Zeros of some bi-orthogonal polynomials

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Abstract. Ercolani and McLaughlin have recently shown that the zeros of the bi-orthogonal polynomials with the weight $w(x,y) = \exp[-(V_1(x)+V_2(y)+2cxy)/2]$, relevant to a model of two coupled hermitian matrices, are real and simple. We show that their argument applies to the more general case of the weight $(w_1 * w_2 * ... * w_j)(x,y)$, a convolution of several weights of the same form. This general case is relevant to a model of several hermitian matrices coupled in a chain. Their argument also works for the weight $W(x,y) = e^{-x-y}/(x+y)$, $0 \le x, y < \infty$, and for a convolution of several such weights.

1. Introduction. For a weight function $w(x,y)$ such that all the moments

$$M_{i,j} := \int w(x,y) x^i y^j dxdy \tag{1.1}$$

exist and

$$D_n := \det[M_{i,j}]_{i,j=0,1,...,n} \neq 0 \tag{1.2}$$

for all $n \ge 0$, unique monic polynomials $p_n(x)$ and $q_n(x)$ of degree n exist satisfying the bi-orthogonality relations (a polynomial is called monic when the coefficient of the highest degree is one)

$$\int w(x,y) p_n(x) q_m(y) dxdy = h_n \delta_{mn}. \tag{1.3}$$

Just like the orthogonal polynomials they can be expressed as determinants, e.g.

$$p_n(x) = \frac{1}{D_{n-1}} \det \begin{bmatrix} M_{0,0} & ... & M_{0,n-1} & 1 \\ M_{1,0} & ... & M_{1,n-1} & x \\ \vdots & \vdots & \vdots & \vdots \\ M_{n,0} & ... & M_{n,n-1} & x^n \end{bmatrix} \tag{1.4}$$

and have integral representations, e.g.

$$p_n(x) \propto \int \Delta_n(\mathbf{x}) \Delta_n(\mathbf{y}) \prod_{j=1}^{n} (x - x_j) w(x_j, y_j) dx_j dy_j \tag{1.5}$$

$$\Delta_n(\mathbf{x}) := \prod_{1 \le i < j \le n} (x_j - x_i), \qquad \Delta_n(\mathbf{y}) := \prod_{1 \le i < j \le n} (y_j - y_i). \tag{1.6}$$

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From limited numerical evidence for the weights

(i) $w(x,y) = \sin(\pi xy), \quad 0 \le x, y \le 1;$

(ii) $w(x,y) = |x-y|, \quad -1 \le x, y \le 1;$

(iii) $w(x,y) = [1/(x+y)] \exp[-x-y], \quad 0 \le x, y < \infty;$

(iv) $w(x,y) = \exp(-x^2 - y^2 - cxy), \quad -\infty < x, y < \infty, \quad 0 < c < 2;$

one might think that the zeros of the bi-orthogonal polynomials are real, simple, lie respectively in the x or y-support of $w(x,y)$, interlace for successive n, ...

Alas, this is not true in general as seen by the following example due to P. Deligne. If one takes

$$w(x,y) = u(x,y) + v(x,y), \tag{1.7}$$

$$u(x,y) = \begin{cases} \delta(x-y), & -1 \le x, y \le 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.8}$$

$$v(x,y) = \frac{1}{8}[\delta(x-1)\delta(y+2) + \delta(x+1)\delta(y-2)]. \tag{1.9}$$

Then the zeros of $p_3(x)$ and $q_3(x)$ are complex.

However, N.M. Ercolani and K.T.-R. Mclaughlin have recently [1] shown that with the weight function

$$w_1(x,y) = \exp\left[-\frac{1}{2}V_1(x) - \frac{1}{2}V_2(y) - c_1 xy\right] \tag{1.10}$$

($-\infty < x, y < \infty$), V_1 and V_2 polynomials of positive even degree, c a small non-zero real constant, all the zeros of the bi-orthogonal polynomials $p_n(x)$ and $q_n(x)$ are real and simple.

In this brief note we will show that their argument works for the following general case encountered for random hermitian matrices coupled in a linear chain. Let $V_j(x)$, $1 \le j \le p$, be polynomials of positive even degree and c_j, $1 \le j < p$, be small real constants, none of them being zero ("small" so that all the moments $M_{i,j}$ defined below, eq.(1.13), exist.) Further let

$$w_k(x,y) := \exp\left[-\frac{1}{2}V_k(x) - \frac{1}{2}V_{k+1}(y) - c_k xy\right] \tag{1.11}$$

$$(w_{i_1} * w_{i_2} * ... * w_{i_k})(\xi_1, \xi_{k+1}) := \int w_{i_1}(\xi_1,\xi_2) w_{i_2}(\xi_2,\xi_3) ... w_{i_k}(\xi_k,\xi_{k+1}) d\xi_2 ... d\xi_k \tag{1.12}$$

Moreover, assume that for all $i, j \ge 0$

$$M_{i,j} := \int x^i (w_1 * w_2 * ... * w_{p-1})(x,\, y) y^j dx dy \tag{1.13}$$

exist.

Theorem. Then monic polynomials $p_j(x)$ and $q_j(x)$ can be uniquely defined by

$$\int p_j(x)(w_1 * w_2 * ... * w_{p-1})(x,y) q_k(y) dx dy = h_j \delta_{jk} \tag{1.14}$$

and all the zeros of $p_j(x)$ and of $q_j(x)$ are real and simple.

The same argument works for any weight $W(x,y)$ such that $\det[W(x_i,y_j)]_{i,j=1,...,n} > 0$ for $x_1 < x_2 < ... < x_n$, $y_1 < y_2 < ... < y_n$ and moments $M_{i,j} = \int W(x,y)x^i y^j dxdy$ exist for all $i,j \geq 0$. For example, if $W(x,y) = [1/(x+y)]\exp[-x-y]$, $0 \leq x, y < \infty$, then monic polynomials $p_j(x)$ can be uniquely defined by

$$\int_0^\infty p_j(x)W(x,y)p_k(y)dxdy = h_j\delta_{jk} \tag{1.15}$$

(here $W(x,y)$ is symmetric in x and y so that $p_j(x) = q_j(x)$) and all the zeros of $p_j(x)$ are real, simple and non-negative.

2. Results and proofs. Here we essentially follow section 3 of reference [1]. With any monic polynomials $p_j(x)$ and $q_j(x)$ of degree j, let us write

$$P_{1,j}(x) := p_j(x) \tag{2.1}$$

$$\begin{aligned} P_{i,j}(x) &:= \int p_j(\xi)(w_1 * w_2 * ... * w_{i-1})(\xi,x)d\xi \\ &:= \int p_j(\xi)U_{Li}(\xi,x)d\xi, \qquad 1 < i \leq p \end{aligned} \tag{2.2}$$

$$Q_{p,j}(x) := q_j(x) \tag{2.3}$$

$$\begin{aligned} Q_{i,j}(x) &:= \int (w_i * w_{i+1} * ... * w_{p-1})(x,\xi)q_j(\xi)d\xi \\ &:= \int U_{Ri}(x,\xi)q_j(\xi)d\xi \qquad 1 \leq i < p \end{aligned} \tag{2.4}$$

Lemma 1. For $x_1 < x_2 < ... < x_n$, $y_1 < y_2 < ... < y_n$,

$$\det\left[w_i(x_j,y_k)\right]_{j,k=1,...,n} > 0. \tag{2.5}$$

This is essentially eq. (40) of reference [1]. This can also be seen as follows. Let $X = [x_i\delta_{ij}]$ and $Y = [y_i\delta_{ij}]$ be two $n\times n$ diagonal matrices with diagonal elements x_1, ..., x_n and y_1, ..., y_n respectively. Then the integral of $\exp[-c\,\mathrm{tr}\,UXU^{-1}Y]$ over the $n\times n$ unitary matrices U is given by [2]

$$K\frac{\det\left[\exp(-c\,x_i y_j)\right]_{i,j=1,...,n}}{\Delta_n(\mathbf{x})\Delta_n(\mathbf{y})} \tag{2.6}$$

where K is a positive constant depending on c and n. Hence

$$\exp\left[-\frac{1}{2}\sum_{j=1}^{n}\left(V_i(x_j)+V_{i+1}(y_j)\right)\right]\int dU e^{-c_i\mathrm{tr}\,UXU^{-1}Y} = K\frac{\det\left[w_i(x_j,y_k)\right]_{j,k=1,...,n}}{\Delta_n(\mathbf{x})\Delta_n(\mathbf{y})} \tag{2.7}$$

The left hand side is evidently positive while on the right hand side the denominator is positive since $x_1 < x_2 < ... < x_n$ and $y_1 < y_2 < ... < y_n$. From this eq. (2.5) follows.

Lemma 2. For $x_1 < x_2 < ... < x_n, \quad y_1 < y_2 < ... < y_n,$

$$\det\left[(w_{i_1} * w_{i_2} * ... * w_{i_\ell})(x_j, y_k)\right]_{j,k=1,...,n} > 0 \tag{2.8}$$

Proof. Binet-Cauchy formula tells us that [3]

$$\det\left[(w_{i_1} * w_{i_2})(x_j, y_k)\right]_{j,k=1,...,n}$$

is equal to

$$\int_{\xi 1 < \xi_2 < ... < \xi_n} \det\left[w_{i_1}(x_j, \xi_k)\right]_{j,k=1,...,n} . \det\left[w_{i_2}(\xi_j, y_k)\right]_{j,k=1,...,n} d\xi_1 ... d\xi_n \tag{2.9}$$

By lemma 1 the integrand is every where positive, so lemma 2 is proved for the case $\ell = 2$. The proof is now completed by induction on ℓ, using again the Binet-Cauchy formula.

Lemma 3. For any monic polynomial $p_j(x)$ of degree j, $P_{i,j}(x)$, $1 \leq i \leq p$, has at most j distinct real zeros. Similarly, for any monic polynomial $q_j(x)$ of degree j, $Q_{i,j}(x)$, $1 \leq i \leq p$, has at most j distinct real zeros.

Proof. Let, if possible, $z_1 < z_2 < ... < z_m$, $m > j$, be the distinct real zeros of $P_{i,j}(x)$. Since

$$P_{i,j}(x) = \sum_{k=0}^{j} a_k T_{i,k}(x), \tag{2.10}$$

with

$$T_{i,k}(x) := \int \xi^k U_{Li}(\xi, x) d\xi, \tag{2.11}$$

$$P_{i,j}(z_\ell) = 0, \qquad \ell = 1, 2, ..., m, \qquad m > j \tag{2.12}$$

implies that

$$\begin{aligned} 0 &= \det \begin{bmatrix} T_{i,0}(z_1) & T_{i,1}(z_1) & ... & T_{i,j}(z_1) \\ ... & ... & ... & ... \\ T_{i,0}(z_{j+1}) & T_{i,1}(z_{j+1}) & ... & T_{i,j}(z_{j+1}) \end{bmatrix} \\ &= \int \det \begin{bmatrix} U_{Li}(\xi_1, z_1) & \xi_2 U_{Li}(\xi_2, z_1) & ... & \xi_{j+1}^j U_{Li}(\xi_{j+1}, z_1) \\ ... & ... & ... & ... \\ U_{Li}(\xi_1, z_{j+1}) & \xi_2 U_{Li}(\xi_2, z_{j+1}) & ... & \xi_{j+1}^j U_{Li}(\xi_{j+1}, z_{j+1}) \end{bmatrix} d\xi_1 ... d\xi_{j+1} \\ &= \int \xi_2 \xi_3^2 ... \xi_{j+1}^j \det\left[U_{Li}(\xi_k, z_\ell)\right]_{k,\ell=1,...,j+1} d\xi_1 ... d\xi_{j+1} \end{aligned} \tag{2.13}$$

or

$$\int \det\left[U_{Li}(\xi_k, z_\ell)\right]_{k,\ell=1,...,j+1} . \det\left[\xi_k^{\ell-1}\right]_{k,\ell=1,...,j+1} d\xi_1 ... d\xi_{j+1} = 0 \tag{2.14}$$

in contradiction to lemma 2. Thus m can not be greater than j.

The proof for $Q_{i,j}(x)$ is similar.

Lemma 4. Let the real constants c_1, ..., c_{p-1}, none of them being zero, be such that

$$M_{i,j} := \int x^i U_{Lp}(x,y) y^j dxdy \equiv \int x^i (w_1 * w_2 * ... * w_{p-1})(x,y) y^j dxdy. \tag{2.15}$$

exist for all $i, j \geq 0$. Then

$$D_n := \det[M_{i,j}]_{i,j=1,...,n} \neq 0 \tag{2.16}$$

for any $n \geq 0$.

Proof. Let, if possible, $D_n = 0$ for some n. Then $\sum_{j=0}^n M_{i,j} q_j = 0$, q_j not all zero, and

$$\int x^i U_{Lp}(x,y) \sum_{j=0}^n q_j y^j dxdy = 0, \qquad i = 0,\ 1, ..., n \tag{2.17}$$

or

$$\int p_i(x) U_{Lp}(x,y) \sum_{j=0}^n q_j y^j dxdy = 0 \tag{2.18}$$

for any polynomial $p_i(x)$ of degree $i \leq n$. But

$$\int U_{Lp}(x,y) \sum_{j=0}^n q_j y^j dy \tag{2.19}$$

has at most n distinct real zeros (lemma 3). So one can choose $p_i(x)$ such that

$$p_i(x) \int U_{Lp}(x,y) \sum_{j=0}^n q_j y^j dy > 0 \tag{2.20}$$

in contradiction to eq. (2.18). So $D_n \neq 0$ and bi-orthogonal polynomials $p_j(x)$, $q_j(x)$ exist, see eqs. (1.4), (1.5).

Lemma 5. Let $p_j(x)$, $q_j(x)$ be the bi-orthogonal polynomials, eq. (1.14); or with the definitions (2.1)-(2.4)

$$\int P_{i,j}(x) Q_{i,k}(x) dx = h_j \delta_{jk}, \qquad 1 \leq i \leq p \tag{2.21}$$

Then $P_{i,j}(x)$ has at least j real distinct zeros of odd multiplicity. So does have $Q_{i,j}(x)$.

Proof. Let, if possible, $z_1 < z_2 < ... < z_m$, $m < j$, be the only real zeros of $P_{i,j}(x)$ of odd multiplicity. Set

$$R(x) = \det \begin{bmatrix} Q_{i,0}(x) & Q_{i,1}(x) & ... & Q_{i,m}(x) \\ Q_{i,0}(z_1) & Q_{i,1}(z_1) & ... & Q_{i,m}(z_1) \\ \vdots & \vdots & & \vdots \\ Q_{i,0}(z_m) & Q_{i,1}(z_m) & ... & Q_{i,m}(z_m) \end{bmatrix} \tag{2.22}$$

$$= \int U_{Ri}(x,\xi) \sum_{k=0}^m \alpha_k \xi^k d\xi \tag{2.23}$$

with some constants α_k depending on z_1, ..., z_m.

Since $m < j$, the bi-orthogonality gives

$$\int P_{i,j}(x)R(x)dx = 0. \tag{2.24}$$

However, $R(x)$ can also be written as

$$\begin{aligned}
R(x) &= \int \det \begin{bmatrix} U_{Ri}(x,\xi_0) & U_{Ri}(x,\xi_1)\xi_1 & \dots & U_{Ri}(x,\xi_m)\xi_m^m \\ U_{Ri}(z_1,\xi_0) & U_{Ri}(z_1,\xi_1)\xi_1 & \dots & U_{Ri}(z_1,\xi_m)\xi_m^m \\ \vdots & \vdots & & \\ U_{Ri}(z_m,\xi_0) & U_{Ri}(z_m,\xi_1)\xi_1 & \dots & U_{Ri}(z_m,\xi_m)\xi_m^m \end{bmatrix} d\xi_0 d\xi_1 \dots d\xi_m \\
&= \int \det \begin{bmatrix} U_{Ri}(x,\xi_0) & U_{Ri}(x,\xi_1) & \dots & U_{Ri}(x,\xi_m) \\ U_{Ri}(z_1,\xi_0) & U_{Ri}(z_1,\xi_1) & \dots & U_{Ri}(z_1,\xi_m) \\ \vdots & \vdots & & \\ U_{Ri}(z_m,\xi_0) & U_{Ri}(z_m,\xi_1) & \dots & U_{Ri}(z_m,\xi_m) \end{bmatrix} \xi_1\xi_2^2\dots\xi_m^m d\xi_0 d\xi_1 \dots d\xi_m \\
&= \frac{1}{(m+1)!}\int \det \begin{bmatrix} U_{Ri}(x,\xi_0) & U_{Ri}(x,\xi_1) & \dots & U_{Ri}(x,\xi_m) \\ U_{Ri}(z_1,\xi_0) & U_{Ri}(z_1,\xi_1) & \dots & U_{Ri}(z_1,\xi_m) \\ \vdots & \vdots & & \\ U_{Ri}(z_m,\xi_0) & U_{Ri}(z_m,\xi_1) & \dots & U_{Ri}(z_m,\xi_m) \end{bmatrix} \prod_{0\le r<s\le m}(\xi_s-\xi_r) d\xi_0 d\xi_1 \dots d\xi_m \\
&= \int_{\xi_0<\xi_1<\dots<\xi_m} \det \begin{bmatrix} U_{Ri}(x,\xi_0) & U_{Ri}(x,\xi_1) & \dots & U_{Ri}(x,\xi_m) \\ U_{Ri}(z_1,\xi_0) & U_{Ri}(z_1,\xi_1) & \dots & U_{Ri}(z_1,\xi_m) \\ \vdots & \vdots & & \\ U_{Ri}(z_m,\xi_0) & U_{Ri}(z_m,\xi_1) & \dots & U_{Ri}(z_m,\xi_m) \end{bmatrix} \prod_{0\le r<s\le m}(\xi_s-\xi_r) d\xi_0 d\xi_1 \dots d\xi_m
\end{aligned} \tag{2.25}$$

Thus $R(x)$ is represented as an integral whose integrand has a fixed sign determined by the relative ordering of the numbers x, z_1, z_2, ..., z_m (lemma 2). It thus follows that $R(x)$ changes sign when x passes through any of the points z_k, $k = 1$, ..., m and at no other value of x. In other words, z_1, ..., z_m are the only real zeros of $R(x)$ having an odd multiplicity. And therefore $P_{i,j}(x)R(x)$ has a constant sign, so that

$$\int P_{i,j}(x)R(x)dx \neq 0 \tag{2.26}$$

in contradiction to (2.24).

The proof for $Q_{i,j}(x)$ is similar.

As a consequence we have the integral representations of $P_{i,j}(x)$ for $i > 1$ and of $Q_{i,j}(x)$ for $i < p$ involving their respective zeros

$$P_{i,j}(x) \propto \int \det \begin{bmatrix} U_{Li}(\xi_0,x) & U_{Li}(\xi_1,x) & \dots & U_{Li}(\xi_j,x) \\ U_{Li}(\xi_0,z_1) & U_{Li}(\xi_1,z_1) & \dots & U_{Li}(\xi_j,z_1) \\ \vdots & \vdots & & \vdots \\ U_{Li}(\xi_0,z_j) & U_{Li}(\xi_1,z_j) & \dots & U_{Li}(\xi_j,z_j) \end{bmatrix} \prod_{0\le r<s\le j}(\xi_s-\xi_r) d\xi_0 d\xi_1 \dots d\xi_j$$

$$(2.27)$$

$$Q_{i,j}(x) \propto \int \det \begin{bmatrix} U_{Ri}(x,\xi_0) & U_{Ri}(x,\xi_1) & ... & U_{Ri}(x,\xi_j) \\ U_{Ri}(z_1,\xi_0) & U_{Ri}(z_1,\xi_1) & ... & U_{Ri}(z_1,\xi_j) \\ \vdots & \vdots & & \vdots \\ U_{Ri}(z_j,\xi_0) & U_{Ri}(z_j,\xi_1) & ... & U_{Ri}(z_j,\xi_j) \end{bmatrix} \prod_{0\le r<s\le j} (\xi_s - \xi_r) d\xi_0 d\xi_1 ... d\xi_j \tag{2.28}$$

Lemmas 3 and 5 tell us that if $p_j(x)$ and $q_j(x)$ are bi-orthogonal polynomials satisfying eq. (1.14), then $P_{i,j}(x)$ and $Q_{i,j}(x)$ each have exactly j distinct real zeros of odd multiplicity. In particular, the zeros of the bi-orthogonal polynomials $p_j(x) \equiv P_{1,j}(x)$ and $q_j(x) \equiv Q_{p,j}(x)$ are real and simple.

With a little more effort one can perhaps show that all the real zeros of $P_{i,j}(x)$ and of $Q_{i,j}(x)$ are simple. Other zeros, if any, must be complex. Whether the zeros of $p_j(x)$ ($q_j(x)$) interlace for successive j, remains an open question.

3. Bi-orthogonal polynomials with another weight.

For the weight $W(x,y) = [1/(x+y)] \exp[-x-y]$, $0 \le x, y < \infty$, one can say as follows.

Lemma 1'. One has [4]

$$\det[W(x_j,y_k)]_{j,k=1,...,n} = \exp\left[-\sum_{j=1}^{n}(x_j+y_j)\right] \Delta_n(\mathbf{x})\Delta_n(\mathbf{y}) \prod_{j,k=1}^{n}(x_j+y_k)^{-1} \tag{3.1}$$

which is evidently positive for $0 \le x_1 < x_2 < ... < x_n$, $0 \le y_1 < y_2 < ... < y_n$.

Lemma 3'. For any monic polynomial $p_j(x)$ of degree j, $P_j(x) := \int_0^\infty W(x,y)p_j(y)dy$ has at most j distinct real non-negative zeros.

In the proof of lemma 3, replace eqs. (2.10)-(2.14) by

$$P_j(x) = \sum_{k=0}^{j} a_k T_k(x), \tag{3.2}$$

$$T_k(x) = \int_0^\infty \xi^k W(\xi,x) d\xi \tag{3.3}$$

$$P_j(z_\ell) = 0, \quad \ell = 1,2,...,m, \quad m > j \tag{3.4}$$

$$\begin{aligned} 0 &= \det \begin{bmatrix} T_0(z_1) & T_1(z_1) & ... & T_j(z_1) \\ ... & ... & ... & ... \\ T_0(z_{j+1}) & T_1(z_{j+1}) & ... & T_j(z_{j+1}) \end{bmatrix} \\ &= \int_0^\infty \det \begin{bmatrix} W(\xi_1,z_1) & \xi_2 W(\xi_2,z_1) & ... & \xi_{j+1}^j W(\xi_{j+1},z_1) \\ ... & ... & ... & ... \\ W(\xi_1,z_{j+1}) & \xi_2 W(\xi_2,z_{j+1}) & ... & \xi_{j+1}^j W(\xi_{j+1},z_{j+1}) \end{bmatrix} d\xi_1 ... d\xi_{j+1} \\ &= \int \xi_2 \xi_3^2 ... \xi_{j+1}^j \det\left[W(\xi_k,z_\ell)\right]_{k,\ell=1,...,j+1} d\xi_1 ... d\xi_{j+1} \end{aligned} \tag{3.5}$$

or

$$\int \det\left[W(\xi_k, z_\ell)\right]_{k,\ell=1,\dots,j+1} . \det\left[\xi_k^{\ell-1}\right]_{k,\ell=1,\dots,j+1} d\xi_1 \dots d\xi_{j+1} = 0 \tag{3.6}$$

in contradiction to lemma 2'. Thus m can not be greater than j.

Lemma 4'. With

$$M_{i,j} := \int_0^\infty x^i W(x,y) y^j dxdy, \tag{3.7}$$

$$D_n := \det[M_{i,j}]_{i,j=0,1,\dots,n} \neq 0 \tag{3.8}$$

for any $n \geq 0$.

In the proof of lemma 4 replace everywhere $\int U_{Lp}(x,y)\dots$ by $\int_0^\infty W(x,y)\dots$.

Lemma 5'. Let $p_j(x)$ be the (bi-orthogonal) polynomials satisfying

$$\int_0^\infty W(x,y) p_j(x) p_k(y) dxdy = h_j \delta_{jk}. \tag{3.9}$$

Then $P_j(x) := \int_0^\infty W(x,y) p_j(y) dy$ and $p_j(x)$ each have at least j distinct real non-negative zeros of odd multiplicity.

Let, if possible, $0 \leq z_1 < z_2 < \dots < z_m$ $m < j$, be the only real non-negative zeros of $P_j(x)$ of odd multiplicity. Set $R(x) = \prod_{j=1}^m (x - z_j)$. Then as $m < j$, one has

$$\int_0^\infty P_j(x) R(x) dx = 0 \tag{3.10}$$

But $P_j(x)$ and $R(x)$ change sign simultaneously as x passes through the values z_1, ..., z_m and at no other real positive value. So the product $P_j(x)R(x)$ never changes sign, in contradiction to (3.10). Therefore $P_j(x)$ has at least j distinct real non-negative zeros of odd multiplicity.

To prove that $p_j(x)$ has at least j distinct real non-negative zeros let if possible, $0 \leq z_1 < z_2 < \dots < z_m$, $m < j$, be the only such zeros. Set

$$R(x) = \det \begin{bmatrix} P_0(x) & P_1(x) & \dots & P_m(x) \\ P_0(z_1) & P_1(z_1) & \dots & P_m(z_1) \\ \vdots & \vdots & & \vdots \\ P_0(z_m) & P_1(z_m) & \dots & P_m(z_m) \end{bmatrix}$$

$$= \int_0^\infty W(x,\xi) \sum_{k=0}^m \alpha_k \xi^k d\xi \tag{3.11}$$

with some constants α_k depending on z_1, ..., z_m.

Since $m < j$, the bi-orthogonality gives

$$\int_0^\infty P_j(x) R(x) dx = 0. \tag{3.12}$$

But

$$R(x) \propto \int_0^\infty \det \begin{bmatrix} W(x,\xi_0) & W(x,\xi_1) & ... & W(x,\xi_m) \\ W(z_1,\xi_0) & W(z_1,\xi_1) & ... & W(z_1,\xi_m) \\ \vdots & \vdots & & \\ W(z_m,\xi_0) & W(z_m,\xi_1) & ... & W(z_m,\xi_m) \end{bmatrix} \prod_{0\le r<s\le m} (\xi_s - \xi_r) d\xi_0 d\xi_1 ... d\xi_m \tag{3.13}$$

which says that $z_1, ..., z_m$ are the only distinct real non-negative zeros of $R(x)$ and therefore $p_j(x)R(x)$ has a constant sign, in contradiction to (3.12).

Conclusion. We have shown with the arguments of Ercolani and McLaughlin that if the weight $w(x,y)$ is such that $\det[w(x_i,y_j)]_{i,j=1,...,n} > 0$ for $x_1 < x_2 < ... < x_n$, $y_1 < y_2 < ... < y_n$ and moments $\int w(x,y)x^i y^j dxdy$ exist for all $i,j \ge 0$, then bi-orthogonal polynomials exist and their zeros are real, simple and lie in the respective supports of the weight $w(x,y)$. The same is true for a weight which is a convolution of several such weights.

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References.

[1] N.M. Ercolani and K.T.-R. Mclaughlin, Asymptotic and integrable structures for bi-orthogonal polynomials associated to a random two matrix model (preprint, to appear)
[2] See e.g. M.L. Mehta, *Random matrices*, Academic Press, new York (1991) Appendix A.5
[3] See e.g. M.L. Mehta, *Matrix theory*, Les Editions de Physique, Z.I. de Courtaboeuf, 91944 Les Ulis Cedex, France, (1989) §3.7
[4] See e.g. reference 3, §7.1.3