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Solution of Maxwell equation in axisymmetric geometry by Fourier series decompostion and by use of $H(\mathrm{rot})$ conforming finite element

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Summary. This study deals with the mathematical and numerical solution of time-harmonic Maxwell equation in axisymmetric geometry. Using Fourier decomposition, we define weighted Sobolev spaces of solution and we prove expected regularity results. A practical contribution of this paper is the construction of a class of finite element conforming with the $H(\mathrm{rot})$ space equipped with the weighted measure $rdrdz$. It appears as an extension of the well-known cartesian mixed finite element of Raviart-Thomas-Nédélec [11]–[15]. These elements are built from classical lagrangian and mixed finite element, therefore no special approximations functions are needed. Finally, following works of Mercier and Raugel [10], we perform an interpolation error estimate for the simplest proposed element.

Mathematics Subject Classification (1991): 65N30

1. Introduction

We consider Maxwell equation in a bounded open domain O of $\mathbb{R}^3$, with a regular frontier ∂O and supposed througout the paper, *axisymmetric*, i.e. with a *symmetry of revolution*. Let ε_0 and μ_0, positive constants, that are electrical properties of domain O. We suppose that the problem has a time-harmonic dependance like $\exp(i\omega t)$. We pose the Maxwell problem at second order in space variable, that is in electric field formulation, with homogeneous Dirichlet conditions. If $(x, y, z) \in O$ related to a basis (e_x, e_y, e_z), we write indifferently the electric complex field $E = (E_x, E_y, E_z)$, or $E(x, y, z)$, and the magnetic flux $B = (B_x, B_y, B_z)$,or $B(x, y, z)$. We need

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rotational operator, noted rot as well in $\mathbb{C}^2$ as in $\mathbb{C}^3$ and defined in $\mathbb{C}^3$ by: $\mathrm{rot}E = (\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y})$. Then the problem is to determine the electromagnetic field (E, B) solution of:

$$\text{(1)} \qquad \begin{cases} -\omega^2\varepsilon_0 E + \mathrm{rot}(\mu_0^{-1}\mathrm{rot}E) = -\mathrm{i}\omega J \text{ in } O \\ \mathrm{i}\omega B + \mathrm{rot}E = 0 \text{ in } O \\ \mathrm{div}B = 0 \text{ in } O \\ \mathrm{div}\varepsilon_0 E = 0 \text{ in } O \\ E \wedge n^O = 0 \text{ on } \partial O \end{cases}$$

where n^O is the outward unit vector to ∂O and where the current density J is given in $(L^2(O))^3$, and verifying the compatibility condition $\mathrm{div}J = 0$. Using the classical functional spaces,

$$H(\mathrm{rot}, O) = \left\{ E, \int_O |E|^2 dxdydz < \infty; \int_O |\mathrm{rot}E|^2 dxdydz < \infty \right\},$$

$$H_0(\mathrm{rot}, O) = \left\{ E \in H(\mathrm{rot}, O), E \wedge n^O|_{\partial O} = 0 \right\},$$

$$H(\mathrm{div}^0, O) = \left\{ B, \int_O |B|^2 dxdydz < \infty; \mathrm{div}B = 0 \right\}.$$

It is well know that problem (1) has one and only one solution $(E, B) \in H_0(\mathrm{rot}, O) \times H(\mathrm{div}^0, O)$, provided that $\varepsilon_0\omega^2$ is not an eigenvalue of the operator $\mathrm{rot}\mu_0{}^{-1}\mathrm{rot}$ for the inside problem in O (see [5]). We are interested in discretizising and approximating this system of equations to take advantage of the particular case of axisymmetric geometry. For this we take as model problem the following: *for J, given function of* $(L^2(O))^3 in H_0(\mathrm{rot}, O)$*, find* $E \in H_0(\mathrm{rot}, O)$*, solution of*

$$\text{(2)} \qquad \begin{cases} -\omega^2\varepsilon_0 E + \mathrm{rot}(\mu_0^{-1}\mathrm{rot}E) = -\mathrm{i}\omega J \text{ in } O \\ E \wedge n^O = 0 \text{ on } \partial O \end{cases}$$

for this, a variational formulation is: *find* $E \in H_0(\mathrm{rot}, O)$*, solution of*

$$\text{(3)} \qquad \begin{aligned} &-\omega^2 \int_O \varepsilon_0 E \cdot F\, dxdydz + \int_O \mu_0^{-1}\mathrm{rot}E \cdot \mathrm{rot}F\, dxdydz \\ &= -\mathrm{i}\omega \int_O J \cdot F\, dxdydz, \end{aligned}$$

$\forall F \in H_0(\mathrm{rot}, O)$.

2. Use of cylindrical coordinate and Fourier series decomposition

2.1. Change from cartesian to cylindrical coordinates

Let be S_1 the unit circle in $\mathbb{R}^2$: $T_1 = [0, 2\pi[\rightarrow S_1$ defined by $\theta \rightarrow (\cos\theta, \sin\theta)$.
We define $\Omega \subset \mathbb{R}^2$ as the meridian of O, and called axisymmetric, (that is a section of $\mathbb{R}^2_+$ generating O) and we posed: $O' = \Omega \times T_1$ and Γ such that $\partial O' = \Gamma \times T_1 \cup (\Omega \times \{0\}) \cup (\Omega \times \{2\pi\})$, as shown below in Fig. 1.

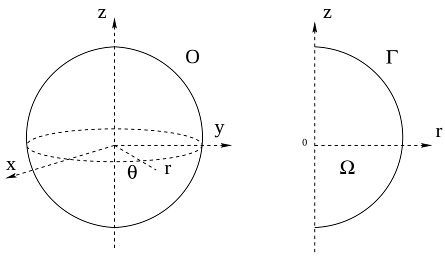

Fig. 1.

At each field E defined in O, we can associate a function $u(r,\theta,z) = \begin{bmatrix} u_r(r,\theta,z) \\ u_\theta(r,\theta,z) \\ u_z(r,\theta,z) \end{bmatrix}$ defined in O' and related to an orthonormalised basis (e_r, e_θ, e_z), so that:

$$\text{(4)} \qquad \begin{bmatrix} E_x(r,\theta,z) \\ E_y(r,\theta,z) \\ E_z(r,\theta,z) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_r(r,\theta,z) \\ u_\theta(r,\theta,z) \\ u_z(r,\theta,z) \end{bmatrix}$$

We recall that implies the following relation
$\mathrm{rot}E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{rot}_r u$
if rot_r denotes the rotational operator in cylindrical coordinates:

$$\text{(5)} \qquad \mathrm{rot}_r \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} = \begin{bmatrix} \dfrac{1}{r}\left(\dfrac{\partial u_z}{\partial \theta} - \dfrac{\partial (r u_\theta)}{\partial z}\right) \\ \dfrac{\partial u_r}{\partial z} - \dfrac{\partial u_z}{\partial r} \\ \dfrac{1}{r}\left(\dfrac{\partial (r u_\theta)}{\partial r} - \dfrac{\partial u_r}{\partial \theta}\right) \end{bmatrix}$$

Then if we give us second right members $j(r,\theta,z) = \begin{bmatrix} j_r(r,\theta,z) \\ j_\theta(r,\theta,z) \\ j_z(r,\theta,z) \end{bmatrix}$, corresponding to J by equation (4), the model problem becomes the following: *for j, given vector in O', find u solution of*

$$(6)\quad \begin{cases} -\omega^2\varepsilon_0\mu_0 u_r + \frac{(u_\theta - u_r)}{r^2} + \frac{\partial^2 u_z}{\partial r\partial z} - \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial u_\theta}{r} \\ \quad = -\mathrm{i}\omega\mu_0 j_r \\ -\omega^2\varepsilon_0\mu_0 u_\theta - \frac{1}{r}\frac{\partial u_z}{\partial z} - \frac{\partial^2 u_\theta}{\partial z^2} - \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{(u_\theta + u_r)}{r^2} - \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{\partial^2 u_\theta}{\partial r^2} \\ \quad = -\mathrm{i}\omega\mu_0 j_\theta \\ -\omega^2\varepsilon_0\mu_0 u_z + \frac{1}{r}\frac{\partial u_r}{\partial z} - \frac{1}{r}\frac{\partial u_z}{\partial r} - \frac{1}{r^2}u_z + \frac{1}{r}\frac{\partial u_\theta}{\partial z} + \frac{\partial^2 u_r}{\partial r\partial z} - \frac{\partial^2 u_z}{\partial r^2} \\ \quad = -\mathrm{i}\omega\mu_0 j_z \end{cases}$$

and verifying a Dirichlet homogeneous condition on $\Gamma \times T_1$ and periodic conditions $u_{|\theta=0} = u_{|\theta=2\pi}$ on Ω. The associated variational formulation is then

$$(7)\quad \begin{aligned} &-\omega^2\varepsilon_0\mu_0 \int_{O'} u \cdot v\, r dr dz d\theta + \int_{O'} \mathrm{rot}_r u \cdot \mathrm{rot}_r v\, r dr dz d\theta \\ &\quad = -\mathrm{i}\omega\mu_0 \int_{O'} j \cdot v\, r dr dz d\theta \end{aligned}$$

2.2. Fourier series decomposition

For a given E in $H(\mathrm{rot}, O)$, we decompose classically the unknown field u, given by (4), in Fourier series, according to the variable θ [5]. But observing that the rotational of a *symmetric field* - with respect to the plane $\{r = 0\}$ - is *antisymmetric*, (and conversely), we rewrite u as the sum of its symmetric part u^{s} and of its antisymmetric part u^{a}:

$$(8)\quad u(r,\theta,z) = u^{\mathrm{s}}(r,\theta,z) + u^{\mathrm{a}}(r,\theta,z)$$

and we posed, omiting the variables:

$$(9)\quad u^{\mathrm{s}} = \begin{bmatrix} u_r^0 \\ 0 \\ u_z^0 \end{bmatrix} + \sum_{n\geq 1} \begin{bmatrix} u_r^n \cos n\theta \\ u_\theta^n \sin n\theta \\ u_z^n \cos n\theta \end{bmatrix}$$

and:

$$(10)\quad u^{\mathrm{a}} = \begin{bmatrix} 0 \\ u_\theta^0 \\ 0 \end{bmatrix} + \sum_{n\geq 1} \begin{bmatrix} u_r^{-n} \sin n\theta \\ u_\theta^{-n} \cos n\theta \\ u_z^{-n} \sin n\theta \end{bmatrix}$$

where u_r^n, u_θ^n, u_z^n, for $n \in \mathbb{Z}$, are complex scalar functions of the variables r and z. In the same manner the rotational is written, as the sum of its symmetric part:

$$\operatorname{rot}_r u^{\mathrm{s}} = \begin{bmatrix} 0 \\ \dfrac{\partial u_r^0}{\partial z} - \dfrac{\partial u_z^0}{\partial r} \\ 0 \end{bmatrix} + \sum_{n\geq 1} \begin{bmatrix} -\left(\dfrac{n}{r}u_z^n + \dfrac{\partial u_\theta^n}{\partial z}\right)\sin n\theta \\ \left(\dfrac{\partial u_r^n}{\partial z} - \dfrac{\partial u_z^n}{\partial r}\right)\cos n\theta \\ \left(\dfrac{\partial u_\theta^n}{\partial r} + \dfrac{1}{r}(u_\theta^n + n u_r^n)\right)\sin n\theta \end{bmatrix} \tag{11}$$

and the rotational of its antisymmetric part:

$$\operatorname{rot}_r u^{\mathrm{a}} = \begin{bmatrix} -\dfrac{\partial u_\theta^0}{\partial z} \\ 0 \\ \dfrac{\partial u_\theta^0}{\partial r} + \dfrac{1}{r}u_\theta^0 \end{bmatrix} + \sum_{n\geq 1} \begin{bmatrix} \left(\dfrac{n}{r}u_z^{-n} - \dfrac{\partial u_\theta^{-n}}{\partial z}\right)\cos n\theta \\ \left(\dfrac{\partial u_r^{-n}}{\partial z} - \dfrac{\partial u_z^{-n}}{\partial r}\right)\sin n\theta \\ \left(\dfrac{\partial u_\theta^{-n}}{\partial r} + \dfrac{1}{r}(u_\theta^{-n} - n u_r^{-n})\right)\cos n\theta \end{bmatrix} \tag{12}$$

We define and use indifferently the notation $(.,.)$ for the inner product of L^2 as well for E as for its equivalent u from (4), and we note

$$\begin{aligned}(E,F) &= \int_O E\cdot F\,dxdydz = (u,v) = \int_{O'} u\cdot v\,rdrdzd\theta \\ &= \int_0^{2\pi}\int_\Omega u\cdot v\,rdrdzd\theta \end{aligned} \tag{13}$$

then we have:

$$(\operatorname{rot}E, \operatorname{rot}F) = (\operatorname{rot}_r u, \operatorname{rot}_r v) = \int_0^{2\pi}\int_\Omega \operatorname{rot}_r u\cdot\operatorname{rot}_r v\,rdrdzd\theta \tag{14}$$

Because vanishes the integral over the interval $]0, 2\pi[$, of the real functions: $\cos n\theta.\cos m\theta$, $\sin n\theta.\sin m\theta$, for integers $m \neq n$, and $\cos n\theta.\sin m\theta$, for $m = n$, we can write the above scalar products:

$$(E,F) = (u,v) = \sum_{n\in\mathbb{Z}} \int_0^{2\pi}\int_\Omega u^n\cdot v^n\,rdrdzd\theta,$$

$$(\operatorname{rot}E, \operatorname{rot}F) = (\operatorname{rot}_r u, \operatorname{rot}_r v) = \sum_{n\in\mathbb{Z}} \int_0^{2\pi}\int_\Omega \operatorname{rot}_r u^n\cdot\operatorname{rot}_r v^n\,rdrdzd\theta$$

If we introduce the classical decomposition for j:

$$j(r,\theta,z) = \sum_{n\in\mathbb{Z}} j^n(r,z)\mathrm{e}^{\mathrm{i}n\theta} = \sum_{n\in\mathbb{Z}} \begin{bmatrix} j_r^n(r,z) \\ j_\theta^n(r,z) \\ j_z^n(r,z) \end{bmatrix} \mathrm{e}^{\mathrm{i}n\theta}$$

the problem associated to (7) can be split into independent problems for each Fourier term, namely for each $n \in \mathbb{Z}$:

$$\begin{aligned} -\omega^2\varepsilon_0\mu_0 \int_{O'} u^n \cdot v^n \, rdrdzd\theta + \int_{O'} \mathrm{rot}_r u^n \cdot \mathrm{rot}_r v^n \, rdrdzd\theta \\ = -\mathrm{i}\omega\mu_0 \int_{O'} j^n \cdot v^n \, rdrdzd\theta \end{aligned}$$

If we suppress in (7) the term equal to π from the integration in the θ variable, we deduce the variational formulation of Maxwell equation for the nth-Fourier term for each $n \in \mathbb{Z}$:

$$\begin{aligned} & -\omega^2\varepsilon_0\mu_0 \int_\Omega (u_r^n \cdot v_r^n + u_\theta^n \cdot v_\theta^n + u_z^n \cdot v_z^n) \, rdrdz \\ & + \int_\Omega \left(\frac{\partial u_\theta^n}{\partial r} + \frac{1}{r}(u_\theta^n + nu_r^n)\right)\left(\frac{\partial v_\theta^n}{\partial r} + \frac{1}{r}(v_\theta^n + nv_r^n)\right) rdrdz \\ & + \int_\Omega \left(\frac{\partial u_r^n}{\partial z} - \frac{\partial u_z^n}{\partial r}\right)\left(\frac{\partial v_r^n}{\partial z} - \frac{\partial v_z^n}{\partial r}\right) rdrdz \\ & + \int_\Omega \left(\frac{n}{r}u_z^n + \frac{\partial u_\theta^n}{\partial z}\right)\left(\frac{n}{r}v_z^n + \frac{\partial v_\theta^n}{\partial z}\right) rdrdz \\ & = -\mathrm{i}\omega\mu_0 \int_\Omega (j_r^n \cdot v_r^n + j_\theta^n \cdot v_\theta^n + j_z^n \cdot v_z^n) \, rdrdz \end{aligned} \tag{15}$$

We call the problem associated to (15) the *Maxwell-Fourier problem* for the *nth*-mode.

3. Definitions of the space of solutions

3.1. Definitions and recalls

We note $\mathcal{D}(\Omega)$, the linear space of C^∞ functions with compact support on Ω, $\mathcal{D}'(\Omega)$ the space of distributions over Ω and $L^2(\Omega)$ the Hilbert space of square integrable functions on Ω with respect to the Lebesgue $drdz$-measure, equipped with the norm:

$$|u| = |u|_{L^2(\Omega)} = \left(\int_\Omega u^2(r,z)\,drdz\right)^{1/2}$$

We define the weighted Sobolev space of distributions on Ω,[1]:

$$L_1^2(\Omega) = \left\{u \in L^2(\Omega), \sqrt{r}u \in L^2(\Omega)\right\} \tag{16}$$

equipped with the following norm:

$$|u\ |_1 = \left(\int_\Omega u^2(r,z)\, r dr dz \right)^{1/2} \tag{17}$$

Remark 1. We shall write this norm, depending on the context: $|u\ |_1 = |u\ |_{L^2_1(\Omega)} = |u\ |_{L^2_1}$ and if necessary: $|u\ |_{(L^2_1(\Omega))^k} = |u\ |_{(L^2_1)^k}$ the corresponding vectorial norms, for $k = 2$ or 3.

$L^2_1(\Omega)$ is an Hilbert space for the scalar product associated to $|\ \ |_1$. The continuity of the function r in the $rdrdz$-measure implies the

Lemma 1. *$\mathcal{D}(\Omega)$ is dense in $L^2_1(\Omega)$ for the norm $|\ \ |_1$.*

We use also the space:
$L^\infty(\Omega) = \{u, drdz-$measurable with $|u| \le C$ a.e., C constant $\}$ with the norm: $|\ \ |_\infty = \inf\{C, |u| \le C$ a.e. on Ω $\}$. We shall need some classical functional Sobolev space:

Definition 1. Let $\alpha \in \mathbb{R}$ and $k, \beta_1, \beta_2 \in \mathbb{N}$, with $\beta = (\beta_1, \beta_2)$ and $|\beta| = \beta_1 + \beta_2$. We define:
$W^k_\alpha(\Omega) = \{u \in \mathcal{D}'(\Omega) : r^\alpha D^\beta u \in L^2(\Omega), |\beta| \le k\}$

equipped with the semi-norm:

$$|u|_{W^k_\alpha(\Omega)} = \left(\sum_{|\beta| \le k} |r^\alpha D^\beta u|^2_{L^2(\Omega)} \right)^{1/2}$$

This is an Hilbert space with the norm:

$$\| u \|_{W^k_\alpha(\Omega)} = \left(\sum_{|\beta| \le k} |r^\alpha D^\beta u|^2_{L^2(\Omega)} \right)^{1/2}$$

We have the following density result.

Lemma 2. *Let $k \in \mathbb{N}, \alpha \in \mathbb{R}$, then*
(i) if $k \ge 1$, the set $\mathcal{D}(\bar{\Omega})$ of the restrictions to Ω of functions of $\mathcal{D}(\mathbb{R}^2_+)$ is dense in $W^k_\alpha(\Omega)$, for $\alpha \le -\frac{1}{2}$ or $k \le \alpha + \frac{1}{2}$.
(ii) if $\alpha + \frac{1}{2} > 0, \mathcal{D}(\bar{\Omega})$ is dense in $W^k_\alpha(\Omega)$

Proof. See [2][8]. □

Theorem 1. *Let $k \in \mathbb{N}, k \ge 2$, we have the following continuous imbedding*

$$W^k_{1/2}(\Omega) \subsetneq L^2(\Omega))$$

Proof. See [10]. □

Defining $\mathcal{P}_k(\Omega)$, as the set of polynomials in the r, z variables of order less or equal to k, we have a result analogous to the classic case, but here for the weighted measure $rdrdz$:

Theorem 2. *Let $k = 2$ or 3, and X a Banach space so that we have the continuous imbedding*

$$W^k_{1/2}(\Omega) \subsetneq X$$

let Π be a linear continuous operator from $W^k_{1/2}(\Omega)$ into X such that

$$(I - \Pi)p = 0, \textit{for all } p \in \mathcal{P}_{k-1}(\Omega),$$

then there exists a constant $C > 0$ such that for all $u \in W^k_{1/2}(\Omega)$, we have

$$|u - \Pi u\ |_X \leq C|u\ |_{W^k_{1/2}(\Omega)}$$

Proof. See [10]. □

3.2. Study of a weighted Sobolev space

We only consider the Maxwell-Fourier problem for the mode $n = 1$, because we shall deduce naturally the properties of the solutions for the others modes $n \geq 1$, and in the same way for the modes $n \leq -1$ and $n = 0$. We consider an axisymmetric domain Ω, that encounters the z axis $\{r = 0\}$, and whose regular boundary is constituted by Γ_0, its intersection with $\{r = 0\}$ and by Γ in the half-plane $\{r \geq 0\}$, itself with a part in $\{z = z_1\}$ and a part in $\{z = z_2\}$, a shown in Fig. 2.

Here we introduce some weighted Sobolev space which provides the right framework of these study. We seek for solutions of (15), with $n = 1$, in the space denoted by $\mathcal{H}(\Omega)$ or $\mathcal{H}$:

$$\begin{aligned}&\mathcal{H}(\Omega) = \{(u_r, u_\theta, u_z) \in \left(L^2_1(\Omega)\right)^3, \text{such that}\\ &\left(\frac{u_z}{r} + \frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \frac{u_r + u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) \in \left(L^2_1(\Omega)\right)^3\}\end{aligned} \tag{18}$$

We can define differently the space of solution. To this end we introduce

$$U = \begin{bmatrix} U_r \\ U_z \end{bmatrix} = \begin{bmatrix} \dfrac{u_r + u_\theta}{r} + \dfrac{\partial u_\theta}{\partial r} \\ \dfrac{u_z}{r} + \dfrac{\partial u_\theta}{\partial z} \end{bmatrix} \text{ and } u_m = \begin{bmatrix} u_r \\ u_z \end{bmatrix} \tag{19}$$

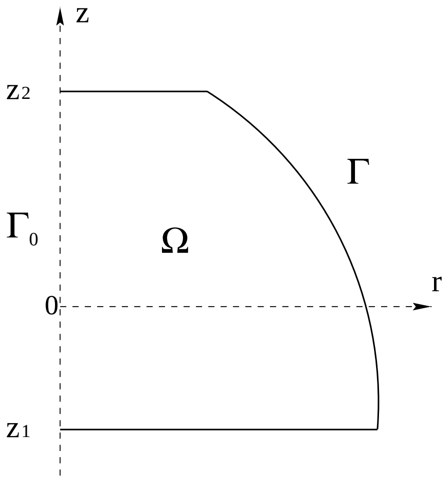

Fig. 2.

from this we deduce certain relations like: $\text{grad}(ru_\theta) = rU - u_m$, or $\text{rot}(rU) = \text{rot}u_m = \frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z}$. (grad and rot considered here in cartesian coordinates in the rz-plane). Then we have also:

$$\mathcal{H}(\Omega) = \left\{(u_r, u_\theta, u_z) \in \left(L^2_1(\Omega)\right)^3\right\}, U \in \left(L^2_1(\Omega)\right)^2, \text{rot}u_m \in L^2_1(\Omega)\}$$

Remark 2. When the closure of the open set Ω is strictly contained in $\mathbb{R}^2_+$, we have:

$$\begin{aligned}\mathcal{H}(\Omega) = \{(u_r, u_\theta, u_z) &\in (L^2(\Omega))^3\}, u_\theta \\ &\in H^1(\Omega), (u_r, u_z) \in H(\text{rot}, \Omega)\}\end{aligned}$$

or also formally (interchanging the first and second component of u)

$$\mathcal{H}(\Omega) = H^1(\Omega) \times H(\text{rot}, \Omega)$$

That means that in the case of a toroidal axisymmetric domain, the Hilbert space $\mathcal{H}$ will separate into a product of classical Sobolev spaces.

Remark 3. We can also imagine to take as principle unknows: (U, u_θ) or (U_r, u_θ, U_z), and work with the space

$$\begin{aligned}\mathcal{H}_1(\Omega) = \{(U_r, u_\theta, U_z) &\in (L^2_1(\Omega))^3, \text{grad}(ru_\theta) \\ &\in (L^2_1(\Omega))^2, \text{rot}(rU) \in L^2_1(\Omega)\}\end{aligned}$$

If we define H^1_1 relative to H^1, as L^2_1 relative to L^2 and:

$$H_1(\text{rot}_r, \Omega) = \{(u_r, u_z) \in (L^2_1(\Omega))^2, \text{rot}_r(u_r, u_z) \in L^2_1(\Omega)\}$$

we even can consider the following space of solutions:

$$\mathcal{H}_1(\Omega) = \{(U_r, u_\theta, U_z) \in (L^2_1(\Omega))^3, ru_\theta \in H^1_1(\Omega), rU \in H_1(\mathrm{rot}_r, \Omega)\}$$

Such a change of unknowns, relieve clearly the notations and transforms the bilinear form of the initial problem in:

$$-\omega^2\varepsilon_0\mu_0 \int_\Omega (U \cdot U' + u_\theta \cdot u'_\theta)\, rdrdz \;+ \int_\Omega (\mathrm{rot}(rU) \cdot \mathrm{rot}(rU') \\ +\mathrm{grad}(ru_\theta) \cdot \mathrm{grad}(ru'_\theta))\, rdrdz$$

We shall use this change of unknowns to determine basis functions of approximation of $\mathcal{H}$ and linear form of interpolation.

We provide $\mathcal{H}$ with the following norm:

(20) $$\| u \|^2_{\mathcal{H}} = |u_r|^2_1 + |u_\theta|^2_1 + |u_z|^2_1 \\ + \left| \frac{u_r + u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right|^2_1 + \left| \frac{u_z}{r} + \frac{\partial u_\theta}{\partial z} \right|^2_1 + \left| \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right|^2_1$$

It is clear that we have

Proposition 1. *$\mathcal{H}$ is an Hilbert space for the norm $\| \, \|_{\mathcal{H}}$.*

We have the following density result:

Proposition 2. *the sub-space of functions of $(\mathcal{D}(\bar{\Omega}))^3$ vanishing near $\{r = 0\}$ is dense in $\mathcal{H}$ equipped with the norm $\| \, \|_{\mathcal{H}}$.*

Proof. It is a consequence of the two following lemmas:

Lemma 3. *the functions of $(L^\infty(\Omega))^3 \cap \mathcal{H}$ that vanish near $\{r = 0\}$, form a sub-space $\mathcal{E}$ dense in $\mathcal{H}$, equipped with the topology of $\mathcal{H}$.*

Proof. The technique of the proof is the same than in Proposition 2.4 of [9]. First we show that the functions of $(L^\infty(\Omega))^3 \cap \mathcal{H}$ are dense in $\mathcal{H}$. Let $u \in (L^\infty(\Omega))^3 \cap \mathcal{H}$. We define: $\Omega_{rk} = \{(r, z), |u_r(r, z)| \leq k\}$, and in the same manner $\Omega_{\theta k}$ and Ω_{zk}, and the following functions:

$$u_k(r,z) = \begin{bmatrix} u_{rk}(r,z) \\ u_{\theta k}(r,z) \\ u_{zk}(r,z) \end{bmatrix} \text{ where } u_{rk}(r,z) = \left| \begin{array}{c} u_r(r,z) \text{ if } (r,z) \in \Omega_{rk} \\ k \text{ if } u_r(r,z) \geq k \\ -k \text{ if } u_r(r,z) \leq -k \end{array} \right.$$

and in the same way $u_{\theta k}(r, z)$ and $u_{zk}(r, z)$. We see that $u_k \in \mathcal{H}$ and by construction:

$$\| u_k \|_{\mathcal{H}} \leq \| u \|_{\mathcal{H}}$$

since on the other hand we have:

$$|u_{rk}(r,z) - u_r(r,z)|^2 \leq 4|u_r(r,z)|^2$$

and that $(u_{rk}(r,z) - u_r(r,z)) \to 0$ a.e. , we see by Lebesgue theorem that $(u_{rk} - u_r) \to 0$ in norm $|\ \ |_1$. And for the same reasons: $(u_{\theta k} - u_\theta) \to 0$, $(u_{zk} - u_z) \to 0$ in L^2_1 as well as the sequences:
$\frac{1}{r}(u_{rk} + u_{\theta k} + r\frac{\partial u_{\theta k}}{\partial r} - u_r - u_\theta - r\frac{\partial u_\theta}{\partial r}) \to 0$ in norm $|\ \ |_1$ when $k \to \infty$
$\frac{1}{r}(u_{zk} + r\frac{\partial u_{\theta k}}{\partial z} - u_z - r\frac{\partial u_\theta}{\partial z}) \to 0$ in norm $|\ \ |_1$ when $k \to \infty$
$\frac{\partial u_{rk}}{\partial z} - \frac{\partial u_{zk}}{\partial r} - \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}) \to 0$ in norm $|\ \ |_1$ when $k \to \infty$
Finally $(u_k - u) \to 0$ in norm $\|\ \|_{\mathcal{H}}$. Now we show the density for the functions belonging to $\mathcal{E}$. We give us $\varepsilon > 0$. Let $\Psi_\varepsilon(r)$ be the function equal to zero for $r < \varepsilon$, equal to 1 for $r > 2\varepsilon$, and equal to $(\frac{r}{\varepsilon} - 1)$ if $\varepsilon \le r \le 2\varepsilon$. For $u \in (L^\infty(\Omega))^3 \cap \mathcal{H}$, we pose: $u_\varepsilon = \Psi_\varepsilon u = (u_{r\varepsilon}, u_{\theta\varepsilon}, u_{z\varepsilon})$. Then $u_\varepsilon \in (L^2_1)^3$ and also the three component of the rotationnal:
$\frac{1}{r}(u_{r\varepsilon} + u_{\theta\varepsilon} + r\frac{\partial u_{\theta\varepsilon}}{\partial r})$, $\frac{1}{r}(u_{z\varepsilon} + r\frac{\partial u_{\theta\varepsilon}}{\partial z})$ and $\frac{\partial u_{r\varepsilon}}{\partial z} - \frac{\partial u_{z\varepsilon}}{\partial r}$.
It is clear that we have $|(u_{r\varepsilon} - u_r)\ |_1 \to 0$, $|(u_{\theta\varepsilon} - u_\theta)\ |_1 \to 0$ and $|(u_{z\varepsilon} - u_z)\ |_1 \to 0$ when $\varepsilon \to 0$ as well as the following limits:

$$\left|\frac{u_{r\varepsilon} + u_{\theta\varepsilon}}{r} + \frac{\partial u_{\theta\varepsilon}}{\partial r} - \frac{u_r + u_\theta}{r} - \frac{\partial u_\theta}{\partial r}\right|_1 \to 0$$

$$\left|\frac{u_{z\varepsilon}}{r} + \frac{\partial u_{\theta\varepsilon}}{\partial z} - \frac{u_z}{r} - \frac{\partial u_\theta}{\partial z}\right|_1 \to 0$$

$$\left|\frac{\partial u_{r\varepsilon}}{\partial z} - \frac{\partial u_{z\varepsilon}}{\partial r} - \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right|_1 \to 0$$

We must establish that: $|u_\theta \frac{\partial \Psi_\varepsilon}{\partial_r}\ |_1 \to 0$ and $|u_z \frac{\partial \Psi_\varepsilon}{\partial_r}\ |_1 \to 0$. It is sufficient to prove the result for the first of the two latter integrals. Now we have:

$$\int_\Omega \left(\frac{\partial \Psi_\varepsilon}{\partial r}\right)^2 u_\theta^2\, r dr dz = \int_{z_2}^{z_1}\int_\varepsilon^{2\varepsilon} \frac{u_\theta^2}{\varepsilon^2}\, r dr dz \le \frac{3(z_2 - z_1)}{2} |u_\theta\ |^2_\infty$$

That proves that the sequence $u_\theta \frac{\partial \Psi_\varepsilon}{\partial_r}$ is bounded in $L^2_1(\Omega)$. Therefore there exists a sub-sequence, also noted $u_\theta \frac{\partial \Psi_\varepsilon}{\partial_r}$, which weakly converges to a function $w \in L^2_1(\Omega)$ when ε tend to 0. This sequence converges to 0 in $\mathcal{D}'(\Omega)$. We have for all Φ in $\mathcal{D}(\Omega)$:

$$\left|< u_\theta \frac{\partial \Psi_\varepsilon}{\partial r}, \Phi >\right| = \left|\int_\Omega u_\theta \frac{\partial \Psi_\varepsilon}{\partial r} \Phi\, dr dz\right| \le \frac{|u_\theta\ |_\infty}{\varepsilon} \left|\int_{z_1}^{z_2}\int_\varepsilon^{2\varepsilon} \Phi\, dr dz\right|$$

and since for $r \in]0, 2\pi[$, we have:

$$|\Phi(r,z)\ | = |\Phi(r,z) - \Phi(0,z)| \le r \sup_{r,z}\left|\frac{\partial \Phi(r,z)}{\partial r}\right|$$

that implies: $| < u_\theta \frac{\partial \Psi_\varepsilon}{\partial r}, \Phi > | \leq |u_\theta|_\infty (z_2 - z_1) \sup_{r,z} \left| \frac{\partial \Phi(r,z)}{\partial r} \right| \frac{3\varepsilon}{2}$
Then $|u_\theta \frac{\partial \Psi_\varepsilon}{\partial r}|_1 \to 0$ and in the same manner $|u_z \frac{\partial \Psi_\varepsilon}{\partial r}|_1 \to 0$
Therefore u_ε tends weakly to u in $\mathcal{H}$, when ε tend to 0. And finally the space $\mathcal{E}$ is dense in $\mathcal{H}$, because it is weakly dense, by vertue of Mazur theorem [3]. □

Lemma 4. *The space* $(\mathcal{D}(\bar{\Omega}))^3 \cap \mathcal{E}$ *is dense in* $\mathcal{E}$ *for the norm* $\| \|_{\mathcal{H}}$.

Proof. According to Remark 2, a function of $(\mathcal{D}(\bar{\Omega}))^3 \cap \mathcal{E}$ is in $H^1(\Omega) \times H(\mathrm{rot}, \Omega)$, if we permute two variables. Therefore the density result is classical [6]. □

3.3. Green formula

To characterise $\mathcal{H}$ we have to employ Green formula. We denoted by $(n_r^\Gamma, 0, n_z^\Gamma)$ the unit outward normal to the boundary ∂O in the plane $\{\theta = 0\}$ of the initial open set O. We can show that for u and $v \in (\mathcal{D}(\bar{\Omega}))^3 \cap \mathcal{H}$, we have the integral by parts formula:

$$\begin{aligned} \int_\Omega \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} \cdot \begin{bmatrix} -\frac{v_z}{r} - \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \\ \frac{\partial v_\theta}{\partial r} + \frac{v_\theta + v_r}{r} \end{bmatrix} rdrdz &= \int_\Omega \begin{bmatrix} -\frac{u_z}{r} - \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\ \frac{\partial u_\theta}{\partial r} + \frac{u_\theta + u_r}{r} \end{bmatrix} \cdot \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} rdrdz \\ &+ \int_\Gamma \left(r \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} \wedge \begin{bmatrix} n_r^\Gamma \\ 0 \\ n_z^\Gamma \end{bmatrix} \right) \cdot \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} d\Gamma \end{aligned} \tag{21}$$

in which disappears the integral term along Γ_0, because of the weight r.

3.4. Trace theorem

We consider an open subset Ω of the $\mathbb{R}_+^2$ plane, which boundary $\Gamma_0 \cup \Gamma$ as in Sect. 3.2 and for which we give us a real $r_0 > 0$ so that $\{r = r_0\}$, $\{z = z_1\}$ and $\{z = z_2\}$, determine the five pieces of frontier that form Γ. So we define: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$, with $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \subset \{r > r_0\}$ as in Fig. 3.

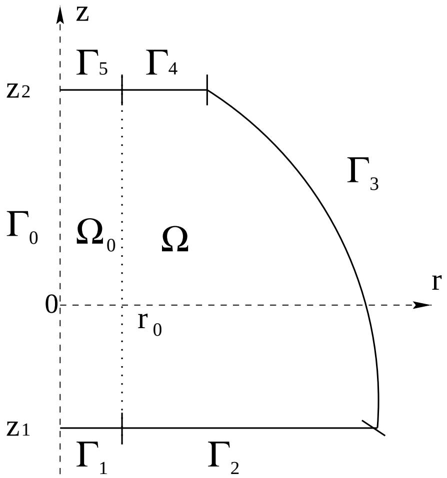

Fig. 3.

Theorem 3. *If u is a function of $\mathcal{H}$, if we note $n^\Gamma = (n_r^\Gamma, n_z^\Gamma)$, we can define the trace of the component $r(u_r, u_z) \wedge n^\Gamma{}_{|\Gamma}$ in $L^2(\Gamma)$ and the trace of the component $ru_{\theta|\Gamma}$ in $L^2(\Gamma)$.*

Proof. The result is obvious for the frontiers Γ_2, Γ_3 and Γ_4 by vertue of Remark 2, of the Green formula of Sect. 3.3 and the classical results of trace theorem about scalar functions with regularity H^1 and vectorial functions with regularity $H(\mathrm{rot}, \Omega)$; then these traces belongs respectively to $H^{1/2}_{\mathrm{loc}}(\Gamma)$ and $H^{-1/2}_{\mathrm{loc}}(\Gamma)$ [6]. Let show the result for frontier Γ_1 or for the similar case of frontier Γ_5. For frontier Γ_1 we prove the following result:

Lemma 5. *If u is a function of $\mathcal{H}$, we can define the trace $ru_{\theta|\Gamma_1}$, as an element of $L^2(\Gamma_1)$.*

Proof. Let $u \in (\mathcal{D}(\bar{\Omega}))^3 \cap \mathcal{H}$. We define first the following domain: $\Omega_0 =]0, r_0[\times]z_1, z_2[$. Let $z \in]z_1, z_2[$, we note $\Omega_{0z} =]0, r_0[\times]z_1, z[$. We have:

$$r^2 u_\theta^2(r, z_1) = r^2 u_\theta^2(r, z) - 2 \int_{z_1}^{z} u_\theta(r, \zeta) \frac{\partial u_\theta(r, \zeta)}{\partial z} r^2 \, d\zeta$$

and then:

$$r^2 u_\theta^2(r, z_1) = r^2 u_\theta^2(r, z) - 2 \int_{z_1}^{z} r u_\theta(r, \zeta) \frac{\partial}{\partial z}(r u_\theta(r, \zeta) + u_z(r, \zeta)) \, d\zeta$$

$$+2 \int_{z_1}^{z} r u_\theta(r, \zeta) u_z(r, \zeta) \, d\zeta$$

thus by integrating with respect to the r variable from 0 to r_0, it comes:

$$\int_0^{r_0} \rho^2 u_\theta^2(\rho, z_1)\, d\rho = \int_0^{r_0} \rho^2 u_\theta^2(\rho, z)\, d\rho + 2\int_{\Omega_{0z}} \rho u_\theta(\rho,\zeta) u_z(\rho,\zeta)\, d\rho d\zeta$$
$$-2\int_{\Omega_{0z}} \rho u_\theta(\rho,\zeta)\frac{\partial}{\partial z}(\rho u_\theta(\rho,\zeta) + u_z(\rho,\zeta))\, d\rho d\zeta$$

We still integrate the previous expression with respect to the z variable from z_1 to z_2, and we use the following estimates, assuming, that is not a restriction, that we have $\rho^2 \leq \rho \leq r_0 \leq 1$:

$$\left|\int_{\Omega_0} \rho^2 u_\theta^2(\rho,\zeta)\, d\rho d\zeta\right| \leq |u_\theta|_1^2$$
$$\left|\int_{\Omega_{0z}} \rho u_\theta(\rho,\zeta)\frac{\partial}{\partial z}(\rho u_\theta(\rho,\zeta) + u_z(\rho,\zeta))\, d\rho d\zeta\right| \leq |u_\theta|_1^2 + \left|\frac{\frac{\partial}{\partial z}(r u_\theta) + u_z}{r}\right|_1^2$$
$$\left|\int_{\Omega_{0z}} \rho u_\theta(\rho,\zeta) u_z(\rho,\zeta)\, d\rho d\zeta\right| \leq |u_\theta|_1^2 + |u_z|_1^2$$

then we assert that there exists a constant C only dependant on Ω_0 and such that:

$$\int_0^{r_0} \rho^2 u_\theta^2(\rho, z_1)\, d\rho \leq C\left(|u_\theta|_1^2 + |u_z|_1^2 + \left|\frac{\frac{\partial}{\partial z}(r u_\theta) + u_z}{r}\right|_1^2\right)$$

that is we have: $\displaystyle\int_0^{r_0} \rho^2 u_\theta^2(\rho, z_1)\, d\rho \leq C \parallel u \parallel_{\mathcal{H}}^2$.
Indeed we can define a trace of $r u_\theta$ on $\{z = z_1\}$ as an element of $L^2(]0, r_0[)$. □

Lemma 6. *If u is a function of $\mathcal{H}$, we can define the trace $u_{r|\Gamma_1}$, that is these of $(r(u_r, u_z) \wedge n^\Gamma)\,|_{\Gamma_1}$, as an element of $L^2(\Gamma_1)$.*

Proof. Let u and v in the space $(\mathcal{D}(\bar{\Omega}))^3 \cap \mathcal{H}$. For r in $]0, r_0[$, in the same way as in Lemma 5, in addition to domain Ω_{0z}, we introduce the domain: $\Omega_{0r} =]0, r[\times]z_1, z_2[$. Let $(r, z) \in \Omega_0$. We have first for all $(\rho, \zeta) \in \Omega_0$:

$$r^2 u_z(r,\zeta) v_\theta(r,\zeta) = \int_0^r \rho^2 v_\theta(\rho,\zeta)\frac{\partial u_z(\rho,\zeta)}{\partial r}\, d\rho$$
$$+ \int_0^r u_z(\rho,\zeta)\frac{\partial}{\partial r}(\rho^2 v_\theta(\rho,\zeta))\, d\rho$$

and also:

$$\rho^2 u_r(\rho, z_1) v_\theta(\rho, z_1) = \rho^2 u_r(\rho, z) v_\theta(\rho, z) - \int_{z_1}^{z} \rho^2 v_\theta(\rho, \zeta) \frac{\partial u_r(\rho, \zeta)}{\partial z}\, d\zeta - \int_{z_1}^{z} u_r(\rho, \zeta) \frac{\partial}{\partial z}(\rho^2 v_\theta(\rho, \zeta))\, d\zeta$$

we integrate the last but one equation, with respect to the ζ variable from z_1 to z, and we integrate the last equation with respect to the ρ variable from 0 to r. Then making the sum and integrating the result, successively with respect to the z variable from z_1 to z_2, and to the r variable from 0 to r_0, it happens the equality:

$$\begin{aligned} & r_0(z_2 - z_1) \int_0^r \rho^2 u_r(\rho, z_1) v_\theta(\rho, z_1)\, d\rho \\ & = \int_0^{r_0}\int_{z_1}^{z_2}\int_0^r \rho^2 u_r(\rho, \zeta) v_\theta(\rho, \zeta)\, d\rho d\zeta dr \\ & \quad - \int_0^{r_0}\int_{z_1}^{z_2}\int_{z_1}^{z} \rho^2 u_z(\rho, \zeta) v_\theta(\rho, \zeta)\, d\rho d\zeta dz \\ & \quad - \int_0^{r_0}\int_{z_1}^{z_2}\int_{\Omega_{0r}\cap\Omega_{0z}} u_r(\rho, \zeta) \frac{\partial}{\partial z}(\rho^2 v_\theta(\rho, \zeta))\, d\rho d\zeta dr dz \\ & \quad + \int_0^{r_0}\int_{z_1}^{z_2}\int_{\Omega_{0r}\cap\Omega_{0z}} u_z(\rho, \zeta) \frac{\partial}{\partial r}(\rho^2 v_\theta(\rho, \zeta))\, d\rho d\zeta dr dz \\ & \quad - \int_0^{r_0}\int_{z_1}^{z_2}\int_{\Omega_{0r}\cap\Omega_{0z}} \rho^2 v_\theta(\rho, \zeta) \mathrm{rot} u(\rho, \zeta)\, d\rho d\zeta dr dz \end{aligned}$$

we write now estimates of each term of the above equality, using Cauchy-Schwarz inequality and using again $\rho^2 \le \rho \le r_0 \le 1$. On one hand for the first:

$$\left| \int_0^{r_0}\int_{z_1}^{z_2}\int_0^r \rho^2 u_r(\rho, \zeta) v_\theta(\rho, \zeta)\, d\rho d\zeta dr \right| \le C_1 |u_r\,|_1^2\, |v_\theta|_1^2$$

and the same for second term:

$$\left| \int_0^{r_0}\int_{z_1}^{z_2}\int_{z_1}^{z} \rho^2 u_z(\rho, \zeta) v_\theta(\rho, \zeta)\, d\rho d\zeta dz \right| \le C_2 |u_z\,|_1^2\, |v_\theta|_1^2$$

for the third term we can write:

$$\begin{aligned} & \left| \int_0^{r_0}\int_{z_1}^{z_2}\int_{\Omega_{0r}\cap\Omega_{0z}} u_r(\rho, \zeta) \frac{\partial}{\partial z}(\rho^2 v_\theta(\rho, \zeta))\, d\rho d\zeta dr dz \right| \\ & \quad \le \left| \int_0^{r_0}\int_{z_1}^{z_2}\int_{\Omega_{0r}\cap\Omega_{0z}} \rho u_r(\rho, \zeta) \left(\frac{\partial \rho v_\theta}{\partial z} + v_z \right)(\rho, \zeta)\, d\rho d\zeta dr dz \right| \end{aligned}$$

$$+\left|\int_0^{r_0}\int_{z_1}^{z_2}\int_{\Omega_{0r}\cap\Omega_{0z}}\rho u_r(\rho,\zeta)v_z(\rho,\zeta)\,d\rho d\zeta drdz\right|$$
$$\leq C_3|u_r\,|_1^2\left(|v_z|_1^2+\left|\frac{\frac{\partial\rho v_\theta}{\partial z}+v_z}{r}\right|_1^2\right)$$

and for the same arguments, the estimates of the fourth term:

$$\left|\int_0^{r_0}\int_{z_1}^{z_2}\int_{\Omega_{0r}\cap\Omega_{0z}}u_z(\rho,\zeta)\frac{\partial}{\partial r}(\rho^2 v_\theta(\rho,\zeta))\,d\rho d\zeta drdz\right|$$
$$\leq C_4|u_z\,|_1^2\left(|v_r\,|_1^2+|v_\theta\,|_1^2+\left|\frac{\frac{\partial\rho v_\theta}{\partial r}+v_r}{r}\right|_1^2\right)$$

finally for the firth term:

$$\left|\int_0^{r_0}\int_{z_1}^{z_2}\int_{\Omega_{0r}\cap\Omega_{0z}}\rho^2 v_\theta(\rho,\zeta)\mathrm{rot}u(\rho,\zeta)\,d\rho d\zeta drdz\right|\leq C_5|v_\theta\,|_1^2\;|\mathrm{rot}u|_1^2$$

where C_1, C_2, C_3, C_4, C_5 are constants only dependant on Ω_0. Finally there exists a constant $C = C(r_0, z_1, z_2)$ only dependant on Ω_0 such that:

$$\left|\int_0^r \rho^2 u_r(\rho,z_1)v_\theta(\rho,z_1)\,d\rho\right|\leq C\parallel u\parallel_{\mathcal{H}}\parallel v\parallel_{\mathcal{H}}.\;\square$$

Remark 4. It is well know that we cannot define traces of functions of $\mathcal{H}_1^1$ (see Remark 3) on the axis Γ_0. Similarly here, we can't define traces on Γ_0 of functions of $\mathcal{H}$. Indeed, the following functions of $\mathcal{H}$, $(v, -v, rw)$, where v and w are regular functions, possesses non identically null trace on Γ_0, if v hav not. That contradict the density result of Proposition 2.

3.5. Characterisation of C^2 functions of $H(\mathrm{rot})$ in axisymmetric geometry

We establish in this paragraph the conditions of equivalence for a C^2-vector function to belong to the Sobolev space $H(\mathrm{rot}, O)$ and for its associated by (4), to belong to space $\mathcal{H}$ in axisymmetric geometry. We study the non-zero Fourier mode. We give before the

Definition 2. We note $\mathcal{H}^n(\Omega)$ the weighted Sobolev space of solution of Maxwell-Fourier problem for the mode $n\in\mathbb{Z}$, as in (18), by the following

$$\mathcal{H}^n(\Omega)=\Bigg\{(u_r,u_\theta,u_z)\in(L_1^2(\Omega))^3\text{such that:}$$
$$\left(\frac{nu_z}{r}+\frac{\partial u_\theta}{\partial z},\frac{\partial u_r}{\partial z}-\frac{\partial u_r}{\partial z},\frac{nu_r+u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)\in(L_1^2(\Omega))^3\Bigg\}$$

We have the following

Proposition 3. *A field E is in $H(\mathrm{rot}, O) \cap (\mathcal{C}^2(O))^3$ if and only if, for all $n \in \mathbb{Z}^*$, $u^n = (u_r^n, u_\theta^n, u_z^n)$ define by (9),(10) is in $\mathcal{H}^n(\Omega)$ and moreover verify the following properties:*

$$\begin{array}{rr} (i) & (nu_r^n + u_\theta^n)_{|\Gamma_0} = 0 \\ (ii) & u^n_{z\,|\Gamma_0} = 0 \end{array} \tag{22}$$

Proof. Let $E \in H(\mathrm{rot}, O) \cap (\mathcal{C}^2(O))^3$. To simplify we make the assumption that E is symmetric and let u associated to E, according to relation (9). Let $F \in \mathcal{D}(\bar{O})^3$ and $v \in \mathcal{D}(\bar{\Omega})^3$ corresponding to F by (9). We have:

$$\int_O \mathrm{rot}E \cdot \mathrm{rot}F \, dxdydz = \int_O \mathrm{rot}\ \mathrm{rot}E \cdot F dxdydz + \int_{\partial O} (\mathrm{rot}E \wedge n^O) \cdot F d(\partial O)$$

The same expression can be written:

$$\int_{O'} \mathrm{rot}_r u \cdot \mathrm{rot}_r v \, r dr dz d\theta = \sum_{n \in \mathbb{Z}} \int_0^{2\pi} \int_\Omega \mathrm{rot}_r^n u \cdot \mathrm{rot}_r v^n \, r dr dz d\theta$$

Let $\varepsilon > 0$. We suppose that Ω is a limit of measurable imbedded open Ω_ε, with boundary $\Gamma \cap \{r > \varepsilon\}$ and bounded on the left by a frontier Γ_ε parallel to the z-axis. Then, following (21) in the case of $\mathcal{H}^n$ all the above equalities becomes in the open Ω of the rz-plane:

$$\begin{aligned} &\int_\Omega \mathrm{rot}_r u^n \cdot \mathrm{rot}_r v \, r dr dz \\ &= \int_\Omega \mathrm{rot}_r \mathrm{rot}_r u^n \cdot v \, r dr dz \\ &+ \int_\Gamma \left(r \begin{bmatrix} -\frac{nu_z^n}{r} - \frac{\partial u_\theta^n}{\partial z} \\ \frac{\partial u_r^n}{\partial z} - \frac{\partial u_z^n}{\partial r} \\ \frac{\partial u_\theta^n}{\partial r} + \frac{u_\theta^n + nu_r^n}{r} \end{bmatrix} \wedge \begin{bmatrix} n_r^\Gamma \\ 0 \\ n_z^\Gamma \end{bmatrix} \right) \cdot \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} d\Gamma \\ &+ \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \left(\varepsilon \begin{bmatrix} -\frac{\partial u_\theta^n}{\partial z} \\ \frac{\partial u_r^n}{\partial z} - \frac{\partial u_z^n}{\partial r} \\ \frac{\partial u_\theta^n}{\partial r} \end{bmatrix} \wedge \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} d\Gamma_\varepsilon \\ &+ \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \begin{bmatrix} -nu_z^n \\ 0 \\ u_\theta^n + nu_r^n \end{bmatrix} \wedge \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) \cdot \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} d\Gamma_\varepsilon \end{aligned}$$

Then it occurs, for all $n \geq 1$ and for all v:

$$\int_{\Gamma_0} \left(\begin{bmatrix} -nu_z^n \\ 0 \\ u_\theta^n + nu_r^n \end{bmatrix} \wedge \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} \, d\Gamma_0 = 0$$

and so we get the two relations (22). □

4. The finite elements of Maxwell-Fourier's equation

We are concerned in this paragraph to construct finite element unisolvent and conforming in the space $\mathcal{H}^n$ for the nth-Fourier mode [7]. To determine such elements, we choose as geometrical elementary domain a triangle K of the rz-plane. We search for approximating space a polynomial space $\mathcal{P}$ of dimension m, and a set of m linear form (or degrees of freedom) and verifying the unisolvence property. Two cases are discussed: the general case $n \geq 1$ and its immediately equivalent $n \leq -1$, and the case $n = 0$ corresponding to the fundamental Fourier mode. For $u^n = (u_r^n, u_\theta^n, u_z^n)$, initial unknown of problem (6), we introduce the two vectors (see (19) and remark 3),

$$U^n = \begin{bmatrix} U_r^n \\ U_z^n \end{bmatrix} = \begin{bmatrix} \dfrac{u_r^n + u_\theta^n}{r} + \dfrac{\partial u_\theta^n}{\partial r} \\ \dfrac{u_z^n}{r} + \dfrac{\partial u_\theta^n}{\partial z} \end{bmatrix} \text{ and } u_m^n = \begin{bmatrix} u_r^n \\ u_z^n \end{bmatrix} \tag{23}$$

We have obviously the propositon, resulting from the definition of $\mathcal{H}^n$

Proposition 4. *A necessary an sufficient condition for u^n to belongs to $\mathcal{H}^n(\Omega)$, for an axisymmetric open Ω, is that the functions u_θ^n and $(\frac{\partial u_r^n}{\partial z} - \frac{\partial u_z^n}{\partial r})$ lie in space $L_1^2(\Omega)$, and that the fields u_m^n and U^n lie in the vectorial space $(L_1^2(\Omega))^2$.*

As a consequence on the Fourier series, we have the fact that for a function or a field to belong to L_1^2, is expressed on the family of coefficient of its Fourier series by the two following conditions: each coefficient of the series belongs to L_1^2, and the numerical series of the square norms in L^2 converge. Therefore the approximation error consists in two imbedded errors. At an upper level the error resulting from the truncature of Fourier series. This estimate is relative to the series that describe both u^n and $\mathrm{rot}_r u^n$, and whom coefficients (exacts) belongs to space L_1^2. We don't consider this in this study. And a lower level of error, those of finite elements. The estimates to be obtain, in the sense of the L_1^2 norm, concern the respective coefficient of u^n and $\mathrm{rot}_r u^n$, that are 6 functions by mode, or also 3 coefficients for u^n and 3 for $\mathrm{rot}_r u^n$. As an assumption we attend to define an approximation of the unknowns with *polynomial* in the r and in the z variables. To this end we introduce the notion of *principal unknowns*.

Definition 3. The unknowns supposed polynomial in the r and z variables, are said *principal*, when the others functions which requires to belong to L^2_1, are also polynomial because of their own expression in the unknowns.

It is not the case for the initial unknowns $(u^n_r, u^n_\theta, u^n_z)$ because of the presence of the r variable at the denominator of $\mathrm{rot}_r u$.

4.1. Modes of rank other than zero

We choose as principal unknowns for theses modes, the functions: $(U^n_r, u^n_\theta, U^n_z)$. For $n \geq 1$, we have

$$U^n_r = \frac{1}{r}\left(nu^n_r + \frac{\partial(ru^n_\theta)}{\partial r}\right) \text{ or } u^n_r = \frac{1}{n}\left(rU^n_r - \frac{\partial(ru^n_\theta)}{\partial r}\right)$$
$$U^n_z = \frac{1}{r}\left(nu^n_z + \frac{\partial(ru^n_\theta)}{\partial z}\right) \text{ or } u^n_z = \frac{1}{n}\left(rU^n_z - \frac{\partial(ru^n_\theta)}{\partial z}\right)$$

In addition we have 3 conditions of belonging to L^2_1 and relative to u^n_r, u^n_z and $(\frac{\partial u^n_r}{\partial z} - \frac{\partial u^n_z}{\partial r})$.
For u^n_r, u^n_z, theses conditions reduce to the belonging to L^2_1 of the gradient of (ru^n_θ). And for the last condition we have

$$\left(\frac{\partial u^n_z}{\partial r} - \frac{\partial u^n_r}{\partial z}\right) = \frac{1}{n}\mathrm{rot}(rU^n)$$

Theses expressions are polynomial as soon as u^n_θ, U^n_r, U^n_z themselves are. It is an analogous situation for the coefficients of rank $n \geq 1$, for antisymmetric part. It is of course still the same for $n \leq -1$.

4.2. Modes of rank zero

Symmetric part of U^0. For this mode the number of initial unknowns is two (u^0_r, u^0_z). There are principals since the only square integrable condition (with weight r) affects the scalar function $(\frac{\partial u^0_r}{\partial z} - \frac{\partial u^0_z}{\partial r})$. Here the practical situation is the same than in classical one dimensional cartesian problem, therefore we shall use standart finite element.
Antisymmetric part of U^0. The unique unknown of the problem is the function u^0_θ since $u^0_r = u^0_z = 0$, and the plane field U^0 reduce to $\mathrm{grad}(u^0_\theta)$. This

unknown is not principal because we have the term $\frac{u_\theta^0}{r}$ that appears in U^0. The natural unknown is then

$$\eta^0 = \frac{1}{r} u_\theta^0$$

since $u_\theta^0 = r\eta^0$ and $U^0 = \frac{1}{r}\mathrm{grad}(r^2\eta^0)$ are polynomial as η^0 is polynomial. It is clear that the previous results provide a general way to construct finite elements in the principal unknowns U^n and u_θ^n. For every one we use respectively standart finite elements for the initial unknown u^n, only with polynomials of order less or equal than 2.

4.3. Construction of finite elements for Maxwell-Fourier problem

Its follows from the previous paragraphs that, in a sense, we have separated the variables of the problem, into a real unknown u_θ and a vectorial unknown U. Therefore a study of finite elements adapted to the problem, results in the approximation of u_θ using finite elements of class H^1 and the approximation of U using finite elements of class $H(\mathrm{rot})$. This is what we propose now by producing finite elements with polynomials in the rz-variables of degrees two, but which can be generalized to any degree.
We consider first the case $|n| \geq 1$. Let K be a triangle of the rz-plane. According to the preeceding statement, we are entitled to consider as set of degrees of freedom, acting on functions of components u_r, u_θ, u_z of an $u \in \mathcal{H}^n(\Omega)$, the following (we omit for convenience in the sequel the index n for u^n and U^n):

(24) $$\sigma_\theta : u \to u_\theta(a)$$

if a is a vertex of the triangle K, associated with

(25) interpolation polynomial $\mathcal{P}_2$ of order ≤ 2

On the other hand if Γ is an edge of K and τ an unit vector to Γ, we are able to consider circulation of U along Γ. That is we can take as degrees of freedom the set of the following linear form:

(26) $$\sigma_\Gamma : u \to \int_\Gamma \begin{bmatrix} \dfrac{nu_r + u_\theta}{r} + \dfrac{\partial u_\theta}{\partial r} \\ \dfrac{nu_z}{r} + \dfrac{\partial u_\theta}{\partial z} \end{bmatrix} \cdot \tau \, d\Gamma$$

In fact it appears more convenient (and equivalent) to use instead of the above circulation the following:

(27) $$\sigma_\Gamma : u \to \int_\Gamma \begin{bmatrix} \dfrac{nu_r + u_\theta}{r} \\ \dfrac{nu_z}{r} \end{bmatrix} \cdot \tau \, d\Gamma$$

Moreover we select as space of polynomial the following second order polynomial set of class $H(\mathrm{rot}, \Omega)$, defined by Nédélec [11]:

$$\mathcal{R} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} r \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ r \end{bmatrix}, \begin{bmatrix} z \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ z \end{bmatrix}, \begin{bmatrix} z^2 \\ -rz \end{bmatrix}, \begin{bmatrix} -rz \\ r^2 \end{bmatrix} \right\} \tag{28}$$

which is a vectorial space of polynomials of dimension 8.
We suppose that u, with components u_r, u_θ, u_z, and U like above (23) are polynomials of degree 2 and accordingly defined by 18 coefficients. We prescribe that:

$$\begin{bmatrix} U_r \\ U_z \end{bmatrix} \in \mathcal{R} \tag{29}$$

So such a polynomial is then defined by 14 coefficients. Finally using Proposition 3 and taking into account conditions (22.i) and (22.ii), that furnished five others relations on the coefficients, we are led to consider as set of polynomial for Maxwell-Fourier equation, vectors u defined by 9 coefficients and of the form:

$$u = \begin{bmatrix} \alpha_1 + \alpha_4 r + \alpha_3 z + \alpha_8 z^2 - \alpha_6 rz \\ -n\alpha_1 + \alpha_2 r - n\alpha_3 z + \alpha_7 r^2 - n\alpha_8 z^2 + \alpha_9 rz \\ \alpha_5 r + \alpha_6 r^2 - \alpha_8 rz \end{bmatrix} \tag{30}$$

where the α_{i} are constants.
Now, we are in a position to introduce the finite element:

Proposition 5. ***(*finite element with 9 d.o.f., for** $|n| \geq 1)$
The following finite element $(\sum^n, K, \mathcal{P}^n)$ *is unisolvent and conforming in* $\mathcal{H}^n(\Omega)$*:*

- *K: triangle with vertices* a_{i}*, with medium point to the edges* b_{i} *and with edges* Γ_{i} *with tangent vector* τ_{i}*, for* $1 \leq i \leq 3$
- $\mathcal{P}^n$*: space of polynomials defined by* (30)
- $\sum^n$*: set of 9 linear form, for* $1 \leq i \leq 3$ *:*

$$\left| \begin{array}{l} \sigma_{\theta_{\mathrm{i}}} : u \to u_\theta(a_{\mathrm{i}}) \\ \bar{\sigma_{\theta_{\mathrm{i}}}} : u \to u_\theta(b_{\mathrm{i}}) \\ \sigma_{\Gamma_{\mathrm{i}}} : u \to \int_{\Gamma_{\mathrm{i}}} \begin{bmatrix} \dfrac{nu_r + u_\theta}{r} \\ \dfrac{nu_z}{r} \end{bmatrix} \cdot \tau_{\mathrm{i}} \, d\Gamma_{\mathrm{i}} \end{array} \right. \tag{31}$$

Proof. We have nothing to prove, because of the construction, except to refer to [4] and [11]. □

If we impose in the set of polynomials of Proposition 5, the relations $\alpha_7 = \alpha_8 = \alpha_9 = 0$, we obtain the more simplest finite elements of Maxwell-Fourier for $|n| \geq 1$:

Proposition 6. **(finite element with 6 d.o.f., for** $|n| \geq 1$)
The following finite element $(\sum^n, K, \mathcal{P}^n)$ *is unisolvent and conforming in* $\mathcal{H}^n(\Omega)$*:*

- *K: triangle with vertices* $a_{\rm i}$*, with edges denoted* $\Gamma_{\rm i}$ *with tangent vector* $\tau_{\rm i}$*, for* $1 \leq i \leq 3$
- $\mathcal{P}^n$*: space of polynomials defined by*

$$u = \begin{bmatrix} \alpha_1 + \alpha_4 r + \alpha_3 z - \alpha_6 rz \\ -n\alpha_1 + \alpha_2 r - n\alpha_3 z \\ \alpha_5 r + \alpha_6 r^2 \end{bmatrix} \tag{32}$$

- $\sum^n$*: set of 6 linear form, for* $1 \leq i \leq 3$ *:*

$$\left| \begin{array}{l} \sigma_{\theta_{\rm i}} : u \to u_\theta(a_{\rm i}) \\ \sigma_{\Gamma_{\rm i}} : u \to \int_{\Gamma_{\rm i}} \begin{bmatrix} \dfrac{nu_r + u_\theta}{r} \\ \dfrac{nu_z}{r} \end{bmatrix} \cdot \tau_{\rm i} \, d\Gamma_{\rm i} \end{array} \right. \tag{33}$$

We define now finite element for the fundamental Fourier mode $n = 0$. This case, corresponding to a purely axisymmetric problem, led to the following relations:

$u = \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix}$ is the unknown and u_r, u_θ, u_z are only functions of the r and z variable. Here we can see that the unknown u allows us to separate Maxwell equation (1), or more precisely our model problem (2), into two distinct problems:

- u_θ is a scalar solution of the wave equation with variables r and z
- (u_r, u_z) is a vector solution of the following system:

$$\begin{cases} -\omega^2\varepsilon_0\mu_0 u_r - \dfrac{\partial}{\partial z}\left(\dfrac{\partial u_r}{\partial z} - \dfrac{\partial u_z}{\partial r}\right) = -\mathrm{i}\omega\mu_0 j_r \\ -\omega^2\varepsilon_0\mu_0 u_z + \dfrac{1}{r}\dfrac{\partial}{\partial r} r\left(\dfrac{\partial u_r}{\partial z} - \dfrac{\partial u_z}{\partial r}\right) = -\mathrm{i}\omega\mu_0 j_z \\ +\text{boundary conditions} \end{cases}$$

The finite element is easily derived for the case $n = 0$. We have the

Proposition 7. *(***finite element with 6 d.o.f., for** $n = 0$)
The following finite element $(\sum^0, K, \mathcal{P}^0)$ *is unisolvent and conforming in* $\mathcal{H}^0(\Omega)$*:*

- *K: triangle with edges* Γ_{i} *and unit tangent vector* τ_{i}*, for* $1 \leq i \leq 3$
- $\mathcal{P}^0$*: space of polynomials defined by*

(34)
$$u = \begin{bmatrix} \alpha_1 r - \alpha_3 rz \\ (\alpha_4 + \alpha_5 r + \alpha_6 z)r \\ \alpha_2 + \alpha_3 r^2 \end{bmatrix}$$

- $\sum^0$*: set of 6 linear form, for* $1 \leq i \leq 3$ *:*

(35)
$$\left| \begin{array}{l} \sigma_{\theta_{\mathrm{i}}} : u \to \dfrac{1}{r} u_\theta(a_{\mathrm{i}}) \\ \sigma_{\Gamma_{\mathrm{i}}} : u \to \displaystyle\int_{\Gamma_{\mathrm{i}}} \begin{bmatrix} u_r \\ u_z \end{bmatrix} \cdot \tau_{\mathrm{i}}\, d\Gamma_{\mathrm{i}} \end{array} \right.$$

5. Interpolation error estimate for finite elements with 6 d.o.f. for $n = 1$

We are going to define explicitly the basis functions and the interpolation operator of finite elements given by Proposition 6. Let us consider a triangular domain K included in Ω. We denote by λ_{i}, for $i = 1$ to 3, the three barycentric coordinates associated to K, and we form the 3 functions $\nu_{\mathrm{i}} = \begin{bmatrix} \nu_{\mathrm{i}}^r \\ \nu_{\mathrm{i}}^z \end{bmatrix} = \lambda_j \mathrm{grad}\lambda_k - \lambda_k \mathrm{grad}\lambda_j$, for (i, j, k) circular permutation of $(1, 2, 3)$.
According to proposition 6, it is straightforward to see that we have the following

Property 1. for $n = 1$, and for the finite element of Proposition 6, we have

- the 3 functions associated to d.o.f. (33) $\sigma_{\theta_{\mathrm{i}}}$, are $\Lambda_{\mathrm{i}} = \begin{bmatrix} \lambda_{\mathrm{i}} \\ -\lambda_i \\ 0 \end{bmatrix}$

– the 3 functions associated to d.o.f. (33) $\sigma_{\Gamma_{\rm i}}$, are $N_{\rm i} = \begin{bmatrix} r\nu_{\rm i}^r \\ 0 \\ r\nu_{\rm i}^z \end{bmatrix}$

and $\varPi$ the interpolate operator over K, defined for all vector field $u \in \mathcal{H}(\varOmega)$, by

$$\varPi u = \sum_{i=1}^{3} \sigma_{\theta_{\rm i}}(u)\varLambda_{\rm i} + \sum_{i=1}^{3} \sigma_{\Gamma_{\rm i}}(u)N_{\rm i}\ . \ \square$$

We need also some notations and recalls. We note $\varPi^{\rm R}$ the interpolate operator of the first order finite element of H^1 in bidimensional cartesian coordinates [4]. For all scalar function of u in H^1, we have

$$\varPi^{\rm R} u = \sum_{i=1}^{3} u(a_{\rm i})\lambda_{\rm i}$$

We note $\varPi^{\rm N}$ the interpolate operator of the first order finite element of $H(\mathrm{rot}, K)$ in bidimensional cartesian coordinates [11]. For all u vector of $H(\mathrm{rot}, K)$, we have

$$\varPi^{\rm N} u = \varPi^{\rm N} \begin{bmatrix} u_r \\ u_z \end{bmatrix} = \sum_{k=1}^{3} \left(\int_{\Gamma_k} \begin{bmatrix} u_r \\ u_z \end{bmatrix} \cdot \tau_k \, d\Gamma_k \right) \nu_k$$

In the sequel, to simplify notations, $\mathrm{grad} u_\theta$ and $\mathrm{rot}(u_r, u_z)$ are respectively the rz-plane operators $(\dfrac{\partial u_\theta}{\partial r}, \dfrac{\partial u_\theta}{\partial z})$ and $\dfrac{\partial u_z}{\partial r} - \dfrac{\partial u_r}{\partial z}$.
We shall use after the following lemma, also use in [12],

Lemma 7. *If u is a scalar function of $\mathcal{H}(K) \cap (\mathcal{C}^1(K))^3$, we have*

$$\sum_{i=1}^{3} \left(\int_{\Gamma_{\rm i}} \mathrm{grad} u_\theta \cdot \tau_{\rm i} \, d\Gamma_{\rm i} \right) \cdot \nu_{\rm i} = \sum_{i=1}^{3} \sigma_{\theta_{\rm i}}(u) \mathrm{grad}\lambda_{\rm i} \tag{36}$$

or equivalently,

$$\varPi^{\rm N}(\mathrm{grad} u) = \mathrm{grad}(\varPi^{\rm R} u) \tag{37}$$

Proof. We have

$$\begin{aligned} \sum_{i=1}^{3} & \left(\int_{\Gamma_{\rm i}} \mathrm{grad} u_\theta \cdot \tau_{\rm i} \, d\Gamma_{\rm i} \right) \cdot \nu_{\rm i} \\ &= \sum_{1 \le i < j \le 3} (\sigma_{\theta_{\rm i}}(u) - \sigma_{\theta_j}(u))(\lambda_j \mathrm{grad}\lambda_{\rm i} - \lambda_{\rm i} \mathrm{grad}\lambda_j) \\ &= \sum_{\text{permutation } (i,j,k)} \sigma_{\theta_k}(u)((\lambda_{\rm i} + \lambda_j)\mathrm{grad}\lambda_k - \lambda_k(\mathrm{grad}\lambda_{\rm i} + \mathrm{grad}\lambda_j)) \end{aligned}$$

and then the result because of: $\sum_{i=1}^{3} \mathrm{grad}\lambda_\mathrm{i} = 0.$ □

We begin by rewriting the distance $(u - \Pi u)$ and $(\mathrm{rot}_r u - \mathrm{rot}_r \Pi u)$ with the norm of L^2_1 with understanding notations as indicated in Remark 1. We have the

Proposition 8. *We suppose that $u \in \mathcal{H}(K) \cap (\mathcal{C}^1(K))^3$. We have the following relations*

$$\begin{aligned} |u - \Pi u\,|^2_{(L^2_1)^3} \leq 2|u_\theta - \Pi^\mathrm{R} u_\theta\,|^2_{L^2_1} &+ |r\mathrm{grad}(u_\theta - \Pi^\mathrm{R} u_\theta)\,|^2_{L^2_1} \\ &+ |r(U - \Pi^\mathrm{N} U)\,|^2_{(L^2_1)^2} \end{aligned} \tag{38}$$

and using notation (19)

$$\begin{aligned} |\mathrm{rot}_r u - \mathrm{rot}_r \Pi u\,|^2_{(L^2_1)^3} = \ &|U - \Pi^\mathrm{N} U\,|^2_{(L^2_1)^2} \\ &+ |\mathrm{rot}(r(U - \Pi^\mathrm{N} U))\,|^2_{(L^2_1)} \end{aligned} \tag{39}$$

Proof. Evaluation of $|u - \Pi u\,|^2_{(L^2_1)^3}$. We have:

$$\begin{aligned} |u - \Pi u\,|^2_{(L^2_1)^3} = &\left| u_\theta - \sum_{i=1}^{3} \sigma_{\theta_\mathrm{i}}(u) \cdot \lambda_\mathrm{i} \right|^2_{L^2_1} \\ &+ \left| \begin{bmatrix} u_r + \sum_{i=1}^{3} \sigma_{\theta_\mathrm{i}}(u) \cdot \lambda_\mathrm{i} \\ u_z \end{bmatrix} - \sum_{i=1}^{3} \sigma_{\Gamma_\mathrm{i}}(u).r\nu_\mathrm{i} \right|^2_{(L^2_1)^2} \end{aligned}$$

and then:

$$\begin{aligned} |u - \Pi u\,|^2_{(L^2_1)^3} \leq 2 &\left| u_\theta - \sum_{i=1}^{3} \sigma_{\theta_\mathrm{i}}(u) \cdot \lambda_\mathrm{i} \right|^2_{L^2_1} \\ &+ \left| \begin{bmatrix} u_r + u_\theta \\ u_z \end{bmatrix} - \sum_{i=1}^{3} \sigma_{\Gamma_\mathrm{i}}(u) \cdot r\nu_\mathrm{i} \right|^2_{(L^2_1)^2} \end{aligned} \tag{40}$$

with some substitutions and using Lemma 7, the last norm of the previous inequality is also written:

$$\begin{aligned} &\left| \begin{bmatrix} u_r + u_\theta + r\frac{\partial u_\theta}{\partial r} \\ u_z + r\frac{\partial u_\theta}{\partial z} \end{bmatrix} - \sum_{i=1}^{3} \left(\int_{\Gamma_\mathrm{i}} \begin{bmatrix} \frac{u_r + u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \\ \frac{u_z}{r} + \frac{\partial u_\theta}{\partial z} \end{bmatrix} \cdot \tau_\mathrm{i}\, d\Gamma_\mathrm{i} \right) r\nu_\mathrm{i} \right. \\ &\qquad \left. - \,r\mathrm{grad}\left(u_\theta - \sum_{i=1}^{3} \sigma_{\theta_\mathrm{i}}(u)\lambda_\mathrm{i} \right) \right|^2_{(L^2_1)^2} \end{aligned}$$

then we report this last expression in (40), and using (19) and the definition of Π^{N}, we obtain inequality (38).
Now we look at the distance of the rotationals:$|\mathrm{rot}_r u - \mathrm{rot}_r \Pi u\ |^2_{(L^2_1)^3}$. We have:

$$\mathrm{rot}_r \Pi u = \sum_{i=1}^{3} \sigma_{\theta_\mathrm{i}}(u) \begin{bmatrix} -\frac{\partial \lambda_\mathrm{i}}{\partial z} \\ -\frac{\partial \lambda_\mathrm{i}}{\partial z} \\ \frac{\partial \lambda_\mathrm{i}}{\partial r} \end{bmatrix} + \sum_{i=1}^{3} \sigma_{\Gamma_\mathrm{i}}(u) \begin{bmatrix} -(\lambda_k \frac{\partial \lambda_j}{\partial z} - \lambda_j \frac{\partial \lambda_k}{\partial z}) \\ -\mathrm{rot}(r\nu_\mathrm{i}) \\ \lambda_k \frac{\partial \lambda_j}{\partial r} - \lambda_j \frac{\partial \lambda_k}{\partial r} \end{bmatrix}$$

Using Lemma 7, we have the equality:

$$\begin{bmatrix} \sum_{i=1}^{3} \sigma_{\Gamma_\mathrm{i}}(u) \left(\lambda_k \frac{\partial \lambda_j}{\partial z} - \lambda_j \frac{\partial \lambda_k}{\partial z} \right) + \sigma_{\theta_\mathrm{i}}(u) \frac{\partial \lambda_\mathrm{i}}{\partial r} \\ \sum_{i=1}^{3} \sigma_{\Gamma_\mathrm{i}}(u) \left(\lambda_k \frac{\partial \lambda_j}{\partial r} - \lambda_j \frac{\partial \lambda_k}{\partial r} \right) + \sigma_{\theta_\mathrm{i}}(u) \frac{\partial \lambda_\mathrm{i}}{\partial z} \end{bmatrix}$$
$$= \sum_{i=1}^{3} \left(\int_{\Gamma_\mathrm{i}} \begin{bmatrix} \frac{u_r + u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \\ \frac{u_z}{r} + \frac{\partial u_\theta}{\partial z} \end{bmatrix} \cdot \tau_\mathrm{i}\, d\Gamma_\mathrm{i} \right) \nu_\mathrm{i}$$

then we can write:

$$|\mathrm{rot}_r u - \mathrm{rot}_r \Pi u\ |^2_{(L^2_1)^3} = |U - \Pi^{\mathrm{N}} U\ |^2_{(L^2_1)^2}$$
$$+ \left| \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} - \sum_{i=1}^{3} \sigma_{\Gamma_\mathrm{i}}(u) \left(\frac{\partial r \nu^r_\mathrm{i}}{\partial z} - \frac{\partial r \nu^z_\mathrm{i}}{\partial r} \right) + \sum_{i=1}^{3} \sigma_{\theta_\mathrm{i}}(u) \frac{\partial \lambda_\mathrm{i}}{\partial z} \right|^2_{L^2_1}$$

The last term of the above relation is also equal to

$$= \mathrm{rot}(rU) - \sum_{i=1}^{3} \left(\int_{\Gamma_\mathrm{i}} (U - \mathrm{grad} u_\theta) \cdot \tau_\mathrm{i}\, d\Gamma_\mathrm{i} \right) \mathrm{rot}(r\nu_\mathrm{i}) + \sum_{i=1}^{3} \sigma_{\theta_\mathrm{i}}(u) \frac{\partial \lambda_\mathrm{i}}{\partial z}$$

and using Lemma 7 for the z-component, and because $\sum_{i=1}^{3} (\int_{\Gamma_\mathrm{i}} \mathrm{grad} u_\theta \cdot \tau_\mathrm{i}\, d\Gamma_\mathrm{i}) \mathrm{rot} \nu_\mathrm{i} = 0$, this term is still equal to

$$\mathrm{rot}(rU - \sum_{i=1}^{3} (\int_{\Gamma_\mathrm{i}} U \cdot \tau_\mathrm{i}\, d\Gamma_\mathrm{i}) r\nu_\mathrm{i}$$

and therefore we have (39). □
Then as a result, it appears, that for the interpolation error, we have to consider the following distances:
on one hand: $|u_\theta - \Pi^{\mathrm{R}} u_\theta\ |^2_{L^2_1(K)}$ and $|r(U - \Pi^{\mathrm{N}} U)\ |^2_{(L^2_1(K))^2}$,

on the other hand: $|r\mathrm{grad}(u_\theta - \Pi^{\mathrm{R}} u_\theta)\ |^2_{(L^2_1(K))^2}$ and $|\mathrm{rot}(r(U - \Pi^{\mathrm{N}} U))\ |^2_{L^2_1(K)}$. To this end we recall some standart interpolation error estimate. We denote by the real number h the diameter of the triangle K. First for the 1-st order finite element of class $H^1(K)$.

Lemma 8. *Let K be a triangle of the rz-plane. Then there exists a constant $C > 0$ such that, for all function u of $H^k(K)$, we have:*

(41)
$$|u - \Pi^{\mathrm{R}} u\ |_{L^2(K)} \leq Ch^k |u\ |_{H^k(K)}$$

(42)
$$|\mathrm{grad}(u - \Pi^{\mathrm{R}} u)\ |_{L^2(K)} \leq Ch^{k-1} |u\ |_{H^k(K)}$$

(43)
$$\|\ u - \Pi^{\mathrm{R}} u\ \|_{H^1(K)} \leq Ch^{k-1} |u\ |_{H^k(K)}$$

Proof. See [4]. □

Lemma 9. *let K be a triangle of the rz-plane. Then there exists a constant $C > 0$ such that, for all function u of $H(\mathrm{rot}, K) \cap (H^k(K))^2$, we have:*

(44)
$$|u - \Pi^{\mathrm{N}} u\ |_{L^2(K)} \leq Ch^k |u\ |_{H^k(K)}$$

(45)
$$|\mathrm{rot}(u - \Pi^{\mathrm{N}} u)\ |_{L^2(K)} \leq Ch^{k-1} |u\ |_{H^k(K)}$$

and then,

(46)
$$\|\ u - \Pi^{\mathrm{N}} u\ \|_{H(\mathrm{rot},K)} \leq Ch^{k-1} |u\ |_{H^k(K)}$$

Proof. See [11]. □

In the sequel Γ_0 is the part of Γ on the axis $\{r = 0\}$ (see figure of Sect. 3.4). We need finally the interpolation error estimate for finite elements approximating $W^k_{1/2}(\Omega)$. This technical result is the following:

Proposition 9. *Let K be a triangle with a vertex or an edge on Γ_0. Then there exists a constant $C > 0$ such that, for all function u of $W^k_{1/2}(K)$, where $k = 2$ in t he case of finite elements of degree 1, $k = 3$ in the case of finite elements of degree 2, we have:*

(47)
$$|u - \Pi^{\mathrm{R}} u\ |_{W^1_{1/2}(K)} \leq Ch^{k-1} |u\ |_{W^k_{1/2}(K)}$$

(48)
$$|u - \Pi^{\mathrm{R}} u\ |_{H^1(K)} \leq Ch^{k-3/2} |u\ |_{W^k_{1/2}(K)}$$

and if in addition, u vanishes on Γ_0,

(49)
$$|r^{-1/2}(u - \Pi^{\mathrm{R}} u)\ |_{L^2(K)} \leq Ch^{k-1} |u\ |_{W^k_{1/2}(K)}\ .$$

Proof. It is the Lemma 6.1 of [10]. □

We use Proposition 9, to prove the next

Proposition 10. *Let K be a triangle with a vertex or an edge on Γ_0. Then there exists a constant $C > 0$ such that, for all function u of $(W^k_{1/2}(K))^2$, where $k = 2$ in the case of finite elements of degree 1, $k = 3$ in the case of finite elements of degree 2, we have:*

$$|\mathrm{rot}(u - \Pi^{\mathrm{N}} u)\ |_{L^2_1(K)} \leq C h^{k-1} |u\ |_{(W^k_{1/2}(K))^2} \tag{50}$$

and if in addition, u vanishes on Γ_0,

$$|r^{-1/2}(u - \Pi^{\mathrm{N}} u)\ |_{(L^2(K))^2} \leq C h^{k-1} |u\ |_{(W^k_{1/2}(K))^2}\ . \tag{51}$$

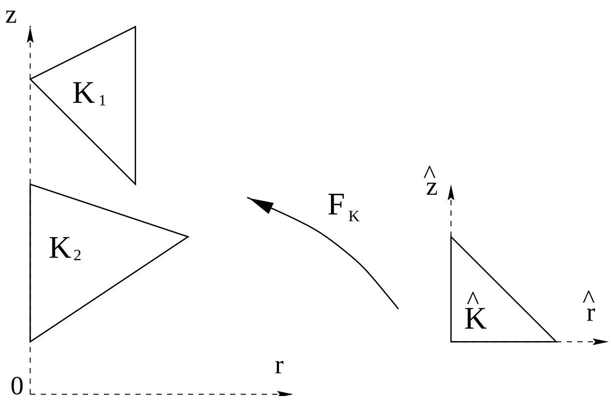

Fig. 4.

Proof. We adapt with few modification the Lemma 6.1 of [10]. Assuming that K is a triangle of the rz-plane, with at least a vertex on the axis $\{r = 0\}$, like on Fig. 4, triangle K_1 or K_2. We denote by F_K the linear mapping that transforms $\hat{K}$ into K, if $\hat{K}$ is the reference triangle of the rz-plane, constituted by points $(0,0),(0,1),(1,0)$. We denote by $(r_{\mathrm{i}}, z_{\mathrm{i}})_{i=1,3}$, the coordinates of the vertices of K, with (r_1, z_1) such that z_1 is less than the ordinate of the eventual second point on the axis. We note B the 2×2 square matrix associated to F_K. So F_{K_1} is define by the matrix $\begin{bmatrix} r_2 & r_3 \\ z_2 - z_1 & z_3 - z_1 \end{bmatrix}$, F_{K_2} by the matrix $\begin{bmatrix} r_2 & 0 \\ z_2 - z_1 & z_3 - z_1 \end{bmatrix}$.
We suppose that the triangle is regular in the sense where if ρ is the radius of the inscribed circle to K, there exists a constant χ such that $h/\rho \leq \chi$.
Let $(\hat{r}, \hat{z})$ be the coordinates of a point of $\hat{K}$ and (r, z) the corresponding coordinates by F_K in K. Then we have: $r = r_2 \hat{r} + r_3 \hat{z}$ for K_1, $r = r_2 \hat{r}$ for

K_2, and there exists a constant C, independant of h, such that we have the inequalities:

(52) $$\frac{1}{C}\rho\,\hat{r} \leq r \leq C\rho\,\hat{r}$$

Let u and $\hat{u}$ be two functions such that: $u(F_K(\hat{r},\hat{z})) = {}^{\mathrm{T}}B^{-1}\hat{u}(\hat{r},\hat{z})$. We define $\hat{\Pi}^{\mathrm{N}}$ on $\hat{K}$ by

$$\Pi^{\mathrm{N}}u = {}^{\mathrm{T}}B^{-1}\hat{\Pi}^{\mathrm{N}}\hat{u}$$

We recall that: $\hat{\mathrm{rot}}\hat{u} = \mathrm{det}B\ \mathrm{rot}u$, where $\hat{\mathrm{rot}}$ is the rotational operator in the $(\hat{r},\hat{z})$ variables.

Let $u \in (W^1_{1/2}(K))^2$. We estimate with (52) and using notations of [4], the following distance

$$|\mathrm{rot}(u - \Pi^{\mathrm{N}}u)\ |_{L^2_1(K)} \leq \frac{C\rho\,\mathrm{det}B}{\parallel B \parallel^2}|\hat{\mathrm{rot}}(\hat{u} - \hat{\Pi}^{\mathrm{N}}\hat{u})\ |_{L^2_1(\hat{K})}$$

and then, since the triangle is regular, we have for another constant still noted C

(53) $$|\mathrm{rot}(u - \Pi^{\mathrm{N}}u)\ |_{L^2_1(K)} \leq C\rho|\hat{\mathrm{rot}}(\hat{u} - \hat{\Pi}^{\mathrm{N}}\hat{u})\ |_{L^2_1(\hat{K})}$$

According to Theorem 1 above, $W^2_{1/2}(\hat{K}) \subsetneq \mathcal{C}^0(\hat{K})$, then $\hat{\Pi}^{\mathrm{N}}$ is an operator defined on space $(W^2_{1/2}(\hat{K}))^2$ and $(I - \hat{\Pi}^{\mathrm{N}})$ is continuous from $(W^k_{1/2}(\hat{K}))^2$ into $(W^1_{1/2}(\hat{K}))^2$ for $k = 2$ or 3:

$$|\hat{\mathrm{rot}}(\hat{u} - \hat{\Pi}^{\mathrm{N}}\hat{u})\ |_{L^2_1(\hat{K})} \leq C \parallel \hat{u} \parallel_{(W^k_{1/2}(\hat{K}))^2}$$

Now we have $(I - \hat{\Pi}^{\mathrm{N}})\hat{p} = 0$, for all $\hat{p} \in \mathcal{P}_{k-1}(\hat{K})$, then $\hat{\Pi}^{\mathrm{N}}$ verify hypothesis of Theorem 2, and we can replace the norm in the preeceding inequality by the corresponding semi-norm, for another constant C,

$$|\hat{\mathrm{rot}}(\hat{u} - \hat{\Pi}^{\mathrm{N}}\hat{u})\ |_{L^2_1(\hat{K})} \leq C|\hat{u}\ |_{(W^k_{1/2}(\hat{K}))^2}$$

then using the first inequality of (52), we obtain the multi-index derivation

(54) $$|\hat{u}\ |^2_{(W^k_{1/2}(\hat{K}))^2} \leq \frac{C}{\rho}\frac{\parallel B \parallel^{2k}}{\mathrm{det}B}|u\ |^2_{(W^k_{1/2}(K))^2}$$

and then because of the regular hypothesis on the triangle

$$|\hat{u}\ |^2_{(W^k_{1/2}(\hat{K}))^2} \leq \frac{C}{\rho}h^{2k-2}|u\ |^2_{(W^k_{1/2}(K))^2}$$

finally by collecting the last inequalities, we have for a constant C

(55) $$|\mathrm{rot}(u - \Pi^{\mathrm{N}}u)\ |_{L^2_1(K)} \leq Ch^{k-1}|u\ |_{(W^k_{1/2}(K))^2}$$

We show now the second statement of Proposition 10.
We recall a not very obvious result from [10] (Theorem 4.4 and Remark 4.1 of [10]),

$$W^k_{1/2}(\hat{K}) \subsetneq H^1(\hat{K})$$

and we recall the following result, immediate for $u \in H^1_0(K)$,

Lemma 10. *There exists a constant $C > 0$ such that for all function $u \in H^1(K)$ and verifying $u_{|\Gamma_0} = 0$, we have*

$$\int_K \frac{u^2}{r^2}\, drdz \le C|u|^2_{H^1(K)} \tag{56}$$

Proof. It is the Corollary 4.1 of [10]. □

We use again Theorem 2, with $X = \left(H^1(\hat{K})\right)^2$, that allow us to obtain

$$|u - \Pi^{\mathrm{N}} u|_{(H^1(K))^2} \le C|\hat{u} - \hat{\Pi}^{\mathrm{N}}\hat{u}|_{(H^1(\hat{K}))^2} \le C|\hat{u}|_{(W^k_{1/2}(\hat{K}))^2}$$

and we deduce as for the first inequality

$$|u - \Pi^{\mathrm{N}} u|_{(H^1(K))^2} \le Ch^{k-3/2}|u|_{(W^k_{1/2}(K))^2}$$

If we suppose that u vanishes on Γ_0, $(u - \Pi^{\mathrm{N}}u)$ also vanishes on Γ_0 and since its belongs to the space $(H^1(K)^2)$, according to Lemma 10, $(u - \Pi^{\mathrm{N}}u)$ belongs to the space $(L^2_{-1}(K))^2$, of square integrable vectorial functions of order 2 for the $(drdz/r)$ measure.
Therefore we can consider the following inequality

$$|r^{-1/2}(u - \Pi^{\mathrm{N}}u)|^2_{(L^2(K))^2} \le \frac{C}{\rho}\det B|\hat{r}^{-1/2}(\hat{u} - \hat{\Pi}^{\mathrm{N}}\hat{u})|^2_{(L^2(\hat{K}))^2} \tag{57}$$

then applying Lemma 10, there exists a constant $C > 0$, such that

$$|\hat{r}^{-1/2}(\hat{u} - \hat{\Pi}^{\mathrm{N}}\hat{u})|^2_{(L^2(\hat{K}))^2} \le |\hat{u} - \hat{\Pi}^{\mathrm{N}}\hat{u}|^2_{(H^1(\hat{K}))^2} \tag{58}$$

from inequalities (54),(57),(58) it ensues that, for a certain C,

$$|r^{-1/2}(u - \Pi^{\mathrm{N}}u)|^2_{(L^2(K))^2} \le \frac{C}{\rho^2} \parallel B \parallel^{2k} |u|^2_{(W^k_{1/2}(K))^2}$$

then finally, since the triangle is regular,

$$|r^{-1/2}(u - \Pi^{\mathrm{N}}u)|_{(L^2(K))^2} \le Ch^{k-1}|u|_{(W^k_{1/2}(K))^2}\,. \quad \square$$

Proposition 11. *Let K be a triangle with a vertex or an edge on Γ_0. Then there exists a constant $C > 0$ such that, for all function u of $\mathcal{H} \cap (W^2_{1/2}(K))^3$, we have the following interpolation error estimate*

$$\| u - \Pi u \|_{\mathcal{H}} \leq Ch^{1/2}(|u_\theta|^2_{W^2_{1/2}(K)} + |U|^2_{(W^2_{1/2}(K))^2})^{1/2} \tag{59}$$

Proof. We are going to estimate successively each encountered term $|u_\theta - \Pi^{\mathrm{R}} u_\theta|_{L^2_1}$, $|r\mathrm{grad}(u_\theta - \Pi^{\mathrm{R}} u_\theta)|_{(L^2_1)^2}$, $|r(U - \Pi^{\mathrm{N}} U)|_{(L^2_1)^2}$ and $|\mathrm{rot}(r(U - \Pi^{\mathrm{N}} U))|_{L^2_1}$.
We consider $u = (u_r, u_\theta, u_z) \in (\mathcal{D}(\Omega))^3$.
Estimate of the term: $|u_\theta - \Pi^{\mathrm{R}} u_\theta|_{L^2_1(K)}$. We have:

$$|u_\theta - \Pi^{\mathrm{R}} u_\theta|_{L^2_1(K)} \leq |u_\theta - \Pi^{\mathrm{R}} u_\theta|_{W^1_{1/2}(K)}$$

then according to (47), it results the inequality:

$$|u_\theta - \Pi^{\mathrm{R}} u_\theta|_{L^2_1(K)} \leq Ch|u_\theta|_{W^2_{1/2}(K)} \tag{60}$$

Estimate of the term: $|r\mathrm{grad}(u_\theta - \Pi^{\mathrm{R}} u_\theta)|_{(L^2_1(K))^2}$. We can write:

$$\begin{aligned} |r\mathrm{grad}(u_\theta - \Pi^{\mathrm{R}} u_\theta)|_{(L^2_1(K))^2} &\leq h|\mathrm{grad}(u_\theta - \Pi^{\mathrm{R}} u_\theta)|_{(L^2_1(K))^2} \\ &\leq h|u_\theta - \Pi^{\mathrm{R}} u_\theta|_{W^1_{1/2}(K)} \end{aligned}$$

then by vertue of the first inequality of Proposition 9, there exists a constant C such that:

$$|r\mathrm{grad}(u_\theta - \Pi^{\mathrm{R}} u_\theta)|_{(L^2_1(K))^2} \leq Ch^2|u_\theta|_{W^2_{1/2}(K)} \tag{61}$$

We suppose for the two following estimates, that u vanishes in an neighbourhood of $\{r = 0\}$ and in addition, we assume that K is included in $\{r < 1\}$.
Estimate of the term: $|r(U - \Pi^{\mathrm{N}} U)|_{(L^2_1(K))^2}$. We have:

$$|r(U - \Pi^{\mathrm{N}} U)|_{(L^2_1(K))^2} \leq |r^{-1/2}(U - \Pi^{\mathrm{N}} U)|_{(L^2(K))^2}$$

then according to the second inequality of Proposition 10, there exists a constant C

$$|r(U - \Pi^{\mathrm{N}} U)|_{(L^2_1(K))^2} \leq Ch|U|_{(W^2_{1/2}(K))^2} \tag{62}$$

Estimate of the term: $|\mathrm{rot}(r(U - \Pi^{\mathrm{N}} U))|_{L^2_1(K)}$. We have the obvious inequality:

$$|\mathrm{rot}(r(U - \Pi^{\mathrm{N}} U))|^2_{L^2_1(K)} \leq |\mathrm{rot}(U - \Pi^{\mathrm{N}} U)|^2_{L^2_1(K)} + |U - \Pi^{\mathrm{N}} U|^2_{(L^2_1(K))^2}$$

then by Proposition 10, there exists a constant C such that:

$$|\mathrm{rot}(U - \Pi^{\mathrm{N}}U)\ |_{L^2_1(K)} \leq Ch|U\ |_{(W^2_{1/2}(K))^2}$$

and finally such that, according to the preeceding estimates,

$$|\mathrm{rot}(r(U - \Pi^{\mathrm{N}}U))\ |_{L^2_1(K)} \leq Ch|U\ |_{(W^2_{1/2}(K))^2} \tag{63}$$

Then inequality (64) is a consequence of, on one hand inequalities (38) an (39), and on the other hand inequalities (60),(61),(62) and (63).
Inequality (64) true for functions in $\mathcal{D}(\bar{\Omega})^3$ vanishing in a neighbourhood of $\{r = 0\}$, is also true for functions of $\mathcal{H}$, by vitue of Lemma 2(ii) and Proposition 2. □

6. Use of Maxwell-Fourier finite elements with 6 d.o.f.

Let Ω be an open of $\mathbb{R}^2_+ = \{(r, z), r > 0\}$. Let $\tau_h = \cup_{l=1}^{N_e} K_l$ be a triangulation of Ω in N_e triangles. We explain how to determine the basis functions in the case of Fourier modes $n \neq 0$. We rewrite the set of d.o.f. define by (33), $\sum = \{\sigma_{\mathrm{i}}$ such that $\sigma_{\mathrm{i}} = \sigma_{\theta_{\mathrm{i}}}$ and $\sigma_{i+3} = \sigma_{\Gamma_{\mathrm{i}}}$ for $i = 1, 3\}$, and we look for functions p_j solutions of the following linear 6×6 system:

$$\sigma_{\mathrm{i}}(p_j) = \delta_{ij},\ \text{(Kronecker symbol), for } 1 \leq i, j \leq 6 \tag{64}$$

Let p_j be a function defined by (32) and for which we search 6 real coefficients $\alpha^j_1, \alpha^j_2, \alpha^j_3, \alpha^j_4, \alpha^j_5, \alpha^j_6$, such that this function verify explicitly the system (64). We note $(r_{\mathrm{i}}, z_{\mathrm{i}})$ and Γ_{i}, for $i = 1, 3$ respectively the three vertices and the three edges of triangle K. We note also $\tau_{\mathrm{i}} = \begin{bmatrix} \lambda_{\mathrm{i}} \\ \mu_{\mathrm{i}} \end{bmatrix}$, the unit vector to the edge Γ_{i} of K. We define the three following 3×3 matrices: the first one corresponding to the d.o.f. relative to the vertices:

$$\mathcal{M} = \begin{bmatrix} -1 & r_1 & -z_1 \\ -1 & r_2 & -z_2 \\ -1 & r_3 & -z_3 \end{bmatrix},$$

the second one corresponding to the d.o.f. linked to the circulation:

$$\mathcal{M}_j = \begin{bmatrix} m'^1_j & 0 & m''^1_j \\ m'^2_j & 0 & m''^2_j \\ m'^3_j & 0 & m''^3_j \end{bmatrix}$$

where $m'^i_j = \lambda_j \int_{\Gamma_{\mathrm{i}}} d\Gamma_{\mathrm{i}}$ and $m''^i_j = \lambda_j \int_{\Gamma_{\mathrm{i}}} z\, d\Gamma_{\mathrm{i}}$,
and the third one:

$$\mathcal{N}_j = \begin{bmatrix} n_j^{\prime 1} & n_j^{\prime\prime 1} & n_j^{\prime\prime\prime 1} \\ n_j^{\prime 2} & n_j^{\prime\prime 2} & n_j^{\prime\prime\prime 2} \\ n_j^{\prime 3} & n_j^{\prime\prime 3} & n_j^{\prime\prime\prime 3} \end{bmatrix}$$

with $n_j^{\prime i} = \lambda_j \int_{\Gamma_{\mathrm{i}}} r\, d\Gamma_{\mathrm{i}}$, $n_j^{\prime\prime i} = \mu_j \int_{\Gamma_{\mathrm{i}}} r\, d\Gamma_{\mathrm{i}}$, $n_j^{\prime\prime\prime i} = \int_{\Gamma_{\mathrm{i}}} (\mu_j r^2 - \lambda_j r z)\, d\Gamma_{\mathrm{i}}$. Then we resolve for each element of number j, the 6×6 systems below:

$$\begin{bmatrix} \mathcal{M} & 0 \\ \mathcal{M}_j & \mathcal{N}_j \end{bmatrix} (\alpha_{\mathrm{i}}^j) = \delta_{ij} \tag{65}$$

which solutions provide the six basis functions whose support encounter the triangle K.

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