Moments of the characteristic polynomial in the three ensembles of random matrices

Madan Lal Mehta, Jean-Marie Normand

Moments of the characteristic polynomial in the three ensembles of random matrices

Madan Lal Mehta* and Jean-Marie Normand**
CEA/Saclay, Service de Physique Théorique, F-91191 Gif-sur-Yvette Cedex, France

Abstract. Moments of the characteristic polynomial of a random matrix taken from any of the three ensembles, orthogonal, unitary or symplectic, are given either as a determinant or a pfaffian or as a sum of determinants. For gaussian ensembles comparing the two expressions of the same moment one gets two remarkable identities, one between an $n \times n$ determinant and an $m \times m$ determinant and another between the pfaffian of a $2n \times 2n$ anti-symmetric matrix and a sum of $m \times m$ determinants.

1. Introduction

Three ensembles of random matrices A have been extensively investigated, namely when A is real symmetric, complex hermitian or quaternion self-dual. Recently the behaviour of their characteristic polynomial $\det(xI - A)$ has been of interest and we study here its integer moments. The same method can be adapted to compute the crossed averages $\langle \prod_{j=1}^{m} \det(x_j I - A) \rangle$.

Consider a non-negative function $w(x)$ with all its moments finite,

$$\int x^m w(x) dx < \infty \qquad m = 0, 1, \dots . \tag{1.1}$$

With this weight function $w(x)$ let us define three scalar products, one symmetric and two anti-symmetric, as [1]

$$\langle f, g \rangle_2 := \int f(x) g(x) w(x) dx = \langle g, f \rangle_2 \tag{1.2}$$

$$\langle f, g \rangle_4 := \int [f(x) g'(x) - f'(x) g(x)] w(x) dx = -\langle g, f \rangle_4 \tag{1.3}$$

$$\langle f, g \rangle_1 := \int \int f(x) g(y) \text{sign}(y - x) \sqrt{w(x) w(y)} dx dy = -\langle g, f \rangle_1 \tag{1.4}$$

and introduce polynomials C_n, Q_n and R_n of degree n, satisfying the orthogonality relations

$$\langle C_n, C_m \rangle_2 = c_n \delta_{n,m} \tag{1.5}$$

$$\langle Q_{2n}, Q_{2m} \rangle_4 = \langle Q_{2n+1}, Q_{2m+1} \rangle_4 = 0 \qquad \langle Q_{2n}, Q_{2m+1} \rangle_4 = q_n \delta_{n,m} \tag{1.6}$$

$$\langle R_{2n}, R_{2m} \rangle_1 = \langle R_{2n+1}, R_{2m+1} \rangle_1 = 0 \qquad \langle R_{2n}, R_{2m+1} \rangle_1 = r_n \delta_{n,m}. \tag{1.7}$$

* Member of CNRS, France. E-mail address: mehta@spht.saclay.cea.fr
** E-mail address: norjm@spht.saclay.cea.fr

We will take these polynomials to be monic, i.e. the coefficient of the highest power will be taken to be one. The above conditions determine completely the $C_n(x)$, $Q_{2n}(x)$ and $R_{2n}(x)$, while $Q_{2n+1}(x)$ can be replaced by $Q_{2n+1}(x)+aQ_{2n}(x)$ with an arbitrary constant a. Similarly, $R_{2n+1}(x)$ can be replaced by $R_{2n+1}(x)+aR_{2n}(x)$ with arbitrary a. We will choose these constants so that the coefficient of x^{2n} in $Q_{2n+1}(x)$ and in $R_{2n+1}(x)$ is zero, thus fixing them also completely.

The subscript $\beta = 1$, 2 or 4 is used to remind that it was convenient to use these polynomials to express the correlation functions of the eigenvalues of real symmetric, complex hermitian and quaternion self-dual random matrices respectively [2]. The polynomials $C_j(x)$ are said to be orthogonal while the polynomials $Q_j(x)$ and $R_j(x)$ are said to be skew-orthogonal.

In what follows n and m are non-negative integers. It can be checked that all the given formulas remain valid on replacing a non-existing sum by 0, a non-existing product, integral or determinant by 1 and forgetting non-existing rows or columns in a matrix.

It has long been known that the orthogonal polynomial $C_n(x)$ can be expressed as a multiple integral [3]

$$C_n(x) \propto \int ... \int \Delta_n^2(\mathbf{y}) \prod_{j=1}^{n} [(x-y_j)w(y_j)dy_j] \tag{1.8}$$

where

$$\mathbf{y} := \{y_1, ..., y_n\} \tag{1.9}$$

and

$$\Delta_n(\mathbf{y}) := \prod_{1\le j<k\le n} (y_k - y_j). \tag{1.10}$$

Similar multiple integral expressions for the skew-orthgonal polynomials $Q_n(x)$ and $R_n(x)$ were recently discovered by B. Eynard [4]

$$Q_{2n}(x) \propto \int ... \int \Delta_n^4(\mathbf{y}) \prod_{j=1}^{n} [(x-y_j)^2 w(y_j)dy_j] \tag{1.11}$$

$$Q_{2n+1}(x) \propto \int ... \int \Delta_n^4(\mathbf{y}) \left(x + 2\sum_{j=1}^{n} y_j \right) \prod_{j=1}^{n} [(x-y_j)^2 w(y_j)dy_j] \tag{1.12}$$

$$R_{2n}(x) \propto \int ... \int |\Delta_{2n}(\mathbf{y})| \prod_{j=1}^{2n} [(x-y_j)\sqrt{w(y_j)}dy_j] \tag{1.13}$$

$$R_{2n+1}(x) \propto \int ... \int |\Delta_{2n}(\mathbf{y})| \left(x + \sum_{j=1}^{n} y_j \right) \prod_{j=1}^{2n} [(x-y_j)\sqrt{w(y_j)}dy_j]. \tag{1.14}$$

It is also known [5] that

$$I_2(n,m;x) := \int ... \int \Delta_n^2(\mathbf{y}) \prod_{j=1}^{n} [(x-y_j)^m w(y_j) dy_j] \tag{1.15}$$

can be expressed as an $m \times m$ determinant. We report here that similar integrals

$$I_\beta(n,m;x) := \int ... \int |\Delta_n(\mathbf{y})|^\beta \prod_{j=1}^{n} \left\{ (x-y_j)^m [w(y_j)]^{1/(1+\delta_{\beta,1})} dy_j \right\} \tag{1.16}$$

for $\beta = 1$ or 4, can as well be expressed as a sum of $m \times m$ determinants. They are the m-th moments of the characteristic polynomial of an $n \times n$ random matrix A

$$\langle \det(xI - A)^m \rangle = \frac{I_\beta(n,m;x)}{I_\beta(n,0;x)} \tag{1.17}$$

the parameter β taking the values 1, 2 or 4 according as A is real symmetric, complex hermitian or quaternion self-dual and the probability density of the eigenvalues $\mathbf{y}$ being $|\Delta_n(\mathbf{y})|^\beta \prod_{j=1}^{n} [w(y_j)]^{1/(1+\delta_{\beta,1})}$. The special case $w(y) = e^{-ay^2}$ arises when the probability density of the random matrix is invariant under any change of basis and the algebraically independent parameters specifying the matrix elements are also statistically independent.

Instead of the real symmetric, complex hermitian and quaternion self-dual matrices some authors considered the corresponding ensembles of unitary matrices [6]. The m-th moment of the characteristic polynomial in $x = \exp(i\alpha)$

$$\int_0^{2\pi} ... \int_0^{2\pi} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{j=1}^{n} \left[|e^{i\alpha} - e^{i\theta_j}|^m d\theta_j \right] \tag{1.18}$$

is a constant independent of α which was evaluated using Selberg's integral [6]. In the limit of large n with fixed m, this constant has a close relation with the moments of the absolute values of the Riemann zeta function $\zeta(z)$ or of the L-functions on the critical line Re $z = 1/2$. Similarly, the moments of the derivative with respect to x of the characteristic polynomial are related to the moments of the absolute values of the derivative $\zeta'(z)$ on the critical line [6].

In case of the real symmetric, complex hermitian and quaternion self-dual matrices, which are the only ones we consider here, the m-th moment of the characteristic polynomial is a polynomial of order mn, and we do not know whether their zeros are related with the Riemann zeta function.

2. Calculation of $\mathbf{I}_\beta(\mathbf{n}, \mathbf{m}; \mathbf{x})$

The case $\beta = 2$ is the simplest involving the better known determinants, while the other two cases $\beta = 1$ and 4 are similar involving the less familiar pfaffians.

The $I_\beta(n,m;x)$ can be expressed in two different forms:
(i) a determinant or a pfaffian of a matrix of size which depends on n; i.e.
- an $n \times n$ determinant in case $\beta = 2$;
- a pfaffian of a $2n \times 2n$ anti-symmetric matrix in case $\beta = 4$;
- a pfaffian of a $n \times n$ (resp. $(n+1)\times(n+1)$) anti-symmetric matrix in case $\beta = 1$ for n even (resp. odd);
(ii) a sum of determinants of matrices of size which depends on m; i.e.
- an $m \times m$ determinant in case $\beta = 2$;
- a sum of $m \times m$ determinants in case $\beta = 4$ for m even;
- a sum of $m \times m$ (resp. $(m+1)\times(m+1)$) determinants in case $\beta = 1$ for n even (resp. odd).

The usefulness of the expressions above, depending on the form of the weight function $w(x)$, may be limited. When the weight is gaussian, two of the forms supplying alternate expressions for $I_2(n,m;x)$ as $n \times n$ and $m \times m$ determinants lead us to an interesting known identity. Similarly, the two expressions of $I_4(n,m;x)$ lead us to an identity between a pfaffian of an $2n \times 2n$ anti-symmetric matrix and a sum of $m \times m$ determinants.

2.1 $\mathbf{I}_\beta(\mathbf{n},\mathbf{m};\mathbf{x})$ as a determinant or a pfaffian of a matrix of size depending on n

Choose $P_j(x)$ and $\pi_j(x)$ any monic polynomials of degree j.

2.1.1 The case $\beta = 2$

Write $\Delta_n(\mathbf{y})$ as an $n \times n$ Vandermonde determinant

$$\begin{aligned}\Delta_n(\mathbf{y}) &= \det\left[y_j^{k-1}\right]_{j,k=1,\dots,n} = \det\left[P_{k-1}(y_j)\right]_{j,k=1,\dots,n} \\ &= \det\left[\pi_{k-1}(y_j)\right]_{j,k=1,\dots,n} \ . \end{aligned} \tag{2.1.1}$$

Then

$$\begin{aligned}\int \dots \int \Delta_n^2(\mathbf{y}) \prod_{j=1}^n [f(y_j)dy_j] &= \int \dots \int \det\left[P_{k-1}(y_j)\right] \det\left[\pi_{k-1}(y_j)\right] \prod_{j=1}^n [f(y_j)dy_j] \\ &= n! \int \dots \int \det\left[\pi_{k-1}(y_j)\right] \prod_{j=1}^n [P_{j-1}(y_j)f(y_j)dy_j] \\ &= n! \int \dots \int \det\left[P_{j-1}(y_j)\pi_{k-1}(y_j)f(y_j)\right] \prod_{j=1}^n dy_j \\ &= n! \ \det\left[\phi_{2;j,k}\right]_{j,k=0,\dots,n-1} \end{aligned} \tag{2.1.2}$$

where

$$\phi_{2;j,k} := \int P_j(y)\pi_k(y)f(y)dy. \tag{2.1.3}$$

In the first three lines of the above equation the indices j, k take the values from 1 to n. In the second line we have replaced the first determinant by a single term which is allowed by the symmetry of the integrand in the y_j; while the last line is obtained by integrating over the y_j each of them occuring only in one column.

Setting $f(y) = (x-y)^m w(y)$, we get a first form of $I_2(n,m;x)$ in terms of an $n \times n$ determinant

$$I_2(n,m;x) = n! \ \det\left[\phi_{2;j,k}(x)\right]_{j,k=0,\dots,n-1} \tag{2.1.4}$$

where

$$\phi_{2;j,k}(x) := \int P_j(y)\pi_k(y)(x-y)^m w(y)dy. \tag{2.1.5}$$

2.1.2 The case $\beta = 4$

Write $\Delta_n^4(\mathbf{y})$ as a $2n \times 2n$ determinant [7]

$$\begin{aligned}\Delta_n^4(\mathbf{y}) &= \det\left[y_j^k, \ ky_j^{k-1}\right]_{j=1,\dots,n;\ k=0,\dots,2n-1} \\ &= \det\left[P_k(y_j), \ P_k'(y_j)\right]_{j=1,\dots,n;\ k=0,\dots,2n-1}. \end{aligned} \tag{2.1.6}$$

Each variable occurs in two columns, a column of monic polynomials and a column of its derivatives. Expand the determinant and integrate to see that [8]

$$\int \dots \int \Delta_n^4(\mathbf{y}) \prod_{j=1}^n f(y_j)dy_j = \sum \pm \phi_{4;j_1,j_2}\phi_{4;j_3,j_4}\dots\phi_{4;j_{2n-1},j_{2n}} \tag{2.1.7}$$

$$= n! \ \mathrm{pf}\left[\phi_{4;j,k}\right]_{j,k=0,\dots,2n-1} \tag{2.1.8}$$

where

$$\phi_{4;j,k} := \int \left[P_j(y)P_k'(y) - P_j'(y)P_k(y)\right] f(y)dy = -\phi_{4;k,j}. \tag{2.1.9}$$

In equation (2.1.7) the sum is over all permutations $\begin{pmatrix} 0 & 1 & \dots & 2n-1 \\ j_1 & j_2 & \dots & j_{2n} \end{pmatrix}$ of the $2n$ indices $(0, 1, \dots, 2n-1)$ with $j_1 < j_2$, ..., $j_{2n-1} < j_{2n}$, the sign being plus or minus according as this permutation is even or odd. In equation (2.1.8) *pf* means the pfaffian.

Setting $f(y) = (x-y)^m w(y)$ as before, we get a first form of $I_4(n,m;x)$ in terms of the pfaffian of a $2n \times 2n$ anti-symmetric matrix

$$I_4(n,m;x) = n! \ \mathrm{pf}\left[\phi_{4;j,k}(x)\right]_{j,k=0,\dots,2n-1} \tag{2.1.10}$$

where

$$\phi_{4;j,k}(x) := \int \left[P_j(y)P_k'(y) - P_j'(y)P_k(y)\right](x-y)^m w(y)dy. \tag{2.1.11}$$

2.1.3 The case $\beta = 1$

The difficulty of the absolute value of $\Delta_n(\mathbf{y})$ can be overcome by ordering the variables. Writing

$$\Delta_n(\mathbf{y}) \prod_{j=1}^{n} f(y_j) = \det \left[P_{k-1}(y_j) f(y_j)\right]_{j,k=1,\dots,n} \tag{2.1.12}$$

with $f(y) = (x-y)^m \sqrt{w(y)}$ and using a result of de Bruijn [9], one gets a first form of $I_1(n,m;x)$ in terms of the pfaffian of an $n \times n$ (resp. $(n+1) \times (n+1)$) anti-symmetric matrix for n even (resp. odd)

$$I_1(n,m;x) = \begin{cases} n! \text{ pf} \left[\phi_{1;j,k}(x)\right]_{j,k=0,\dots,n-1} & n \text{ even} \\ n! \text{ pf} \begin{bmatrix} \phi_{1;j,k}(x) & \alpha_j(x) \\ -\alpha_k(x) & 0 \end{bmatrix}_{j,k=0,\dots,n-1} & n \text{ odd} \end{cases} \tag{2.1.13}$$

where

$$\phi_{1;j,k}(x) := \int\int P_j(z) P_k(y)(x-z)^m (x-y)^m \sqrt{w(z)w(y)} \text{sign}(y-z) dz dy \tag{2.1.14}$$

and

$$\alpha_j(x) := \int P_j(y)(x-y)^m \sqrt{w(y)} dy. \tag{2.1.15}$$

Another way to get the same result is [8] to order the variables, integrate over the alternate ones, remove the ordering over the remaining alternate variables and observe that the result is given by equations (2.1.13)-(2.1.15).

2.1.4 An alternative of the expressions above for $\beta = 1$, 2 and 4

If one replaces the monic polynomials $P_j(y)$ and $\pi_j(y)$ by $P_j(y-x)$ and $\pi_j(y-x)$, equations (2.1.4), (2.1.10) and (2.1.13) become

$$I_2(n,m;x) = (-1)^{mn} n! \det \left[\varphi_{2;j,k}(x)\right]_{j,k=0,\dots,n-1} \tag{2.1.16}$$

$$I_4(n,m;x) = (-1)^{mn} n! \text{ pf} \left[\varphi_{4;j,k}(x)\right]_{j,k=0,\dots,2n-1} \tag{2.1.17}$$

$$I_1(n,m;x) = \begin{cases} n! \text{ pf} \left[\varphi_{1;j,k}(x)\right]_{j,k=0,\dots,n-1} & n \text{ even} \\ (-1)^m n! \text{ pf} \begin{bmatrix} \varphi_{1;j,k}(x) & \theta_j(x) \\ -\theta_k(x) & 0 \end{bmatrix}_{j,k=0,\dots,n-1} & n \text{ odd} \end{cases} \tag{2.1.18}$$

where

$$\varphi_{2;j,k}(x) := \int P_j(y-x) \pi_k(y-x)(y-x)^m w(y) dy \tag{2.1.19}$$

$$\varphi_{4;j,k}(x) := \int \left[P_j(y-x) P_k'(y-x) - P_j'(y-x) P_k(y-x)\right] (y-x)^m w(y) dy \tag{2.1.20}$$

$$\varphi_{1;j,k}(x) := \int\int P_j(z-x) P_k(y-x) [(z-x)(y-x)]^m \sqrt{w(z)w(y)} \text{sign}(y-z) dz dy \tag{2.1.21}$$

$$\theta_j(x) := \int P_j(y-x)(y-x)^m \sqrt{w(y)} dy. \tag{2.1.22}$$

2.2 $\mathbf{I}_\beta(\mathbf{n},\mathbf{m};\mathbf{x})$ as determinants of size depending on m

2.2.1 The case $\beta = 2$

For $I_2(n,m;x)$, equation (1.15), let us write the integrand as the product of two determinants: $\Delta_n(\mathbf{y})$ given in equation (2.1.1) and

$$\Delta_n(\mathbf{y})\prod_{j=1}^{n}(x-y_j)^m = b\ \det\left[y_j^{k-1}, \left(\frac{d}{dx}\right)^l x^{k-1}\right]_{\substack{j=1,\dots,n\\ k=1,\dots,n+m\\ l=0,\dots,m-1}}$$

$$= b\ \det\left[P_{k-1}(y_j), P_{k-1}^{(l)}(x)\right]_{\substack{j=1,\dots,n\\ k=1,\dots,n+m\\ l=0,\dots,m-1}} \quad (2.2.1)$$

where

$$b = \left(\prod_{l=0}^{m-1} l!\right)^{-1} \quad (2.2.2)$$

$P_k(x)$ is any monic polynomial of degree k and $P_k^{(l)}(x)$ is its l-th derivative. Expression (2.2.1) is an $(m+n)\times(m+n)$ determinant the last m columns of which are $P_k(x)$ and its successive derivatives [7]. As the integrand in equation (1.15) is symmetric in the y_j, we can replace the first $n\times n$ determinant by a single term and multiply the result by $n!$

$$I_2(n,m;x) = b\ n!\int\dots\int \det\left[P_{k-1}(y_j), P_{k-1}^{(l)}(x)\right]_{\substack{j=1,\dots,n\\ k=1,\dots,n+m\\ l=0,\dots,m-1}} \prod_{j=1}^{n}\left[P_{j-1}(y_j)w(y_j)dy_j\right]$$

$$= b\ n!\int\dots\int \det\left[P_{j-1}(y_j)P_{k-1}(y_j)w(y_j), P_{k-1}^{(l)}(x)\right]_{\substack{j=1,\dots,n\\ k=1,\dots,n+m\\ l=0,\dots,m-1}} \prod_{j=1}^{n} dy_j. \quad (2.2.3)$$

As each variable y_j occurs only in one column, we can integrate over them independently

$$I_2(n,m;x) = b\ n! \det\left[\int P_{j-1}(y)P_{k-1}(y)w(y)dy, P_{k-1}^{(l)}(x)\right]_{\substack{j=1,\dots,n\\ k=1,\dots,n+m\\ l=0,\dots,m-1}}. \quad (2.2.4)$$

If we choose the polynomials $P_j(x)$ to be the orthogonal polynomials $C_j(x)$, equation (1.5), then one gets a second form for $I_2(n,m;x)$

$$I_2(n,m;x) = b\ n!\ c_0\dots c_{n-1} \det\left[C_{n+k}^{(l)}(x)\right]_{k,l=0,\dots,m-1} \quad (2.2.5)$$

an $m \times m$ determinant whose first column consists of $C_{n+k}(x)$, $k = 0, 1, ..., m-1$, and the other $m-1$ columns are the successive derivatives of the first column. Therefore from equation (1.17)

$$\langle \det(xI - A)^m \rangle = b \ \det \left[C_{n+k}^{(l)}(x) \right]_{k,l=0,\ldots,m-1} . \tag{2.2.6}$$

This result appears in reference [5] as equation (15) apart from a sign which seems to be wrong.

2.2.2 The case $\beta = 4$

This method applies only for even moments. For $I_4(n, 2m; x)$, equation (1.16), one can write the integrand as a single determinant [7]

$$\begin{aligned} \Delta_n^4(\mathbf{y}) \prod_{j=1}^{n} (x-y)^{2m} &= b \ \det \left[y_j^k, k y_j^{k-1}, \left(\frac{d}{dx}\right)^l x^k \right]_{\substack{j=1,\ldots,n \\ k=0,\ldots,n+m-1 \\ l=0,\ldots,m-1}} \\ &= b \ \det \left[P_k(y_j), P'_k(y_j), P_k^{(l)}(x) \right]_{\substack{j=1,\ldots,n \\ k=0,\ldots,n+m-1 \\ l=0,\ldots,m-1}} \end{aligned} \tag{2.2.7}$$

where b is given by equation (2.2.2). Each variable y_j occurs in two columns. Expanding the determinant and integrating one sees that the result is a sum of products of the form

$$\pm b \ a_{s_1,s_2} \ldots a_{s_{2n-1},s_{2n}} \det \left[P_{j_k}^{(l-1)}(x) \right]_{k,l=1,\ldots,m} \tag{2.2.8}$$

with

$$a_{j,k} := \int \left[P_j(y) P'_k(y) - P'_j(y) P_k(y) \right] w(y) dy = \langle P_j, P_k \rangle_4 \tag{2.2.9}$$

the indices $s_1, ..., s_{2n}, j_1, ..., j_m$ are all distinct, chosen from $0, 1, ..., 2n+m-1$ and the sign is plus or minus according as the permutation

$$\begin{pmatrix} 0 & 1 & \ldots & 2n-1 & 2n & \ldots & 2n+m-1 \\ s_1 & s_2 & \ldots & s_{2n} & j_1 & \ldots & j_m \end{pmatrix} \tag{2.2.10}$$

is even or odd. If we choose the polynomials $P_j(x)$ to be the skew-orthogonal polynomials $Q_j(x)$, equation (1.6), then all the $a_{s,j}$ except $a_{2s,2s+1} = q_s$ will be zero and we get a second form of $I_4(n, 2m; x)$ in terms of a sum of $m \times m$ determinants

$$I_4(n, 2m; x) = b \ n! \sum_{(s)} q_{s_1} \ldots q_{s_n} \det \left[Q_{j_k}^{(l-1)}(x) \right]_{k,l=1,\ldots,m} \tag{2.2.11}$$

where the sum is over all choices of $s_1 < \ldots < s_n$ such that $2s_1, 2s_1+1, ..., 2s_n, 2s_n+1, j_1, ..., j_m$ are all the indices from 0 to $2n+m-1$ and moreover $j_1 < \ldots < j_m$.

2.2.3 The case $\beta = 1$

For $I_1(n, m; x)$, equation (1.16), the absolute value sign of $\Delta_n(\mathbf{y})$ is the main difficulty. Ordering the variables and using the same method as in section 2.2.2 above [7], one has

$$\begin{aligned} I_1(n, m; x) &= \int ... \int |\Delta_n(\mathbf{y})| \prod_{j=1}^{n} \left[(x - y_j)^m \sqrt{w(y_j)} dy_j \right] \\ &= n! \int ... \int_{y_1 \le ... \le y_n} \Delta_n(\mathbf{y}) \prod_{j=1}^{n} \left[(x - y_j)^m \sqrt{w(y_j)} dy_j \right] \\ &= b\ n! \int ... \int_{y_1 \le ... \le y_n} \det \left[P_k(y_j), P_k^{(l)}(x) \right]_{\substack{j=1,...,n \\ k=0,...,n+m-1 \\ l=0,...,m-1}} \\ &\quad \cdot \prod_{j=1}^{n} \left[\sqrt{w(y_j)} dy_j \right] \end{aligned} \tag{2.2.12}$$

with b as in equation (2.2.2). Integrating successively over the alternate variables y_1, y_3, y_5, ... and then removing the restriction over the remaining variables y_2, y_4, ... as indicated at the end of section 2.1.3 above, one gets for even n

$$\begin{aligned} I_1(n, m; x) &= b\ \frac{n!}{[n/2]!} \int ... \int \det \left[G_k(y_{2j}), P_k(y_{2j}), P_k^{(l)}(x) \right]_{\substack{j=1,...,[n/2] \\ k=0,...,n+m-1 \\ l=0,...,m-1}} \\ &\quad \cdot \prod_{j=1}^{[n/2]} \left[\sqrt{w(y_{2j})} dy_{2j} \right] \end{aligned} \tag{2.2.13}$$

with

$$G_k(y_2) := \int^{y_2} P_k(y_1) \sqrt{w(y_1)} dy_1. \tag{2.2.14}$$

In case n is odd, one more column of the numbers $g_k := \int P_k(y)\sqrt{w(y)}dy$ appears in the determinant just after the column of $P_k(y_{n-1})$.

The present situation is exactly as in the case $\beta = 4$ with each variable occuring in two columns. Expanding the determinant one sees that the result contains the expressions

$$\begin{aligned} \int [G_j(y)P_k(y) - G_k(y)P_j(y)] \sqrt{w(y)} dy &= \int \int P_j(x)P_k(y) \text{sign}(y - x) \sqrt{w(x)w(y)} dx dy \\ &= \langle P_j, P_k \rangle_1. \end{aligned} \tag{2.2.15}$$

If we choose $P_j(x) = R_j(x)$, equation (1.7), then writing the result separately for even and odd n for clarity, one has

$$I_1(2n, m; x) = b\ (2n)! \sum_{(s)} r_{s_1} ... r_{s_n} \det \left[R_{j_k}^{(l-1)}(x) \right]_{k,l=1,...,m} \tag{2.2.16}$$

where the sum is taken over all choices of $s_1 < ... < s_n$ such that $2s_1$, $2s_1+1$, ..., $2s_n$, $2s_n+1$, j_1, ..., j_m are all the indices from 0 to $2n+m-1$ and moreover $j_1 < ... < j_m$ and

$$I_1(2n+1,m;x) = b\ (2n+1)! \sum_{(s)} r_{s_1}...r_{s_n} \det\left[g_{j_k}, R_{j_k}^{(l-1)}(x)\right]_{k=1,...,m+1;\ l=1,...,m} \quad (2.2.17)$$

where now the sum is taken over all choices of $s_1 < ... < s_n$ such that $2s_1$, $2s_1+1$, ..., $2s_n$, $2s_n+1$, j_1, ..., j_m, j_{m+1} are all the indices from 0 to $2n+m$ and moreover $j_1 < ... < j_{m+1}$.

Note that equations (1.8), (1.11) and (1.13) are particular cases ($m=1$) of equations (2.2.5), (2.2.11) and (2.2.16) respectively. If we shift the index of the last row in the right hand side of equation (2.2.7) for $m=1$ from n to $n+1$, i.e. replace the last row

$$[P_n(y_j), P_n'(y_j), P_n(x)]$$

by the row

$$[P_{n+1}(y_j), P_{n+1}'(y_j), P_{n+1}(x)]$$

we get the integrand of equation (1.12). Following the procedure which leads to equation (2.2.11) we get equation (1.12). Similarly, if we shift the index of the last row from n to $n+1$ in the right hand side of equation (2.2.12) for $m=1$ and follow the procedure leading to equation (2.2.16) we get equation (1.14).

3. Special case of the gaussian weight

These formulas for $I_\beta(n,m;x)$ will have little use if one does not know the polynomials $C_n(x)$, $Q_n(x)$ or $R_n(x)$. Fortunately one knows them for almost all classical weights. As an example we give them here for the gaussian weight $w(x) = e^{-x^2}$ over $[-\infty,\infty]$ in terms of Hermite polynomials $H_n(x) := e^{x^2}(-d/dx)^n e^{-x^2}$. Their verification is straightforward. One has

$$C_n(x) = 2^{-n} H_n(x) \quad (3.1)$$

$$c_n = 2^{-n} n! \sqrt{\pi} \quad (3.2)$$

$$Q_{2n}(x) = \sum_{j=0}^{n} 2^{-2j} \frac{n!}{j!} H_{2j}(x) \qquad Q_{2n+1}(x) = 2^{-2n-1} H_{2n+1}(x) \quad (3.3)$$

$$q_n = 2^{-2n}(2n+1)!\sqrt{\pi} \quad (3.4)$$

$$R_{2n}(x) = 2^{-2n} H_{2n}(x) \qquad R_{2n+1}(x) = 2^{-2n}[xH_{2n}(x) - H_{2n}'(x)] \quad (3.5)$$

$$r_n = 2^{1-2n}(2n)!\sqrt{\pi}. \quad (3.6)$$

In this case some results can be given other forms. In particular, using the recurrence relation

$$2xH_n(x) = H_{n+1}(x) + H_n'(x) \quad (3.7)$$

or

$$xC_n(x) = C_{n+1}(x) + \frac{1}{2}C'_n(x) \tag{3.8}$$

one can replace the $C^{(l)}_{n+k}$ in the determinant (2.2.5) by $C_{n+k+l}(x)$ so that

$$\begin{aligned} I_2(n,m;x) &= b\ n!\ c_0 ... c_{n-1}(-2)^{m(m-1)/2} \det\left[C_{n+j+k}(x)\right]_{j,k=0,1,\ldots,m-1} \\ &= \pi^{n/2} \frac{2^{-n(n-1)/2}\prod_{j=0}^{n} j!}{(-2)^{-m(m-1)/2}\prod_{j=0}^{m-1} j!} \det\left[C_{n+j+k}(x)\right]_{j,k=0,1,\ldots,m-1}. \end{aligned} \tag{3.9}$$

Also for any non-negative integer j [10]

$$\int_{-\infty}^{\infty}(y-x)^j e^{-y^2}\,dy = \sqrt{\pi}\ i^j\ C_j(ix) \tag{3.10}$$

with $i=\sqrt{-1}$. Therefore, choosing $P_j(y-x) = \pi_j(y-x) = (y-x)^j$ in equations (2.1.19) and (2.1.20), we get

$$\varphi_{2;j,k}(x) = \sqrt{\pi}\ i^{m+j+k}\ C_{m+j+k}(ix) \tag{3.11}$$

$$\varphi_{4;j,k}(x) = \sqrt{\pi}\ i^{m+j+k-1}\ (k-j)C_{m+j+k-1}(ix) \tag{3.12}$$

and equations (2.1.16) and (2.1.17) then give

$$I_2(n,m;x) = (-i)^{mn}\ n!\ \pi^{n/2}(-1)^{n(n-1)/2}\det\left[C_{m+j+k}(ix)\right]_{j,k=0,\ldots,n-1} \tag{3.13}$$

$$I_4(n,m;x) = (-i)^{mn}\ n!\ \pi^{n/2}\mathrm{pf}\left[(k-j)C_{m+j+k-1}(ix)\right]_{j,k=0,\ldots,2n-1}. \tag{3.14}$$

Equations (3.9) and (3.13) give a relation almost symmetric in n and m. Writing them again

$$\frac{I_2(n,m;x)}{I_2(m,n;ix)} = (-i)^{mn}\frac{\pi^{n/2}2^{-n(n-1)/2}\prod_{j=0}^{n} j!}{\pi^{m/2}2^{-m(m-1)/2}\prod_{j=0}^{m} j!} \tag{3.15}$$

or equivalently

$$\frac{\det\left[C_{n+j+k}(x)\right]_{j,k=0,\ldots,m-1}}{\det\left[C_{m+j+k}(ix)\right]_{j,k=0,\ldots,n-1}} = (-i)^{mn}\frac{(-2)^{n(n-1)/2}\prod_{j=0}^{n-1} j!}{(-2)^{m(m-1)/2}\prod_{j=0}^{m-1} j!}. \tag{3.16}$$

Equations (3.15) and (3.16) appear in [11] as equations (4.43) and (4.44).

Equations (2.2.11) and (3.14) give another identity relating the pfaffian

$$\mathrm{pf}[(k-j)C_{2m+j+k-1}]_{j,k=0,\ldots,2n-1}$$

to a sum of $m\times m$ determinants.

Forrester and Witte [11] state that the identity

$$\frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\Delta_n(\mathbf{y})|^{2\gamma}\prod_{j=1}^{n}\left[(x-y_j)^m e^{-y_j^2}\,dy_j\right]}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\Delta_m(\mathbf{y})|^{2/\gamma}\prod_{j=1}^{m}\left[(ix-y_j)^n e^{-y_j^2}\,dy_j\right]} = \text{const.} \tag{3.17}$$

is valid for any γ. This is not true for $\gamma \neq 1$ as verified below for $\{n,m\}=\{n,1\}$, $\{2,2\}$ and $\{3,2\}$. For values of a and γ such that the integrals below exist, letting

$$\int d\mu(\mathbf{y})f(\mathbf{y}) := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\Delta_n(\mathbf{y})|^{2\gamma}\prod_{j=1}^{n}\left(e^{-ay_j^2}\,dy_j\right)f(\mathbf{y}) \tag{3.18}$$

$$\langle f(\mathbf{y})\rangle := \int d\mu(\mathbf{y})f(\mathbf{y}) \div \int d\mu(\mathbf{y}) \tag{3.19}$$

$$\mathcal{I}(\gamma,a,n,m;x) := \int d\mu(\mathbf{y})\prod_{j=1}^{n}(x-y_j)^m = \mathcal{I}(\gamma,a,n,0;x)\langle\prod_{j=1}^{n}(x-y_j)^m\rangle \tag{3.20}$$

where $\langle\prod_{j=1}^{n}(x-y_j)^m\rangle$ is a polynomial in x whose coefficients can be evaluated using the expressions given in [12].

For any n and $m=1$, one has

$$\begin{aligned}\frac{\mathcal{I}(\gamma,a,n,1;x)}{\mathcal{I}(\gamma,a,n,0;x)} &= \langle\prod_{j=1}^{n}(x-y_j)\rangle = \sum_{j=0}^{n}\binom{n}{j}(-1)^j\langle y_1\ldots y_j\rangle x^{n-j} \\ &= \sum_{j=0}^{[n/2]}\binom{n}{2j}\left(-\frac{\gamma}{2a}\right)^j\frac{(2j)!}{2^j j!}x^{n-2j} \\ &= \left(\frac{\gamma}{4a}\right)^{n/2}H_n\left(x\sqrt{\frac{a}{\gamma}}\right)\end{aligned} \tag{3.21}$$

whereas for $n=1$ and any m

$$\begin{aligned}\frac{\mathcal{I}(\gamma,a,1,m;x)}{\mathcal{I}(\gamma,a,1,0;x)} &= \langle(x-y_1)^m\rangle = \sum_{j=0}^{m}\binom{m}{j}(-1)^j\langle y_1^j\rangle x^{m-j} \\ &= \sum_{j=0}^{[m/2]}\binom{m}{2j}\frac{(2j)!}{2^{2j}j!}a^{-j}x^{m-2j} \\ &= (4a)^{-m/2}i^{-m}H_m(ix\sqrt{a}).\end{aligned} \tag{3.22}$$

Therefore for any γ, γ', a and n, one verifies that

$$\frac{\mathcal{I}(\gamma,a,n,1;x)}{\mathcal{I}(\gamma',a/\gamma,1,n;ix)} = (-i)^n\frac{\mathcal{I}(\gamma,a,n,0;x)}{\mathcal{I}(\gamma',a/\gamma,1,0;ix)} \tag{3.23}$$

is a constant independent of x.
For $n = m = 2$, one has

$$\frac{\mathcal{I}(\gamma, a, 2, 2; x)}{\mathcal{I}(\gamma, a, 2, 0; x)} = \langle \prod_{j=1}^{2} (x - y_j)^2 \rangle = x^4 + 2(\langle y_1^2 \rangle + 2\langle y_1 y_2 \rangle)x^2 + \langle y_1^2 y_2^2 \rangle$$
$$= x^4 + \frac{1}{a}(1-\gamma)x^2 + \frac{1}{4a^2}(1+\gamma+\gamma^2)$$
$$= \frac{\mathcal{I}(1/\gamma, a/\gamma, 2, 2; ix)}{\mathcal{I}(1/\gamma, a/\gamma, 2, 0; ix)}. \tag{3.24}$$

For $n = 3$ and $m = 2$, one has

$$\frac{\mathcal{I}(\gamma, a, 3, 2; x)}{\mathcal{I}(\gamma, a, 3, 0; x)} = \langle \prod_{j=1}^{3} (x - y_j)^2 \rangle$$
$$= x^6 + 3(\langle y_1^2 \rangle + 4\langle y_1 y_2 \rangle)x^4 + 3(\langle y_1^2 y_2^2 \rangle + 4\langle y_1^2 y_2 y_3 \rangle)x^2 + \langle y_1^2 y_2^2 y_3^2 \rangle$$
$$= x^6 + \frac{3}{2a}(1-2\gamma)x^4 + \frac{3}{4a^2}(1-\gamma+3\gamma^2)x^2 + \frac{1}{8a^3}(1+3\gamma+5\gamma^2) \tag{3.25}$$

whereas for $n = 2$ and $m = 3$

$$\frac{\mathcal{I}(\gamma, a, 2, 3; x)}{\mathcal{I}(\gamma, a, 2, 0; x)} = \langle \prod_{j=1}^{2} (x - y_j)^3 \rangle$$
$$= x^6 + 3(2\langle y_1^2 \rangle + 3\langle y_1 y_2 \rangle)x^4 + 3(2\langle y_1^3 y_2 \rangle + 3\langle y_1^2 y_2^2 \rangle)x^2 + \langle y_1^3 y_2^3 \rangle$$
$$= x^6 + \frac{3}{2a}(2-\gamma)x^4 + \frac{3}{4a^2}(3-\gamma+\gamma^2)x^2 - \frac{\gamma}{8a^3}(5+3\gamma+\gamma^2) \tag{3.26}$$

hence

$$\frac{\mathcal{I}(\gamma, a, 3, 2; x)}{\mathcal{I}(\gamma, a, 3, 0; x)} = -\frac{\mathcal{I}(1/\gamma, a/\gamma, 2, 3; ix)}{\mathcal{I}(1/\gamma, a/\gamma, 2, 0; ix)}. \tag{3.27}$$

Selberg's integral also gives the constant [12]

$$\mathcal{I}(\gamma, a, n, 0; x) = \pi^{n/2} 2^{-\gamma n(n-1)/2} a^{-\gamma n(n-1)/2 - n/2} \prod_{j=1}^{n} \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)}. \tag{3.28}$$

So, what is probably true is

$$\frac{\mathcal{I}(\gamma, a, n, m; x)}{\mathcal{I}(1/\gamma, a/\gamma, m, n; ix)} = (-i)^{mn} \frac{\mathcal{I}(\gamma, a, n, 0; x)}{\mathcal{I}(1/\gamma, a/\gamma, m, 0; ix)} \tag{3.29}$$

a constant depending on γ, a, n and m. i.e. the gaussians in the numerator and denominator of equation (3.17) should have different variances. In particular,

$$\frac{I_4(n,m;x)}{I_1(m,n;ix)} = (-i)^{mn} \frac{\pi^{n/2} 2^{-n^2} \prod_{j=0}^{n} (2j)!}{2^{3m/2} \prod_{j=0}^{m} \Gamma(1+j/2)}. \tag{3.30}$$

4. Conclusion

The m-th moment of the characteristic polynomial of an $n \times n$ random matrix is expressed as: an $n \times n$, equations (2.1.4 and 2.1.16), or as an $m \times m$, equation (2.2.5), determinant, for the unitary ensemble ($\beta = 2$); as a pfaffian of a $2n \times 2n$ anti-symmetric matrix, equations (2.1.10 and 2.1.17), or a sum of $m \times m$ determinants in case m is even, equation (2.2.11), for the symplectic ensemble ($\beta = 4$); as a pfaffian of an $n \times n$ (resp. $(n+1) \times (n+1)$) anti-symmetric matrix, equations (2.1.13 and 2.1.18), or a sum of $m \times m$ (resp. $(m+1) \times (m+1)$) determinants, equation (2.2.16, resp. 2.2.17), for the orthogonal ensemble ($\beta = 1$) for n even (resp. odd).
In the gaussian case $w(y) = e^{-y^2}$ this leads to the remarkable identity almost symmetric in m and n, equation (3.15),

$$I_2(n,m;x) = \text{const. } I_2(m,n;ix) \tag{4.1}$$

or, equation (3.16),

$$\det \left[H_{m+j+k}(x) \right]_{j,k=0,\ldots,n-1} = \text{const. } \det \left[H_{n+j+k}(ix) \right]_{j,k=0,\ldots,m-1} \tag{4.2}$$

and another identity expressing $\text{pf} \left[(k-j) H_{2m+j+k}(ix) \right]_{j,k=0,\ldots,2n-1}$, equation (3.14), as a sum of $m \times m$ determinants, equation (2.2.11). Also probably the remarkable general identity (3.29) and in particular (3.30) holds.

Acknowlegements

One of us (MLM) is thankful to A. Edelman for asking the question about the moments of the characteristic polynomial. We thank P.J. Forrester for sending us the preprint with N.S. Witte and indicating the relevant place in his paper with Baker. In particular, he indicated that in [11] equations (5.30a), (5.1) and (1.3a) relate the integral $\mathcal{I}(\beta/2, \beta/2, n, \beta; x)$, with our notations in equation (3.20), to a generalized Hermite polynomial which appears in equation (5.31) of [11] as the integral $\mathcal{I}(2/\beta, 1, \beta, n; ix)$. Thus it provides a proof of conjecture (3.29) in the special case $m = 2\gamma = 2a = \beta$. We are grateful to our colleague R. Balian for critically reading and commenting the manuscript. In particular he noted that if in equation (1.4) one replaces $\text{sign}(y-x)$ by $\text{sign}(y-x+\epsilon/2)$ and expands in powers of ϵ, then the three scalar products are just the first three terms in this expansion

$$\int\int f(x) g(y) \text{sign}(y-x+\epsilon/2) \sqrt{w(x)w(y)}\, dx dy = \langle f,g\rangle_1 + \epsilon \langle f,g\rangle_2 + \frac{\epsilon^2}{8} \langle f,g\rangle_4 + \ldots \tag{4.3}$$

Does this have some deeper significance and/or applications?

We are thankful to the referees for bringing to our attention the references 6 and 9.

References

[1] See for example, *Matrix theory*, M.L. Mehta, Les Editions de Physique, 91944 Les Ulis Cedex, France (1989) appendix A.14

[2] G. Mahoux and M.L. Mehta, A method of integration over matrix variables IV, J. Phys. I France 1 (1991) 1093-1108

[3] See for example, *Higher transcendental functions* A. Erdélyi et al. (ed.), Bateman Manuscript project, MacGraw Hill, New York (1953), vol. 2, §10.3 (5); or G. Szegö, *Orthogonal polynomials*, American Mathematical Society, Providence, R.I. (1939), equation (2.2.10)

[4] B. Eynard, private communication and http://xxx.lanl.cond-mat/0012046

[5] E. Brézin and S. Hikami, Characteristic polynomials of random matrices, Comm. Math. Phys. 214 (2000) 111-135

[6] J.P. Keating and N.C. Snaith, Random matrix theory and $\zeta(1/2+it)$, Comm. Math. Phys. 214 (2000) 57-89
J.P. Keating and N.C. Snaith, Random matrix theory and L-functions at $s = 1/2$, Comm. Math. Phys. 214 (2000) 91-110
C.P. Hughes, J.P. Keating and N. O'Connell, Random matrix theory and the derivative of the Riemann zeta function, Proc. Royal Soc. London A: Math. 456 (2000) 2611-2627

[7] See for example, reference 1, chapter 7.1.2

[8] See for example, *Random matrices*, M.L. Mehta, Academic Press, New york (1991) chapter 6.5 and appendix A.18

[9] N.G. de Bruijn, On some multiple integrals involving determinants, J. Indian Math. Soc. 19 (1955) 133-151; equations (1.2), (4.2)-(4.4) with $\phi_k(y_j) = P_{k-1}(y_j)(x - y_j)^m\sqrt{w(y_j)}$

[10] See for example, *Higher transcendental functions* Bateman manuscript project, A. Erdéyli et al. (eds.), MacGraw Hill, New York (1953) vol 2, §10.13 (30)-(31)

[11] P.J. Forrester and N.S. Witte, Application of the τ-function theory of Painlevé equations to random matrices: PIV, PII and GUE, (preprint), equation (4.43) and the paragraph following equation (4.44). See also T.H. Baker and P.J. Forrester, The Calogero-Sutherland model and generalized classical polynomials, Comm. Math. Phys. 188 (1997) 175-216, around equations (5.30)-(5.33)

[12] See for example, reference 8, chapter 17, equations (17.8.5)-(17.8.9), (17.6.7)