a discrete delta function to regularize and to transfer to the Cartesian grid the forces exerted on the interface. The method is first-order accurate due to the smearing of the boundary layers by the discrete delta function. However, for some one dimensional problems, the use of a non smooth discrete delta function [20] gives second-order-accurate solutions. The immersed interface method (IIM) [21] (see an overview in [22]) is an extension of the IBM for problems with discontinuous coefficients and solutions across the immersed boundary. A finite difference scheme, with a modified stencil near the immersed boundary and additional unknowns on the immersed boundary, is derived to account for the immersed jumps. Then, the method is proved to be formally second order accurate in the L^2 norm for Dirichlet problems [23, 24, 25].

In the Fat Boundary Method recently developed by Maury [26], two subproblems are solved: a global problem on the fictitious domain and a local subproblem in a narrow domain around the immersed boundary. For a Dirichlet problem, Bertoluzza *al.* [27] prove that an optimal rate of convergence in the H^1 norm is obtained locally in a domain strictly included in Ω. A convergence study of a homogeneous Neumann problem is made in [28]: the strong convergence of the approximate solution to the analytical solution is demonstrated for the H^1 norm.

Glowinski *et al.*[29] introduce regularization techniques in the form of penalty parameters for a Neumann problem and this approach is of order h for the H^1 norm and of order h^2 for the L^2 norm. Another study of convergence for a fictitious domain approach dedicated to Neumann problems can be found in [30]. This approach deals with a difference scheme. When the solution of the original problem belongs to the space $W^{k,2}, k = 3, 4$, the order of accuracy of the approximate solution is $\mathcal{O}(h^{(k-2)/3})$ in the H^1(or $W^{1,2}$) norm.

In this paper, rectangular elements discretize the approximate domain Ω_h. A Lagrangian Q_1 finite element method (see for example [1]) is used. As Ω_h is not included in Ω, the finite element space V_h is not a subspace of the Hilbert space V to which the original solution belongs. Thus, we have a Q_1-nonconforming finite element discretization. For each kind of boundary conditions (Dirichlet, Robin, Neumann), the error estimates are proved to be of $\mathcal{O}(h^{1/2})$ for the H^1 norm. For Robin or Neumann boundary conditions, a local correction is devised to take account of the relative surface ratio which allows the convergence order to be preserved. Moreover, for Dirichlet boundary conditions, duality arguments enable us to prove the $\mathcal{O}(h)$ accuracy for the L^2 norm by using the Aubin-Nitsche theorem. These estimates confirm the order of convergence that is "naturally" expected and are consistent with the years of experience in finite element methods. However these estimates, specially for Robin problems, have never been theoretically demonstrated before.

The decay of the order of convergence (first-order accuracy for the L^2 norm compared to second-order accuracy for boundary-fitted methods), due to the poor approximation of the boundary by stair-steps, is compensated for the use of a simple structured Cartesian mesh. The cost of the mesh generation is then significantly reduced compared to the use of boundary conforming meshes. As fast solvers and efficient preconditioners carried out the resolution of the problem, this kind of approach offers a good ratio of the precision of the approximate solution over the CPU time. In practical computations, specially in a fictitious domain context, this ratio can be more reduced by the combination of this approach with a multigrid [31] process using local nested structured meshes in the vicinity of the stair-case boundary, see some recent developments in this way in [2, 5]. This leads to an efficient first-order method: the accuracy varies like $\mathcal{O}(h_l)$ in the L^2 norm, where h_l is the mesh step of the finest local grids.

An outline of the paper is as follows. Section 2 is devoted to the convergence analysis of the Dirichlet problem. A first analysis of a *semi-conforming* mesh (see the definition later) is presented to facilitate the exposition of the convergence analysis of the general nonconforming mesh. In Section 3, the Robin (or Neumann) problem is studied. Then, in Section 4, numerical results are presented.

2. Dirichlet problem

2.1. Definition of the problem

Let Ω be an open bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz-continuous boundary Γ. The approximation using non-boundary fitted meshes of the solution to an elliptic problem over Ω with Dirichlet boundary conditions on Γ is studied in this section. By sake of simplicity, we limit this study to homogeneous Dirichlet problems. Some possible extensions to nonhomogeneous problems are discussed in the concluding remark of this section (see page 23). The following hypotheses $(\mathcal{H})$ hold:

- $\mathbf{a} = (a_{ij})_{1\leqslant i,j\leqslant d}$ with $a_{ij} \in L^\infty(\Omega)$ verifying the *ellipticity assumptions*:

$$\exists\, a_0 > 0,\ \forall \boldsymbol{\xi} \in \mathbb{R}^d,\ \sum_{i,j=1}^{d} a_{ij}(x)\xi_j\xi_i \geqslant a_0|\boldsymbol{\xi}|^2 \text{ a.e. in } \Omega,$$

  where $|.|$ *is the Euclidean norm in* $\mathbb{R}^d$,
- f belongs to $L^2(\Omega)$.

We consider the second-order elliptic problem

$$\left\{ \begin{array}{rcl} -\mathrm{div}(\mathbf{a}\nabla u) & = & f \quad \text{in } \Omega, \\ u|_\Gamma & = & 0. \end{array} \right. \tag{1}$$

The variational formulation $(\mathcal{P})$ of (1) writes :

$$(\mathcal{P}) \quad \text{find } u \in V = \{v \in H^1(\Omega), v|_\Gamma = 0\} = H_0^1(\Omega), \ \text{ such that } a(u,v) = L(v), \forall v \in V \tag{2}$$

where

$$\left\{ \begin{array}{rcl} a(u,v) & = & \displaystyle\sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,\mathrm{d}x\,, \\ L(v) & = & \displaystyle\int_\Omega f v \,\mathrm{d}x. \end{array} \right. \tag{3}$$

Under the hypotheses $(\mathcal{H})$, the bilinear form $a(.,.)$ is continuous on $V \times V$ and V-elliptic. Moreover, the linear form $L(.)$ is continuous on V. Lax-Milgram's theorem (e.g. [1]) enables us to conclude that the solution $u \in V$ to the problem $(\mathcal{P})$ exists and is unique.

2.2. Discretization by finite elements

2.2.1. Definitions and notations

From now on, we assume that Ω is convex for sake of simplicity. If Ω is not convex, the further H^1 error estimates still probably hold by using more technical arguments.

We focus on Dirichlet problems in $\mathbb{R}^2$ ($d = 2$). Let $(\mathcal{T}_h)_h$ denote a family of meshes of Ω by (closed) finite elements K in the sense of [1]. We introduce the following notations:

$$h_K = \operatorname{diam}(K) \quad \text{(maximum of the Euclidean distances between two points of } K\text{)}, \tag{4}$$

$$h = \max\{h_K,\, K \in \mathcal{T}_h\} \quad \text{is the mesh size,} \tag{5}$$

$$\rho_K = \sup\{\operatorname{diam}(S); S \text{ is a ball contained in } K\}. \tag{6}$$

Definition 2.1 (Regular family) *A family $(\mathcal{T}_h)_h$ of finite elements is regular if the following two conditions are satisfied*

i) There exists a constant $\sigma \geqslant 1$ such that

$$\forall h,\ \forall K \in \mathcal{T}_h,\ \frac{h_K}{\varrho_K} \leqslant \sigma. \tag{7}$$

ii) The diameters h_K approach zero:

$$h = \max_{K \in \mathcal{T}_h} h_K \to 0. \tag{8}$$

Each mesh $\mathcal{T}_h$ defines an approximate polygonal open domain Ω_h such that

$$\overline{\Omega}_h = \bigcup_{K \in \mathcal{T}_h} K,$$

with the boundary $\Gamma_h = \partial\Omega_h$.

Let $\partial\mathcal{T}_h$ denote the set of elements $K \in \mathcal{T}_h$ such that the intersection of K with Γ_h has a positive measure in $\mathbb{R}^{d-1}$ (see Fig. 2) and $\partial\mathcal{T}_h^0 = \{K \in \mathcal{T}_h; \partial K \cap \Gamma_h \neq \emptyset\}$ denote the set of rectangles K which have at least a node on Γ_h.

A Lagrangian Q_1 finite element method is used to approximate the solution u of the problem $(\mathcal{P})$. The discretization nodes $(a_{i,K})_{1 \leqslant i \leqslant 4}$ are located on the vertices of the rectangular element K and the shape functions $(q_{i,K})_{1 \leqslant i \leqslant 4}$ associated to the nodes $a_{i,K}$ belong to $Q_1(K)$ (see e.g. [1]), where Q_k stands for the space of polynomials of degree for each variable less than or equal to k. In $\mathbb{R}^2$, $Q_1 = \operatorname{span}\{1, x, y, xy\}$.

For a homogeneous Dirichlet case, the approximation space V_h is defined by

$$V_h = \{v \in \mathcal{C}^0(\overline{\Omega}_h);\ v|_{\Gamma_h} = 0, \forall K \in \mathcal{T}_h,\ v|_K \in Q_1(K)\} \ \subset H_0^1(\Omega_h), \tag{9}$$

The spaces V and V_h are endowed with the H^1 semi-norm

$$\forall v \in H^1(\Omega), \quad |v|^2_{H^1(\Omega)} = \int_\Omega |\nabla v|^2, \qquad \forall v_h \in H^1(\Omega_h), \quad |v_h|^2_{H^1(\Omega_h)} = \int_{\Omega_h} |\nabla v_h|^2, \tag{10}$$

which is a Hilbert norm on H_0^1 equivalent to the H^1 norm.

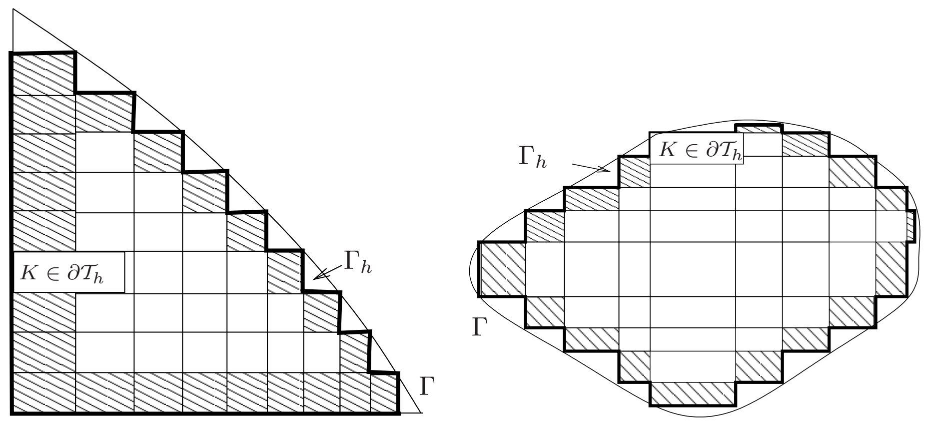
In what follows, to condense the notations, the subscript K is omitted when no ambiguity lies on the rectangle K under consideration. Moreover, C stands for a generic constant that may vary from one line to the other but is always independent of h.

2.2.2. Semi-conforming mesh
As a preparatory step of the general result, we first approximate the original domain Ω by a polygonal domain Ω_h such that some vertices of the boundary Γ_h of Ω_h are located on Γ (see Fig. 2). Even if this condition can appear seriously restrictive, this step enables us to introduce some classical arguments to prove the convergence of the error and to enlighten some key points of the general demonstration for nonconforming structured meshes.

Definition 2.2 (Semi-conforming mesh) *A mesh $\mathcal{T}_h = \{K\}$ of $\overline{\Omega}$ is referred to as semi-conforming if*

i) The open bounded domain Ω is approximated by the open polygonal domain Ω_h such that $\overline{\Omega}_h = \bigcup_{K\in\mathcal{T}_h} K \subset \overline{\Omega}$ with usual assumptions on K (see e.g. [1]).

ii) Each element $K \in \partial\mathcal{T}_h$ has at least one vertex on Γ.



(a) Regular mesh (b) Non regular mesh

Figure 2. Examples of semi-conforming meshes composed by rectangles K.

The domain Ω is discretized by a Q_1 semi-conforming mesh $\mathcal{T}_h$ composed of rectangles K with diameter $h_K \leqslant h$. The family $(\mathcal{T}_h)_h$ is regular, in the sense of definition 2.1, if the non-straight parts of the boundary of the domain Ω accept neither a vertical nor a horizontal tangent (see examples in Fig. 2). This restricts the domains under study. However, as we underlined before, this first analysis is carried out with the only intention of preparing the exposition of the general non-conforming case, where no domain restriction occurs.
The family $(\mathcal{T}_h)_h$ is chosen regular. As the open domain Ω is supposed to be regular, then there exists a constant C depending on the curvature of Γ only such that

$$\forall x \in \Gamma_h \cap \partial K, \quad \mathrm{dist}(x, \Gamma) \leqslant C h_K. \tag{11}$$

The hypotheses $(\mathcal{H})$ on the data $\mathbf{a}$ and f enable us to conclude to the existence and the uniqueness of the solution $u_h \in V_h$ of the discrete problem $(\mathcal{P}_h)$

$$(\mathcal{P}_h) \qquad \forall v_h \in V_h,\ \sum_{i,j=1}^{2} \int_{\Omega_h} a_{ij} \frac{\partial u_h}{\partial x_j} \frac{\partial v_h}{\partial x_i}\, \mathrm{d}x = \int_{\Omega_h} f v_h \, \mathrm{d}x. \tag{12}$$

Since $V_h \not\subset V$, we introduced the space $\tilde{V}_h$ defined in Ω by

$$\tilde{V}_h = \{v \in \mathcal{C}^0(\overline{\Omega});\ v|_{\Omega_h} \in V_h,\ v|_{\Omega \setminus \Omega_h} = 0\}. \tag{13}$$

The space $\tilde{V}_h$ is then a finite dimensional subspace of $V = H_0^1(\Omega)$.
Let $\tilde{u}_h$ be the extension by 0 in $\Omega \setminus \Omega_h$ of the solution u_h of the discrete problem $(\mathcal{P}_h)$ (cf. Eq.(12)):

$$\tilde{u}_h = \begin{cases} u_h & \text{in } \Omega_h \\ 0 & \text{in } \Omega \setminus \Omega_h \end{cases} \in \tilde{V}_h.$$

The function $\tilde{u}_h \in \tilde{V}_h$ can be characterized as the solution to following variational problem:

$$\forall \tilde{v}_h \in \tilde{V}_h,\ \sum_{i,j=1}^{2} \int_{\Omega} a_{ij} \frac{\partial \tilde{u}_h}{\partial x_j} \frac{\partial \tilde{v}_h}{\partial x_i}\, \mathrm{d}x = \int_{\Omega} f \tilde{v}_h \, \mathrm{d}x.$$

Thus,

$$\forall \tilde{v}_h \in \tilde{V}_h \subset H_0^1(\Omega), \quad a(\tilde{u}_h, \tilde{v}_h) = L(\tilde{v}_h). \tag{14}$$

Lemma 2.1 (Céa's lemma in the semi-conforming case) *There exists a constant $C > 0$ independent of the subspace $\tilde{V}_h$ (resp. V_h) such that:*

$$|u - \tilde{u}_h|_{H^1(\Omega)} \leqslant C \inf_{\tilde{v}_h \in \tilde{V}_h} |u - \tilde{v}_h|_{H^1(\Omega)}. \tag{15}$$

$$|u - u_h|_{H^1(\Omega_h)} \leqslant C \inf_{v_h \in V_h} |u - v_h|_{H^1(\Omega_h)}. \tag{16}$$

Proof. We use the same proof as for adapted meshes (see e.g. [1, 32]).
From (2) and (14), we have

$$\begin{aligned} a(u, v) &= L(v) \quad \forall v \in V = H_0^1(\Omega). \\ a(\tilde{u}_h, \tilde{v}_h) &= L(\tilde{v}_h) \quad \forall \tilde{v}_h \in \tilde{V}_h \subset H_0^1(\Omega). \end{aligned}$$

The function $\tilde{w}_h = \tilde{v}_h - \tilde{u}_h \in \tilde{V}_h$ is used as test function. As $\tilde{V}_h$ is a subspace of V, $\tilde{w}_h$ belongs to V. Then,

$$a(u, \tilde{w}_h) = a(\tilde{u}_h, \tilde{w}_h).$$

Hence

$$a(u - \tilde{u}_h, \tilde{w}_h) = 0, \tag{17}$$

$$\forall \tilde{v}_h \in \tilde{V}_h, \quad a(u - \tilde{u}_h, u - \tilde{u}_h) = a(u - \tilde{u}_h, u - \tilde{v}_h). \tag{18}$$

The V-ellipticity and the continuity of the bilinear form $a(.,.)$ lead to

$$\begin{aligned} a_0|u-\tilde{u}_h|^2_{H^1(\Omega)} \leqslant a(u-\tilde{u}_h,u-\tilde{u}_h) &= a(u-\tilde{u}_h,u-\tilde{v}_h)\leqslant M|u-\tilde{u}_h|_{H^1(\Omega)}|u-\tilde{v}_h|_{H^1(\Omega)},\\ |u-\tilde{u}_h|^2_{H^1(\Omega)} &\leqslant \frac{M}{a_0}|u-\tilde{u}_h|_{H^1(\Omega)}|u-\tilde{v}_h|_{H^1(\Omega)}, \end{aligned}$$

where M is the continuity constant. Inequality (15) follows with $C=\dfrac{M}{a_0}\geqslant 1$.
From Eq.(17), we can also write

$$\forall v_h\in V_h,\quad \sum_{i,j=1}^{2}\int_{\Omega_h} a_{ij}\frac{\partial}{\partial x_j}(u-u_h)\frac{\partial}{\partial x_i}(u-u_h)\,\mathrm{d}x=\sum_{i,j=1}^{2}\int_{\Omega_h} a_{ij}\frac{\partial}{\partial x_j}(u-u_h)\frac{\partial}{\partial x_i}(u-v_h)\,\mathrm{d}x.$$

Inequality (16) is then deduced from the properties of $\mathbf{a}=(a_{ij})_{1\leqslant i,j\leqslant 2}$. □
Moreover,

$$\forall \tilde{v}_h\in\tilde{V}_h,\quad |u-\tilde{v}_h|_{H^1(\Omega)}=\Big(|u-\tilde{v}_h|^2_{H^1(\Omega_h)}+|u|^2_{H^1(\Omega\backslash\Omega_h)}\Big)^{1/2}.$$

Then for $\tilde{v}_h=\tilde{u}_h$

$$\boxed{|u-\tilde{u}_h|_{H^1(\Omega)}=\Big(|u-u_h|^2_{H^1(\Omega_h)}+|u|^2_{H^1(\Omega\backslash\Omega_h)}\Big)^{1/2}.} \tag{19}$$

We introduce the dual problem $(\mathcal{P}^\star)$: For $g\in L^2(\Omega)$, find $\varphi_g\in V$ such that

$$(\mathcal{P}^\star)\qquad \forall v\in V,\quad a(v,\varphi_g)=\int_\Omega gv\mathrm{d}x. \tag{20}$$

With the Lax-Milgram's theorem and the hypotheses $(\mathcal{H})$, the problem $(\mathcal{P}^\star)$ admits an unique solution.
If for any $g\in L^2(\Omega)$, the solution φ_g of (20) belongs to $H^2(\Omega)\cap V$, the problem $(\mathcal{P}^\star)$ is said to be *regular*. It then follows that there exists a constant $C^\star$ such that

$$||\varphi_g||_{H^2(\Omega)}\leqslant C^\star||g||_{L^2(\Omega)}\qquad \forall g\in L^2(\Omega). \tag{21}$$

<u>Remark.</u> *Since Ω is convex, using [33] yields that, if $\mathbf{a}\in W^{1,\infty}(\Omega)=\{v\in L^\infty(\Omega),\partial v\in L^\infty(\Omega)\}$, the problems $(\mathcal{P})$ and $(\mathcal{P}^\star)$ are regular.*

Theorem 2.2 (Error estimate with a Q_1 semi-conforming mesh - Dirichlet case) *Let Ω be a regular convex open bounded domain. We assume that the solution u of the Dirichlet problem $(\mathcal{P})$ (see Eq.(2)) is in the space $H^2(\Omega)$ and that u_h is the solution to the discrete problem $(\mathcal{P}_h)$ (see Eq.(12)). For any regular family $(\mathcal{T}_h)_h$ of Q_1 semi-conforming meshes to Ω, there exists some constants C such that*

$$|u-u_h|_{H^1(\Omega_h)}\leqslant Ch^{1/2}\|u\|_{H^2(\Omega)}, \tag{22}$$

and if the dual problem $(\mathcal{P}^\star)$ (see Eq. (20)) is regular,

$$\|u-u_h\|_{L^2(\Omega_h)}\leqslant Ch\|u\|_{H^2(\Omega)}. \tag{23}$$

In order to demonstrate theorem 2.2, two lemmas are introduced.

Lemma 2.3 (Trace inequality) *Let σ be an edge of a simplex $\omega \subset \mathbb{R}^d$ then*

$$\forall u \in H^1(\omega), \quad \|u\|_{L^2(\sigma)} \leqslant \left(d\,\frac{\text{meas}(\sigma)}{\text{meas}(\omega)} \right)^{1/2} (\|u\|_{L^2(\omega)} + \text{diam}(\omega)|u|_{H^1(\omega)}). \tag{24}$$

A proof of this lemma can be found in [34].

Lemma 2.4. *If the property (11) holds, there exists a constant C such that any function $u \in H^2(\Omega)$ satisfies*

$$|u|_{H^1(\Omega\setminus\Omega_h)} \leqslant C h^{1/2} \|u\|_{H^2(\Omega)}. \tag{25}$$

Proof of Lemma 2.4. With the same reasoning as in [32, Lemma 5.2-3], we will demonstrate that there exists a constant C such that

$$\forall v \in H^1(\Omega), \quad \|v\|_{L^2(\Omega\setminus\Omega_h)} \leqslant C \left(h^{1/2}\|v\|_{L^2(\Gamma)} + h|v|_{H^1(\Omega\setminus\Omega_h)} \right). \tag{26}$$

Due to the trace theorem (e.g. [1]), the estimation (26) yields

$$\forall v \in H^1(\Omega), \quad \|v\|_{L^2(\Omega\setminus\Omega_h)} \leqslant C h^{1/2} \|v\|_{H^1(\Omega)}. \tag{27}$$

Then, the estimate (25) is deduced from (27) where $v = \dfrac{\partial u}{\partial x}$ and $v = \dfrac{\partial u}{\partial y}$.

To prove inequality (26), let first introduce some additional notations for the rectangles $K \in \partial\mathcal{T}_h$ (cf. Fig. 3):

- $a^\star_{i,K}$, $1 \leqslant i \leqslant 4$, is the orthogonal projection of $a_{i,K}$ onto Γ, ($|a_{i,K} a^\star_{i,K}|$ is then the distance from $a_{i,K}$ to Γ);
- $(a_{i,K}, \xi_{i,K}, \eta_{i,K})$ is the orthonormal basis such that the axis $\eta_{i,K}$ coincide with the directed line $(a_{i,K} a^\star_{i,K})$;
- The vertices of K are numbered so that the segment $[a_{2,K} a_{3,K}]$ belongs to Γ_h with $a_{3,K} \in \Gamma_h \cap \Gamma$ and $a_{2,K} \notin \Gamma$. The open domain $\mathcal{O}_K$ is delimited by the side $[a_{2,K} a_{3,K}]$, the paren $a_{3,K} a_{j,K'}$ and the side $[a_{2,K} a_{j,K'}]$, where $a_{j,K'}$ denotes the nearest node of $a_{2,K}$ which is on $\Gamma_h \cap \Gamma$ and distinct of $a_{3,K}$.

Let $\mathcal{D}(\overline{\Omega})$ denote the space of infinitely differentiable functions with compact support in $\overline{\Omega}$. As $\mathcal{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$, we will prove inequality (26) for any function $v \in \mathcal{D}(\overline{\Omega})$.
In the orthonormal basis $(a_{2,K}, \xi_{2,K}, \eta_{2,K})$ (simpler named $(a_{2,K}, \xi, \eta)$), let $\eta = \varphi(\xi)$ be the equation of the paren $a_{3,K} a_{j,K'}$ and (ξ, η) be a point in $\mathcal{O}_K$. Then for all $v \in D(\overline{\Omega})$,

$$v(\xi,\eta) = v(\xi, \varphi(\xi)) + \int_{\varphi(\xi)}^{\eta} \frac{\partial v}{\partial s}(\xi, s)\mathrm{d}s.$$

So,

$$v^2(\xi,\eta) \leqslant 2v^2(\xi,\varphi(\xi)) + 2\left(\int_{\varphi(\xi)}^{\eta} \frac{\partial v}{\partial s}(\xi,s)\mathrm{d}s \right)^2.$$

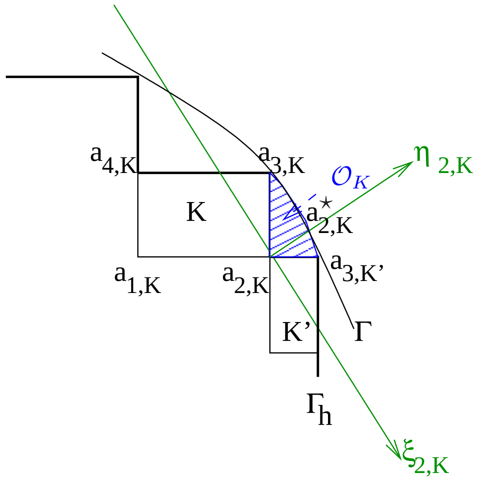


Figure 3. Example of an open $\mathcal{O}_K$.

The Cauchy-Schwarz inequality leads to

$$v^2(\xi,\eta) \leqslant 2v^2\left(\xi,\varphi(\xi)\right) + 2|\varphi(\xi)-\eta| \int_\eta^{\varphi(\xi)} \left(\frac{\partial v}{\partial s}(\xi,s)\right)^2 \mathrm{d}s.$$

Since $(\xi,\eta)\in\mathcal{O}_K$, inequality (11) yields

$$v^2(\xi,\eta) \leqslant 2v^2\left(\xi,\varphi(\xi)\right) + Ch_K \int_\eta^{\varphi(\xi)} \left(\frac{\partial v}{\partial s}(\xi,s)\right)^2 \mathrm{d}s.$$

Integrating on $\mathcal{O}_K$, we have

$$\forall K \in \partial\mathcal{T}_h, \qquad \|v\|^2_{L^2(\mathcal{O}_K)} \leqslant C\left(h_K\|v\|^2_{L^2(\Gamma\cap\partial\mathcal{O}_K)} + h_K^2\left\|\frac{\partial v}{\partial\eta}\right\|^2_{L^2(\mathcal{O}_K)}\right).$$

Summing over all the rectangles $K\in\partial\mathcal{T}_h$,

$$\|v\|^2_{L^2(\Omega\backslash\Omega_h)} \leqslant C\left(h\|v\|^2_{L^2(\Gamma)} + h^2|v|^2_{H^1(\Omega\backslash\Omega_h)}\right).$$

Then, the boundedness (26) is obtained. □

<u>REMARK.</u>

- *When $v|_\Gamma = 0$, inequality (26) becomes*

$$\|v\|_{L^2(\Omega\backslash\Omega_h)} \leqslant Ch|v|_{H^1(\Omega\backslash\Omega_h)}. \tag{28}$$

  Then, if $\boldsymbol{\nabla} u|_\Gamma = \mathbf{0}$ (for example if u admits an extrema for all points of Γ), Lemma 2.4 becomes

$$|u|_{H^1(\Omega\backslash\Omega_h)} \leqslant Ch\|u\|_{H^2(\Omega\backslash\Omega_h)}. \tag{29}$$

- *Following the same arguments as in the above proof, we can easily prove that*

$$\forall v \in H^1(\Omega), \quad \|v\|_{L^2(\Omega\backslash\Omega_h)} \leqslant C\left(h^{1/2}\|v\|_{L^2(\Gamma_h)} + h|v|_{H^1(\Omega\backslash\Omega_h)}\right), \tag{30}$$

and then, for $v|_{\Gamma_h} = 0$*, we have*

$$\|v\|_{L^2(\Omega\backslash\Omega_h)} \leqslant Ch|v|_{H^1(\Omega\backslash\Omega_h)}. \tag{31}$$

Proof of theorem 2.2. Let Π_h be the Lagrange Q_1-interpolation operator, defined for any continuous function v on K by

$$\forall K \in \mathcal{T}_h, \quad \Pi_h v|_K = \sum_{i=1}^{4} v(a_i) q_i, \tag{32}$$

Then

$$\forall K \in \mathcal{T}_h,\ \Pi_h v|_K \in Q_1(K),\ \Pi_h v(a_i) = v(a_i).$$

With a semi-conforming mesh, the function $\Pi_h v$ does not vanish on Γ_h. Consequently, $\Pi_h v$ does not belong to the discretization space V_h. Another Q_1-interpolation operator Π_h^0, which belongs to V_h, is then introduced by

$$\forall K \in \mathcal{T}_h, \quad \Pi_h^0 v|_K = \sum_{i=1}^{4} \tilde{v}(a_i) q_i \tag{33}$$

with

$$\tilde{v}(a_i) = \begin{cases} 0 & \text{if } a_i \in \Gamma_h, \\ v(a_i) & \text{otherwise.} \end{cases} \tag{34}$$

So we can write,

$$\boxed{\inf_{v_h \in V_h} |u - v_h|_{H^1(\Omega_h)} \leqslant |u - \Pi_h^0 u|_{H^1(\Omega_h)} \leqslant |u - \Pi_h u|_{H^1(\Omega_h)} + |\Pi_h u - \Pi_h^0 u|_{H^1(\Omega_h)}.} \tag{35}$$

Each of the right terms in inequality (35) is then bounded as follows:

- **Boundedness of** $|u - \Pi_h u|_{H^1(\Omega_h)}$

The interpolation theory (e.g. [1]) gives an estimate of $v - \Pi_h v$ on any Q_1 finite element K and for any function $v \in H^2(K)$,

$$|v - \Pi_h v|_{H^1(K)} \leqslant C \frac{h_K^2}{\varrho_K} |v|_{H^2(K)}, \tag{36}$$

Then, for $u \in H^2(\Omega) \cap V$,

$$|u - \Pi_h u|_{H^1(\Omega_h)} = \Big(\sum_{K \in \mathcal{T}_h} |u - \Pi_h u|^2_{H^1(K)}\Big)^{1/2} \leqslant C\, h \max_{K \in \mathcal{T}_h} \Big(\frac{h_K}{\varrho_K}\Big) |u|_{H^2(\Omega_h)}.$$

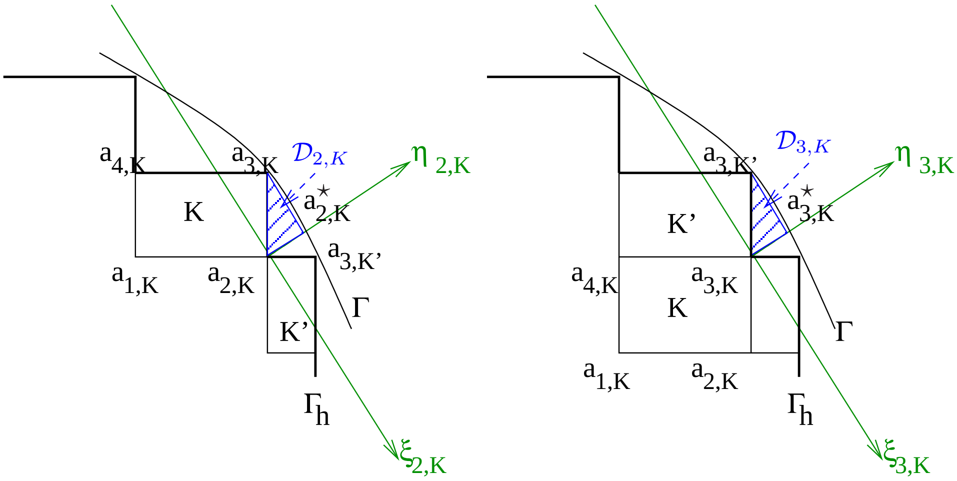
Using the regularity of $(\mathcal{T}_h)_h$

$$\boxed{|u - \Pi_h u|_{H^1(\Omega_h)} \leqslant C\, h\, |u|_{H^2(\Omega)}} \tag{37}$$

- **Boundedness of** $|\Pi_h u - \Pi_h^0 u|_{H^1(\Omega_h)}$

By definition, the function $\Pi_h u - \Pi_h^0 u$ does not vanish on the rectangles $K \in \partial\mathcal{T}_h^0$ only. For sake of clarity, we suppose that each rectangle $K \in \partial\mathcal{T}_h^0$ has either 2 sides lying on Γ_h (rectangle of type 1) or a node on Γ_h only (rectangle of type 2). For each $K \in \partial\mathcal{T}_h^0$ the same notations as in the proof of Lemma 2.4 are used, completed by the following notations (see Fig. 4):

- the vertices $a_{i,K}$, $1\leqslant i\leqslant 4$ of the rectangle K are such that
  the sides $[a_{2,K}a_{3,K}]$ and $[a_{3,K}a_{4,K}]$ are located on Γ_h for a rectangle of type 1,
  the point $a_{3,K}$ is located on Γ_h for a rectangle of type 2;
- for i such that $a_{i,K}\in\Gamma_h\setminus\Gamma$, let $\mathcal{D}_{i,K}$ be the triangle delimited by the sides $[a_{i,K}a^\star_{i,K}]$, $[a^\star_{i,K}a_{j,K'}]$ and $[a_{i,K}a_{j,K'}]$, where $a_{j,K'}$ denotes a node belonging to $\Gamma_h\cap\Gamma$ which is a direct neighbor of $a^\star_{i,K}$. Among the direct neighbors of $a^\star_{i,K}$, the node $a_{j,K'}$ is chosen such that the measure of the triangle $\mathcal{D}_{i,K}$ is as large as possible. As Ω is convex, $\mathcal{D}_{i,K}\subset(\Omega\setminus\Omega_h)$.



(a) rectangle of type 1 (b) rectangle of type 2

Figure 4. Notations for the two types of rectangles $K\in\partial\mathcal{T}_h^0$.

For any rectangle $K\in\partial\mathcal{T}_h^0$ of type 1, $(\Pi_h u-\Pi_h^0 u)$ is defined by :

- $(\Pi_h u-\Pi_h^0 u)|_K\in Q_1(K)$.
- $\begin{cases}(\Pi_h u-\Pi_h^0 u)(a_i) = 0 & \text{if } i=1 \text{ or } 3,\\ (\Pi_h u-\Pi_h^0 u)(a_i) = u(a_i) & \text{if } i=2 \text{ or } 4.\end{cases}$

Then,

$$(\Pi_h u-\Pi_h^0 u)|_K = u(a_2)q_2+u(a_4)q_4. \tag{38}$$

For a rectangle K of type 2, we have :

$$(\Pi_h u-\Pi_h^0 u)|_K = u(a_3)q_3. \tag{39}$$

For any $K\in\partial\mathcal{T}_h^0$, the estimation of $|\Pi_h u-\Pi_h^0 u|_{H^1(K)}$ is given by bounding the term $|u(a_i)q_i|_{H^1(K)}$ for a node $a_i\in\Gamma_h\setminus\Gamma$.

One focuses on the product $u(a_2)q_2$ for a rectangle K of type 1. For sake of clarity, let ξ and η denote the axis of the orthonormal basis (a_2, ξ_2, η_2).
We have

$$|u(a_2)q_2|_{H^1(K)} \leqslant |u(a_2)|\,|q_2|_{H^1(K)}.$$

As

$$u(a_2) = u(a_2^\star) + \int_{a_2^\star}^{a_2} \frac{\partial u}{\partial \eta}(\xi,\eta)\mathrm{d}\eta,$$

the definition of the problem (1) gives

$$u(a_2) = \int_{a_2^\star}^{a_2} \frac{\partial u}{\partial \eta}(\xi,\eta)\mathrm{d}\eta.$$

On account of the Cauchy-Schwarz inequality,

$$|u(a_2)| \leqslant |a_2 a_2^\star|^{1/2} \left\| \frac{\partial u}{\partial \eta} \right\|_{L^2([a_2,a_2^\star])} \tag{40}$$

We apply the trace inequality (Lemma 2.3) to $\dfrac{\partial u}{\partial \eta}$ on the edge $[a_2\, a_2^\star]$ of the triangle $\mathcal{D}_{2,K}$.

$$\|\frac{\partial u}{\partial \eta}\|_{L^2([a_2,a_2^\star])} \leqslant C \left(\frac{|a_2 a_2^\star|}{\mathrm{meas}(\mathcal{D}_{2,K})} \right)^{1/2} \left(\|\frac{\partial u}{\partial \eta}\|_{L^2(\mathcal{D}_{2,K})} + h_K |\frac{\partial u}{\partial \eta}|_{H^1(\mathcal{D}_{2,K})} \right).$$

By construction of the regular semi-conforming mesh and since Ω is convex, the measure of $\mathcal{O}_K$ is of order h^2. Then, the measure of triangle $\mathcal{D}_{i,K}$ can be bounded by

$$\mathrm{meas}(\mathcal{D}_{i,K}) \geqslant C h^2.$$

The property (11) yields

$$|u(a_2)| \leqslant C \left(|u|_{H^1(\mathcal{D}_{2,K})} + h_K |u|_{H^2(\mathcal{D}_{2,K})} \right). \tag{41}$$

Using the expression of each $q_i, 1 \leqslant i \leqslant 4$, on a rectangle K, we can show that

$$\forall K \in \mathcal{T}_h,\ \forall\, 1 \leqslant i \leqslant 4,\ |q_i|_{H^1(K)} \leqslant C \frac{h_K}{\varrho_K}, \tag{42}$$

with $C = \sqrt{\dfrac{1}{3}}$.
For each $K \in \partial \mathcal{T}_h^0$ of type 1 (resp. type 2), we have

$$|\Pi_h u - \Pi_h^0 u|_{H^1(K)} \leqslant \left(C_1 \left(|u|_{H^1(\mathcal{D}_{2,K})} + h_K |u|_{H^2(\mathcal{D}_{2,K})} \right) + C_2 \left(|u|_{H^1(\mathcal{D}_{4,K})} + h_K |u|_{H^2(\mathcal{D}_{4,K})} \right) \right) \frac{h_K}{\varrho_K}.$$

$$(\text{resp. } |\Pi_h u - \Pi_h^0 u|_{H^1(K)} \leqslant C \left(|u|_{H^1(\mathcal{D}_{3,K})} + h_K |u|_{H^2(\mathcal{D}_{3,K})} \right) \frac{h_K}{\varrho_K}.)$$

Summing over all $K \in \partial \mathcal{T}_h^0$ and using the regularity of $(\mathcal{T}_h)_h$ lead to

$$|\Pi_h u - \Pi_h^0 u|_{H^1(\Omega_h)} \leqslant C \left(|u|_{H^1(\Omega \setminus \Omega_h)} + h |u|_{H^2(\Omega \setminus \Omega_h)} \right).$$

Thanks to Lemma 2.4, we obtain

$$|\Pi_h u - \Pi_h^0 u|_{H^1(\Omega_h)} \leqslant C\left(h^{1/2}\|u\|_{H^2(\Omega)} + h|u|_{H^2(\Omega\backslash\Omega_h)}\right).$$

To finish,

$$\boxed{|\Pi_h u - \Pi_h^0 u|_{H^1(\Omega_h)} \leqslant C h^{1/2}\|u\|_{H^2(\Omega)}} \tag{43}$$

Combining the bounds (37) and (43), inequality (35) yields

$$\inf_{v_h\in V_h} |u - v_h|_{H^1(\Omega_h)} \leqslant C h^{1/2}\|u\|_{H^2(\Omega)}. \tag{44}$$

This conclude the proof of (22).
<u>REMARK.</u>

- *In the particular case where $\boldsymbol{\nabla} u|_\Gamma = \mathbf{0}$, Eq. (43) is bounded by h instead of $h^{1/2}$ thanks to Eq. (29). The optimal first-order convergence is then yielded for the H^1 norm. .*
- *Combining equation (19) to estimations (22) and (25) lead to the following estimate*

$$\boxed{|u - \tilde{u}_h|_{H^1(\Omega)} \leqslant C h^{1/2}\|u\|_{H^2(\Omega)}.} \tag{45}$$

As usual, for the L^2-norm error estimate, we suppose the regularity of the dual problem $(\mathcal{P}^\star)$. From (20), taking v equal to $u - \tilde{u}_h$, we can write

$$a(u - \tilde{u}_h, \varphi_g) = \int_\Omega g(u - \tilde{u}_h)\mathrm{d}x. \tag{46}$$

Combining the variational formulation (2) and (14) lead to

$$a(u - \tilde{u}_h, \tilde{v}_h) = 0 \quad \forall \tilde{v}_h \in \tilde{V}_h \subset V,$$

Hence $\forall \tilde{\varphi}_h \in \tilde{V}_h$,

$$a(u - \tilde{u}_h, \varphi_g - \tilde{\varphi}_h) = \int_\Omega g(u - \tilde{u}_h)\mathrm{d}x. \tag{47}$$

As

$$\|u - \tilde{u}_h\|_{L^2(\Omega)} = \sup_{0\neq g\in L^2(\Omega)} \frac{\int_\Omega g(u - \tilde{u}_h)\mathrm{d}x}{\|g\|_{L^2(\Omega)}}, \tag{48}$$

the continuity of the bilinear form $a(.,.)$ gives the inequality of Aubin-Nitsche (see [35, 36, 1]) for $u - \tilde{u}_h \in V$

$$\|u - \tilde{u}_h\|_{L^2(\Omega)} \leqslant M|u - \tilde{u}_h|_{H^1(\Omega)} \sup_{0\neq g\in L^2(\Omega)} \inf_{\tilde{\varphi}_h\in\tilde{V}_h} \frac{|\varphi_g - \tilde{\varphi}_h|_{H^1(\Omega)}}{\|g\|_{L^2(\Omega)}}. \tag{49}$$

Using the same proof used to find estimates (44) and (45), we obtain

$$\inf_{\tilde{\varphi}_h\in\tilde{V}_h} |\varphi_g - \tilde{\varphi}_h|_{H^1(\Omega)} \leqslant C h^{1/2}\|\varphi_g\|_{H^2(\Omega)},$$

The regularity of the dual problem $(\mathcal{P}^{\star})$ leads to

$$\|u-\tilde{u}_h\|_{L^2(\Omega)}\leqslant Ch^{1/2}|u-\tilde{u}_h|_{H^1(\Omega)}. \tag{50}$$

Hence, using (45)

$$\boxed{\|u-\tilde{u}_h\|_{L^2(\Omega)}\leqslant Ch\|u\|_{H^2(\Omega)}.} \tag{51}$$

As $\Omega_h \subset \Omega$ and $\tilde{u}_h|_{\Omega_h} = u_h$, this concludes the proof. □

<u>Remark.</u> *The following step could be to consider a structured regular Q_1 non-boundary fitted mesh (mainly a uniform Cartesian mesh) which is* **strictly interior** *to the domain Ω. In this case, the solution u_h can also be extended by zero in $\Omega \setminus \Omega_h$, and then the Céa Lemma is the same as in the semi-conforming case. The solution u has to be estimated on all the nodes of the boundary Γ_h. Since all these boundary nodes are interior to Ω, the demonstration of the error convergence is very similar to the previous one. So we will directly study the general nonconforming case.*

2.2.3. Nonconforming mesh

Definition 2.3 (Nonconforming mesh) *A mesh $\mathcal{T}_h = \{K\}$ is denoted as nonconforming to Ω if*

i) The open bounded domain Ω is approximated by the open polygonal domain Ω_h such that $\overline{\Omega}_h = \underset{K\in\mathcal{T}_h}{\cup} K$ with usual assumptions on K (see e.g. [1]).

ii) The approximate domain Ω_h is not boundary-fitted to Ω (see Fig. 1).

As the approximate domain Ω_h is not boundary-fitted to Ω, the discretization space V_h related to Ω_h is not a subspace of V (see problem $(\mathcal{P})$-Eq.(2)). The terminology *nonconforming* used here comes from the analogy with an adapted mesh to Ω composed by finite elements that are not in $\mathcal{C}^0(\overline{\Omega})$ implying $V_h \not\subset V$ too (see e.g. the Wilson's brick in [1]).

<u>Remark.</u> *A semi-conforming mesh (cf. Definition 2.2) is a particular case of nonconforming meshes.*

We want to discretize the original domain Ω with a structured regular nonconforming mesh $\mathcal{T}_h$ (e.g. a uniform Cartesian mesh) composed of Q_1 rectangular finite elements K, such that if h is the mesh diameter, the approximate domains $(\Omega_h)_h$ verify

$$\text{meas}((\Omega \cup \Omega_h) \setminus (\Omega \cap \Omega_h)) = \mathcal{O}(h).$$

The domain Ω being supposed to be regular, there exists a constant C depending on the curvature of Γ only, such that

$$\forall x \in \Gamma_h \cap \partial K, \quad \text{dist}(x,\Gamma)\leqslant Ch_K, \tag{52}$$

where $\Gamma_h = \partial\Omega_h$.

The boundary Γ of Ω is supposed to be Lipschitz-continuous, then there exists (see e.g. [37] or [38]) an extension operator

$$E : H^2(\Omega) \to H^2(\mathbb{R}^d),$$

such that $\forall v \in H^2(\Omega)$, the function $Ev \in H^2(\mathbb{R}^d)$ verifies $Ev|_\Omega = v$. Moreover, the operator E is continuous

$$\exists C(\Omega) \text{ such that}, \quad \forall v \in H^2(\Omega), \quad \|Ev\|_{H^2(\mathbb{R}^d)}\leqslant C(\Omega)\|v\|_{H^2(\Omega)}. \tag{53}$$

Let $\tilde{\Omega}$ be an open bounded domain such that

$$\Omega \subset \tilde{\Omega} \qquad \text{and} \qquad \forall h,\ \Omega_h \subset \tilde{\Omega}.$$

Assume that the solution u of the problem $(\mathcal{P})$ belongs to $H^2(\Omega) \cap H_0^1(\Omega)$. Let $\tilde{u} = Eu|_{\tilde{\Omega}}$ be an extension of the function u in $\tilde{\Omega}$. Then $\tilde{u} \in H^2(\tilde{\Omega})$ and thanks to (53):

$$\forall h,\ \|\tilde{u}\|_{H^2(\Omega_h)} \leqslant \|\tilde{u}\|_{H^2(\tilde{\Omega})} \leqslant C\|u\|_{H^2(\Omega)}. \tag{54}$$

This concept of encapsulating domain and extension operator is extensively used and studied in optimal-shape design [39, 40]. In this field, the encapsulating domain $\tilde{\Omega}$ is referred to as *security set* or *hold-all domain.*

We consider the following variational problem $(\mathcal{P}_h)$ on Ω_h: find $u_h \in V_h$ such that

$$(\mathcal{P}_h) \qquad \forall v_h \in V_h,;\ \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial u_h}{\partial x_j} \frac{\partial v_h}{\partial x_i}\, \mathrm{d}x = \int_{\Omega_h} \tilde{f} v_h \, \mathrm{d}x, \tag{55}$$

where $\tilde{\mathbf{a}}$ (resp. $\tilde{f}$) is an extension of $\mathbf{a}$ (resp. f) in $L^\infty(\tilde{\Omega})$ (resp. $L^2(\tilde{\Omega})$). Moreover $\tilde{\mathbf{a}}$ verifies the ellipticity assumptions on $\tilde{\Omega}$. Then using Lax-Milgram's theorem, the solution $u_h \in V_h$ of the equation (55) exists and is unique.
REMARK. *The extension $\tilde{f}$ of f in $L^2(\tilde{\Omega})$ may be chosen as:*

$$\tilde{f} = -\mathrm{div}(\tilde{\mathbf{a}}\boldsymbol{\nabla}\tilde{u}).$$

However in practical computations, the function u is the unknown of the problem $(\mathcal{P})$ and then its extension $\tilde{u}$ is not known either. So the extension $\tilde{f}$ of f, which is a given data of the variational problem $(\mathcal{P}_h)$, cannot be defined by the previous equation. In the sequel we will denote by $\hat{f}$ the particular extension of f verifying

$$\hat{f} = -\mathrm{div}(\tilde{\mathbf{a}}\boldsymbol{\nabla}\tilde{u}).$$

Lemma 2.5. *There exists a constant $C > 0$ such that*

$$|\tilde{u} - u_h|_{H^1(\Omega_h)} \leqslant C \left(\inf_{v_h \in V_h} |\tilde{u} - v_h|_{H^1(\Omega_h)} + \sup_{w_h \in V_h} \frac{\left| \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j} \frac{\partial w_h}{\partial x_i}\, \mathrm{d}x - \int_{\Omega_h} \tilde{f} w_h \, \mathrm{d}x \right|}{|w_h|_{H^1(\Omega_h)}} \right). \tag{56}$$

Proof. For any $v_h \in V_h$, the V_h-ellipticity property leads to

$$\begin{aligned}
\tilde{a}_0|u_h - v_h|^2_{H^1(\Omega_h)} &\leqslant \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial(u_h - v_h)}{\partial x_j} \frac{\partial(u_h - v_h)}{\partial x_i} \,\mathrm{d}x, \\
&\leqslant \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial(\tilde{u} - v_h)}{\partial x_j} \frac{\partial(u_h - v_h)}{\partial x_i} \,\mathrm{d}x + \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial(u_h - \tilde{u})}{\partial x_j} \frac{\partial(u_h - v_h)}{\partial x_i} \,\mathrm{d}x, \\
&\leqslant C|\tilde{u} - v_h|_{H^1(\Omega_h)}|u_h - v_h|_{H^1(\Omega_h)} + \int_{\Omega_h} \tilde{f}(u_h - v_h)\,\mathrm{d}x - \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j} \frac{\partial(u_h - v_h)}{\partial x_i} \,\mathrm{d}x,
\end{aligned}$$

where $\tilde{a}_0$ is the V_h-ellipticity constant.
Then,

$$|u_h - v_h|_{H^1(\Omega_h)} \leqslant C \left(|\tilde{u} - v_h|_{H^1(\Omega_h)} + \sup_{w_h \in V_h} \frac{\left| \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j} \frac{\partial w_h}{\partial x_i} \,\mathrm{d}x - \int_{\Omega_h} \tilde{f} w_h \,\mathrm{d}x \right|}{|w_h|_{1,\Omega_h}} \right).$$

The triangle inequality concludes the proof. □

<u>REMARK.</u> *The inequality of Lemma 2.5 is analogous to the second Strang's lemma (see [41, 42, 1]) in our case of nonconforming elements.*

Theorem 2.6 (Error estimate with a Q_1 nonconforming mesh - Dirichlet case)
Let Ω and $\tilde{\Omega}$ be two regular convex open bounded domains such that $\Omega \subset \tilde{\Omega}$. Let u be the solution to the Dirichlet problem $(\mathcal{P})$ (see Eq. (2)) and u_h be the solution to the discrete problem $(\mathcal{P}_h)$ (see Eq. (55)). The solution u is supposed to belong to $H^2(\Omega) \cap H^1_0(\Omega)$. Thus, there exists an extension $\tilde{u} \in H^2(\tilde{\Omega})$ of u such that $\|\tilde{u}\|_{H^2(\tilde{\Omega})} \leqslant C(\Omega)\|u\|_{H^2(\Omega)}$. Let $\tilde{f}$ denote an L^2-extension of f in $\tilde{\Omega}$ and $\tilde{\mathbf{a}} \in W^{1,\infty}(\tilde{\Omega})$ be an extension of $\mathbf{a}$ that verifies ellipticity assumptions. We introduce

$$\hat{f} = -\mathrm{div}(\tilde{\mathbf{a}}\nabla\tilde{\mathrm{u}}) \in \mathrm{L}^2(\tilde{\Omega}).$$

For any regular family $(\mathcal{T}_h)_h$ of Q_1 nonconforming meshes to Ω such that

$$\Omega_h \subset \tilde{\Omega} \quad \textit{and} \quad \mathrm{meas}((\Omega \cup \Omega_h) \setminus (\Omega \cap \Omega_h)) = \mathcal{O}(h),$$

there exists some constants C such that

$$|\tilde{u} - u_h|_{H^1(\Omega_h)} \leqslant C h^{1/2}(\|u\|_{H^2(\Omega)} + h^{1/2}\|\hat{f} - \tilde{f}\|_{L^2(\Omega_h)}), \tag{57}$$

and if the dual problem $(\tilde{\mathcal{P}}^\star_h)$ (see Eq. (65)) on the convex hull $\tilde{\Omega}_h$ of $\Omega \cup \Omega_h$ is regular,

$$\|\tilde{u} - u_h\|_{L^2(\Omega_h)} \leqslant C h(\|u\|_{H^2(\Omega)} + h^{1/2}\|\hat{f} - \tilde{f}\|_{L^2(\Omega_h)}). \tag{58}$$

Proof. We begin by estimating the two terms of inequality (56).

• **Boundedness of** $\inf_{v_h \in V_h} |\tilde{u} - v_h|_{1,\Omega_h}$

Following the same idea as in the semi-conforming case, the first term of the right hand side of (56) is bounded by

$$\inf_{v_h\in V_h}|\tilde{u}-v_h|_{H^1(\Omega_h)}\leqslant|\tilde{u}-\Pi_h^0\tilde{u}|_{H^1(\Omega_h)}\leqslant|\tilde{u}-\Pi_h\tilde{u}|_{H^1(\Omega_h)}+|\Pi_h\tilde{u}-\Pi_h^0\tilde{u}|_{H^1(\Omega_h)},$$

where Π_h is the Lagrange Q_1 interpolation operator over $(\mathcal{T}_h)$(see Eq. (32) and [1]) and Π_h^0 is defined by Eqs. (33-34).
By definition of Π_h, we have

$$\boxed{|\tilde{u}-\Pi_h\tilde{u}|_{H^1(\Omega_h)}\leqslant Ch|\tilde{u}|_{H^2(\Omega_h)}.}$$

The function $\Pi_h u-\Pi_h^0 u$ does not vanish on the rectangles $K\in\partial{\mathcal{T}_h}^0$ only.

$$\forall K\in\partial{\mathcal{T}_h}^0,\quad (\Pi_h\tilde{u}-\Pi_h^0\tilde{u})|_K=\sum_{i;a_i\in\Gamma_h}\tilde{u}(a_i)q_i.$$

Then

$$|\Pi_h\tilde{u}-\Pi_h^0\tilde{u}|_{H^1(K)}\leqslant\sum_{i;a_i\in\Gamma_h}|\tilde{u}(a_i)||q_i|_{H^1(K)}$$

From estimate (42), we have

$$\forall K\in\mathcal{T}_h,\ \forall\, 1\leqslant i\leqslant 4,\ |q_i|_{H^1(K)}\leqslant C\frac{h_K}{\varrho_K}$$

For each $K\in\partial\mathcal{T}_h^0$, we use the same notations as these introduced in the proof of Lemma 2.4. For each $a_{i,K}\in\Gamma_h$, the point $\hat{a}_{i,K'}$ is defined such that (see Fig. 5):

- the triangle $\mathcal{D}_{i,K}$, delimited by the sides $[a_{i,K}a_{i,K}^\star]$, $[a_{i,K}^\star\hat{a}_{i,K'}]$ and $[a_{i,K}\hat{a}_{i,K'}]$, is included in $\Omega\setminus\Omega_h$ (resp. in $\Omega_h\setminus\Omega$) for $a_{i,K}\in\Omega$ (resp. $a_{i,K}\notin\Omega$),
- the ratio $\dfrac{|a_{i,K}a_{i,K}^\star|^2}{\text{meas}(\mathcal{D}_{i,K})}\leqslant C$ independently of h.

By definition of the regular nonconforming mesh $(\mathcal{T}_h)_h$ and since Ω is convex, such a point $\hat{a}_{i,K'}$ always exists.
By the same reasoning as in the semi-conforming case, we can prove that:

$$|\tilde{u}(a_i)|\leqslant C\left(|u|_{H^1(\mathcal{D}_{i,K})}+h_K|u|_{H^2(\mathcal{D}_{i,K})}\right).$$

Let Ω_i be the open defined by:

$$\Omega_i=(\Omega\cup\Omega_h)\setminus(\Omega\cap\Omega_h). \tag{59}$$

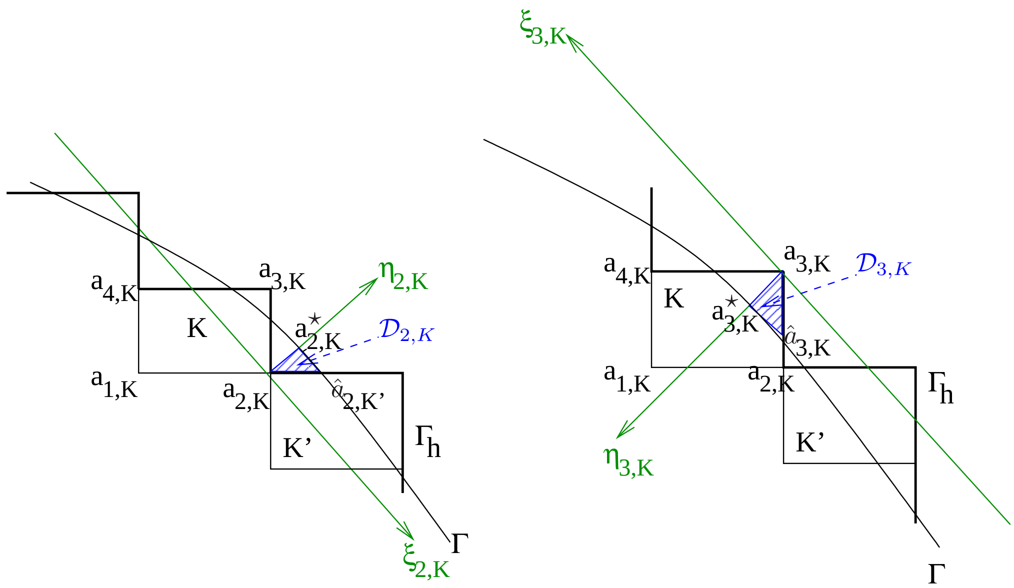
Then,

$$|\Pi_h\tilde{u}-\Pi_h^0\tilde{u}|_{H^1(\Omega_h)}\leqslant C\left(|\tilde{u}|_{H^1(\Omega_i)}+h|\tilde{u}|_{H^2(\Omega_i)}\right).$$

Moreover, by the same proof as in Lemma 2.4, we obtain that

$$|\tilde{u}|_{H^1(\Omega\setminus\Omega_h)}\leqslant Ch^{1/2}\|\tilde{u}\|_{H^2(\tilde{\Omega})}, \tag{60}$$

$$|\tilde{u}|_{H^1(\Omega_h\setminus\Omega)}\leqslant Ch^{1/2}\|\tilde{u}\|_{H^2(\tilde{\Omega})}. \tag{61}$$



(a) $a_{i,K} \in \Omega$ (b) $a_{i,K} \notin \Omega$

Figure 5. Definition of $\hat{a}_{i,K'}$ according to the position of $a_{i,K}$.

Hence,

$$|\tilde{u}|_{H^1(\Omega_i)} \leqslant C h^{1/2} \|\tilde{u}\|_{H^2(\tilde{\Omega})}. \tag{62}$$

Finally,

$$\boxed{|\Pi_h \tilde{u} - \Pi_h^0 \tilde{u}|_{H^1(\Omega_h)} \leqslant C h^{1/2} \|\tilde{u}\|_{H^2(\tilde{\Omega})}.}$$

The extension theorem (53) gives

$$\boxed{\inf_{v_h \in V_h} |\tilde{u} - v_h|_{H^1(\Omega_h)} \leqslant C h^{1/2} \|u\|_{H^2(\Omega)}.} \tag{63}$$

• **Boundedness of** $\displaystyle \sup_{w_h \in V_h} \frac{\left| \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j} \frac{\partial w_h}{\partial x_i} \, \mathrm{d}x - \int_{\Omega_h} \tilde{f} w_h \, \mathrm{d}x \right|}{|w_h|_{H^1(\Omega_h)}}$

Let $D_h(\tilde{u}, w_h) = \displaystyle \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j} \frac{\partial w_h}{\partial x_i} \, \mathrm{d}x - \int_{\Omega_h} \tilde{f} w_h \, \mathrm{d}x, \quad \forall w_h \in V_h \subset H_0^1(\Omega_h).$

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The Green formula leads to

$$\sum_{i,j=1}^{2} \int\limits_{\Omega_h} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j} \frac{\partial w_h}{\partial x_i} \,\mathrm{d}x = -\int\limits_{\Omega_h} \mathrm{div}(\tilde{\mathbf{a}}.\nabla \tilde{u})\, w_h \,\mathrm{d}x, \\ = \int\limits_{\Omega_h} \hat{f} w_h \,\mathrm{d}x.$$

Since $\hat{f}$ and $\tilde{f}$ are extensions of f in $\tilde{\Omega}$, we obtain:

$$D_h(\tilde{u}, w_h) = \int\limits_{\Omega_h \setminus \Omega} (\hat{f} - \tilde{f}) w_h, \qquad \forall w_h \in V_h \subset H_0^1(\Omega_h).$$

According to Cauchy-Schwartz inequality and the generalized Poincaré inequality (28) on $\Omega_h \setminus \Omega$, we obtain

$$D_h(\tilde{u}, w_h) \leqslant Ch \|\hat{f} - \tilde{f}\|_{L^2(\Omega_h \setminus \Omega)} |w_h|_{H^1(\Omega_h \setminus \Omega)} \tag{64}$$

To conclude, the estimate (57) is obtained combining (63) and (64) in the inequality of Lemma 2.5.

<u>Remark.</u> *As in the semi-conforming case, the additional condition* $\boldsymbol{\nabla} u|_\Gamma = \mathbf{0}$ *yields the optimal first-order convergence for the* H^1 *norm. Indeed, the non-optimal* $\mathcal{O}(h^{1/2})$ *estimate (see Theorem 2.6, Eq. (57)) follows from Lemma 2.4, or its equivalent in the nonconforming case (see Eq. (62)). In the particular case where* $\boldsymbol{\nabla} u|_\Gamma = \mathbf{0}$*, this Lemma provides an* $\mathcal{O}(h)$ *estimate.*

Concerning the L^2-norm estimate, we introduce the open bounded domain $\tilde{\Omega}_h$ which is the convex hull of $\Omega \cup \Omega_h$ (cf. Figure 6). By definition $\tilde{\Omega}_h \subset \tilde{\Omega}$. The dual problem $(\tilde{\mathcal{P}}_h^\star)$ is then defined on $\tilde{\Omega}_h$ by: given $\tilde{g}$ in $L^2(\tilde{\Omega}_h)$, find $\tilde{\varphi}_g$ in $H_0^1(\tilde{\Omega}_h)$ such that

$$(\tilde{\mathcal{P}}_h^\star) \quad \forall \tilde{v} \in H_0^1(\tilde{\Omega}_h), \; \sum_{i,j=1}^{2} \int\limits_{\tilde{\Omega}_h} \tilde{a}_{ij} \frac{\partial \tilde{v}}{\partial x_j} \frac{\partial \tilde{\varphi}_g}{\partial x_i} \,\mathrm{d}x = \int\limits_{\tilde{\Omega}_h} \tilde{g} \tilde{v} \mathrm{d}x \tag{65}$$

Hypotheses $(\mathcal{H})$ and the assumptions on the extended coefficients $\tilde{\mathbf{a}}$ and $\tilde{f}$ enable us to conclude that the problem $(\tilde{\mathcal{P}}_h^\star)$ admits a unique solution $\tilde{\varphi}_g \in H^2(\tilde{\Omega}_h) \cap H_0^1(\tilde{\Omega}_h)$.

The dual problem $(\tilde{\mathcal{P}}_h^\star)$ is supposed to be regular, that implies

$$\|\tilde{\varphi}_g\|_{H^2(\tilde{\Omega}_h)} \leqslant C^\star \|\tilde{g}\|_{L^2(\tilde{\Omega}_h)} \tag{66}$$

<u>Remark.</u> *Let us remark that the constant* $C^\star = C^\star(\tilde{\Omega}_h)$ *of inequality (66) depends only on the diameter of* $\tilde{\Omega}_h$ *(see [33]) such that*

$$C^\star(\tilde{\Omega}_h)^2 \leqslant 1 + K(\tilde{\Omega}_h)^2 + K(\tilde{\Omega}_h)^4$$

where $K(\tilde{\Omega}_h)$ *is the constant of the Poincaré inequality (see e.g. [38]) which depends only on the diameter of* $\tilde{\Omega}_h$*. Since the diameter of* $\tilde{\Omega}_h$ *tends to the diameter of* Ω *when* h *tends to 0,*

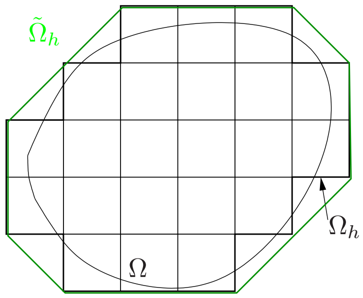


Figure 6. Convex hull $\tilde{\Omega}_h$ of $\Omega \cup \Omega_h$

this constant $C^\star$ may be bounded independently of h.
Let $\hat{u}$ and $\hat{u}_h$ be the functions of $H^1_0(\tilde{\Omega}_h)$ defined:

$$\hat{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \tilde{\Omega}_h \setminus \Omega \end{cases}, \qquad \hat{u}_h = \begin{cases} u_h & \text{in } \Omega_h \\ 0 & \text{in } \tilde{\Omega}_h \setminus \Omega_h \end{cases}$$

Then the function $\hat{u} - \hat{u}_h$ vanishes on $\tilde{\Omega}_h \setminus (\Omega \cup \Omega_h)$. Introducing $\hat{u} - \hat{u}_h$ as test function in the dual problem $(\tilde{\mathcal{P}}^\star_h)$ yields

$$\sum_{i,j=1}^{2} \int_{\Omega\cup\Omega_h} \tilde{a}_{ij} \frac{\partial(\hat{u} - \hat{u}_h)}{\partial x_j} \frac{\partial \tilde{\varphi}_g}{\partial x_i} \, \mathrm{d}x = \int_{\Omega\cup\Omega_h} \tilde{g}(\hat{u} - \hat{u}_h)\mathrm{d}x. \tag{67}$$

Let $\Omega_h^{int} = \{\cup K; K \subset (\Omega \cap \Omega_h)\}$ be the interior domain to $(\Omega \cap \Omega_h)$ of boundary Γ_h^{int} (cf. Fig. 7).

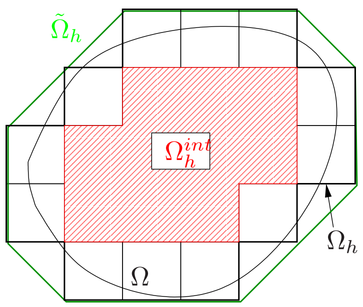


Figure 7. Definition of the domain $\Omega_h^{int} \subset (\Omega \cap \Omega_h)$

By definition of Ω_h^{int} and of $\tilde{\Omega}_h$, the following property holds

$$\forall x \in \Gamma_h^{int} \cap \partial K, \quad \mathrm{dist}(x, \partial\tilde{\Omega}_h) \leqslant C h_K \tag{68}$$

Let $\Pi_h^{0,int}$ be the Q_1 interpolation operator Π_h^0 introduced in Eqs. (33-34) defined on Ω_h^{int}:

$$\forall K \in \mathcal{T}_h, K \subset \Omega_h^{int}, \quad \Pi_h^{0,int} v|_K = \sum_{i=1}^{4} \tilde{v}(a_i) q_i,$$

with

$$\tilde{v}(a_i) = \begin{cases} 0 & \text{if } a_i \in \Gamma_h^{int}, \\ v(a_i) & \text{otherwise.} \end{cases}$$

Let us still denote by $\Pi_h^{0,int}\tilde{\varphi}_g$ the extension by 0 on $\tilde{\Omega}_h \setminus \Omega_h^{int}$ of the function $\Pi_h^{0,int}\tilde{\varphi}_g$, then:

$$\sum_{i,j=1}^{2} \int_{\Omega\cup\Omega_h} \tilde{a}_{ij} \frac{\partial(\hat{u}-\hat{u}_h)}{\partial x_j} \frac{\partial \Pi_h^{0,int}\tilde{\varphi}_g}{\partial x_i} \,\mathrm{d}x = \sum_{i,j=1}^{2} \int_{\Omega_h^{int}} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \Pi_h^{0,int}\tilde{\varphi}_g}{\partial x_i} \,\mathrm{d}x - \sum_{i,j=1}^{2} \int_{\Omega_h^{int}} a_{ij} \frac{\partial u_h}{\partial x_j} \frac{\partial \Pi_h^{0,int}\tilde{\varphi}_g}{\partial x_i} \,\mathrm{d}x \tag{69}$$

The first term of the right hand side (69) is transformed using the Green formula on Ω_h^{int} with $\Pi_h^{0,int}\tilde{\varphi}_g|_{\Gamma_h^{int}} = 0$, while the second term is transformed using the variational problem $(\mathcal{P}_h)$-Eq.(55) with $\Pi_h^{0,int}\tilde{\varphi}_g|_{\Omega_h} \in V_h$. We finally obtain

$$\sum_{i,j=1}^{2} \int_{\Omega\cup\Omega_h} \tilde{a}_{ij} \frac{\partial(\hat{u}-\hat{u}_h)}{\partial x_j} \frac{\partial \Pi_h^{0,int}\tilde{\varphi}_g}{\partial x_i} \,\mathrm{d}x = 0. \tag{70}$$

Hence,

$$\sum_{i,j=1}^{2} \int_{\Omega\cup\Omega_h} \tilde{a}_{ij} \frac{\partial(\hat{u}-\hat{u}_h)}{\partial x_j} \frac{\partial(\tilde{\varphi}_g - \Pi_h^{0,int}\tilde{\varphi}_g)}{\partial x_i} \,\mathrm{d}x = \int_{\Omega\cup\Omega_h} \tilde{g}(\hat{u}-\hat{u}_h)\mathrm{d}x. \tag{71}$$

As usual, we have

$$\|\hat{u}-\hat{u}_h\|_{L^2(\tilde{\Omega}_h)} = \sup_{\substack{g \in L^2(\tilde{\Omega}_h) \\ g \neq 0}} \frac{\int_{\tilde{\Omega}_h} \tilde{g}(\hat{u}-\hat{u}_h)\mathrm{d}x}{\|\tilde{g}\|_{L^2(\tilde{\Omega}_h)}}.$$

The function $(\hat{u}-\hat{u}_h)$ vanishing on $\tilde{\Omega}_h \setminus (\Omega\cup\Omega_h)$, we obtain

$$\|\hat{u}-\hat{u}_h\|_{L^2(\Omega\cup\Omega_h)} = \sup_{\substack{g \in L^2(\tilde{\Omega}_h) \\ g \neq 0}} \frac{\int_{\Omega\cup\Omega_h} \tilde{g}(\hat{u}-\hat{u}_h)\mathrm{d}x}{\|\tilde{g}\|_{L^2(\tilde{\Omega}_h)}}. \tag{72}$$

Thanks to the properties of $\tilde{\mathbf{a}}$, Eq. (71) yields

$$\int_{\Omega\cup\Omega_h} \tilde{g}(\hat{u}-\hat{u}_h) \leqslant C|\hat{u}-\hat{u}_h|_{H^1(\Omega\cup\Omega_h)}\, |\tilde{\varphi}_g - \Pi_h^{0,int}\tilde{\varphi}_g|_{H^1(\Omega\cup\Omega_h)}. \tag{73}$$

The first term of the right hand side of (73) is bounded thanks to the triangular inequality, inequalities (60-61), the extension property 54 and the H^1 error estimate (57)

$$\begin{aligned} |\hat{u}-\hat{u}_h|_{H^1(\Omega\cup\Omega_h)} &\leqslant |\hat{u}-\tilde{u}|_{H^1(\Omega\cup\Omega_h)} + |\tilde{u}-\hat{u}_h|_{H^1(\Omega\cup\Omega_h)}, \\ &\leqslant |\tilde{u}|_{H^1(\Omega_h\setminus\Omega)} + |\tilde{u}-u_h|_{H^1(\Omega_h)} + |\tilde{u}|_{H^1(\Omega\setminus\Omega_h)}, \\ &\leqslant Ch^{1/2}(\|u\|_{2,\Omega} + h^{1/2}\|\hat{f}-\tilde{f}\|_{0,\Omega_h}). \end{aligned} \tag{74}$$

Concerning the second term of Eq. (73), we have:

$$\begin{array}{rcl} |\tilde{\varphi}_g - \Pi_h^{0,int}\tilde{\varphi}_g|_{H^1(\Omega\cup\Omega_h)} & \leqslant & |\tilde{\varphi}_g - \Pi_h^{0,int}\tilde{\varphi}_g|_{H^1(\tilde{\Omega}_h)}, \\ & \leqslant & |\tilde{\varphi}_g - \Pi_h^{0,int}\tilde{\varphi}_g|_{H^1(\Omega_h^{int})} + |\tilde{\varphi}_g - \Pi_h^{0,int}\tilde{\varphi}_g|_{H^1(\tilde{\Omega}_h\setminus\Omega_h^{int})}. \end{array}$$

By definition of Ω_h^{int} and $\tilde{\Omega}_h$, inequality (68) and estimation of the error convergence in the semi-conforming case easily yield

$$\forall\tilde{\varphi}_g \in H_0^1(\tilde{\Omega}_h), \qquad |\tilde{\varphi}_g - \Pi_h^{0,int}\tilde{\varphi}_g|_{H^1(\Omega_h^{int})} \leqslant Ch^{1/2}\|\tilde{\varphi}_g\|_{H^2(\tilde{\Omega}_h)}.$$

As previously, by the same proof as in Lemma 2.4, the following bound is obtained

$$\begin{array}{rcl} |\tilde{\varphi}_g - \Pi_h^{0,int}\tilde{\varphi}_g|_{H^1(\tilde{\Omega}_h\setminus\Omega_h^{int})} & = & |\tilde{\varphi}_g|_{H^1(\tilde{\Omega}_h\setminus\Omega_h^{int})}, \\ & \leqslant & Ch^{1/2}\|\tilde{\varphi}_g\|_{H^2(\tilde{\Omega}_h)}. \end{array}$$

Hence,

$$|\tilde{\varphi}_g - \Pi_h^{0,int}\tilde{\varphi}_g|_{H^1(\Omega\cup\Omega_h)} \leqslant Ch^{1/2}\|\tilde{\varphi}_g\|_{H^2(\tilde{\Omega}_h)}. \tag{75}$$

Substituting inequalities (74) and (75) into inequality (73) and using the regularity of the dual problem $(\tilde{\mathcal{P}}_h^\star)$ lead to

$$\boxed{\|\hat{u} - \hat{u}_h\|_{L^2(\Omega\cup\Omega_h)} \leqslant Ch(\|u\|_{H^2(\Omega)} + h^{1/2}\|\hat{f} - \tilde{f}\|_{L^2(\Omega_h)})} \tag{76}$$

The triangular inequality

$$\|\tilde{u} - u_h\|_{L^2(\Omega_h)} \leqslant \|\tilde{u} - \hat{u}\|_{L^2(\Omega_h)} + \|\hat{u} - u_h\|_{L^2(\Omega_h)},$$

combined with the generalized Poincaré inequality (31) and the extension theorem (53) conclude the proof. □

<u>Remark.</u> *(Comments about estimates for nonhomogeneous Dirichlet problems)*
Like in most of convergence analyses, we demonstrated error estimates for a homogeneous Dirichlet problem. The most straightforward way of approximating the solution of a nonhomogeneous Dirichlet problem is to consider the "modified" variational problem whose the solution $u \in H^1(\Omega)$ satisfies (see [43, 42, 1])

$$\left\{ \begin{array}{l} (u - \tilde{u}_D) \in H_0^1(\Omega), \\ \forall v \in H_0^1(\Omega), a(u,v) = L(v), \end{array} \right. \tag{77}$$

where $\tilde{u}_D$ is the lift in $H^1(\Omega)$ of the Dirichlet data $u|_\Gamma = u_D \in H^{1/2}(\Gamma)$, and the forms $a(.,.)$ and $L(.)$ are defined in (3). An associated discrete problem is then deduced. Some arguments introduced in [42] may be extended to nonconforming meshes in order to prove the $h^{1/2}$ error convergence in the H^1 norm.
However, the discrete problem used for this theoretical analysis does not correspond to the discrete problem solved in practical computations. The computed problem corresponds to the usual variational problem with the approximation space

$$V_h = \{v \in \mathcal{C}^0(\overline{\Omega}_h);\ v|_{\Gamma_h} = \tilde{u}_D|_{\Gamma_h}, \forall K \in \mathcal{T}_h,\ v|_K \in Q_1(K)\} \subset H^1(\Omega_h).$$

The boundary condition u_D is then transfered to the stair-step approximate boundary Γ_h through the lifted function $\tilde{u}_D$. Some arguments of the convergence analysis for Robin problems (see next section), where the coefficients have also to be lifted on the stair-case boundary, may be used to obtain error estimates. Numerical results (see section 4) confirm that the discretization error of the nonhomogeneous Dirichlet problem behaves like $h^{1/2}$ in the H^1 norm and h in the L^2 norm.
Another interesting way is to use penalty methods (e.g. [44, 45]). One can show that the nonhomogeneous Dirichlet convergence analysis returns to the Robin problem analysis ($-(\mathbf{a}\nabla u).\mathbf{n} = \alpha u + g$ on Γ) with penalized coefficients such that $\alpha = \dfrac{1}{\eta}$ and $g = -\dfrac{1}{\eta} u_D$, the penalty parameter $0 < \eta << 1$ being sufficiently small ($\eta = 10^{-12}$ in practice). The $\mathcal{O}(h^{1/2})$ error estimate in the H^1 norm is then straightforward deduced from the convergence analysis of Robin problems. Furthermore, a combined convergence study with respect to the discretization step h and penalty parameter η would enable us to appreciate the influence of the modelling error.

3. Robin problem

3.1. Definition of the problem

We proceed under the assumptions on the domain Ω as in section 2.1 and the hypotheses $(\mathcal{H})$. The Robin problem under study writes

$$\begin{cases} -\text{div}(\mathbf{a}\nabla u) &= f & \text{in } \Omega, \\ -(\mathbf{a}\nabla u).\mathbf{n} &= \alpha u + g & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{78}$$

where $\mathbf{n}$ is the outward unit normal vector on Γ and $0 \leqslant \alpha \in W^{1,\infty}(\Gamma)$ and $g \in H^{1/2}(\Gamma)$.
<u>REMARK.</u> *To obtain a solution $u \in H^1(\Omega)$, the assumptions $0 \leqslant \alpha \in L^\infty(\Gamma)$ and $g \in L^2(\Gamma)$ are sufficient. However to estimate the order of the convergence of the discretization error, the solution must be in $H^2(\Omega)$ and then α and g are assumed to belong to $W^{1,\infty}(\Gamma)$ and $H^{1/2}(\Gamma)$ respectively.* Then, the variational formulation $(\mathcal{P})$ of (78) is

$$(\mathcal{P}) \quad \text{find } u \in V = H^1(\Omega) \text{ such that } a(u, v) = L(v) \ \forall v \in V, \tag{79}$$

where

$$\begin{cases} a(u, v) &= \displaystyle\sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, \mathrm{d}x + \int_\Gamma \alpha \, u \, v \, \mathrm{d}s \,, \\ L(v) &= \displaystyle\int_\Omega f v \, \mathrm{d}x - \int_\Gamma g \, v \, \mathrm{d}s \,. \end{cases} \tag{80}$$

Under the hypotheses $(\mathcal{H})$, with the usual techniques, we prove the continuity of the bilinear form $a(.,.)$ on $V \times V$ and the V-ellipticity. Moreover, the linear form $L(.)$ is continuous on V. Lax-Milgram's theorem enables us to conclude that the solution $u \in V$ to the problem $(\mathcal{P})$ exists and is unique.
<u>REMARK.</u> *A Neumann condition is obtained setting $\alpha = 0$ in the problem (78). In this case,*

an additional compatibility condition is required for the solution u to exist. Then, we usually suppose that the non unique solution belongs to the space of the functions with a zero mean over Ω.

3.2. Nonconforming Q_1 finite element mesh

We assume that the domain $\Omega \in \mathbb{R}^2$ is convex. A finite element discretization with Q_1 rectangular elements is used. The same notations as in section 2.2.1 are used. For a Robin (or Neumann) problem, the approximation space V_h is defined by

$$V_h = \{v \in \mathcal{C}^0(\overline{\Omega}_h);\ \forall K \in \mathcal{T}_h,\ v|_K \in Q_1(K)\} \ \subset H^1(\Omega_h), \tag{81}$$

The space V and V_h are equipped with the H^1 Sobolev norm (on Ω and Ω_h respectively).

We use a nonconforming mesh of Ω (see definition 2.3 - section 2.2.3) with the same assumptions as in section 2.2.3.
Let $\tilde{\alpha}$ and $\tilde{g}$ denote the lifts in Ω [38, 33] of α and g respectively. Then, we have $0\leqslant\tilde{\alpha} \in W^{1,\infty}(\Omega)$ and $\tilde{g} \in H^1(\Omega)$. We still denote by $0\leqslant\tilde{\alpha} \in W^{1,\infty}(\tilde{\Omega})$ and $\tilde{g} \in H^1(\tilde{\Omega})$ extensions of these lifts over $\tilde{\Omega}$. The variational problem $(\mathcal{P}_h)$ on Ω_h follows: find $u_h \in V_h$ such that

$$(\mathcal{P}_h) \quad \forall v_h \in V_h,\ \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial u_h}{\partial x_j}\frac{\partial v_h}{\partial x_i}\,\mathrm{d}x + \int_{\Gamma_h} \frac{\tilde{\alpha}}{\epsilon_h} u_h\, v_h\,\mathrm{d}s = \int_{\Omega_h} \tilde{f} v_h\,\mathrm{d}x - \int_{\Gamma_h} \frac{\tilde{g}}{\epsilon_h} v_h\,\mathrm{d}s, \tag{82}$$

where $\tilde{\mathbf{a}}$ (resp. $\tilde{f}$) is an extension of $\mathbf{a}$ (resp. f) in $L^\infty(\tilde{\Omega})$ (resp. $L^2(\tilde{\Omega})$). Moreover, $\tilde{\mathbf{a}}$ verifies the ellipticity assumptions on $\tilde{\Omega}$. A piecewise constant correction parameter $\epsilon_h > 0$ is incorporated to ensure the local conservativity between the original flux integral on Γ in the problem $(\mathcal{P})$ and the approximate flux integral on Γ_h in $(\mathcal{P}_h)$. As it will be specified in the sequel, the choice of this parameter allows the mesh convergence to be preserved. The solution $u_h \in V_h$ of the discrete problem $(\mathcal{P}_h)$ exists and is unique.

Lemma 3.1. *There exists a constant $C > 0$ such that*

$$\|\tilde{u}-u_h\|_{H^1(\Omega_h)}\leqslant C\left(\inf_{v_h\in V_h}\|\tilde{u}-v_h\|_{H^1(\Omega_h)} + \sup_{w_h\in V_h}\frac{\left|\sum_{i,j=1}^{2}\int_{\Omega_h}\tilde{a}_{ij}\frac{\partial\tilde{u}}{\partial x_j}\frac{\partial w_h}{\partial x_i}\,\mathrm{d}x - \int_{\Omega_h}\tilde{f}w_h\,\mathrm{d}x + \int_{\Gamma_h}\frac{\tilde{\alpha}\,\tilde{u}+\tilde{g}}{\epsilon_h}w_h\,\mathrm{d}s\right|}{\|w_h\|_{H^1(\Omega_h)}}\right). \tag{83}$$

Proof. The proof is the same as for Lemma 2.5. □

Theorem 3.2 (Error estimate with a Q_1 nonconforming mesh - Robin case)
Let Ω and $\tilde{\Omega}$ be two regular convex open bounded domains such that $\Omega \subset \tilde{\Omega}$. Let u be the solution to the Robin problem $(\mathcal{P})$ (see Eq. (79)) and u_h be the solution to the problem $(\mathcal{P}_h)$ (see Eq. (82)). The solution u is supposed to belong to $H^2(\Omega)$. Thus, there exists an

extension $\tilde{u} \in H^2(\tilde{\Omega})$ *of* u *such that* $\|\tilde{u}\|_{H^2(\tilde{\Omega})} \leqslant C(\Omega)\|u\|_{H^2(\Omega)}$. *For any regular family* $(\mathcal{T}_h)_h$ *of* Q_1 *nonconforming meshes to* Ω *such that*

$$\Omega_h \subset \tilde{\Omega} \quad \textit{and} \quad \mathrm{meas}((\Omega \cup \Omega_h) \setminus (\Omega \cap \Omega_h)) = \mathcal{O}(h),$$

there exist a constant C *such that*

$$\|\tilde{u} - u_h\|_{H^1(\Omega_h)} \leqslant Ch^{1/2}(\|u\|_{H^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\tilde{\alpha}\|_{W^{1,\infty}(\Omega)}\|u\|_{H^1(\Omega)} + \|\tilde{g}\|_{H^1(\Omega)}) \tag{84}$$

Proof. To obtain an estimation of $\|\tilde{u} - u_h\|_{H^1(\Omega_h)}$ we will estimate each of the right hand term of (83).

•Boundedness of $\inf\limits_{v_h \in V_h} \|\tilde{u} - v_h\|_{H^1(\Omega_h)}$

If Π_h denotes the Lagrange Q_1-interpolation operator over $(\mathcal{T}_h)$, then the usual interpolation property and the extension theorem (53) lead to

$$\inf_{v_h \in V_h} \|\tilde{u} - v_h\|_{H^1(\Omega_h)} \leqslant \|\tilde{u} - \Pi_h \tilde{u}\|_{H^1(\Omega_h)} \leqslant Ch\|u\|_{H^2(\Omega)}$$

Now, we want to estimate the second term of the right hand side of (83). Let

$$D_h(\tilde{u}, w_h) = \sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j} \frac{\partial w_h}{\partial x_i} \, \mathrm{d}x - \int_{\Omega_h} \tilde{f} w_h \, \mathrm{d}x + \int_{\Gamma_h} \frac{\tilde{\alpha}\,\tilde{u} + \tilde{g}}{\epsilon_h} w_h \, \mathrm{d}s.$$

•Boundedness of $\sup\limits_{w_h \in V_h} \dfrac{|D_h(\tilde{u}, w_h)|}{\|w_h\|_{H^1(\Omega_h)}}$

Assume that the extension $\tilde{\mathbf{a}}$ of $\mathbf{a}$ belongs to $W^{1,\infty}(\tilde{\Omega})$, then

$$\hat{f} = -\mathrm{div}(\tilde{\mathbf{a}} \nabla \tilde{\mathrm{u}}) \in \mathrm{L}^2(\tilde{\Omega}). \tag{85}$$

Integrating by part Eq. (85) on Ω_h with $w_h \in V_h$ as test function gives

$$\sum_{i,j=1}^{2} \int_{\Omega_h} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j} \frac{\partial w_h}{\partial x_i} \, \mathrm{d}x + \int_{\Gamma_h} \varphi(\tilde{u}) w_h \, \mathrm{d}s = \int_{\Omega_h} \hat{f} w_h \, \mathrm{d}x,$$

where $\varphi(\tilde{u}) = -\tilde{\mathbf{a}} \nabla \tilde{u}.\mathbf{n} = \sum\limits_{i,j=1}^{2} \tilde{a}_{ij} \dfrac{\partial \tilde{u}}{\partial x_j} n_i$, $\mathbf{n}$ being the outward unit normal vector on Γ_h.

Hence,

$$D_h(\tilde{u}, w_h) = \int_{\Gamma_h} \frac{\tilde{\alpha}\,\tilde{u} + \tilde{g}}{\epsilon_h} w_h \, \mathrm{d}s - \int_{\Gamma_h} \varphi(\tilde{u}) \, w_h \, \mathrm{d}s + \int_{\Omega_h \setminus \Omega} (\hat{f} - \tilde{f}) w_h \, \mathrm{d}x.$$

Let $\mathcal{T}_h^{ext}$ denote an external structured regular nonconforming mesh of Ω composed by Q_1 finite elements such that

$$\mathcal{T}_h^{ext} = \{\cup K; K|_{\Omega_h} \in \mathcal{T}_h\}$$

and

$$\overline{\Omega}_h^{ext} = \bigcup_{K \in \mathcal{T}_h^{ext}} K, \quad \Omega \subset \Omega_h^{ext}.$$

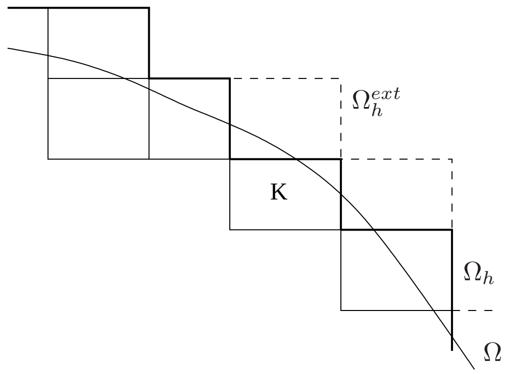


Figure 8. Definition of the external nonconforming mesh of Ω.

The extension domain $\tilde{\Omega}$ is now chosen such that

$$\forall h, \qquad \Omega_h^{ext} \subset \tilde{\Omega}.$$

Among the external nonconforming meshes of Ω, the mesh $\mathcal{T}_h^{ext}$ is such that the domain Ω_h^{ext} is the smallest possible (see Fig. 8).

Let $\omega_{h,\Gamma} = \{\cup K,\ K \in \mathcal{T}_h^{ext}, K \cap \Gamma \neq \emptyset\}$ be the union of the finite elements crossed by Γ. For each $K \subset \omega_{h,\Gamma}$, $K \in \mathcal{T}_h^{ext}$, we define the open set $\mathcal{O}_K$ by

$$\mathcal{O}_K = \begin{cases} K \cap (\Omega_h \setminus \Omega) & \text{if } K \in \mathcal{T}_h, \\ K \cap (\Omega \setminus \Omega_h) & \text{otherwise.} \end{cases} \tag{86}$$

There exists an extension $w_h \in H^1(\Omega_h^{ext})$ of $w_h \in V_h$ over Ω_h^{ext} such that

$$\|w_h\|_{H^1(\Omega_h^{ext})} \leqslant C \|w_h\|_{H^1(\Omega_h)} \tag{87}$$

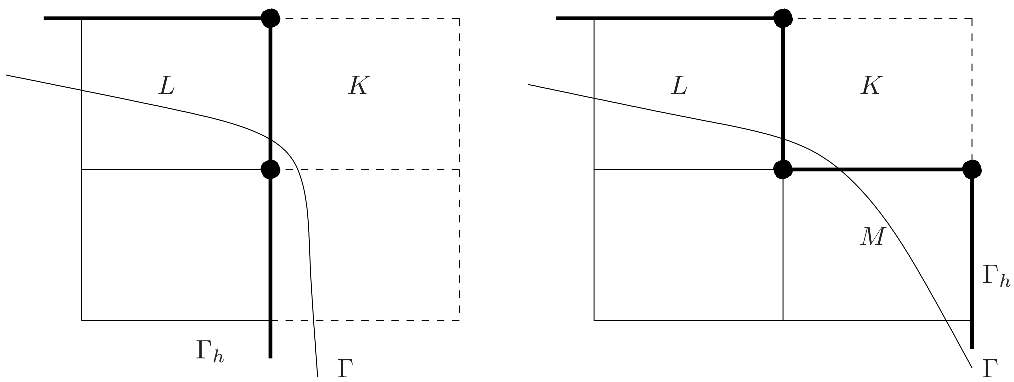
<u>Remark.</u> *This extension can be a discrete extension $w_h \in Q_1$ over Ω_h^{ext}. For each $K \in \mathcal{T}_h^{ext}, K \in \Omega_h^{ext} \setminus \Omega_h$, this extension is constructed as follows:*

- *if only two nodes of K belong to $\partial K \cap \Gamma_h$ (see Fig. 9(a)), $w_h|_K$ is chosen to be the symmetric with respect to $\partial K \cap \Gamma_h$ of $w_h|_L$, where L is the cell in Ω_h, $L \in \mathcal{N}(K)$, sharing the same nodes than K on Γ_h. Then:*

$$\|w_h\|_{H^1(K)} = \|w_h\|_{H^1(L)}.$$

- *if three nodes of K belong to $\partial K \cap \Gamma_h$ (see Fig. 9(b)), the value of w_h on the fourth node is calculated in order to minimize $\|w_h\|_{H^1(K)}$. Let L and M be the cells of Ω_h having each one two nodes on $\partial K \cap \Gamma_h$, then*

$$\|w_h\|_{H^1(K)} \leqslant C(\|w_h\|_{H^1(L)} + \|w_h\|_{H^1(M)}).$$



(a) Two nodes on $\partial K \cap \Gamma_h$.

(b) Three nodes on $\partial K \cap \Gamma_h$.

Figure 9. Configuration of the nodes of a cell $K \in \Omega_h^{ext} \setminus \Omega_h$.

The term $\displaystyle\int_{\Gamma_h} \varphi(\tilde{u}) w_h \, \mathrm{d}s$ is transformed by invoking integration by parts on Eq. (85) on each open $\mathcal{O}_K$, $K \subset \omega_{h,\Gamma}$. Then by summing over the elements $K \subset \omega_{h,\Gamma}$ we obtain

$$\begin{aligned} \int_{\Gamma_h} \varphi(\tilde{u}) w_h \, \mathrm{d}s &= \int_{\Gamma} (\alpha u + g)\, w_h \, \mathrm{d}s + \int_{\Omega_h \setminus \Omega} \hat{f} w_h \, \mathrm{d}x - \int_{\Omega \setminus \Omega_h} f w_h \, \mathrm{d}x \\ &- \sum_{i,j=1}^{2} \int_{\Omega_h \setminus \Omega} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j} \frac{\partial w_h}{\partial x_i} \, \mathrm{d}x + \sum_{i,j=1}^{2} \int_{\Omega \setminus \Omega_h} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial w_h}{\partial x_i} \, \mathrm{d}x. \end{aligned}$$

We finally have

$$\begin{aligned} D_h(\tilde{u}, w_h) &= \int_{\Gamma_h} \frac{\tilde{\alpha}\, \tilde{u} + \tilde{g}}{\epsilon_h} w_h \, \mathrm{d}s - \int_{\Gamma} (\alpha u + g)\, w_h \, \mathrm{d}s - \int_{\Omega_h \setminus \Omega} \tilde{f} w_h \, \mathrm{d}x + \int_{\Omega \setminus \Omega_h} f w_h \, \mathrm{d}x \\ &+ \sum_{i,j=1}^{2} \int_{\Omega_h \setminus \Omega} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial x_j} \frac{\partial w_h}{\partial x_i} \, \mathrm{d}x - \sum_{i,j=1}^{2} \int_{\Omega \setminus \Omega_h} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial w_h}{\partial x_i} \, \mathrm{d}x. \end{aligned}$$

The properties of $\tilde{\mathbf{a}}$, estimations (60-61), and the analogue to (27) in the nonconforming case, yield

$$\begin{aligned} |D_h(\tilde{u}, w_h)| &\leqslant \left| \int_{\Gamma_h} \frac{\tilde{\alpha}\, \tilde{u} + \tilde{g}}{\epsilon_h} w_h \, \mathrm{d}s - \int_{\Gamma} (\alpha u + g)\, w_h \, \mathrm{d}s \right| \\ &+ C h^{1/2} (\|u\|_{H^2(\tilde{\Omega})} + \|\tilde{f}\|_{L^2(\Omega_h \setminus \Omega)} + \|f\|_{L^2(\Omega \setminus \Omega_h)}) \|w_h\|_{H^1(\Omega_h^{ext})}. \end{aligned}$$

Using extension properties, we can write

$$
\begin{aligned}
|D_h(\tilde{u}, w_h)| \leqslant & \left| \int\limits_{\Gamma_h} \frac{\tilde{\alpha}\,\tilde{u} + \tilde{g}}{\epsilon_h}\, w_h \,\mathrm{d}s - \int\limits_{\Gamma} (\alpha u + g)\, w_h \,\mathrm{d}s \right| \\
& + C h^{1/2} (\|u\|_{H^2(\Omega)} + \|f\|_{L^2(\Omega)}) \|w_h\|_{H^1(\Omega_h)}.
\end{aligned} \tag{88}
$$

Then we have to estimate $\int\limits_{\Gamma_h} \frac{\tilde{\alpha}\,\tilde{u} + \tilde{g}}{\epsilon_h}\, w_h \,\mathrm{d}s - \int\limits_{\Gamma} (\alpha u + g)\, w_h \,\mathrm{d}s$.

We have to define the correction parameter ϵ_h in order to keep at least the $\mathcal{O}(h^{1/2})$ accuracy. First, let introduce the following additional notations on $\mathcal{T}_h^{ext}$. Let $\mathcal{E}$ be the family of edges of the elements of $\mathcal{T}_h^{ext}$. Then, $\sigma = K|L \in \mathcal{E}$ denote the edge between two distinct elements K and L of $\mathcal{T}_h^{ext}$: $\overline{\sigma} = \partial K \cap \partial L$ has a non zero Lebesgue measure in $\mathbb{R}$. The set of neighbors of K is denoted by $\mathcal{N}(K)$, that is $\mathcal{N}(K) = \{L \in \mathcal{T}_h^{ext};\, \exists \sigma \in \mathcal{E},\, \overline{\sigma} = \partial K \cap \partial L\}$.
For all $K \subset \omega_{h,\Gamma}$, we denote by $\hat{\mathcal{E}}_K = \{\sigma \in \mathcal{E};\, \sigma \subset \Gamma_h,\, \sigma \in \partial K \cap \partial \Omega_h^{ext}$ or $\sigma = K|L$ with either $L \not\subset \Omega_h$ or $L \not\subset \omega_{h,\Gamma}\}$ the set of sides lying on Γ_h associated to K. Otherwise, for $K \not\subset \omega_{h,\Gamma}$, $\hat{\mathcal{E}}_K = \emptyset$. Moreover, we denote by

$$
\Gamma_{h,K} = \bigcup_{\sigma \in \hat{\mathcal{E}}_K} \overline{\sigma}
$$

the part of Γ_h associated to the element K (see Fig. 10).
Some cells $K \not\subset \omega_{h,\Gamma}$ may have $\hat{\mathcal{E}}_K = \emptyset$ (for example if any side of K belongs to Γ_h), then the part Γ_K of Γ included in this king of cells may be not considered in the calculation of ϵ_h. In order to take account of the entire measure of the original interface Γ, for the cells $K \subset \omega_{h,\Gamma}$ such that $\hat{\mathcal{E}}_K = \emptyset$, we associated K to one of its neighbor $K' \in \mathcal{N}(K)$ such that $\hat{\mathcal{E}}_{K'} \neq \emptyset$. The choice of the neighbor K' (called in the sequel *"chosen" neighbor*) can be made by different ways, for example arbitrarily among all the neighbors $L \in \mathcal{N}(K)$ such that $\hat{\mathcal{E}}_L \neq \emptyset$. Another choice is detailed in [5]. For all $K \subset \omega_{h,\Gamma}$ such that $\hat{\mathcal{E}}_K \neq \emptyset$, let $\mathcal{C}(K) = \{L \in \mathcal{T}_h^{ext};\, L \in \mathcal{N}(K),\, L \subset \omega_{h,\Gamma},\, I(N) = \emptyset,\, K$ is the "chosen" neighbor of $L\}$ be the set of cells of which K is the "chosen" neighbor.
Then, we define $\hat{\Gamma}_K$ (see Fig. 10) such that :

$$
\hat{\Gamma}_K = \Gamma_K + \sum_{L \in \mathcal{C}(K)} \Gamma_L \tag{89}
$$

where $\Gamma_K = \Gamma \cap \partial K$.
The local correction ϵ_K introduced in [5], which is the value of ϵ_h over each element $K \subset \omega_{h,\Gamma}$, reads

$$
\forall K \subset \omega_{h,\Gamma} \text{ such that } \hat{\mathcal{E}}_K \neq \emptyset, \quad \epsilon_K = \frac{\mathrm{meas}(\Gamma_{h,K})}{\mathrm{meas}(\hat{\Gamma}_K)}. \tag{90}
$$

And

$$
\forall \sigma \subset \Gamma_h,\; \sigma \in \hat{\mathcal{E}}_K, \quad \epsilon_\sigma = \epsilon_K, \tag{91}
$$

which defines the piecewise constant function ϵ_h on Γ_h.
By definition of ϵ_K, for each $K \subset \omega_{h,\Gamma}$ such that $\hat{\mathcal{E}}_K \neq \emptyset$, for any constant C we have:

$$
\frac{1}{\epsilon_K} \int\limits_{\Gamma_{h,K}} C \mathrm{d}s - \int\limits_{\hat{\Gamma}_K} C \mathrm{d}s = 0
$$

Let $\overline{w_h}$ and $\overline{(\tilde{\alpha}\tilde{u}+\tilde{g})}$ be the mean values over K of w_h and $(\tilde{\alpha}\tilde{u}+\tilde{g})$ respectively, then we get

$$\begin{aligned}\int_{\Gamma_{h,K}} \frac{\tilde{\alpha}\,\tilde{u}+\tilde{g}}{\epsilon_h} w_h \,\mathrm{d}s - \int_{\hat{\Gamma}_K} (\alpha u+g)\, w_h \,\mathrm{d}s =& \frac{1}{\epsilon_K}\int_{\Gamma_{h,K}} [(\tilde{\alpha}\,\tilde{u}+\tilde{g})\, w_h - \overline{(\tilde{\alpha}\tilde{u}+\tilde{g})}.\overline{w_h}]\,\mathrm{d}s \\ &- \int_{\hat{\Gamma}_K} [(\alpha u+g)\, w_h - \overline{(\tilde{\alpha}\tilde{u}+\tilde{g})}.\overline{w_h}]\,\mathrm{d}s.\end{aligned} \tag{92}$$

Assume that $\hat{\mathcal{O}}_K = \mathcal{O}_K \cup (\underset{L\in\mathcal{C}(K)}{\cup} \mathcal{O}_L)$ (see Fig. 10). By the same reasoning as in the proof of Lemma 2.4, we can obtain the following estimate:
There exists a constant C such that $\forall K \subset \omega_{h,\Gamma}, \hat{\mathcal{E}}_K \neq \emptyset$

$$\begin{aligned}\int_{\hat{\Gamma}_K} [(\alpha u+g)\, w_h - \overline{(\tilde{\alpha}\tilde{u}+\tilde{g})}.\overline{w_h}]\,\mathrm{d}s \leqslant C\Big(& \int_{\Gamma_{h,K}} [(\tilde{\alpha}\,\tilde{u}+\tilde{g})\, w_h - \overline{(\tilde{\alpha}\tilde{u}+\tilde{g})}.\overline{w_h}]\,\mathrm{d}s \\ &+ \|\tilde{\alpha}\,\tilde{u}+\tilde{g}\|_{H^1(\hat{\mathcal{O}}_K)}\|w_h\|_{L^2(\hat{\mathcal{O}}_K)} \\ &+ \|\tilde{\alpha}\,\tilde{u}+\tilde{g}\|_{L^2(\hat{\mathcal{O}}_K)}\|w_h\|_{H^1(\hat{\mathcal{O}}_K)}\Big).\end{aligned} \tag{93}$$

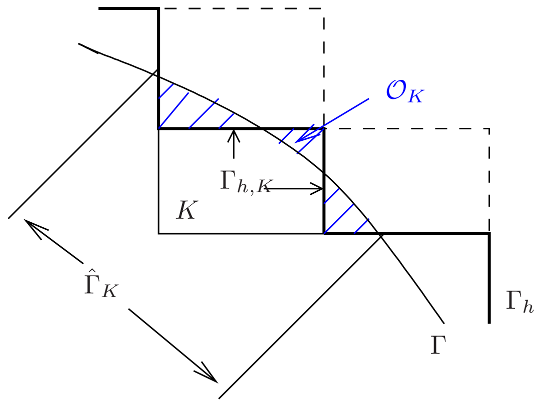


Figure 10. Definition of $\Gamma_{h,K}$, $\hat{\Gamma}_K$ and $\mathcal{O}_K$.

As $\dfrac{1}{\epsilon_K}\leqslant C$, we obtain

$$\begin{aligned}\frac{1}{\epsilon_K}\int_{\Gamma_{h,K}} (\tilde{\alpha}\,\tilde{u}+\tilde{g})\, w_h \,\mathrm{d}s - \int_{\hat{\Gamma}_K} (\alpha u+g)\, w_h \,\mathrm{d}s \leqslant C\Big(& \int_{\Gamma_{h,K}} [(\tilde{\alpha}\,\tilde{u}+\tilde{g})\, w_h - \overline{(\tilde{\alpha}\tilde{u}+\tilde{g})}.\overline{w_h}]\,\mathrm{d}s \\ &+ \|\tilde{\alpha}\,\tilde{u}+\tilde{g}\|_{H^1(\hat{\mathcal{O}}_K)}\|w_h\|_{L^2(\hat{\mathcal{O}}_K)} \\ &+ \|\tilde{\alpha}\,\tilde{u}+\tilde{g}\|_{L^2(\hat{\mathcal{O}}_K)}\|w_h\|_{H^1(\hat{\mathcal{O}}_K)}\Big).\end{aligned} \tag{94}$$

Considering the first term of the right hand side of (94), the Cauchy-Schwarz inequality gives

$$\begin{aligned}\int_{\Gamma_{h,K}} [(\tilde\alpha\,\tilde u+\tilde g)\,w_h-\overline{(\tilde\alpha\tilde u+\tilde g)}.\overline{w_h}]\,\mathrm{d}s &= \int_{\Gamma_{h,K}} [(\tilde\alpha\,\tilde u+\tilde g)\,w_h-(\tilde\alpha\,\tilde u+\tilde g)\,\overline{w_h}+(\tilde\alpha\,\tilde u+\tilde g)\,\overline{w_h}-\overline{(\tilde\alpha\tilde u+\tilde g)}.\overline{w_h}]\,\mathrm{d}s,\\ &= \int_{\Gamma_{h,K}} (\tilde\alpha\,\tilde u+\tilde g)\,(w_h-\overline{w_h})+\int_{\Gamma_{h,K}} [(\tilde\alpha\,\tilde u+\tilde g)-\overline{(\tilde\alpha\tilde u+\tilde g)}]\,\overline{w_h}\,\mathrm{d}s,\\ &\leqslant \sum_{\sigma\in\hat{\mathcal{E}}_K}\big(\|\tilde\alpha\,\tilde u+\tilde g\|_{L^2(\sigma)}\,\|w_h-\overline{w_h}\|_{L^2(\sigma)}+\\ &\qquad \|(\tilde\alpha\,\tilde u+\tilde g)-\overline{(\tilde\alpha\tilde u+\tilde g)}\|_{L^2(\sigma)}\|\overline{w_h}\|_{L^2(\sigma)}\big).\end{aligned}$$

On each edge $\sigma\in\hat{\mathcal{E}}_K$, the L^2 norm of $(w_h-\overline{w_h})$ is estimated by the trace inequality (24) of Lemma 2.3 on the rectangle K:

$$\|w_h-\overline{w_h}\|_{L^2(\sigma)}\leqslant Ch_K^{-1/2}\left(\|w_h-\overline{w_h}\|_{L^2(K)}+h_K\|w_h\|_{H^1(K)}\right).$$

Since $\overline{w_h}=\dfrac{1}{\mathrm{meas}(K)}\displaystyle\int_K w_h\,\mathrm{d}x$, the Poincaré-Wirtinger inequality (see e.g. [46]) enables us to conclude that

$$\|w_h-\overline{w_h}\|_{L^2(\sigma)}\leqslant Ch_K^{1/2}\|w_h\|_{H^1(K)}. \tag{95}$$

By the same arguments, we have

$$\|(\tilde\alpha\,\tilde u+\tilde g)-\overline{(\tilde\alpha\tilde u+\tilde g)}\|_{L^2(\sigma)}\leqslant Ch_K^{1/2}\|\tilde\alpha\,\tilde u+\tilde g\|_{H^1(K)}. \tag{96}$$

Moreover, on each cell $K\subset\omega_{h,\Gamma},\hat{\mathcal{E}}_K\neq\emptyset$, by definition of the mean value (which is a projection on the L^2 space):

$$\forall\sigma\in\hat{\mathcal{E}}_K,\quad \|\overline{w_h}\|_{L^2(\sigma)}\leqslant Ch_K^{-1/2}\|\overline{w_h}\|_{L^2(K)}\leqslant Ch_K^{-1/2}\|w_h\|_{L^2(K)}$$

Combining all the previous inequalities leads to

$$\begin{aligned}\int_{\Gamma_{h,K}} [(\tilde\alpha\,\tilde u+\tilde g)\,w_h-\overline{(\tilde\alpha\tilde u+\tilde g)}.\overline{w_h}]\,\mathrm{d}s\leqslant C\big(&h_K^{1/2}\|\tilde\alpha\tilde u+\tilde g\|_{L^2(\Gamma_{h,K})}\,\|w_h\|_{H^1(K)}\\ &+\|\tilde\alpha\tilde u+\tilde g\|_{H^1(K)}\,\|w_h\|_{L^2(K)}\big).\end{aligned}$$

On each $K\subset\omega_{h,\Gamma},\hat{\mathcal{E}}_K\neq\emptyset$, we have

$$\begin{aligned}\frac{1}{\epsilon_K}\int_{\Gamma_{h,K}}(\tilde\alpha\,\tilde u+\tilde g)\,w_h\,\mathrm{d}s-\int_{\hat\Gamma_K}(\alpha u+g)\,w_h\,\mathrm{d}s\leqslant C\big(&h_K^{1/2}\|\tilde\alpha\tilde u+\tilde g\|_{L^2(\Gamma_{h,K})}\,\|w_h\|_{H^1(K)}\\ &+\|\tilde\alpha\tilde u+\tilde g\|_{H^1(K)}\,\|w_h\|_{L^2(K)}\\ &+\|\tilde\alpha\tilde u+\tilde g\|_{H^1(\hat{\mathcal{O}}_K)}\|w_h\|_{L^2(\hat{\mathcal{O}}_K)}\\ &+\|\tilde\alpha\tilde u+\tilde g\|_{L^2(\hat{\mathcal{O}}_K)}\|w_h\|_{H^1(\hat{\mathcal{O}}_K)}\big).\end{aligned} \tag{97}$$

Summing over all the $K \subset \omega_{h,\Gamma}, \hat{\mathcal{E}}_K \neq \emptyset$:

$$\int_{\Gamma_h} \frac{\tilde{\alpha}\,\tilde{u} + \tilde{g}}{\epsilon_h}\, w_h \mathrm{ds} - \int_{\Gamma} (\alpha u + g)\, w_h \mathrm{ds} \leqslant C\big(h^{1/2} \|\tilde{\alpha}\tilde{u} + \tilde{g}\|_{L^2(\Gamma_h)} \|w_h\|_{H^1(\omega_{h,\Gamma})} + \|\tilde{\alpha}\tilde{u} + \tilde{g}\|_{H^1(\omega_{h,\Gamma})} \|w_h\|_{L^2(\omega_{h,\Gamma})} + \|\tilde{\alpha}\tilde{u} + \tilde{g}\|_{L^2(\omega_{h,\Gamma})} \|w_h\|_{H^1(\omega_{h,\Gamma})}\big). \tag{98}$$

Then, as in the proof of Lemma 2.4-Eq. (27), we have the following estimates $\forall v \in H^1(\Omega_h^{ext})$

$$\begin{aligned} \|v\|_{L^2(\Omega\setminus\Omega_h)} &\leqslant Ch^{1/2}\|v\|_{H^1(\Omega)}, \\ \|v\|_{L^2(\Omega_h^{ext}\setminus\Omega)} &\leqslant Ch^{1/2}\|v\|_{H^1(\Omega_h^{ext})}. \end{aligned}$$

Hence,

$$\forall v \in H^1(\Omega_h^{ext}), \qquad \|v\|_{L^2(\omega_{h,\Gamma})} \leqslant Ch^{1/2}\|v\|_{H^1(\Omega_h^{ext})}. \tag{99}$$

Finally,

$$\int_{\Gamma_h} \frac{\tilde{\alpha}\,\tilde{u} + \tilde{g}}{\epsilon_h}\, w_h \mathrm{ds} - \int_{\Gamma} (\alpha u + g)\, w_h \mathrm{ds} \leqslant Ch^{1/2} \|\tilde{\alpha}\tilde{u} + \tilde{g}\|_{H^1(\Omega_h^{ext})} \|w_h\|_{H^1(\Omega_h^{ext})}. \tag{100}$$

Then the extension properties conclude the proof. □

REMARK 1. *For the Robin problem, the L^2 norm error estimate in $\mathcal{O}(h)$ still surely holds as well as for the Dirichlet problem in Theorem 2.6. However, some technical issues have to be overcome to apply the Aubin-Nitsche trick. This is beyond the scope of the present work. But the numerical results in Section 4.3 do confirm the first-order convergence for the L^2 norm.*

REMARK 2. *From Eq. (88) and by the same arguments as in the previous proof, we clearly see that the choice of a global parameter $\epsilon_h = \dfrac{\text{meas}(\Gamma_h)}{\text{meas}(\Gamma)}$ does not ensure the $\mathcal{O}(h^{1/2})$ accuracy for the H^1-norm. The error convergence with respect to the mesh step h is lost.*

4. Numerical experiments

4.1. Fictitious domain methodology

To validate the previous theoretical estimates, we present some numerical results obtained with the fictitious domain method with spread interface introduced by Ramière *et al.* [2]. Diffusion problems are under study in two kinds of domain: a quarter disk domain (see Fig. 11) and a corner domain (see Fig. 17). In the fictitious domain approach, each original domain Ω is immersed into the unit square $\Omega_f =]0,1[\times]0,1[$ (see Figs. 11(b) and 17(a)) on which a uniform Cartesian mesh composed by Q_1 square finite elements is defined. Approximate staircase immersed interfaces are then lying on sides of this Cartesian mesh. Thus, the resulting approximate domains Ω_h are such that: $\text{meas}((\Omega \cup \Omega_h) \setminus (\Omega \cap \Omega_h)) = \mathcal{O}(h)$.

Problems with Dirichlet or Robin boundary conditions are under study in the quarter disk domain while a mixed problem (Dirichlet and Robin conditions on different parts of the boundary) in the corner domain is presented.

4.1.1. Dirichlet boundary conditions

For a Dirichlet boundary condition, a penalization of the exterior fictitious domain (see [2]) enables us to impose the Dirichlet boundary condition $u_h \simeq u_e$ on Γ_h, where u_e is an H^1 extension of the Dirichlet value $u|_\Gamma = u_D$ in the exterior domain such that the trace $\gamma_0(u_e) = u_e|_\Gamma = u_D$. The penalization method consists in adding the reaction term $\frac{1}{\eta}(u - u_e)$ in the equation solved in the exterior fictitious domain, where the so-called penalty coefficient η is likely to tend to zero ($\eta = 10^{-12}$ in our simulations).

Another similar approach consists in penalizing the cells crossed by the immersed interface, which define the spread interface $\omega_{h,\Gamma}$, instead of the exterior domain. In this case $u_h \simeq u_e$ in the whole spread interface. This approach is mainly used in case of mixed boundary conditions (different kinds of conditions on different parts of the boundary).

4.1.2. Robin boundary conditions

Using standard Q_1 finite elements over the fictitious domain Ω_f, neither a jump of solution nor a jump of flux is allowed on the approximate immersed interface Γ_h. Then, we cannot impose directly the Robin boundary condition on Γ_h as it is the case for the Dirichlet condition with a penalization.

Hence, in the fictitious domain method with spread interface of [2], a volume correction coefficient ϵ_K is calculated on each cell $K \subset \omega_{h,\Gamma}$ to impose the Robin flux on Γ as a source term carried by the spread interface $\omega_{h,\Gamma}$. This volume correction can be seen as a volume extension of the surface correction ϵ_σ (see Eq. (91)) analyzed here. Then, we have:

$$\forall K \subset \omega_{h,\Gamma}, \qquad \epsilon_K = \frac{\text{meas(K)}}{\text{meas}(\Gamma_\text{K})}.$$

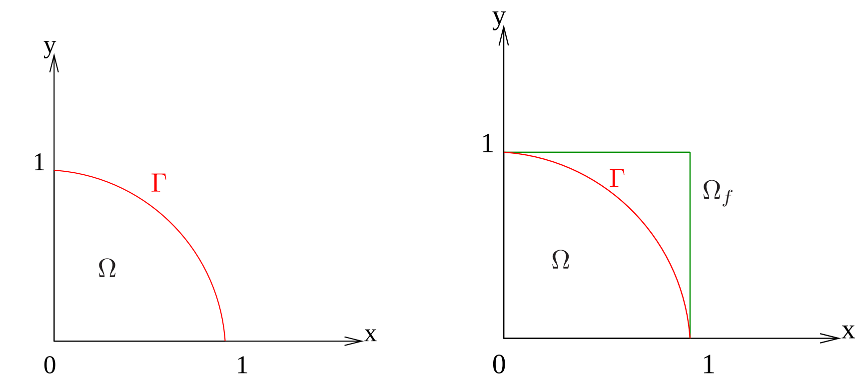
Results obtained with the fictitious domain approach with a volume correction are presented in complement of the results obtained within the theoretical analysis framework of this paper where the approximate Robin conditions are directly imposed on the stair-case approximate domain boundary.

4.1.3. Discrete norms

Let $e_h = \tilde{u} - u_h$ denote the error between the extended analytic solution and the approximate solution in Ω_h. The following results focus on the H^1 seminorm and the L^2 norm of e_h in Ω_h. Numerically speaking, these norms are evaluated by summing the integrals over each cell $K \in \mathcal{T}_h$. In order to avoid any phenomenon of superconvergence, each cell integral is evaluated by a Gauss-Legendre quadrature formula which is exact for a Q_5 function (see for example [1]).

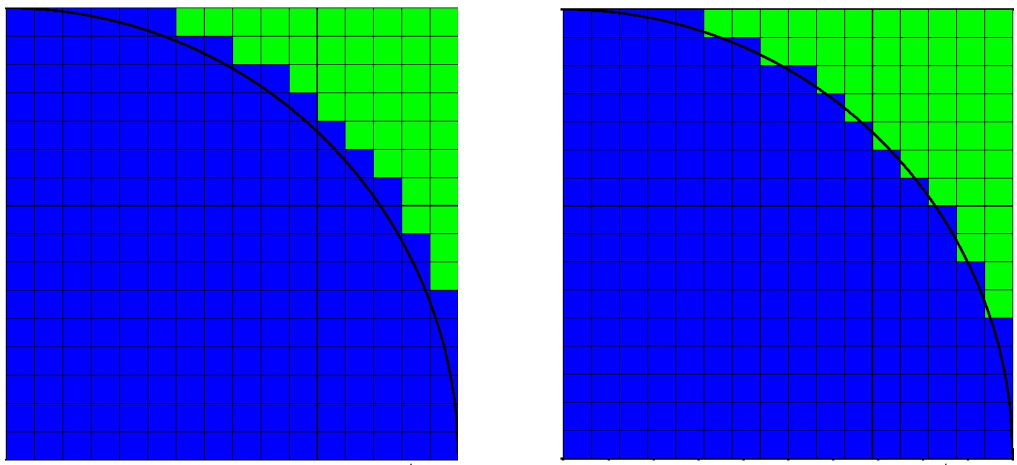
4.2. First study domain: a quarter disk domain

We first consider diffusion problems in a quarter of the unit disk Ω with symmetry conditions on the x and y axises. The fictitious domain is the unit square $\Omega_f =]0,1[\times]0,1[$, see Fig. 11(b). The domain Ω_f is meshed by uniform square cells K with a grid step varying from $h = \frac{1}{4}$ to $h = \frac{1}{256}$. This defines two kinds of nonconforming meshes for the original domain Ω: the exterior approximate domain Ω_h^{ext} (see Fig. 12(a)) such that $\Omega \subset \Omega_h^{ext}$ or the cut approximate domain Ω_h^{cut} (see Fig. 12(b)), the boundary of which may cross the original immersed interface.



(a) Original quarter disk domain Ω.　(b) Fictitious domain Ω_f.

Figure 11. Immersion of the quarter of the unit disk into the unit square.



(a) Exterior approximate domain Ω_h^{ext}.　(b) Cut approximate domain Ω_h^{cut}.

Figure 12. Nonconforming meshes and approximate domains Ω_h for an original quarter disk domain.

4.2.1. Dirichlet case

Homogeneous problem

To illustrate the theoretical estimates obtained for a homogeneous Dirichlet boundary condition, we consider an original homogeneous Dirichlet problem $-\text{div}\left[\left(\frac{2}{3}(x^2+y^2)+\frac{1}{3}\right)\nabla u\right] = -\frac{8}{3}$ with the analytical solution $u = \ln\left(\frac{2}{3}(x^2+y^2)+\frac{1}{3}\right)$ in Ω. A penalization technique in the exterior fictitious domain is computed with $u_e = 0$, see section 4.1.1. The curves of the errors in the H^1 seminorm and the L^2 norm are represented in Figure 13.

The numerical results are in agreement with the estimates obtained in Section 2. Indeed, for the two nonconforming meshes of Ω, the error varies like $\mathcal{O}(h^{1/2})$ for the H^1 norm and like $\mathcal{O}(h)$ for the L^2 norm. As expected, the choice of the cut approximate boundary yields smaller errors than the exterior approximate boundary.

Nonhomogeneous problem

In order to numerically confirm that a nonhomogeneous Dirichlet boundary condition yields the same estimates than a homogeneous condition, we study the following diffusion problem $-\triangle u = 2\cos x \sin(\frac{\pi}{2} - y)$ with the nonhomogeneous Dirichlet data $u|_\Gamma = u_D = \cos x \sin(\frac{\pi}{2} - \sqrt{1-x^2})$ on the circle. The analytical solution of this problem is $u = \cos x \sin(\frac{\pi}{2} - y)$ in Ω.

Here again, the exterior domain penalization technique is computed with $u_e = \cos x \sin(\frac{\pi}{2} - \sqrt{1-x^2})$ on the whole exterior domain. Fig. 14 shows that the excepted $\mathcal{O}(h^{1/2})$ accuracy (resp. $\mathcal{O}(h)$ accuracy) is reached for the H^1 norm (resp. L^2 norm) for both approximate immersed interfaces.

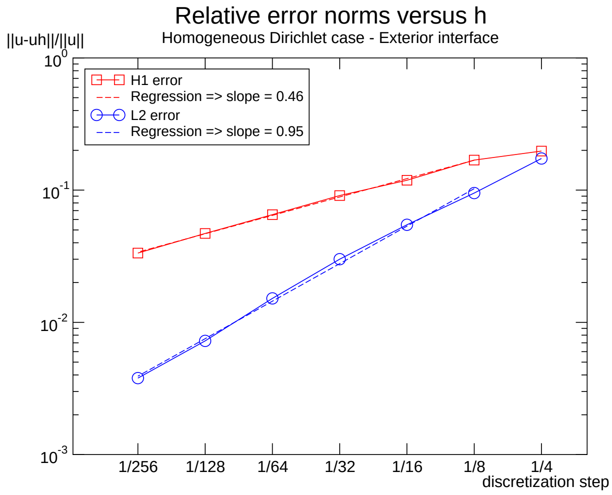
4.3. Robin case

We now consider a Robin diffusion problem $-\triangle u = -(2+4x^2)\exp(x^2)$ with the condition $-\nabla u.\mathbf{n} = \cos^2\theta\,(u-3) - 3\cos^2\theta\exp(\cos^2\theta)$ on the circle, where $\theta = \arctan(\frac{x}{y})$. This problem admits the analytical solution $u = 3 + \exp(x^2)$ in Ω.

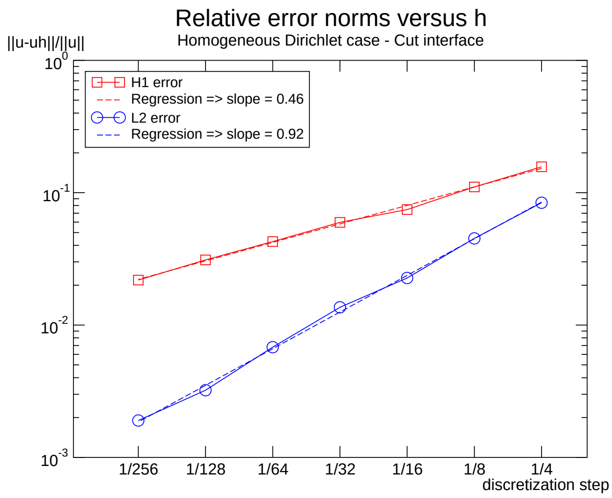
In this case, as in most of practical computations, the Robin coefficients α and g (here, $\alpha = \cos^2\theta$ and $g = -3\cos^2\theta(1+\exp(\cos^2\theta))$ are straightforward lifted. Since the original Robin coefficients are used to be constant or their expression in $H^{1/2}(\Gamma)$ is also a function of $H^1(\Omega)$, the lifted coefficients have the same expression than the original coefficients.

We first validate the theoretical estimates of section 3 in solving the Robin problem directly in an approximate Cartesian nonconforming mesh without any fictitious domain. Hence, the Robin boundary condition with surface correction is imposed as boundary condition on the approximate domain boundary. Figure 15 illustrates the order of convergence for the two kinds of nonconforming meshes Ω_h^{ext} and Ω_h^{cut}. As expected, the accuracy of the method is $\mathcal{O}(h^{1/2})$ for the H^1 norm. We can observe that we also still obtain an accuracy of $\mathcal{O}(h)$ for the L^2 norm as for the Dirichlet case. Here again, the approximate cut domain Ω_h^{cut} gives a better precision than the approximate exterior domain Ω_h^{ext}.

We are also interested in the behaviour of the fictitious domain method described in section 4.1.2, with a volume correction parameter on the cells crossed by the immersed interface Γ. In Figure 16, we can observe that this fictitious domain approach leads to the same rates of



(a) Exterior approximate domain Ω_h^{ext}.

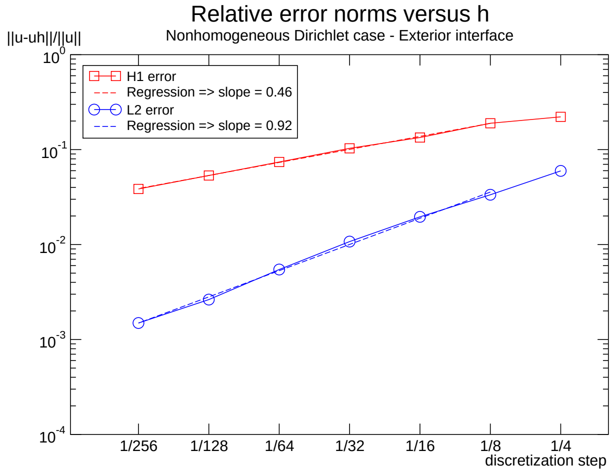


(b) Cut approximate domain Ω_h^{cut}.

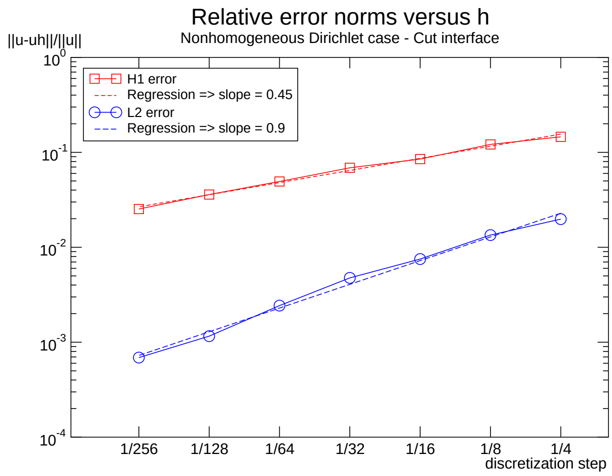
Figure 13. Convergence of the discretization error with h for the homogeneous Dirichlet problem with a fictitious domain approach.

convergence than the nonconforming approach with a surface correction term: the error varies like $\mathcal{O}(h^{1/2})$ for the H^1 norm and like $\mathcal{O}(h)$ for the L^2 norm.

Remark. *Another fictitious domain approach which deals with a thin approximate interface and immersed jumps has been recently introduced and tested in [3, 4, 5]. The approximate*



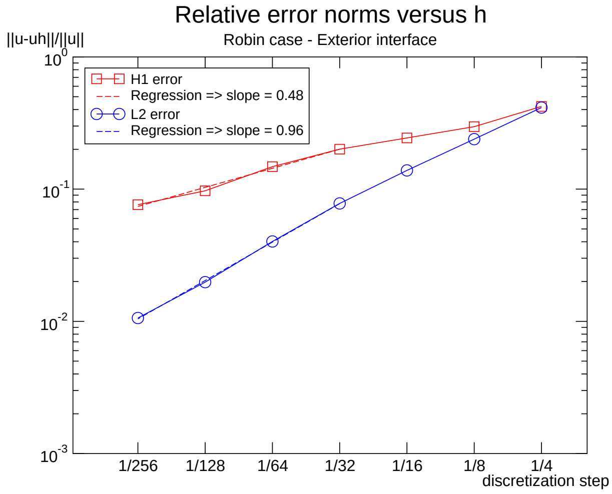
(a) Exterior approximate domain Ω_h^{ext}.



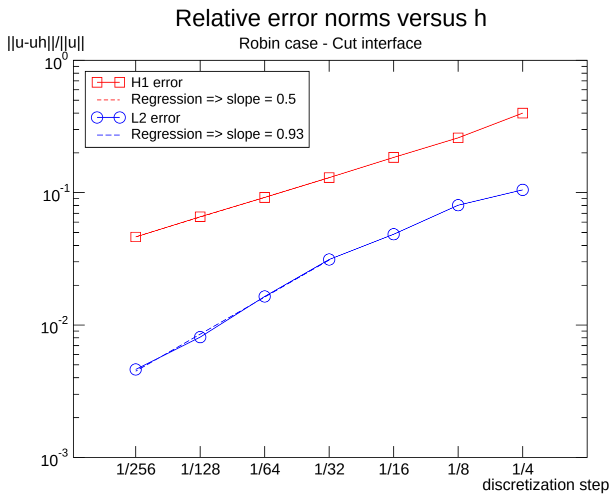
(b) Cut approximate domain Ω_h^{cut}.

Figure 14. Convergence of the discretization error with h for the nonhomogeneous Dirichlet problem with a fictitious domain approach.

interface lies on sides of control volumes and a cell-centered finite volume scheme is derived to compute this approach. Then, for a Robin B.C., the surface correction ϵ_K analyzed here is applied on the approximate interface. The numerical results in [4, 5] confirm the first order accuracy of the error for the L^2 norm.



(a) Exterior approximate domain Ω_h^{ext}.



(b) Cut approximate domain Ω_h^{cut}.

Figure 15. Convergence of the discretization error with h for the Robin problem with a nonconforming mesh.

4.4. Second study domain: a corner domain

We now consider an original polygonal domain Ω that defines a corner boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, see Fig. 17(a). To go further in the numerical validation, we illustrate that the estimates of Theorem 2.6 and Theorem 3.2 still hold if the boundary Γ is the union of some parts, each of

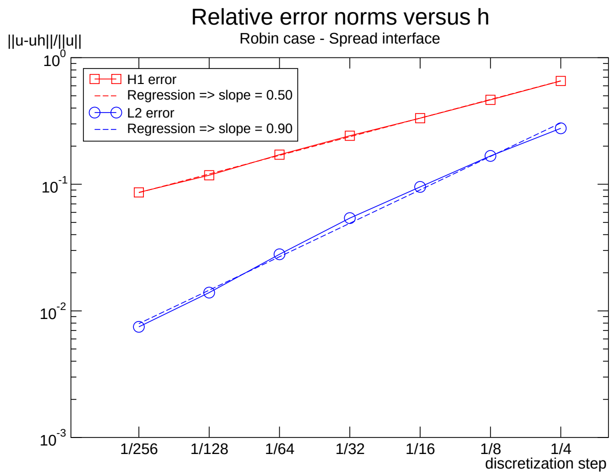


Figure 16. Convergence of the discretization error with h for the Robin problem which a fictitious domain approach (volume correction parameter).

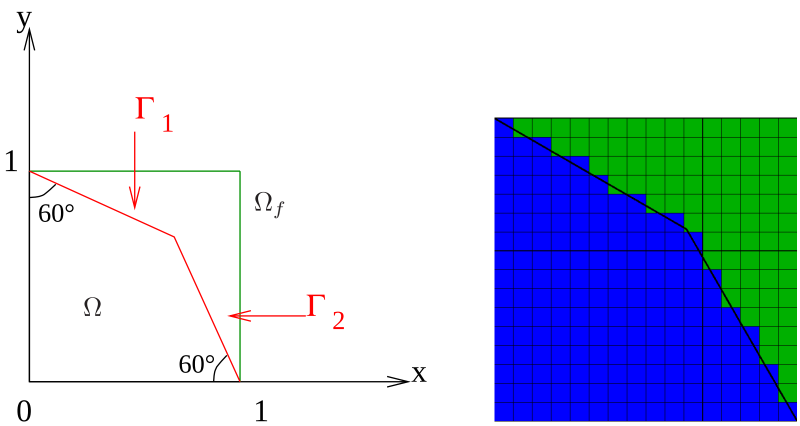
them supporting a different boundary condition (Dirichlet, Robin or Neumann).
We solve the **mixed** problem with symmetry conditions on the Cartesian coordinate axises

$$\left\{\begin{array}{rcll} -\mathrm{div}\,(\exp(2y-x)\nabla\mathbf{u}) &=& f & \text{in } \Omega \quad (a=\exp(2y-x)),\\ \multicolumn{4}{l}{\quad\text{with } f=(\frac{8}{3}x^2y^2+8x^2y-4xy^2-4x^2-4y^2)\sin(y+\frac{x}{\sqrt{3}}-1)+}\\ \multicolumn{4}{l}{\qquad(\frac{12-2\sqrt{3}}{3}x^2y^2-8x^2y-\frac{8}{\sqrt{3}}xy^2)\cos(y+\frac{x}{\sqrt{3}}-1),}\\ u|_{\Gamma_1} &=& 0 & \text{on } \Gamma_1\ (u_D=0),\\ -\exp(2y-x)\nabla\mathbf{u}.\mathbf{n}|_{\Gamma_2} &=& \alpha\, u|_{\Gamma_2}+g & \text{on } \Gamma_2,\\ \multicolumn{4}{l}{\quad\text{where } \alpha=\exp(2\sqrt{3}-(1+2\sqrt{3})x),}\\ \multicolumn{4}{l}{\text{and } g=(1-x)\left[\sqrt{3}x(3x^2+x-6)\sin(\sqrt{3}-1-\frac{2}{\sqrt{3}}x)-6x^2(1-x)\cos(\sqrt{3}-1-\frac{2}{\sqrt{3}}x)\right].} \end{array}\right.$$

which has the analytic solution $u=2x^2y^2\sin(y+\frac{x}{\sqrt{3}}-1)\exp(x-2y)$ in Ω.
As one part of the boundary supports a Robin boundary condition, we present results for the two kinds of approaches that preserve the local flux conservativity:

- the surface correction technique where the Robin condition is directly imposed on the boundary of the nonconforming mesh. In this case, the Dirichlet boundary condition is also directly imposed on the approximate boundary. This approach enables us to numerically validate the theoretical estimates obtained in this paper.
  In view of previous results (see section 4.2), the approximate nonconforming Cartesian mesh Ω_h is chosen to be the cut domain Ω_h^{cut}, see Fig. 17(b). As for the quarter disk domain, the mesh step h varies from $h=\frac{1}{4}$ to $h=\frac{1}{256}$. The error norms are plotted in Fig. 18(a).

- the volume correction technique within a fictitious domain context with uniform Q_1 finite elements. The Robin boundary condition is imposed through the corrected source term carried by the spread interface while the Dirichlet boundary condition is imposed by penalization in the spread interface. The original domain Ω is immersed in the unit square fictitious domain Ω_f (see Fig 17(a)), on which a Cartesian mesh with a step h varying from $h = \dfrac{1}{4}$ to $h = \dfrac{1}{256}$ is defined. Fig. 18(b) represents the H^1-norm error and the L^2-norm error versus the discretization step for the fictitious domain approach.



(a) Original corner domain Ω and associated fictitious square domain Ω_f.

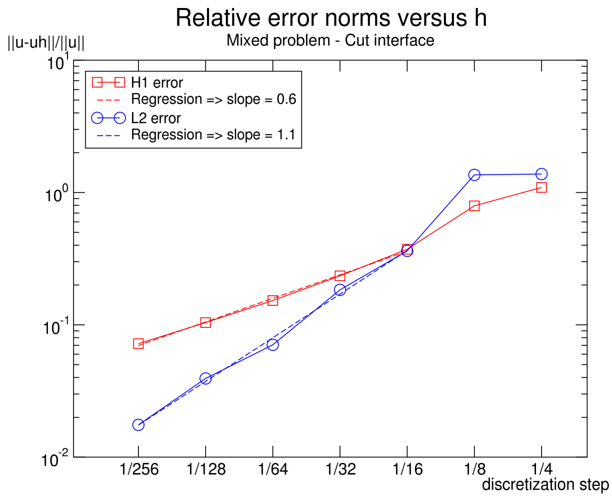
(b) Cut approximate domain Ω_h^{cut}

Figure 17. Description of the original corner domain and the associated nonconforming cut mesh.

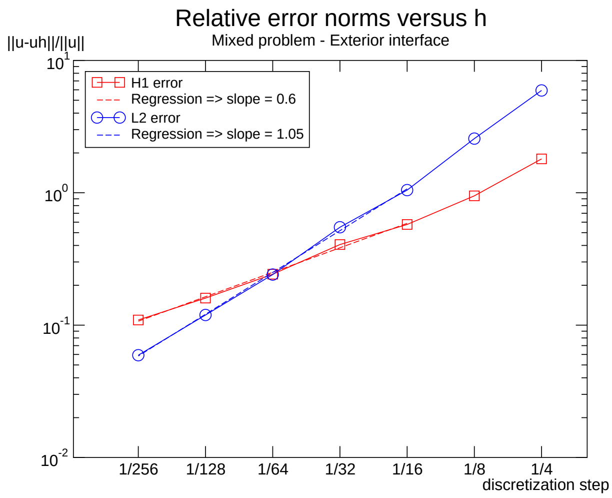
Both approaches exhibit an $\mathcal{O}(h^{1/2})$ accuracy for the H^1 norm and an $\mathcal{O}(h)$ accuracy for the L^2 norm (see Fig. 18). This test case enables us to conclude our theoretical analysis can be straightforward extended to mixed problem and remains valid for more singular domains (with corners for examples). Moreover, we can also remark that the fictitious domain approach with spread interface of Ramière *et al.* [2] performed here, easily deals with mixed boundary conditions.

5. Conclusion and perspectives

Space discretization error estimates for the Q_1 Finite Element Method based on nonconforming meshes are demonstrated in this paper. This approach has been rarely considered previously from the numerical analysis point of view, although nonconforming meshes are widely used in practice. The convergence analysis carried out for second-order elliptic problems yields $\mathcal{O}(h^{1/2})$ estimates in the H^1 norm for any general boundary conditions: Dirichlet or Robin (and hence Neumann). The most innovative result of this paper is the theoretical study of the error



(a) Surface correction approach (Nonconforming cut mesh).



(b) Volume correction approach (Fictitious domain approach).

Figure 18. Convergence of the discretization error with h for the mixed problem with two kinds of nonconforming approaches.

convergence for Robin conditions imposed on a non-boundary fitted mesh. The main point of this analysis lies in the incorporation of a correction term into the approximate problem in order to respect the local flux conservativity. Another interesting feature of this paper is the proof of the $\mathcal{O}(h)$ convergence in the L^2 norm for a Dirichlet boundary value problem.

This proof uses classical arguments but it involves some technical difficulties due to the non-conformity of the mesh to be overcome.
Numerical results illustrate the previous theoretical estimates. Various problems and configurations are under study. In particular, to replace this study in an actual framework, we present results computed within a fictitious domain approach. These practical computations exhibit the same orders of convergence those theoretically obtained in this paper. It confirms that the analysis led in this paper is the core of the convergence study of the fictitious domain methods with stair-case approximate boundaries.

By sake of simplicity, the Dirichlet estimates have been demonstrated for homogeneous problems. Although the numerical tests confirm that these estimates still hold for nonhomogeneous problems, it should be interesting to study more thorougly the convergence analysis of nonhomogeneous problems. In the same meaning, the technical issue of the L^2 convergence estimate for Robin problems should be studied in some details. However, the main goal of this article was to build a first step in the convergence study of general fictitious domain methods that do not involve boundary-fitted meshes. So the next step will consists in the complete convergence analysis of such kind of methods, linking the discretization error due to the non-boundary conforming mesh and the modelling error introduced by imposing the immersed boundary conditions.

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