DE LA RECHERCHE À L'INDUSTRIE

TrioCFD: code & numerical schemes

POEMs 2019

CIRM

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Commissariat à l'Energie Atomique et
aux Energies Alternatives
Centre de Saclay

CEA/DEN/DANS/DM2S/STMF/LMSF

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Outline

TrioCFD code (previously Trio_U code)

- Developed by the CEA/DEN since the early 1990s.
- Dedicated to unsteady, low Mach number, turbulent flows.
- C++, designed for HPC, open source since 2015.

- Cartesian and triangular/tetrahedral meshes.
- Explicit and semi-implicit time integration.
- Turbulence models: RANS, LES, Wall models.

- Participation in the FVCA 8 benchmark session.

- PhD student: MsFEM (Q. Feng, P. Omnes, G. Allaire).
- Post-docs: ALE (R. Pegonen, A. Puscas) and sensitivity analysis (C. Fiorini, A. Puscas).

- Website (under evolution): `www-trio-u.cea.fr`
  Download: `https://sourceforge.net/projects/triocfd/`

- Our objectives: support polygonal/polyhedral meshes and high order methods.

Navier-Stokes equations

- Find $(\vec{u}, p) \in \vec{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that:

$$\left\{\begin{array}{rcl} \partial_t \vec{u} - \nu \Delta \vec{u} + (\vec{u} \cdot \vec{\nabla}\,)\vec{u} + \vec{\nabla}\,p & = & \vec{f}, \\ \vec{\nabla} \cdot \vec{u} & = & 0. \end{array}\right.$$

- Time discretization (semi-implicit scheme):

1. **Prediction step:** Compute U^* the solution of

$$\delta t^{-1} MU^* + AU^* + L(U^n)U^* + {}^tBP^n = F^{n+1} + \delta t^{-1} MU^n.$$

At this step, we may have $BU^* \neq 0$.

2. **Pressure computation step:** Compute P' the solution of

$$BM^{-1\,t}BP' = \delta t^{-1} BU^*, \quad P^{n+1} = P' + P^n.$$

3. **Correction step:** Compute U^{n+1} the solution of

$$MU^{n+1} = MU^* - \delta t\,{}^tBP'.$$

Stokes Problem

- Find $(\vec{u}, p) \in \vec{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that $\left\{ \begin{array}{rcl} -\nu\Delta\vec{u} + \vec{\nabla}\, p & = & \vec{f}, \\ \vec{\nabla}\cdot\vec{u} & = & 0. \end{array} \right.$
- Find $(\vec{u}, p) \in \vec{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\left\{ \begin{array}{lrcl} \forall \vec{v} \in \vec{H}_0^1(\Omega), & a(\vec{u}, \vec{v}) + b(\vec{v}, p) & = & \langle \vec{f}, \vec{v} \rangle, \\ \forall q \in L_0^2(\Omega), & b(\vec{u}, q) & = & 0. \end{array} \right.$$

where $\quad a(\vec{u}, \vec{v}) = (\, \nu\vec{\nabla}\,\vec{u}\,,\, \vec{\nabla}\,\vec{v}\,)_0 \quad$ and $\quad b(\vec{v}, p) = -(\, q\,,\, \vec{\nabla}\cdot\vec{v}\,)_0.$

Well-posedness: a and b continuous, a coercive on $\vec{H}_0^1(\Omega)$ and the *inf-sup condition* holds

$$\forall q \in L_0^2(\Omega), \qquad \sup_{\vec{v}\in\vec{H}_0^1(\Omega)} \frac{b(\vec{v}, q)}{|\vec{v}|_1} \geq \beta \|q\|_0.$$

Stokes Problem

$$(\vec{u},p) \in \vec{H}_0^1(\Omega) \times L_0^2(\Omega) \mid \left\{ \begin{array}{lrcl} \forall \vec{v} \in \vec{H}_0^1(\Omega), & a(\vec{u},\vec{v}) + b(\vec{v},p) & = & \langle \vec{f},\vec{v} \rangle, \\ \forall q \in L_0^2(\Omega), & b(\vec{u},q) & = & 0. \end{array} \right.$$

Well-posedness: a and b continuous, a coercive on $\vec{H}_0^1(\Omega)$ and the *inf-sup condition* holds

$$\forall q \in L_0^2(\Omega), \qquad \sup_{\vec{v}\in\vec{H}_0^1(\Omega)} \frac{b(\vec{v},q)}{|\vec{v}|_1} \geq \beta \|q\|_0.$$

$$\vec{H}_0^1(\Omega) = \vec{V} \oplus \vec{V}^\perp, \quad \left\{ \begin{array}{lcl} \vec{V} & := & \{\vec{v} \in \vec{H}_0^1(\Omega); \quad \vec{\nabla}\cdot\vec{v} = 0\}, \\ \vec{V}^\perp & = & \{(-\Delta)^{-1}\vec{\nabla} q; \quad q \in L^2(\Omega)\}. \end{array} \right.$$

Abstract tools: Girault-Raviart'86, chap I, cor. 2.4.

1°) The operator $\vec{\nabla}$ is an isomorphism of $L_0^2(\Omega)$ onto $\vec{V}^0$ such that

$$\vec{V}^0 := \left\{ \vec{y} \in \vec{H}^{-1}(\Omega); \quad \langle \vec{y},\vec{\phi} \rangle = 0 \quad \forall \vec{\phi} \in \vec{V} \right\}.$$

2°) The operator $\vec{\nabla}\cdot$ is an isomorphism of $\vec{V}^\perp$ onto $L_0^2(\Omega)$.

Space discretizations

- Let $\mathcal{T}_h$ a conforming triangulation of Ω, let $\mathcal{F}_h$ be the set of facets.
- Conforming discretization: $\vec{X}_h \subset \vec{H}_0^1(\Omega)$.

  Find $(\vec{u}_h, p_h) \in \vec{X}_h \times M_h$ such that

$$\left\{ \begin{array}{lrcl} \forall \vec{v}_h \in \vec{X}_h, & a(\vec{u}_h, \vec{v}_h) + b(\vec{v}_h, p_h) & = & (\vec{f}, \vec{v}_h)_0, \\ \forall q_h \in M_h, & b(\vec{u}_h, q_h) & = & 0. \end{array} \right.$$

  Nodal finite element method: Taylor-Hood, conforming Crouzeix-Raviart...

  Advantage: Well-posedness of the discrete pb (check ISC).

  Drawback: Couplings between the vertices.
- Non-conforming discretization: $\vec{X}_h \not\subset \vec{H}_0^1(\Omega)$.

  Find $(\vec{u}_h, p_h) \in \vec{X}_h \times M_h$ such that

$$\left\{ \begin{array}{lrcl} \forall \vec{v}_h \in \vec{X}_h, & a_h(\vec{u}_h, \vec{v}_h) + b_h(\vec{v}_h, p_h) & = & (\vec{f}, \vec{v}_h)_0, \\ \forall q_h \in M_h, & b_h(\vec{u}_h, q_h) & = & 0. \end{array} \right.$$

  Advantage: Couplings between the facets.

  Drawback: Check continuity of a_h, b_h, ellipticity of a_h and ISC.

Space discretizations

- Find $(\vec{u}_h, p_h) \in \vec{X}_h \times M_h$ such that

$$\left\{ \begin{array}{lrcl} \forall \vec{v}_h \in \vec{X}_h, & a_{(h)}(\vec{u}_h, \vec{v}_h) + b_{(h)}(\vec{v}_h, p_h) & = & (\vec{f}, \vec{v}_h)_0, \\ \forall q_h \in M_h, & b_{(h)}(\vec{u}_h, q_h) & = & 0. \end{array} \right.$$

- In both cases, the main technical difficulty is that $\vec{V}_h \not\subset \vec{V}$ where:

$$\vec{V}_h := \{\vec{v}_h \in \vec{X}_h \,|\, \forall q_h \in M_h, \quad b_{(h)}(\vec{v}_h, q_h) = 0\}.$$

- Conforming discretization, a priori error estimates:

$$\left\{ \begin{array}{lcl} \|\vec{u} - \vec{u}_h\|_1 & \lesssim & \inf_{\vec{v}_h \in \vec{X}_h} \|\vec{u} - \vec{v}_h\|_1 + \inf_{q_h \in M_h} \|p - q_h\|_0, \\ \|p - p_h\|_0 & \lesssim & \inf_{\vec{v}_h \in \vec{X}_h} \|\vec{u} - \vec{v}_h\|_1 + \inf_{q_h \in M_h} \|p - q_h\|_0. \end{array} \right.$$

- Non-conforming discretization, a priori error estimates:

$$\left\{ \begin{array}{lcl} \|\vec{u} - \vec{u}_h\|_{\vec{X}_h} & \lesssim & \inf_{\vec{v}_h \in \vec{X}_h} \|\vec{u} - \vec{v}_h\|_{\vec{X}_h} + \inf_{q_h \in M_h} \|p - q_h\|_0, \\ \|p - p_h\|_{M_h} & \lesssim & \inf_{\vec{v}_h \in \vec{X}_h} \|\vec{u} - \vec{v}_h\|_{\vec{X}_h} + \inf_{q_h \in M_h} \|p - q_h\|_0 + \|r_h(u,p)\|_h^*. \end{array} \right.$$

Non-conforming space discretization

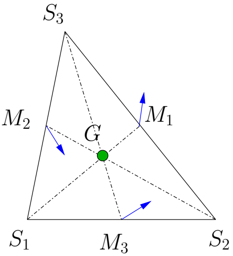
- $(\vec{P}_1^{NC},\, P_0)$ proposed in Crouzeix-Raviart'73 $\quad \|\vec{u}-\vec{u}_h\|_0 \lesssim h\,(|\vec{u}|_1 + \|p\|_0)$.

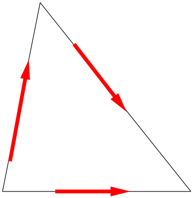
  $\vec{X}_h := \vec{H}_0^1(\Omega) \cap \vec{P}_1^{NC}$, $M_h := L_0^2(\Omega) \cap P_0$:

$$\begin{array}{rcl} \vec{P}_1^{NC} & := & \{\vec{v}_h \,|\, \forall T \in \mathcal{T}_h\ \vec{v}_{h|T} \in (P_1(T))^d \text{ and } \forall f \in \mathcal{F}_h\ [\vec{v}_h](\vec{x}_f) = 0\}, \\ P_0 & := & \{q_h \,|\, \forall T \in \mathcal{T}_h,\ q_h \in \mathbb{R}\}. \end{array}$$

$$a_h(\vec{u}_h, \vec{v}_h) = \textstyle\sum_T (\nu \vec{\nabla}\, \vec{u}_h, \vec{\nabla}\, \vec{v}_h)_{0,T}, \quad b_h(\vec{v}_h, q_h) = -\textstyle\sum_T (\vec{\nabla} \cdot \vec{v}_h, q_h)_{0,T}.$$

Drawback: Spurious discrete-divergence free vectors.





$p_h \in L_0^2(\Omega) \cap P_0$

$\vec{u}_h \in \vec{H}_0^1(\Omega) \cap \vec{P}_1^{NC}$

$$\vec{u}_h.\vec{n}_{|\partial T} = 0 \Rightarrow \int_T \vec{\nabla} \cdot \vec{u}_h \mathrm{d}\,T = 0.$$

Non-conforming space discretization

- Less degrees of freedom in $\vec{X}_h$ Hecht'81.
- More degrees of freedom in M_h. In $2D$:
- $(\vec{P}_1^{NC}, P_1 + P_b)$ Bernardi-Hecht'00 $M_h = L^2_0(\Omega) \cap P_1 + L^2_0(\Omega) \cap P_b$

$$(\vec{u}, p) \in \vec{H}^{s+1}(\Omega) \times H^s(\Omega) \quad \|\vec{u} - \vec{u}_h\|_{\vec{X}_h} \lesssim h^s \left(\|\vec{u}\|_{s+1} + \|p\|_s\right)$$

Drawback: Convergence of the discrete pressure ?

- $(\vec{P}_1^{NC}, P_1)$ Heib'03 $M_h = L^2_0(\Omega) \cap P_1$

$$(\vec{u}, p) \in \vec{H}^{s+1}(\Omega) \times H^s(\Omega) \quad \|\vec{u} - \vec{u}_h\|_{\vec{X}_h} + \|p - p_h\|_0 \lesssim h^s \left(\|\vec{u}\|_{s+1} + \|p\|_s\right)$$

Drawback: Spurious discrete divergence-free vectors.

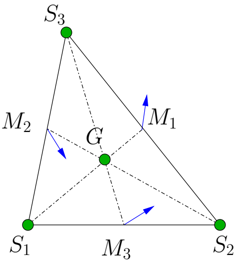
- $(\vec{P}_1^{NC}, P_0 + P_1)$ Heib'03 $M_h = L^2_0(\Omega) \cap P_0 + L^2_0(\Omega) \cap P_1$

$$(\vec{u}, p) \in \vec{H}^{2}(\Omega) \times H^1(\Omega) \quad \|\vec{u} - \vec{u}_h\|_{\vec{X}_h} + \|p - p_h\|_0 \lesssim h \left(\|\vec{u}\|_{2} + \|p\|_1\right).$$

$(\vec{P}_1^{NC},\, P_0 + P_1)$ discretization

- $M_h = L^2_0(\Omega) \cap P_0 \,+\, L^2_0(\Omega) \cap P_1\,, \quad P_1 := \{q_h \in \mathcal{C}^0(\Omega) \,|\, \forall T \in \mathcal{T}_h,\, q_h \in P_1(T)\}$

- $b_h\left(\vec{v}_h, q_{h,0} + q_{h,1}\right) = -\sum_T \left(\vec{\nabla}\cdot\vec{v}_h\,,\, q_{h,0}\right)_{0,T} + \sum_T \left(\vec{v}_h\,,\, \vec{\nabla}\, q_{h,1}\right)_{0,T}.$



$p_h \in L^2_0(\Omega) \cap P_0 \,+\, L^2_0(\Omega) \cap P_1$

$\vec{u}_h \in \vec{H}^1_0(\Omega) \cap \vec{P}_1^{NC}$

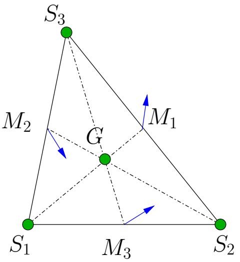
$\forall \vec{v}_h \in \vec{V}_h$:

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{v}_{h|T} &= 0, \\
\sum_T \left(\vec{\nabla}\, q_{h,2}\,,\, \vec{v}_h\right)_{0,T} &= 0 \qquad \forall q_{h,2} \in P_2, \\
\sum_T \int_{\partial T} q_{h,2}\, \vec{v}_h \cdot \vec{n}\, \mathrm{d}\,\sigma &= 0 \qquad \forall q_{h,2} \in P_2.
\end{aligned}
$$

$P_2 := \left\{q_{h,2} \in \mathcal{C}^0(\Omega) \,|\, \forall T \in \mathcal{T}_h,\, q_{h,2} \in P_2(T)\right\}$

$(\vec{P}_1^{NC},\, P_0 + P_1)$ discretization

- $M_h = L^2_0(\Omega) \cap P_0 \,+\, L^2_0(\Omega) \cap P_1\,, \quad P_1 := \{q_h \in \mathcal{C}^0(\Omega) \,|\, \forall T \in \mathcal{T}_h,\, q_h \in P_1(T)\}$
- $b_h(\vec{v}_h, q_{h,0} + q_{h,1}) = -\sum_T \left(\vec{\nabla}\cdot\vec{v}_h, q_{h,0}\right)_{0,T} + \sum_T \left(\vec{v}_h, \vec{\nabla}\, q_{h,1}\right)_{0,T}.$



Proof based on Simpson's rule,
exact for order 3 polynomials on a segment:

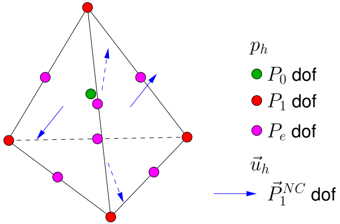
$$\int_e q_{h,3}\,\mathrm{d}\,\sigma = \frac{|e|}{6}\left(q_{h,3}(S_{1,e}) + 4\, q_{h,3}(M_e) + q_{h,3}(S_{2,e})\right)$$

 $p_h \in L^2_0(\Omega) \cap P_0 \,+\, L^2_0(\Omega) \cap P_1$

 $\vec{u}_h \in \vec{H}^1_0(\Omega) \cap \vec{P}_1^{NC}$

Non-conforming space discretization

- More degrees of freedom in M_h. In $3D$:
  - ▷ $(\vec{P}_1^{NC}, (P_0 + P_1 + P_e))$ T. Fortin'06.



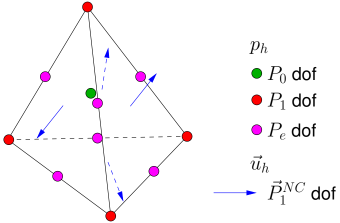
  - ▷ Numerical integration scheme on a triangle, exact for order 3 polynomials:

$$\int_T q_{h,3} \mathrm{d}\, T = \frac{|T|}{5} \left(\sum_{i=1}^{3} \frac{1}{4} q_{h,3}(S_{i,T}) + \frac{2}{3} \sum_{i=1}^{3} q_{h,3}(M_{i,T}) + \frac{3}{4} q_{h,3}(G_T) \right)$$

  - ▷ Numerous degrees of freedom.

Non-conforming space discretization

- More degrees of freedom in M_h. In $3D$:
  - $(\vec{P}_1^{NC},\ (P_0 + P_1 + P_e)\,)$ T. Fortin'06.

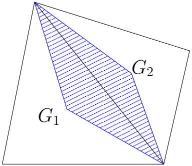


  - $M_h^e := \{q_h \in \mathcal{C}^0(\Omega)\,|\,\forall T \in \mathcal{T}_h,\ q_h \in P_e(T)\}$.
    For $e = [S_i S_j]$, $\gamma_{e|T} := 4\,\lambda_{i,T}\lambda_{j,T}$.
  - $M_h = \left(\,(M_h^e / M_h^{1,p} + M_h^1) \cap L_0^2(\Omega) + M_h^0 \cap L_0^2(\Omega)\,\right)$
  - For Stokes, if $\vec{f} = \vec{\nabla}\,p$, where $p \in H^4(\Omega) \cap L_0^2(\Omega)$ and $|p|_4 \neq 0$, then

$$\|\vec{u}_h\|_{\vec{X}_h} \lesssim h^5 \quad \text{and} \quad \|p_h - p\|_0 \lesssim h^2.$$

TrioCFD: a *finite volume-element* code.

- Control volumes of the FVM. Emonot'92



- FVM: the mass matrix is diagonal in $2D$ and $3D$.

  FEM: the mass matrix is diagonal in $2D$ only.

  In TrioCFD code, the mass matrix is diagonal in $2D$ and $3D$ as in FVM.
- FVM and FEM: Diffusion matrix is the same.
- FVM and FEM: Coupling matrix is the same.
- FVM or FEM: Convection schemes (centered, upwind, muscl, stab)
- FVM and FEM: Source term is different ($\approx h^2$).

  In TrioCFD code, the discrete source term is coded as in FEM.

Helmholtz decomposition

- $\vec{L}^2(\Omega)$-Helmholtz decomposition and $\vec{L}^2(\Omega)$-Helmholtz projector (Ω connected):

$$\forall \vec{f} \in \vec{L}^2(\Omega), \quad \exists\,(\psi, \vec{w}) \in H^1(\Omega) \times \vec{H}_0(\mathrm{div}0, \Omega) \,|\quad \vec{f} = \vec{\nabla}\,\psi + \vec{w},$$

$$\vec{L}^2(\Omega) = \vec{H}^\perp \oplus \vec{H} : \quad (\vec{\nabla}\,\psi, \vec{w})_0 = -(\psi, \vec{\nabla}\cdot\vec{w})_0 = 0.$$

$$\mathbb{P} : \vec{L}^2(\Omega) \to \vec{L}^2(\Omega), \quad \mathbb{P}\left(\vec{f}\right) = \vec{w}.$$

- Navier-Stokes variational formulation:

Find $(\vec{u}, p) \in \vec{H}^1_0(\Omega) \times L^2_0(\Omega)$ s.t. $\forall(\vec{v}, q) \in \vec{H}^1_0(\Omega) \times L^2_0(\Omega)$

$$\begin{aligned} \frac{d}{dt}(\vec{u}, \vec{v})_0 + (\nu\vec{\nabla}\,\vec{u}\,, \vec{\nabla}\,\vec{v})_0 + \left((\vec{u}\cdot\vec{\nabla}\,)\,\vec{u}\,, \vec{v}\right)_0 - (p, \vec{\nabla}\cdot\vec{v})_0 &= (\vec{f}\,, \vec{v})_0, \\ (q, \vec{\nabla}\cdot\vec{u})_0 &= 0. \end{aligned}$$

- $\vec{u} = \mathbb{P}\,(\vec{u})\,, \quad \mathbb{P}\left(\vec{\nabla}\,\vec{u}\right) = 0.$

Helmholtz decomposition

- $\vec{L}^2(\Omega)$-Helmholtz decomposition and $\vec{L}^2(\Omega)$-Helmholtz projector (Ω connected):

$$\forall \vec{f} \in \vec{L}^2(\Omega), \quad \exists\,(\psi, \vec{w}) \in H^1(\Omega) \times \vec{H}_0(\mathrm{div}0, \Omega) \mid \quad \vec{f} = \vec{\nabla}\,\psi + \mathbb{P}\left(\vec{f}\right),$$

$$\vec{L}^2(\Omega) = \vec{H}^\perp \oplus \vec{H} : \quad \left(\vec{\nabla}\,\psi, \mathbb{P}\left(\vec{f}\right)\right)_0 = -\left(\psi, \mathbb{P}\left(\vec{f}\right)\right)_0 = 0.$$

- Divergence-free momentum balance: $\forall \vec{v} \in \vec{V}$,

$$\begin{aligned}\frac{d}{dt}(\vec{u}, \vec{v})_0 + \left(\nu \vec{\nabla}\,\vec{u}\,,\, \vec{\nabla}\,\vec{v}\right)_0 + \left(\mathbb{P}\left(\,(\vec{\nabla}\cdot\vec{u})\,\vec{u}\,\right),\, \vec{v}\right)_0 \quad &= \quad \left(\mathbb{P}\left(\vec{f}\right)\,,\, \vec{v}\right)_0, \\ &= \quad \left(\mathbb{P}\left(\vec{f} + \vec{\nabla}\,\phi\right)\,,\, \vec{v}\right)_0,\end{aligned}$$

- *Irrotational* momentum balance: $\forall \vec{v}^\perp \in \vec{V}^\perp$,

$$\frac{d}{dt}\left(\vec{u}, \vec{v}^\perp\right)_0 - \left(p\,,\, \vec{\nabla}\cdot\vec{v}^\perp\right)_0 + \left((\vec{\nabla}\cdot\vec{u})\,\vec{u}\,,\, \vec{v}^\perp\right)_0 \quad = \quad \left(\vec{f}, \vec{v}^\perp\right)_0.$$

- Possible discrete biorthogonalization.

Raviart-Thomas FE

- $\vec{H}(\mathrm{div},\Omega)$-conforming FE Raviart-Thomas'77:

$$\vec{X}_h^{RT_0} := \{\vec{v}_h \in \vec{H}(\mathrm{div},\Omega)\ :\ \forall T \in \mathcal{T}_h, \quad \vec{v}_{h|T}(\vec{x}) = \vec{a}_T + b_T\vec{x}\}$$

- Raviart-Thomas projection operator (lowest order):

$$\Pi^{RT_0}(\vec{v}_{h|T})(\overrightarrow{OM}_i)\cdot\vec{n}_i = |F_i|^{-1}\int_{F_i}\vec{v}_h\cdot\vec{n}_i\,\mathrm{d}\vec{x}, \quad \forall i \in \{1,\cdots,d+1\}.$$

- $\forall\vec{v}_h \in \vec{X}_h, \quad \vec{\nabla}\cdot\vec{v}_{h|T} = \vec{\nabla}\cdot\left(\Pi^{RT_0}\vec{v}_{h|T}\right).$
- Let $(\vec{\phi}_i^{\alpha})_{i,\alpha}$ be the basis of $\vec{X}_h$:

  For $i \in \{1,\cdots,N_f\}, \alpha \in \{1,\cdots,d\}, \quad \vec{\phi}_{i|T}^{\alpha} := (1 - d\lambda_{i|T})\vec{e}_{\alpha}$. One can show that:

$$\Pi^{RT_0}\left(\vec{\phi}_{i|T}^{\alpha}\right) = \frac{\vec{S}_{i|T}\cdot\vec{e}_{\alpha}}{d|T|}\left(\vec{x} - \overrightarrow{OS}_{i|T}\right).$$

Raviart-Thomas projection

- *Towards a better constistency* Linke et al: Use Raviart-Thomas projection.
- Source term:

$$\forall \vec{v}_h \in \vec{P}_1^{NC} : \begin{cases} \left(\vec{f}, \vec{v}_h\right)_0 & \neq & -\sum_T \left(\psi, \vec{\nabla} \cdot \vec{v}_h\right)_{0,T} + (\vec{w}, \vec{v}_h)_0 \, , \\ \left(\vec{f}, \Pi^{RT_0} \vec{v}_h\right)_0 & = & -\sum_T \left(\psi, \vec{\nabla} \cdot \vec{v}_h\right)_{0,T} + \left(\vec{w}, \Pi^{RT_0} \vec{v}_h\right)_0 . \end{cases}$$

- Convection term: $(\vec{u} \cdot \vec{\nabla})\vec{u} = (\vec{\nabla} \times \vec{u}) \times \vec{u} + \frac{1}{2} \vec{\nabla} (|\vec{u}|^2)$

$$\left((\vec{u}^{n-1} \cdot \vec{\nabla})\vec{u}^n, \vec{v}\right)_0 = \left(\vec{\nabla} \times \vec{u}^n, \vec{u}^{n-1} \times \vec{v}\right)_0 - \frac{1}{2} \left(\vec{u}^n \cdot \vec{u}^{n-1}, \vec{\nabla} \cdot \vec{v}\right)_0 .$$

$$\sum_T \vec{\nabla} \times \vec{u}^n_{h|T} \cdot \int_T \vec{u}_h^{n-1} \times \vec{v}_h \mathrm{d}T \quad \to \quad \sum_T \vec{\nabla} \times \vec{u}^n_{h|T} \cdot \int_T \Pi^{RT_0} \vec{u}_h^{n-1} \times \Pi^{RT_0} \vec{v}_h \mathrm{d}T$$

- Mass term: $(\vec{u}_h, \vec{v}_h)_0 \quad \to \quad \left(\Pi^{RT_0} \vec{u}_h, \Pi^{RT_0} \vec{v}_h\right)_0 .$

Numerical results in $2D$: Stokes equation, no flow

- $\Omega = (0,1)^2$, $\nu = 1$, $\vec{f} = {}^t(0, Ra(1 - y + 3y^2))$, $Ra > 0$.

$$\vec{u} = \vec{0}, \quad p = Ra\left(y^3 - \frac{y^2}{2} + y - \frac{7}{12}\right)$$

- Values of $\|\vec{u}_h\|_0$ for $Ra = 1$

$1/h$	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_1$	$\vec{P}_1^{NC} - P_0P_1$
20	$2.74\,e-04$	$9.01\,e-13$	$1.21\,e-04$	$1.12\,e-08$
40	$6.53\,e-05$	$1.30\,e-13$	$1.39\,e-05$	$6.97\,e-10$
80	$1.66\,e-05$	$4.44\,e-13$	$1.77\,e-06$	$4.56\,e-11$
160	$4.13\,e-06$	$4.41\,e-13$	$2.21\,e-07$	$2.82\,e-12$
τ	2	–	3	4

- Values of $\|\vec{u}_h\|_0$ for $Ra = 100$

$1/h$	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_1$	$\vec{P}_1^{NC} - P_0P_1$
20	$2.74\,e-02$	$9.01\,e-13$	$1.21\,e-06$	$1.12\,e-06$
40	$6.53\,e-03$	$1.30\,e-13$	$1.39\,e-07$	$6.97\,e-08$
80	$1.66\,e-03$	$4.44\,e-13$	$1.77\,e-08$	$4.56\,e-09$
160	$4.13\,e-04$	$4.41\,e-13$	$2.21\,e-09$	$2.82\,e-10$
τ	2	–	3	4

Numerical results in $2D$: Navier-Stokes equation, stationary vortex

- $\Omega = (-1,1)^2$, $\nu = 1$, $\vec{f} = (Re - 1)\,{}^t(x,y)$, $Re > 0$.

$$\vec{u} = {}^t(-y\,,\,x), \quad p = Re\left(\frac{x^2+y^2}{2} - \frac{4}{3}\right)$$

- Values of $\|\vec{u} - \vec{u}_h\|_0$ for $Re = 1$

$1/h$	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_0P_1$
20	$1.32\,e-04$	$8.80\,e-12$	$3.36\,e-05$
40	$3.00\,e-05$	$1.27\,e-11$	$7.52\,e-06$
80	$7.86\,e-06$	$3.70\,e-11$	$2.17\,e-06$
160	$1.92\,e-06$	$1.25\,e-11$	$5.04\,e-07$
τ	2	–	4

- Values of $\|\vec{u} - \vec{u}_h\|_0$ for $Re = 10$

$1/h$	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_0P_1$
20	$8.97\,e-04$	$8.77\,e-11$	$3.36\,e-05$
40	$2.01\,e-04$	$1.26\,e-10$	$7.52\,e-06$
80	$5.29\,e-05$	$3.70\,e-10$	$2.17\,e-06$
160	$1.12\,e-05$	$1.25\,e-10$	$5.04\,e-07$
τ	2	–	4

Numerical results in $3D$: Navier-Stokes equation, stationary vortex

- $\Omega = (0,1)^3, \quad \nu = 1, \quad \vec{f} = (Re-1)\,{}^t(0, 2y-1, 2z-1).$

$$\vec{u} = {}^t(0\,,\ -2z+1\,,\ 2y-1), \quad p = Re\left(\frac{(2y-1)^2+(2z-1)^2}{2} - \frac{1}{3}\right)$$

- Values of $\|\vec{u}-\vec{u}_h\|_0$ for $Re = 1$

# DoFs	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_0P_1$
$7.71e+03$	$1.00\,e-03$	$3.22\,e-05$	$6.08\,e-03$
$1.53e+04$	$5.62\,e-03$	$1.77\,e-05$	$3.80\,e-03$
$3.05e+04$	$3.16\,e-04$	$8.77\,e-06$	$2.45\,e-03$
$6.11e+04$	$1.78\,e-04$	$4.42\,e-06$	$1.72\,e-03$
τ	2.5	2.8	1.7

- Values of $\|\vec{u}-\vec{u}_h\|_0$ for $Re = 100$

# DoFs	$\vec{P}_1^{NC} - P_0$	$\vec{P}_1^{NC} - P_0 + \Pi^{RT_0}$	$\vec{P}_1^{NC} - P_0P_1$
$7.71e+03$	$1.02\,e-01$	$3.19\,e-05$	$6.10\,e-03$
$1.53e+04$	$5.58\,e-01$	$1.75\,e-05$	$3.83\,e-03$
$3.05e+04$	$3.15\,e-02$	$8.75\,e-06$	$2.43\,e-03$
$6.11e+04$	$1.76\,e-02$	$4.39\,e-06$	$1.74\,e-03$
τ	2.5	2.8	1.7

Fortin-Soulie with Π^{RT_1}

- Matlab tests (Stokes, $2D$) and Fortin-Soulie'86 with M. Rihani (M2 internship last year).
- No flow in $\Omega = (0,1)^2$ with $\vec{f} = \vec{\nabla}(x^3)$ $\quad (h = 0.05, \quad N_{\vec{u}} = 6\,486, \quad N_p = 3\,152)$:

	$\Vert\vec{u} - \vec{u} - h\Vert_0$	$\Vert\vec{u} - \vec{u}_h\Vert_h$	$\Vert p - p_h\Vert_h$
$F - S$	$2.6\,e-07$	$3.7\,e-05$	$2.9\,e-03$
$F - S + \Pi^{RT_1}$	$9.7\,e-19$	$3.3\,e-17$	$2.9\,e-03$

- Vortex in $\Omega = (0,1)^2$ with varying ν $\quad (h = 0.1, \quad N_{\vec{u}} = 1\,534, \quad N_p = 726)$:
$\vec{u} = \overrightarrow{\text{curl}}\left(x^2(1-x)^2y^2(1-y^2)\right), \quad p = x^3 + y^3 - 1/2, \quad \vec{f} = -\nu\Delta\vec{u} + \vec{\nabla}\,p.$

	ν	$\Vert\vec{u} - \vec{u} - h\Vert_0$	$\Vert\vec{u} - \vec{u}_h\Vert_h$	$\Vert p - p_h\Vert_h$
$F - S$	$1.0\,e-0$	$1.5\,e-5$	$9.2\,e-4$	$2.1\,e-3$
	$1.0\,e-2$	$2.8\,e-4$	$2.0\,e-2$	$1.9\,e-3$
	$1.0\,e-4$	$9.4\,e-2$	$1.9\,e-0$	$1.9\,e-3$

	ν	$\Vert\vec{u} - \vec{u} - h\Vert_0$	$\Vert\vec{u} - \vec{u}_h\Vert_h$	$\Vert p - p_h\Vert_h$
$F - S + \Pi^{RT_1}$	$1.0\,e-0$	$1.7\,e-5$	$1.1\,e-4$	$2.8\,e-3$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$1.0\,e-8$	$1.7\,e-5$	$1.1\,e-4$	$2.8\,e-3$

Outlook

- Ongoing work:

  Using Raviart-Thomas projection:

  ▷ Convective form: $\left((\Pi^{RT_0}(\vec{u}_h^{n+1}) \cdot \vec{\nabla}\,)\,\vec{u}_h^{n}\,,\,\Pi^{RT_0}(\vec{v}_h)\right)_0$.

  ▷ Unsteady state with $\Pi^{RT_0}(M)$.

  Splines for INS using GeoPDEs with L. Dray, M2 student.

- Future work:

  Handle polyhedral meshes in particular prism or hexahedra and tetrahedra.

  ▷ PolyMAC code (A. Gershenfeld et al).

  Order 2 method.

  ▷ Preliminary work with M. Rihani, former M2 student.

  Matlab tests for Stokes in $2D$ and Fortin-Soulie'86 with Π^{RT_1}.

Thank you for your attention!