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Cécile Monthus. Properties of the simplest inhomogeneous and homogeneous Tree-Tensor-States for Long-Ranged Quantum Spin Chains with or without disorder. 2020. cea-02571099

HAL Id: cea-02571099

<https://hal-cea.archives-ouvertes.fr/cea-02571099>

Preprint submitted on 12 May 2020

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Properties of the simplest inhomogeneous and homogeneous Tree-Tensor-States for Long-Ranged Quantum Spin Chains with or without disorder

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The simplest Tree-Tensor-States (TTS) respecting the Parity and the Time-Reversal symmetries are studied in order to describe the ground states of Long-Ranged Quantum Spin Chains with or without disorder. Explicit formulas are given for the one-point and two-point reduced density matrices that allow to compute any one-spin and two-spin observable. For Hamiltonians containing only one-body and two-body contributions, the energy of the TTS can be then evaluated and minimized in order to obtain the optimal parameters of the TTS. This variational optimization of the TTS parameters is compared with the traditional block-spin renormalization procedure based on the diagonalization of some intra-block renormalized Hamiltonian.

I. INTRODUCTION

The entanglement between the different regions of many-body quantum systems (see the reviews [1–6] and references therein) has emerged as an essential physical property that should be taken into account in their descriptions. In the field of Tensor Networks (see the reviews [7–17] and references therein), the ground-state wavefunction is decomposed into elementary small tensors that can be assembled in various ways in order to adapt to the geometry, to the symmetries, and to the entanglement properties of the problem under focus. In particular, various previous real-space renormalization procedures for the ground states of quantum spin chains have been reinterpreted and possibly improved within this new perspective. For instance the Density-Matrix-Renormalization-Group [18–20] was reformulated as a variational problem based on Matrix-Products-States that are well adapted to describe non-critical states displaying area-law entanglement. The traditional block-spin renormalization for critical points corresponds to scale-invariant Tree-Tensor-States, and has been improved via the multi-scale-entanglement-renormalization-ansatz (MERA) [21, 22], where ‘disentangles’ between blocks are introduced besides the block-coarse-graining isometries already present in Tree-Tensor-States. Finally in the field of disordered spin chains, the Strong Disorder Renormalization approach (see the reviews [23, 24]) has been reformulated either as a Matrix-Product-Operator-Renormalization or as a self-assembling Tree-Tensor-Network, and various improvements have been proposed [25–30].

However, even in the second example where the ‘old’ block-spin renormalization procedure and the ‘new’ Tree-Tensor-State variational approach share the same entanglement architecture, the precise choice of the elementary isometries remains different. Indeed in the traditional block-spin renormalization, the isometries are determined via the diagonalization of some ‘intra-block’ Hamiltonian involving a few renormalized spins, so that one can usually obtain explicit RG flows for the parameters of the renormalized Hamiltonian. The two main criticisms levelled against this procedure can be summarized as follows : (i) at the level of principles, the choice of the ground state of the ‘intra-block Hamiltonian’ does not take at all into account the ‘environment’ of the neighboring blocks; (ii) in practice, there is usually some arbitrariness in the decomposition of the Hamiltonian into the ‘intra-block’ and the ‘extra-block’ contributions that can lead to completely different outputs, so that the quality of the results strongly depends on the cleverness of the choice of the intra-block Hamiltonian. In the Tree-Tensor-Network perspective, one considers instead the whole ground-state wavefunction as a variational tree-tensor-state involving isometries, and the optimization of each isometry is based on the minimization of the total energy of the Tree-Tensor-State. At the level of principles, the theoretical advantage is clearly that the output corresponds to the optimal Tree-Tensor-State, i.e. to the best renormalization procedure within the class of all renormalization procedures of a given dimension. In practice, the drawback is that this global optimization is more complicated and can usually be done only numerically, unless the isometries are completely fixed by the very strong quantum symmetries of the model [31].

In the present paper, the goal is to analyze the explicit properties of the simplest Tree-Tensor-States of the smallest bond dimension $D = 2$ in the context of Long-Ranged quantum spin chains with or without disorder, in order to analyze more precisely the improvement given by the global optimization of the isometries with respect to the traditional block-spin procedure.

The paper is organized as follows. In section II, we introduce the notations for Long-Ranged quantum spin chains with Parity and Time-Reversal symmetries. In section III, we describe the simplest inhomogeneous Tree-Tensor-States respecting these two symmetries and write the corresponding ascending and descending superoperators. In section IV, the explicit forms of the one-point and two-point reduced density matrices are derived in order to analyze the structure of magnetizations and two-points correlations. In section V, we focus on the energy of the Tree-Tensor-State for disordered Long-Ranged Spin Chains in order to discuss the optimization with respect to the Tree-Tensor-States parameters. In section VI, we turn to the case of pure Long-Ranged Spin Chains in order to take into account the

supplementary symmetries in the Tree-Tensor-States. Finally in section VII, we study the properties of the scale-invariant Tree-Tensor-States for the critical points of pure models. Our conclusions are summarized in section VIII. The appendix A contains the traditional block-spin determination of the parameters of the Tree-Tensor-State, in order to compare with the variational optimization discussed in the text.

II. LONG-RANGED SPIN CHAINS WITH PARITY AND TIME-REVERSAL SYMMETRIES

Within the Tensor-Network perspective, the symmetries play an essential role in order to restrict the form of the possible isometries. In the present paper, we focus on quantum spin chains with Parity and Time-Reversal Symmetries.

A. Parity and Time-Reversal operators

For a quantum spin chain of N spins described by Pauli matrices $\sigma_n^{a=0,x,y,z}$, the Parity operator

$$\mathcal{P} = \prod_{n=1}^N \sigma_n^z \quad (1)$$

and the Time-Reversal operator \mathcal{T} whose action can be defined via

$$\begin{aligned} \mathcal{T}i\mathcal{T}^{-1} &= -i \\ \mathcal{T}\sigma_n^x\mathcal{T}^{-1} &= \sigma_n^x \\ \mathcal{T}\sigma_n^y\mathcal{T}^{-1} &= -\sigma_n^y \\ \mathcal{T}\sigma_n^z\mathcal{T}^{-1} &= \sigma_n^z \end{aligned} \quad (2)$$

are among the most important possible symmetries. It is then useful to decompose the space of operators \mathcal{O} into sectors that commute $P = +1$ or anticommute $P = -1$ with the Parity operator \mathcal{P} , and that commute $T = +1$ or anticommute $T = -1$ with the Time-Reversal operator \mathcal{T}

$$\begin{aligned} \mathcal{P}\mathcal{O} &= P\mathcal{O}\mathcal{P} \\ \mathcal{T}\mathcal{O} &= T\mathcal{O}\mathcal{T} \end{aligned} \quad (3)$$

Let us now describe the classification of one-body and two-body operators with respect to these four symmetry sectors ($P = \pm 1, T = \pm 1$).

B. Classification of one-body operators $\sigma_n^{a=0,x,y,z}$ with respect to the four sectors ($P = \pm 1, T = \pm 1$)

The four operators $\sigma_n^{a=0,x,y,z}$ of the Pauli basis can be classified as follows :

- (1) the sector ($P = +1, T = -1$) is empty
- (2) the sector ($P = -1, T = +1$) contains only σ_n^x
- (3) the sector ($P = -1, T = -1$) contains only σ_n^y
- (4) the sector ($P = +1, T = +1$) contains the two operators σ_n^0 and σ_n^z .

C. Classification of two-body operators $\sigma_n^{a=0,x,y,z}\sigma_{n'}^{b=0,x,y,z}$ with respect to the four sectors ($P = \pm 1, T = \pm 1$)

The 16 two-body operators $\sigma_n^{a=0,x,y,z}\sigma_{n'}^{b=0,x,y,z}$ of the Pauli basis can be classified as follows :

- (1) the sector ($P = +1, T = -1$) contains the two operators

$$\sigma_n^x\sigma_{n'}^y, \sigma_n^y\sigma_{n'}^x \quad (4)$$

- (2) the sector ($P = -1, T = +1$) contains the four operators

$$\sigma_n^x\sigma_{n'}^0, \sigma_n^0\sigma_{n'}^x, \sigma_n^x\sigma_{n'}^z, \sigma_n^z\sigma_{n'}^x \quad (5)$$

(3) the sector ($P = -1, T = -1$) contains the four operators

$$\sigma_n^y \sigma_{n'}^0, \sigma_n^0 \sigma_{n'}^y, \sigma_n^y \sigma_{n'}^z, \sigma_n^z \sigma_{n'}^y, \quad (6)$$

(4) the sector ($P = +1, T = +1$) contains the six operators

$$\sigma_n^0 \sigma_{n'}^0, \sigma_n^z \sigma_{n'}^0, \sigma_n^0 \sigma_{n'}^z, \sigma_n^z \sigma_{n'}^z, \sigma_n^x \sigma_{n'}^x, \sigma_n^y \sigma_{n'}^y, \quad (7)$$

D. Hamiltonians containing only one-body and two-body terms respecting the Parity and Time-Reversal

The Hamiltonians commuting with the Parity and Time-Reversal operators belong to the sector ($P = +1, T = +1$). If they contain only one-body and two-body terms, the above classification yields that they can be parametrized in terms of fields h_n and in terms of couplings $\mathcal{J}_{n,n'}^{a=x,y,z}$ as

$$\mathcal{H}_N = - \sum_{n=1}^N h_n \sigma_n^z - \sum_{1 \leq n < n' \leq N} (\mathcal{J}_{n,n'}^x \sigma_n^x \sigma_{n'}^x + \mathcal{J}_{n,n'}^y \sigma_n^y \sigma_{n'}^y + \mathcal{J}_{n,n'}^z \sigma_n^z \sigma_{n'}^z) \quad (8)$$

Then one needs to choose whether the fields h_n are uniform or random, whether the couplings $\mathcal{J}_{n,n'}^{a=x,y,z}$ are Short-Ranged or Long-Ranged, with or without disorder.

In the present paper, we will focus on the cases where the couplings $\mathcal{J}_{n,n'}^{a=x,y,z}$ are Long-Ranged with some power-law dependence with respect to the distance $r(n, n') = |n - n'|$ between the two sites n and n'

$$\mathcal{J}_{n,n'}^a = \frac{J_{n,n'}^a}{[r(n, n')]^{1+\omega_a}} = \frac{J_{n,n'}^a}{|n - n'|^{1+\omega_a}} \quad (9)$$

where the amplitudes $J_{n,n'}^a$ are of order unity, while the exponents ω_a governing the decays with the distance are positive $\omega_a > 0$ (in order to ensure the extensivity of the energy when the couplings have all the same sign).

We should stress here that the Parity and the Time-Reversal are the only symmetries that will be taken into account in the present paper, while the models displaying further symmetries like magnetization conservation (corresponding to the identity between x and y couplings $\mathcal{J}_{n,n'}^x = \mathcal{J}_{n,n'}^y$) or $SU(2)$ invariance (corresponding to the identity between x, y and z couplings $\mathcal{J}_{n,n'}^x = \mathcal{J}_{n,n'}^y = \mathcal{J}_{n,n'}^z$) would require other isometries in order to take into account these stronger symmetry properties.

Since the quantum Ising model is the basic short-ranged model in the field of zero-temperature quantum phase transitions [32], its Long-Ranged version

$$\mathcal{H}_{pure}^{QI} = - \sum_n h \sigma_n^z - \sum_{n < n'} \frac{J^x}{|n - n'|^{1+\omega_x}} \sigma_n^x \sigma_{n'}^x \quad (10)$$

has been also much studied in order to analyze how the critical properties depend upon the exponent ω_x [33], as well as in relation with the other problem of the dissipative short-ranged quantum spin chain [34–37]. The effects of random transverse-fields h_n

$$\mathcal{H}_{random}^{QI} = - \sum_n h_n \sigma_n^z - \sum_{n < n'} \frac{J^x}{|n - n'|^{1+\omega_x}} \sigma_n^x \sigma_{n'}^x \quad (11)$$

has also been studied recently [38–40] (see also the related work [41] concerning Long-Ranged epidemic models) via the Strong Disorder Renormalization approach (see the reviews [23, 24]).

E. Dyson hierarchical version of Long-Ranged quantum spin chains

Dyson hierarchical models are based on the following binary tree structure. The generation $g = 0$ contains the single site called the root. The generation $g = 1$ contains its two children labelled by the index $i_1 = 1, 2$. The generation $g = 2$ contains the two children $i_2 = 1, 2$ of each site $i_1 = 1, 2$ of generation $g = 1$, and so on. So the generation g contains $N_g = 2^g$ sites labelled by the g binary indices (i_1, i_2, \dots, i_g) that indicate the whole line of ancestors up to the root at $g = 0$.

The Dyson hierarchical version of the Long-Ranged Hamiltonian of Eqs 8 and 9 is then defined for a chain of $N_G = 2^G$ spins σ_I labelled by the positions $I = (i_1, i_2, \dots, i_G)$ of the last generation G of the tree structure by

$$\mathcal{H}^{[G]} = - \sum_{I=(i_1, i_2, \dots, i_G)} h_I \sigma_I^z - \sum_{I=(i_1, i_2, \dots, i_G) < I'=(i'_1, i'_2, \dots, i'_G)} \sum_{a=x, y, z} \mathcal{J}_{I, I'}^a \sigma_I^a \sigma_{I'}^a, \quad (12)$$

where the couplings $\mathcal{J}_{I, I'}^a$ have exactly the same power-law dependence as in Eq 9

$$\mathcal{J}_{I, I'}^a = \frac{J_{I, I'}^a}{[r(I, I')]^{1+\omega_a}} \quad (13)$$

but with respect to the ultrametric distance $r(I, I')$ on the tree defined in terms of the generation of their Last Common Ancestor as follows. Two sites $I = (I_c, i_{G-k} = 1, F)$ and $I' = (I_c, i_{G-k} = 2, F')$ that have in common the first $(G - k - 1)$ indices $I_c = (i_1, \dots, i_{G-k-1})$, while they have different indices $i_{G-k} = 1$ and $i_{G-k} = 2$ at generation $(G - k)$, are separated by the distance

$$r(I = (I_c, i_{G-k} = 1, F), I' = (I_c, i_{G-k} = 2, F')) \equiv 2^k \quad (14)$$

for any values of the remaining indices $F = (i_{G-k+1}, \dots, i_G)$ and $F' = (i'_{G-k+1}, \dots, i'_G)$. The minimal value $k = 0$ corresponds to the distance $r = 2^0 = 1$ between spins that have the same ancestor at position $I_c = (i_1, i_2, \dots, i_{G-1})$ of the generation $(G - 1)$ while they differ $i_G = 1, 2$ at generation G (here F and F' are empty). The maximal value $k = G - 1$ corresponds to the distance $r = 2^{G-1} = \frac{N_G}{2}$ between any spin belonging to the first half $i_1 = 1$ and any spin belonging to the second half $i_1 = 2$ (here I_c is empty and their Last Common ancestor is the root 0). As a consequence, the Dyson Hamiltonian of Eq. 12 can be rewritten more explicitly as a sum over the index $k = 0, \dots, G - 1$ that labels the possible distance $r_k = 2^k$ as

$$\mathcal{H}^{[G]} = - \sum_{I=(i_1, i_2, \dots, i_G)} h_I \sigma_I^z - \sum_{k=0}^{G-1} \sum_{I_c=(i_1, i_2, \dots, i_{G-k-1})} \sum_{F=(i_{G-k+1}, \dots, i_G)} \sum_{F'=(i'_{G-k+1}, \dots, i'_G)} \sum_{a=x, y, z} \frac{J_{I_c 1 F, I_c 2 F'}^a}{2^{k(1+\omega_a)}} \sigma_{I_c 1 F}^a \sigma_{I_c 2 F'}^a \quad (15)$$

The Dyson hierarchical version of the pure quantum Ising model of Eq. 10 has been already studied via block-spin renormalization in order to analyze its critical properties [42] and its entanglement properties for various bipartite partitions [43]. The block-spin renormalization has also been used to analyze the Dyson random transverse field Ising model [42] and the quantum spin-glass in uniform transverse field [44].

More generally, the Dyson hierarchical versions of many long ranged models have been considered since the original Dyson hierarchical classical ferromagnetic Ising model [45] that has been much studied by both mathematicians [46–49] and physicists [50–53], including the properties of the dynamics [54, 55]. In the field of classical disordered systems, equilibrium properties have been analyzed for random fields Ising models [56, 57] and for spin-glasses [58–63], while the dynamical properties are discussed in Refs [55, 64]. Finally, let us mention that Dyson hierarchical models have been also considered for Anderson localization models [65–72] and for Many-Body-Localization [73].

III. SIMPLEST TREE-TENSOR-STATES WITH PARITY AND TIME-REVERSAL SYMMETRIES

In this section, the goal is to construct the simplest inhomogeneous Tree-Tensor-States for disorder spin chains with Parity and Time-Reversal symmetries, while the case of homogeneous Tree-Tensor-States for pure spin chains is postponed to the sections VI and VII where their supplementary symmetries will be taken into account.

A. Isometries based on blocks of two spins preserving the (P, T) symmetries

The traditional block-spin renormalization procedure based on blocks of two sites can be summarized as follows in terms of the tree notations introduced in the subsection II E. The initial chain of $N_G = 2^G$ spins $\sigma_{i_1, i_2, \dots, i_G}^{[G]}$ belonging to the last generation G will be first renormalized into a chain of $N_{G-1} = 2^{G-1} = \frac{N_G}{2}$ spins $\sigma_{i_1, i_2, \dots, i_{G-1}}^{[G-1]}$ of generation $(G - 1)$. This procedure will be then iterated up to the last RG step where there will be a single spin $\sigma^{[g=0]}$ at the root corresponding to generation $g = 0$.

The basic block-spin RG step is implemented by the elementary coarse-graining isometry $w^{[g, I]}$, where $I = (i_1, \dots, i_g)$ labels the possible positions at generation g , between the two-dimensional Hilbert space of the renormalized spin

$|\sigma_I^{[g]z} = \pm\rangle$ and the two relevant states $|\psi_{I_1, I_2}^{[g+1]\pm}\rangle$ that are kept out of the four-dimensional Hilbert space of its two children $|\sigma_{I_1}^{[g+1]z} = \pm, \sigma_{I_2}^{[g+1]z} = \pm\rangle$

$$w^{[g, I]} \equiv |\psi_{I_1, I_2}^{[g+1]+}\rangle \langle \sigma_I^{[g]z} = +| + |\psi_{I_1, I_2}^{[g+1]-}\rangle \langle \sigma_I^{[g]z} = -| \quad (16)$$

So the product

$$\left(w^{[g, I]}\right)^\dagger \left(w^{[g, I]}\right) = |\sigma_I^{[g]z} = +\rangle \langle \sigma_I^{[g]z} = +| + |\sigma_I^{[g]z} = -\rangle \langle \sigma_I^{[g]z} = -| = \sigma_I^{[g]0} \quad (17)$$

is simply the identity operator $\sigma_I^{[g]0}$ of the Hilbert space of the ancestor spin, while the product

$$\left(w^{[g, I]}\right) \left(w^{[g, I]}\right)^\dagger = |\psi_{I_1, I_2}^{[g+1]+}\rangle \langle \psi_{I_1, I_2}^{[g+1]+}| + |\psi_{I_1, I_2}^{[g+1]-}\rangle \langle \psi_{I_1, I_2}^{[g+1]-}| \equiv \Pi_{\psi_{I_1, I_2}^{[g+1]\pm}} \quad (18)$$

corresponds to the projector $\Pi_{\psi_{I_1, I_2}^{[g+1]\pm}}$ onto the subspace spanned by the two states $\psi_{I_1, I_2}^{[g+1]\pm}$ that are kept out of the four-dimensional Hilbert space of the two children.

In order to preserve the Parity, the normalized ket $|\psi_{I_1, I_2}^{[g+1]+}\rangle$ will be chosen as some linear combination of the two states of positive parity $|\sigma_{I_1}^{[g+1]z} = +, \sigma_{I_2}^{[g+1]z} = +\rangle$ and $|\sigma_{I_1}^{[g+1]z} = -, \sigma_{I_2}^{[g+1]z} = -\rangle$ for the block of the two children. Since the Time-Reversal-Symmetry imposes real coefficients, the parametrization involves only a single angle $\theta^{[g, I]+}$

$$|\psi_{I_1, I_2}^{[g+1]+}\rangle = \cos(\theta^{[g, I]+}) |\sigma_{I_1}^{[g+1]z} = +, \sigma_{I_2}^{[g+1]z} = +\rangle + \sin(\theta^{[g, I]+}) |\sigma_{I_1}^{[g+1]z} = -, \sigma_{I_2}^{[g+1]z} = -\rangle \quad (19)$$

Similarly, the ket $|\psi_{I_1, I_2}^{[g+1]-}\rangle$ will be chosen as some linear combination of the two states of negative parity $|\sigma_{I_1}^{[g+1]z} = +, \sigma_{I_2}^{[g+1]z} = -\rangle$ and $|\sigma_{I_1}^{[g+1]z} = -, \sigma_{I_2}^{[g+1]z} = +\rangle$ for the block of the two children and involves only another single angle $\theta^{[g, I]-}$

$$|\psi_{I_1, I_2}^{[g+1]-}\rangle = \cos(\theta^{[g, I]-}) |\sigma_{I_1}^{[g+1]z} = +, \sigma_{I_2}^{[g+1]z} = -\rangle + \sin(\theta^{[g, I]-}) |\sigma_{I_1}^{[g+1]z} = -, \sigma_{I_2}^{[g+1]z} = +\rangle \quad (20)$$

From the ascending block-spin renormalization perspective, Eqs 19 and 20 parametrize the representative states that are kept in each two-dimensional parity sector $P = \pm$ respectively. From the descending perspective, Eqs 19 and 20 can be interpreted as the Schmidt decompositions of the kept state of parity $P = \pm$ in terms of the states of its two children

$$|\psi_{I_1, I_2}^{[g+1]P}\rangle = \sum_{\alpha=\pm} \Lambda_{\alpha}^{[g, I]P} |\sigma_{I_1}^{[g+1]z} = \alpha\rangle \otimes |\sigma_{I_2}^{[g+1]z} = \alpha P\rangle \quad (21)$$

where the two Schmidt singular values are given by

$$\begin{aligned} \Lambda_{\alpha=+}^{[g, I]P} &= \cos(\theta^{[g, I]P}) \\ \Lambda_{\alpha=-}^{[g, I]P} &= \sin(\theta^{[g, I]P}) \end{aligned} \quad (22)$$

while the kets $|\sigma_{I_1}^{[g+1]z} = \alpha\rangle$ and $|\sigma_{I_2}^{[g+1]z} = \alpha P\rangle$ correspond to the associated Schmidt eigenvectors of the first child and the second child respectively. Indeed, the partial traces over a single child of the projector associated to the state of parity P of Eq. 21 is diagonal for these eigenvectors

$$\begin{aligned} \text{Tr}_{\{I_2\}} \left(|\psi_{I_1, I_2}^{[g+1]P}\rangle \langle \psi_{I_1, I_2}^{[g+1]P}| \right) &= \sum_{\alpha=\pm} \left(\Lambda_{\alpha}^{[g, I]P} \right)^2 |\sigma_{I_1}^{[g+1]z} = \alpha\rangle \langle \sigma_{I_1}^{[g+1]z} = \alpha| \\ \text{Tr}_{\{I_1\}} \left(|\psi_{I_1, I_2}^{[g+1]P}\rangle \langle \psi_{I_1, I_2}^{[g+1]P}| \right) &= \sum_{\alpha=\pm} \left(\Lambda_{\alpha}^{[g, I]P} \right)^2 |\sigma_{I_2}^{[g+1]z} = \alpha P\rangle \langle \sigma_{I_2}^{[g+1]z} = \alpha P| \end{aligned} \quad (23)$$

and $\left(\Lambda_{\alpha=\pm}^{[g, I]P}\right)^2$ are the two common weights normalized to unity as it should

$$\sum_{\alpha=\pm} \left(\Lambda_{\alpha}^{[g, I]P}\right)^2 = \cos^2(\theta^{[g, I]P}) + \sin^2(\theta^{[g, I]P}) = 1 \quad (24)$$

B. Local ascending and descending super-operators $\mathcal{A}^{[g,I]}$ and $\mathcal{D}^{[g,I]}$

The local ascending superoperator $\mathcal{A}^{[g,I]}$ describes how the the 16 two-spin operators $\sigma_{I_1}^{[g+1]a_1=0,x,y,z} \sigma_{I_2}^{[g+1]a_2=0,x,y,z}$ of the two children are projected onto the four Pauli operators $\sigma_I^{[g]a=0,x,y,z}$ of their ancestor via the isometry $w^{[g,I]}$

$$\mathcal{A}^{[g,I]} \left[\sigma_{I_1}^{[g+1]a_1} \sigma_{I_2}^{[g+1]a_2} \right] \equiv (w^{[g,I]})^\dagger \left(\sigma_{I_1}^{[g+1]a_1} \sigma_{I_2}^{[g+1]a_2} \right) w^{[g,I]} = \sum_{a=0,x,y,z} F_{a_1,a_2}^{[g,I]a} \sigma_I^{[g]a} \quad (25)$$

where the fusion coefficients

$$F_{a_1,a_2}^{[g,I]a} = \frac{1}{2} \text{Tr}_{\{I\}} \left(\left[\mathcal{A}^{[g,I]} \left[\sigma_{I_1}^{[g+1]a_1} \sigma_{I_2}^{[g+1]a_2} \right] \right] \sigma_I^{[g]a} \right) = \frac{1}{2} \text{Tr}_{\{I\}} \left(\left[(w^{[g,I]})^\dagger \left(\sigma_{I_1}^{[g+1]a_1} \sigma_{I_2}^{[g+1]a_2} \right) w^{[g,I]} \right] \sigma_I^{[g]a} \right) \quad (26)$$

can be rewritten as

$$F_{a_1,a_2}^{[g,I]a} = \frac{1}{2} \text{Tr}_{\{I_1,I_2\}} \left(\sigma_{I_1}^{[g+1]a_1} \sigma_{I_2}^{[g+1]a_2} \left[w^{[g,I]} \sigma_I^{[g]a} (w^{[g,I]})^\dagger \right] \right) \quad (27)$$

As a consequence, the local descending superoperator $\mathcal{D}^{[g,I]}$ that translates the four ancestor spin operators $\sigma_I^{[g]a=0,x,y,z}$ into operators for its two children involves the same fusion coefficients

$$\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]a} \right] \equiv w^{[g,I]} \sigma_I^{[g]a} (w^{[g,I]})^\dagger = \frac{1}{2} \sum_{a_1=0,x,y,z} \sum_{a_2=0,x,y,z} F_{a_1,a_2}^{[g,I]a} \sigma_{I_1}^{[g+1]a_1} \sigma_{I_2}^{[g+1]a_2} \quad (28)$$

Since the isometry $w^{[g,I]}$ preserves the Parity and the Time-Reversal symmetries, the fusion rules respect the four symmetry sectors ($P = \pm 1, T = \pm 1$) of operators described in the subsections II B and II C. As a consequence, the two operators of Eq. 4 corresponding to the sector ($P = +1, T = -1$) are projected out

$$\begin{aligned} \mathcal{A}^{[g,I]} \left[\sigma_{I_1}^{[g+1]x} \sigma_{I_2}^{[g+1]y} \right] &= 0 \\ \mathcal{A}^{[g,I]} \left[\sigma_{I_1}^{[g+1]y} \sigma_{I_2}^{[g+1]x} \right] &= 0 \end{aligned} \quad (29)$$

and will never be produced by $\mathcal{D}^{[g,I]}$.

The fusion rules in the three other non-trivial symmetry sectors are described in the following subsections in terms of the two angles

$$\begin{aligned} \phi^{[g,I]} &\equiv \frac{\pi}{2} - \theta^{[g,I] +} - \theta^{[g,I] -} \\ \tilde{\phi}^{[g,I]} &\equiv -\theta^{[g,I] +} + \theta^{[g,I] -} \end{aligned} \quad (30)$$

with the following notations for their cosinus and sinus

$$\begin{aligned} c^{[g,I]} &\equiv \cos \left(\phi^{[g,I]} \right) \\ s^{[g,I]} &\equiv \sin \left(\phi^{[g,I]} \right) \\ \tilde{c}^{[g,I]} &\equiv \cos \left(\tilde{\phi}^{[g,I]} \right) \\ \tilde{s}^{[g,I]} &\equiv \sin \left(\tilde{\phi}^{[g,I]} \right) \end{aligned} \quad (31)$$

in order to obtain simpler explicit expressions.

C. Local fusion rules for operators in the symmetry sector ($P = -1, T = +1$)

The action of the ascending superoperator $\mathcal{A}^{[g,I]}$ on the four operators of Eq. 5 corresponding to the symmetry sector ($P = -1, T = +1$) can only involve the operator $\sigma_I^{[g]x}$ and the explicit computation yields the fusion coefficients

$$\begin{aligned}\mathcal{A}^{[g,I]} \left[\sigma_{I1}^{[g+1]x} \sigma_{I2}^{[g+1]0} \right] &= c^{[g,I]} \sigma_I^{[g]x} \\ \mathcal{A}^{[g,I]} \left[\sigma_{I1}^{[g+1]0} \sigma_{I2}^{[g+1]x} \right] &= \tilde{c}^{[g,I]} \sigma_I^{[g]x} \\ \mathcal{A}^{[g,I]} \left[\sigma_{I1}^{[g+1]x} \sigma_{I2}^{[g+1]z} \right] &= \tilde{s}^{[g,I]} \sigma_I^{[g]x} \\ \mathcal{A}^{[g,I]} \left[\sigma_{I1}^{[g+1]z} \sigma_{I2}^{[g+1]x} \right] &= s^{[g,I]} \sigma_I^{[g]x}\end{aligned}\quad (32)$$

Reciprocally, the four operators of Eq. 5 will appear in the application of the descending superoperator $\mathcal{D}^{[g,I]}$ to $\sigma_I^{[g]x}$ with the same fusion coefficients given by the duality of Eq. 28

$$\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]x} \right] = \frac{1}{2} \left[c^{[g,I]} \sigma_{I1}^{[g+1]x} \sigma_{I2}^{[g+1]0} + \tilde{c}^{[g,I]} \sigma_{I1}^{[g+1]0} \sigma_{I2}^{[g+1]x} + \tilde{s}^{[g,I]} \sigma_{I1}^{[g+1]x} \sigma_{I2}^{[g+1]z} + s^{[g,I]} \sigma_{I1}^{[g+1]z} \sigma_{I2}^{[g+1]x} \right] \quad (33)$$

while the partial traces over a single child reduce to

$$\begin{aligned}\text{Tr}_{\{I2\}} \left(\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]x} \right] \right) &= c^{[g,I]} \sigma_{I1}^{[g+1]x} \\ \text{Tr}_{\{I1\}} \left(\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]x} \right] \right) &= \tilde{c}^{[g,I]} \sigma_{I2}^{[g+1]x}\end{aligned}\quad (34)$$

It is thus convenient to introduce the following notation

$$\lambda_{i_{g+1}}^{[g,I]x} \equiv c^{[g,I]} \delta_{i_{g+1},1} + \tilde{c}^{[g,I]} \delta_{i_{g+1},2} \quad (35)$$

to denote the local scaling property of the single child operator $\sigma_{I_{i_{g+1}}}^{[g+1]x}$ with respect to its ancestor operator $\sigma_I^{[g]x}$.

D. Local fusion rules for operators in the symmetry sector ($P = -1, T = -1$)

Similarly, the action of the ascending superoperator $\mathcal{A}^{[g,I]}$ on the four operators of Eq. 6 corresponding to the symmetry sector ($P = -1, T = -1$) can only involve the operator $\sigma_I^{[g]y}$ and the explicit computation yields the fusion coefficients

$$\begin{aligned}\mathcal{A}^{[g,I]} \left[\sigma_{I1}^{[g+1]y} \sigma_{I2}^{[g+1]0} \right] &= \tilde{s}^{[g,I]} \sigma_I^{[g]y} \\ \mathcal{A}^{[g,I]} \left[\sigma_{I1}^{[g+1]0} \sigma_{I2}^{[g+1]y} \right] &= s^{[g,I]} \sigma_I^{[g]y} \\ \mathcal{A}^{[g,I]} \left[\sigma_{I1}^{[g+1]y} \sigma_{I2}^{[g+1]z} \right] &= c^{[g,I]} \sigma_I^{[g]y} \\ \mathcal{A}^{[g,I]} \left[\sigma_{I1}^{[g+1]z} \sigma_{I2}^{[g+1]y} \right] &= \tilde{c}^{[g,I]} \sigma_I^{[g]y}\end{aligned}\quad (36)$$

Reciprocally, the four operators of Eq. 6 will appear in the application of the descending superoperator to $\sigma_I^{[g]y}$ with the same fusion coefficients given by the duality of Eq. 28

$$\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]y} \right] = \frac{1}{2} \left[\tilde{s}^{[g,I]} \sigma_{I1}^{[g+1]y} \sigma_{I2}^{[g+1]0} + s^{[g,I]} \sigma_{I1}^{[g+1]0} \sigma_{I2}^{[g+1]y} + c^{[g,I]} \sigma_{I1}^{[g+1]y} \sigma_{I2}^{[g+1]z} + \tilde{c}^{[g,I]} \sigma_{I1}^{[g+1]z} \sigma_{I2}^{[g+1]y} \right] \quad (37)$$

while the partial traces over a single child reduce to

$$\begin{aligned}\text{Tr}_{\{I2\}} \left(\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]y} \right] \right) &= \tilde{s}^{[g,I]} \sigma_{I1}^{[g+1]y} \\ \text{Tr}_{\{I1\}} \left(\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]y} \right] \right) &= s^{[g,I]} \sigma_{I2}^{[g+1]y}\end{aligned}\quad (38)$$

Again it is convenient to introduce the following notation

$$\lambda_{i_{g+1}}^{[g,I]y} \equiv \tilde{s}^{[g,I]} \delta_{i_{g+1},1} + s^{[g,I]} \delta_{i_{g+1},2} \quad (39)$$

to denote the local scaling property of the single child operator $\sigma_{I_{i_{g+1}}}^{[g+1]y}$ with respect to its ancestor operator $\sigma_I^{[g]y}$.

E. Local fusion rules for operators in the symmetry sector ($P = +1, T = +1$)

The action of the ascending superoperator $\mathcal{A}^{[g,I]}$ on the six operators of Eq. 7 corresponding to the symmetry sector ($P = +1, T = +1$) can only involve the two operators $\sigma_I^{[g]a=0,z}$. The identity $\sigma_{I_1}^{[g+1]0} \sigma_{I_2}^{[g+1]0}$ of the children space is projected onto the identity $\sigma_I^{[g]0}$ of the ancestor space as a consequence of Eq 17

$$\mathcal{A}^{[g,I]} \left[\sigma_{I_1}^{[g+1]0} \sigma_{I_2}^{[g+1]0} \right] = \sigma_I^{[g]0} \quad (40)$$

while the parity $\sigma_{I_1}^{[g+1]z} \sigma_{I_2}^{[g+1]z}$ of the block of the two children is projected onto the parity $\sigma_I^{[g]z}$ of the ancestor

$$\mathcal{A}^{[g,I]} \left[\sigma_{I_1}^{[g+1]z} \sigma_{I_2}^{[g+1]z} \right] = \sigma_I^{[g]z} \quad (41)$$

The remaining operators are projected onto the following linear combinations of the two operators $\sigma_I^{[g]0}$ and $\sigma_I^{[g]z}$

$$\begin{aligned} \mathcal{A}^{[g,I]} \left[\sigma_{I_1}^{[g+1]z} \sigma_{I_2}^{[g+1]0} \right] &= s^{[g,I]} \tilde{c}^{[g,I]} \sigma_I^{[g]0} + c^{[g,I]} \tilde{s}^{[g,I]} \sigma_I^{[g]z} \\ \mathcal{A}^{[g,I]} \left[\sigma_{I_1}^{[g+1]0} \sigma_{I_2}^{[g+1]z} \right] &= c^{[g,I]} \tilde{s}^{[g,I]} \sigma_I^{[g]0} + s^{[g,I]} \tilde{c}^{[g,I]} \sigma_I^{[g]z} \\ \mathcal{A}^{[g,I]} \left[\sigma_{I_1}^{[g+1]x} \sigma_{I_2}^{[g+1]x} \right] &= c^{[g,I]} \tilde{c}^{[g,I]} \sigma_I^{[g]0} - s^{[g,I]} \tilde{s}^{[g,I]} \sigma_I^{[g]z} \\ \mathcal{A}^{[g,I]} \left[\sigma_{I_1}^{[g+1]y} \sigma_{I_2}^{[g+1]y} \right] &= s^{[g,I]} \tilde{s}^{[g,I]} \sigma_I^{[g]0} - c^{[g,I]} \tilde{c}^{[g,I]} \sigma_I^{[g]z} \end{aligned} \quad (42)$$

The duality of Eq. 28 yields that the application of the descending superoperator $\mathcal{D}^{[g,I]}$ to $\sigma_I^{[g]z}$ involves five operators of the list of Eq. 7 (only the block identity $\sigma_{I_1}^{[g+1]0} \sigma_{I_2}^{[g+1]0}$ does not appear)

$$\begin{aligned} \mathcal{D}^{[g,I]} \left[\sigma_I^{[g]z} \right] &= \frac{1}{2} \left[c^{[g,I]} \tilde{s}^{[g,I]} \sigma_{I_1}^{[g+1]z} \sigma_{I_2}^{[g+1]0} + s^{[g,I]} \tilde{c}^{[g,I]} \sigma_{I_1}^{[g+1]0} \sigma_{I_2}^{[g+1]z} \right] \\ &+ \frac{1}{2} \left[\sigma_{I_1}^{[g+1]z} \sigma_{I_2}^{[g+1]z} - s^{[g,I]} \tilde{s}^{[g,I]} \sigma_{I_1}^{[g+1]x} \sigma_{I_2}^{[g+1]x} - c^{[g,I]} \tilde{c}^{[g,I]} \sigma_{I_1}^{[g+1]y} \sigma_{I_2}^{[g+1]y} \right] \end{aligned} \quad (43)$$

and the partial traces over a single child reduce to

$$\begin{aligned} \text{Tr}_{\{I_2\}} \left(\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]z} \right] \right) &= c^{[g,I]} \tilde{s}^{[g,I]} \sigma_{I_1}^{[g+1]z} \\ \text{Tr}_{\{I_1\}} \left(\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]z} \right] \right) &= s^{[g,I]} \tilde{c}^{[g,I]} \sigma_{I_2}^{[g+1]z} \end{aligned} \quad (44)$$

Again it is convenient to introduce the following notation

$$\lambda_{i_{g+1}}^{[g,I]z} \equiv c^{[g,I]} \tilde{s}^{[g,I]} \delta_{i_{g+1},1} + s^{[g,I]} \tilde{c}^{[g,I]} \delta_{i_{g+1},2} \quad (45)$$

to denote the local scaling property of the single child operator $\sigma_{I_{i_{g+1}}}^{[g+1]z}$ with respect to its ancestor operator $\sigma_I^{[g]z}$.

The duality of Eq. 28 yields that the application of the descending superoperator $\mathcal{D}^{[g,I]}$ to the identity $\sigma_I^{[g]0}$ of the ancestor space involves five operators of the list of Eq. 7 (only the block parity $\sigma_{I_1}^{[g+1]z} \sigma_{I_2}^{[g+1]z}$ does not appear)

$$\begin{aligned} \mathcal{D}^{[g,I]} \left[\sigma_I^{[g]0} \right] &= \frac{1}{2} \left[\sigma_{I_1}^{[g+1]0} \sigma_{I_2}^{[g+1]0} + s^{[g,I]} \tilde{c}^{[g,I]} \sigma_{I_1}^{[g+1]z} \sigma_{I_2}^{[g+1]0} + c^{[g,I]} \tilde{s}^{[g,I]} \sigma_{I_1}^{[g+1]0} \sigma_{I_2}^{[g+1]z} \right] \\ &+ \frac{1}{2} \left[c^{[g,I]} \tilde{c}^{[g,I]} \sigma_{I_1}^{[g+1]x} \sigma_{I_2}^{[g+1]x} + s^{[g,I]} \tilde{s}^{[g,I]} \sigma_{I_1}^{[g+1]y} \sigma_{I_2}^{[g+1]y} \right] \end{aligned} \quad (46)$$

and the partial traces over a single child reduce to

$$\begin{aligned} \text{Tr}_{\{I_2\}} \left(\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]0} \right] \right) &= \sigma_{I_1}^{[g+1]0} + s^{[g,I]} \tilde{c}^{[g,I]} \sigma_{I_1}^{[g+1]z} \\ \text{Tr}_{\{I_1\}} \left(\mathcal{D}^{[g,I]} \left[\sigma_I^{[g]0} \right] \right) &= \sigma_{I_2}^{[g+1]0} + c^{[g,I]} \tilde{s}^{[g,I]} \sigma_{I_2}^{[g+1]z} \end{aligned} \quad (47)$$

Although the meaning is different from the three scaling factors $\lambda_{i_{g+1}}^{[g,I]a=x,y,z}$ introduced above, it will be convenient to introduce

$$\lambda_{i_{g+1}}^{[g,I]0} \equiv s^{[g,I]} \tilde{c}^{[g,I]} \delta_{i_{g+1},1} + c^{[g,I]} \tilde{s}^{[g,I]} \delta_{i_{g+1},2} \quad (48)$$

to denote the local scaling property of the single child operator $\sigma_{I_{i_{g+1}}}^{[g+1]z}$ with respect to its ancestor identity $\sigma_I^{[g]0}$.

F. Assembling the elementary isometries to build the whole Tree-Tensor-State of parity $P = +$

The correspondence between the ket $|\Psi^{[g]}\rangle$ for the chain of generation g containing $N_g = 2^g$ spins and the ket $|\Psi^{[g+1]}\rangle$ for the chain of generation $(g+1)$ containing $N_{g+1} = 2^{g+1}$ spins

$$|\Psi^{[g+1]}\rangle = W^{[g]} |\Psi^{[g]}\rangle \quad (49)$$

is described by the global isometry $W^{[g]}$ made of the tensor product over the 2^g positions $I = (i_1, i_2, \dots, i_g)$ of the elementary isometries of Eq. 16

$$W^{[g]} = \prod_{I=(i_1, i_2, \dots, i_g)} w^{[g, I]} \quad (50)$$

At generation $g = 0$, the state of the single spin $\sigma^{[0]}$ at the root of the tree represents the Parity of the whole chain. We will focus on the positive parity sector $P = +$ corresponding to the initial ket

$$|\Psi^{[0]}\rangle = |\sigma^{[0]z} = P = +\rangle \quad (51)$$

The iteration of the rule of Eq. 49 will then generate a Tree-Tensor-State of parity $P = +$ for the chain of generation g containing $N_g = 2^g$ spins

$$|\Psi^{[g]}\rangle = W^{[g-1]} |\Psi^{[g-1]}\rangle = W^{[g-1]} W^{[g-2]} \dots W^{[1]} W^{[0]} |\Psi^{[0]}\rangle \quad (52)$$

Since the elementary isometry $w^{[g', I']}$ at generation g' and position I' involves only the two angles $\theta^{[g', I']\pm}$, the global isometry $W^{[g']}$ for the $2^{g'}$ sites I' of the generation g' involves $2 \times 2^{g'}$ angles, except for the generation $g' = 0$ where only the angle $\theta^{[0]\pm}$ will appear for the initial condition of Eq. 51. So the total number of parameters involved in the Tree-Tensor-State $|\Psi^{[g]}\rangle$ of generation g containing $N_g = 2^g$ spins grows only linearly with respect to N_g

$$\mathcal{N}_g^{Parameters} = 1 + 2 \times \sum_{g'=1}^{g-1} 2^{g'} = 2(2^g - 1) - 1 = 2N_g - 3 \quad (53)$$

As explained around Eq 21, these angles $\theta^{[g', I']\pm}$ parametrize the hierarchical entanglement at different levels labelled by the generation g' and different positions labelled by the positions I' . The consequences of this tree-tensor structure for the entanglement of various bipartite partitions have been studied in detail in [43] on the specific case of the pure Dyson quantum Ising model. In the following section, we will thus focus instead on the consequences for the one-point and two-point reduced density matrices that allow to compute any one-spin and two-spin observable.

IV. EXPLICIT FORMS FOR ONE-POINT AND TWO-POINT REDUCED DENSITY MATRICES

The hierarchical structure of the inhomogeneous Tree-Tensor-States described in the previous section allows to write simple recursions for the corresponding one-point and two-point reduced density matrices.

A. Recursion for the full density matrices via the descending super-operator $\mathcal{D}^{[g]}$

The full density matrix for the chain at generation g

$$\rho^{[g]} \equiv |\Psi^{[g]}\rangle \langle \Psi^{[g]}| \quad (54)$$

satisfies the recurrence involving the global descending superoperator $\mathcal{D}^{[g]}$

$$\rho^{[g+1]} = W^{[g]} \rho^{[g]} (W^{[g]})^\dagger \equiv \mathcal{D}^{[g]} [\rho^{[g]}] \quad (55)$$

while the initial condition at generation $g = 0$ reads (Eq 51)

$$\rho^{[0]} \equiv |\Psi^{[0]}\rangle \langle \Psi^{[0]}| = |\sigma^{[0]z} = +\rangle \langle \sigma^{[0]z} = +| = \frac{\sigma^{[0]0} + \sigma^{[0]z}}{2} \quad (56)$$

Since the density matrix $\rho^{[g]}$ of generation g can be expanded in the Pauli basis of the 2^g spins $\sigma_I^{[g]a_I=0,x,y,z}$, one just needs to know how to apply the descending superoperator to products of Pauli matrices

$$\mathcal{D}^{[g]} \left[\prod_{I=(i_1, I_2, \dots, i_g)} \sigma_I^{[g]a_I} \right] = \prod_{I=(i_1, I_2, \dots, i_g)} \mathcal{D}^{[g, I]} \left[\sigma_I^{[g]a_I} \right] \quad (57)$$

where the properties of the local descending superoperator $\mathcal{D}^{[g, I]}$ of Eq. 28 have been described in detail in the previous section.

B. Parametrization of one-spin and two-spins reduced density matrices

In order to compute all the one-spin and two-spins observables, one just needs the one-spin and two-spins reduced density matrices. Since the initial condition of Eq. 56 belongs to the symmetry sector ($P = +, T = +$), the full density matrices of Eq. 55 obtained by the successive application of the global descending superoperator $\mathcal{D}^{[g]}$ are also in the sector ($P = +, T = +$), and the partial traces over some spins will also preserve this symmetry sector. As a consequence, the single-spin reduced density matrices can be parametrized as

$$\rho_I^{[g]} = \frac{\sigma_I^{[g]0} + m_I^{[g]} \sigma_I^{[g]z}}{2} \quad (58)$$

where the coefficient $1/2$ of the identity $\sigma_I^{[g]0}$ is fixed by the normalization

$$\text{Tr}_{\{I\}}(\rho_I^{[g]}) = 1 \quad (59)$$

while $m_I^{[g]}$ represents the magnetization at site I of generation g

$$m_I^{[g]} = \text{Tr}_{\{I\}}(\sigma_I^{[g]z} \rho_I^{[g]}) \quad (60)$$

Similarly, the reduced density matrices $\rho_{I, I'}^{[g]}$ of two spins at positions I and I' of generation g can only involve the six two-spin operators of Eq. 7 of the sector ($P = +, T = +$) and can be thus parametrized as

$$\rho_{I, I'}^{[g]} = \frac{\sigma_I^{[g]0} \sigma_{I'}^{[g]0}}{4} + m_I^{[g]} \frac{\sigma_I^{[g]z} \sigma_{I'}^{[g]0}}{4} + m_{I'}^{[g]} \frac{\sigma_I^{[g]0} \sigma_{I'}^{[g]z}}{4} + C_{I, I'}^{[g]z} \frac{\sigma_I^{[g]z} \sigma_{I'}^{[g]z}}{4} + C_{I, I'}^{[g]x} \frac{\sigma_I^{[g]x} \sigma_{I'}^{[g]x}}{4} + C_{I, I'}^{[g]y} \frac{\sigma_I^{[g]y} \sigma_{I'}^{[g]y}}{4} \quad (61)$$

where the three first coefficients are fixed by the compatibility with the one-point reduced density matrices of Eq. 58, while the three last coefficients $C_{I, I'}^{[g]a=x,y,z}$ represent the two-points xx , yy and zz correlations

$$C_{I, I'}^{[g]a} = \text{Tr}_{\{I, I'\}}(\sigma_I^{[g]a} \sigma_{I'}^{[g]a} \rho_{I, I'}^{[g]}) \quad (62)$$

C. Recursions for the one-point magnetizations and the two-point correlations

The application of the local descending superoperator $\mathcal{D}^{[g, I]}$ to the reduced density matrix $\rho_I^{[g]}$ of the single site I of generation g of Eq. 58 produces the following reduced density matrix of its two children ($I1, I2$) of generation $(g+1)$ using Eqs 43 and 46

$$\begin{aligned} \rho_{I1, I2}^{[g+1]} &= \mathcal{D}^{[g, I]}[\rho_I^{[g]}] = \frac{1}{2} \left[\mathcal{D}^{[g, I]}[\sigma_I^{[g]0}] + m_I^{[g]} \mathcal{D}^{[g, I]}[\sigma_I^{[g]z}] \right] \\ &= \frac{\sigma_{I1}^{[g+1]0} \sigma_{I2}^{[g+1]0}}{4} + \left(s^{[g, I]} \tilde{c}^{[g, I]} + m_I^{[g]} c^{[g, I]} \tilde{s}^{[g, I]} \right) \frac{\sigma_{I1}^{[g+1]z} \sigma_{I2}^{[g+1]0}}{4} + \left(c^{[g, I]} \tilde{s}^{[g, I]} + m_I^{[g]} s^{[g, I]} \tilde{c}^{[g, I]} \right) \frac{\sigma_{I1}^{[g+1]0} \sigma_{I2}^{[g+1]z}}{4} \\ &+ m_I^{[g]} \frac{\sigma_{I1}^{[g+1]z} \sigma_{I2}^{[g+1]z}}{4} + \left(c^{[g, I]} \tilde{c}^{[g, I]} - m_I^{[g]} s^{[g, I]} \tilde{s}^{[g, I]} \right) \frac{\sigma_{I1}^{[g+1]x} \sigma_{I2}^{[g+1]x}}{4} + \left(s^{[g, I]} \tilde{s}^{[g, I]} - m_I^{[g]} c^{[g, I]} \tilde{c}^{[g, I]} \right) \frac{\sigma_{I1}^{[g+1]y} \sigma_{I2}^{[g+1]y}}{4} \end{aligned} \quad (63)$$

The identification with the parametrization of Eq. 61 yields the following affine recursions for the magnetizations in terms of the coefficients $\lambda_{i_{g+1}}^{[g, I]a=0,z}$ introduced in Eqs 45 and 48

$$m_{I_{g+1}}^{[g+1]} = \lambda_{i_{g+1}}^{[g, I]0} + \lambda_{i_{g+1}}^{[g, I]z} m_I^{[g]} \quad (64)$$

and gives how the correlations between the two children of the same ancestor appear in terms of the coefficients $\lambda_{i_{g+1}}^{[g,I]a=x,y}$ introduced in Eqs 35 and 39

$$\begin{aligned} C_{I1,I2}^{[g+1]x} &= \lambda_1^{[g,I]x} \lambda_2^{[g,I]x} - \lambda_1^{[g,I]y} \lambda_2^{[g,I]y} m_I^{[g]} \\ C_{I1,I2}^{[g+1]y} &= \lambda_1^{[g,I]y} \lambda_2^{[g,I]y} - \lambda_1^{[g,I]x} \lambda_2^{[g,I]x} m_I^{[g]} \\ C_{I1,I2}^{[g+1]z} &= m_I^{[g]} \end{aligned} \quad (65)$$

The application of the descending superoperator $\mathcal{D}^{[g]}$ to the reduced density matrix $\rho_{I,I'}^{[g]}$ of two different sites $I \neq I'$ of generation g of Eq. 61 will produce the four-sites reduced density matrix for their children $(I1, I2)$ and $(I'1, I'2)$ of generation $(g+1)$

$$\begin{aligned} \rho_{I1,I2,I'1,I'2}^{[g+1]P} &= \mathcal{D}^{[g]}[\rho_{I,I'}^{[g]P}] \\ &= \frac{1}{4} \left[\mathcal{D}^{[g,I]}[\sigma_I^{[g]0}] \mathcal{D}^{[g,I']}[\sigma_{I'}^{[g]0}] + m_I^{[g]} \mathcal{D}^{[g,I]}[\sigma_I^{[g]z}] \mathcal{D}^{[g,I']}[\sigma_{I'}^{[g]0}] + m_{I'}^{[g]} \mathcal{D}^{[g,I]}[\sigma_I^{[g]0}] \mathcal{D}^{[g,I']}[\sigma_{I'}^{[g]z}] \right] \\ &\quad + \frac{1}{4} \left[C_{I,I'}^{[g]z} \mathcal{D}^{[g,I]}[\sigma_I^{[g]z}] \mathcal{D}^{[g,I']}[\sigma_{I'}^{[g]z}] + C_{I,I'}^{[g]x} \mathcal{D}^{[g,I]}[\sigma_I^{[g]x}] \mathcal{D}^{[g,I']}[\sigma_{I'}^{[g]x}] + C_{I,I'}^{[g]y} \mathcal{D}^{[g,I]}[\sigma_I^{[g]y}] \mathcal{D}^{[g,I']}[\sigma_{I'}^{[g]y}] \right] \end{aligned} \quad (66)$$

One then needs to take the trace over one child in each block to obtain the reduced density matrices of the two remaining children

$$\begin{aligned} \rho_{I1,I'1}^{[g+1]} &= \text{Tr}_{\{I2,I'2\}} \left(\mathcal{D}^{[g]}[\rho_{I,I'}^{[g]}] \right) \\ \rho_{I1,I'2}^{[g+1]} &= \text{Tr}_{\{I2,I'1\}} \left(\mathcal{D}^{[g]}[\rho_{I,I'}^{[g]}] \right) \\ \rho_{I2,I'1}^{[g+1]} &= \text{Tr}_{\{I1,I'2\}} \left(\mathcal{D}^{[g]}[\rho_{I,I'}^{[g]}] \right) \\ \rho_{I2,I'2}^{[g+1]} &= \text{Tr}_{\{I1,I'1\}} \left(\mathcal{D}^{[g]}[\rho_{I,I'}^{[g]}] \right) \end{aligned} \quad (67)$$

Using the partial traces over a single child in each block computed before in Eqs 34 38 44 47, one obtains the following rules for the two-point correlations between the children of different blocks. The xx and yy correlations are governed by the following multiplicative factorized rules as long as $I \neq I'$

$$C_{Ii_{g+1},I'i'_{g+1}}^{[g+1]x} = \lambda_{i_{g+1}}^{[g,I]x} \lambda_{i'_{g+1}}^{[g,I']x} C_{I,I'}^{[g]x} \quad (68)$$

$$C_{Ii_{g+1},I'i'_{g+1}}^{[g+1]y} = \lambda_{i_{g+1}}^{[g,I]y} \lambda_{i'_{g+1}}^{[g,I']y} C_{I,I'}^{[g]y} \quad (69)$$

The recursions for the zz correlations involve four terms

$$C_{Ii_{g+1},I'i'_{g+1}}^{[g+1]z} = \lambda_{i_{g+1}}^{[g,I]0} \lambda_{i'_{g+1}}^{[g,I']0} + \lambda_{i_{g+1}}^{[g,I]z} \lambda_{i'_{g+1}}^{[g,I']0} m_I^{[g]} + \lambda_{i_{g+1}}^{[g,I]0} \lambda_{i'_{g+1}}^{[g,I']z} m_{I'}^{[g]} + \lambda_{i_{g+1}}^{[g,I]z} \lambda_{i'_{g+1}}^{[g,I']z} C_{I,I'}^{[g]z} \quad (70)$$

so that it is more convenient to consider the simpler multiplicative factorized rule satisfied by the connected correlations for $I \neq I'$ using Eq. 64

$$\left(C_{Ii_{g+1},I'i'_{g+1}}^{[g+1]z} - m_{Ii_{g+1}}^{[g+1]} m_{I'i'_{g+1}}^{[g+1]} \right) = \lambda_{i_{g+1}}^{[g,I]z} \lambda_{i'_{g+1}}^{[g,I']z} \left(C_{I,I'}^{[g]z} - m_I^{[g]} m_{I'}^{[g]} \right) \quad (71)$$

D. Explicit solutions for the one-point magnetizations

The initial condition for the magnetization at generation $g=0$ is given by the parity $P=+$ (Eq 56)

$$m^{[0]} = +1 \quad (72)$$

The first iterations of the affine recursion of Eq. 64 give for the generations $g=1$ and $g=2$

$$\begin{aligned} m_{i_1}^{[1]} &= \lambda_{i_1}^{[0]0} + \lambda_{i_1}^{[0]z} \\ m_{i_1,i_2}^{[2]} &= \lambda_{i_2}^{[1,i_1]0} + \lambda_{i_2}^{[1,i_1]z} \left(\lambda_{i_1}^{[0]0} + \lambda_{i_1}^{[0]z} \right) \end{aligned} \quad (73)$$

More generally, one obtains that the magnetization at position (i_1, \dots, i_g) of generation g reads

$$m_{i_1, \dots, i_g}^{[g]} = \lambda_{i_g}^{[g-1, (i_1, \dots, i_{g-1})]0} + \sum_{g''=0}^{g-2} \left(\prod_{g'=g''+1}^{g-1} \lambda_{i_{g'+1}}^{[g', (i_1, \dots, i_{g'})]z} \right) \lambda_{i_{g''+1}}^{[g'', (i_1, \dots, i_{g''])0} + \left(\prod_{g'=0}^{g-1} \lambda_{i_{g'+1}}^{[g', (i_1, \dots, i_{g'})]z} \right) \quad (74)$$

The first term involving a single scaling factor $\lambda_{i_g}^{[g-1, (i_1, \dots, i_{g-1})]0}$ is already enough to produce a finite magnetization, while the last term involving the g scaling factors up to the initial condition of the root will be exponentially small.

E. Explicit solutions for the two-points correlations

The xx correlations between two sites $(I, 1, i_{G-k+1}, \dots, i_G)$ and $(I, 2, i'_{G-k+1}, \dots, i'_G)$ that have their Last Common Ancestor at the position $I = (i_1, \dots, i_{G-k-1})$ of generation $g = G - k - 1$ and that are thus at distance $r_k = 2^k$ (Eq 14) on the tree satisfy the recursion of Eq 68 as long as they are apart

$$C_{(I,1,i_{G-k+1},\dots,i_G),(I,2,i'_{G-k+1},\dots,i'_G)}^{[G]x} = \left(\prod_{g'=G-k}^{G-1} \lambda_{i_{g'+1}}^{[g',(I,1,i_{G-k+1},\dots,i_{g'})]x} \lambda_{i'_{g'+1}}^{[g',(I,2,i'_{G-k+1},\dots,i'_{g'})]x} \right) C_{(I1),(I2)}^{[G-k]x} \quad (75)$$

while the remaining correlation at generation $(G-k)$ is given by Eq. 65 in terms of the magnetization $m_{I=(i_1, \dots, i_{G-k-1})}^{[G-k-1]}$ of their Last Common Ancestor

$$C_{I1,I2}^{[G-k]x} = \lambda_1^{[G-k-1,I]x} \lambda_2^{[G-k-1,I]x} - \lambda_1^{[G-k-1,I]y} \lambda_2^{[G-k-1,I]y} m_I^{[G-k-1]} \quad (76)$$

and will thus be finite. As a consequence, the decay of the correlation of Eq. 75 with respect to the distance $r = 2^k$ will be governed by the two strings of the k scaling factors $\lambda_{i_{g'+1}}^{[g', \cdot]x}$ leading to their Last Common Ancestor.

Similarly, the yy correlations are given by

$$C_{(I,1,i_{G-k+1},\dots,i_G),(I,2,i'_{G-k+1},\dots,i'_G)}^{[G]y} = \left(\prod_{g'=G-k}^{G-1} \lambda_{i_{g'+1}}^{[g',(I,1,i_{G-k+1},\dots,i_{g'})]y} \lambda_{i'_{g'+1}}^{[g',(I,2,i'_{G-k+1},\dots,i'_{g'})]y} \right) C_{(I1),(I2)}^{[G-k]y} \quad (77)$$

with

$$C_{I1,I2}^{[G-k]y} = \lambda_1^{[G-k-1,I]y} \lambda_2^{[G-k-1,I]y} - \lambda_1^{[G-k-1,I]x} \lambda_2^{[G-k-1,I]x} m_I^{[G-k-1]} \quad (78)$$

Again, the decay of the correlation of Eq. 77 with respect to the distance $r = 2^k$ will be governed by the two strings of the k scaling factors $\lambda_{i_{g'+1}}^{[g', \cdot]y}$ leading to their Last Common Ancestor.

Finally, the zz connected correlations satisfying Eq 71 read

$$\begin{aligned} & \left(C_{(I,1,i_{G-k+1},\dots,i_G),(I,2,i'_{G-k+1},\dots,i'_G)}^{[G]z} - m_{(I,1,i_{G-k+1},\dots,i_G)}^{[G]} m_{(I,2,i'_{G-k+1},\dots,i'_G)}^{[G]} \right) \\ &= \left(\prod_{g'=G-k}^{G-1} \lambda_{i_{g'+1}}^{[g',(I,1,i_{G-k+1},\dots,i_{g'})]z} \lambda_{i'_{g'+1}}^{[g',(I,2,i'_{G-k+1},\dots,i'_{g'})]z} \right) \left(C_{(I1),(I2)}^{[G-k]z} - m_{I1}^{[G-k]} m_{I2}^{[G-k]} \right) \end{aligned} \quad (79)$$

where the remaining connected correlation at generation $(G-k)$ is given by Eqs 65 and 64

$$C_{I1,I2}^{[G-k]z} - m_{I1}^{[G-k]} m_{I2}^{[G-k]} = m_I^{[G-k-1]} - \left(\lambda_1^{[G-k-1,I]0} + \lambda_1^{[G-k-1,I]z} m_I^{[G-k-1]} \right) \left(\lambda_2^{[G-k-1,I]0} + \lambda_2^{[G-k-1,I]z} m_I^{[G-k-1]} \right) \quad (80)$$

in terms of the magnetization $m_{I=(i_1, \dots, i_{G-k-1})}^{[G-k-1]}$ of their last common ancestor.

V. ENERGY OF THE TREE-TENSOR-STATE AND OPTIMIZATION OF ITS PARAMETERS

Up to now, we have only used the Parity and the Time-Reversal symmetries to build the simplest inhomogeneous Tree-Tensor-States and analyze its general properties. In the present section, we take into account the specific form of the Hamiltonian, in order to evaluate the energy of the Tree-Tensor-State and to optimize its parameters.

A. Energy of the Tree-Tensor-State in terms of the magnetizations and the correlations at generation G

For the Dyson Hamiltonian $\mathcal{H}^{[G]}$ of Eq. 12 that contains only one-body and two-body terms, the energy of the Tree-Tensor-State $|\Psi^{[G]}\rangle$

$$\mathcal{E}^{[G]} \equiv \langle \Psi^{[G]} | \mathcal{H}^{[G]} | \Psi^{[G]} \rangle = \text{Tr}_{\{G\}}(\mathcal{H}^{[G]} \rho^{[G]}) \quad (81)$$

involves only the one-body and the two-body reduced density matrices of Eqs 58 and 61. It can be thus rewritten in terms of the magnetizations $m_I^{[G]}$ and of the two-point correlations $C_{I,I'}^{[G]a=x,y,z}$ as

$$\mathcal{E}^{[G]} = - \sum_{I=(i_1, i_2, \dots, i_G)} h_I m_I^{[G]} - \sum_{I=(i_1, i_2, \dots, i_G) < I'=(i'_1, i'_2, \dots, i'_G)} \sum_{a=x,y,z} \frac{J_{I,I'}^a}{[r(I, I')]^{1+\omega_a}} C_{I,I'}^{[G]a} \quad (82)$$

All the Tree-Tensor-State parameters are contained in the magnetizations $m_I^{[G]}$ and of the two-point correlations $C_{I,I'}^{[G]a=x,y,z}$ computed in the previous section, and one could thus try to write the optimization equations by deriving Eq. 82 with respect to the various parameters. However, in order to isolate more clearly the role of each parameter, it is more convenient to consider how the energy of the Tree-Tensor-State of Eq. 82 can be rewritten in terms of the variables associated to any other generation g .

B. Energy of the Tree-Tensor-State in terms of the properties at generation g

Using the recurrence for the density matrices of Eq. 54, the energy of Eq. 81 can be rewritten via the duality between ascending and descending operator as the energy for the spin chain at generation $(G-1)$

$$\mathcal{E}^{[G]} = \text{Tr}_{\{G\}} \left(\mathcal{H}^{[G]} \mathcal{D}^{[G-1]} \left[\rho^{[G-1]} \right] \right) = \text{Tr}_{\{G-1\}}(\mathcal{H}^{[G-1]} \rho^{[G-1]}) \equiv \mathcal{E}^{[G-1]} \quad (83)$$

of the renormalized Hamiltonian

$$\mathcal{H}^{[G-1]} = \mathcal{A}^{[G-1]} \left[\mathcal{H}^{[G]} \right] = (W^{[G-1]})^\dagger \mathcal{H}^{[G]} W^{[G-1]} \quad (84)$$

obtained via the application of the ascending superoperator $\mathcal{A}^{[G-1]}$ to the initial Hamiltonian $\mathcal{H}^{[G]}$. More generally, it is convenient to introduce the renormalized Hamiltonian $\mathcal{H}^{[g]}$ at any generation g via the recurrence

$$\mathcal{H}^{[g]} = \mathcal{A}^{[g]}[\mathcal{H}^{[g+1]}] \quad (85)$$

Since the ascending superoperators preserve the Parity and Time-Reversal symmetries, the renormalized Hamiltonian $\mathcal{H}^{[g]}$ can be parametrized in terms renormalized fields $h_I^{[g]}$ and renormalized couplings $J_{I,I'}^{[g]a}$, as well as a constant term $E^{[g]}$

$$\mathcal{H}^{[g]} = E^{[g]} - \sum_{I=(i_1, i_2, \dots, i_g)} h_I^{[g]} \sigma_I^{[g]z} - \sum_{I=(i_1, i_2, \dots, i_g) < I'=(i'_1, i'_2, \dots, i'_g)} \sum_{a=x,y,z} \frac{J_{I,I'}^{[g]a}}{[r(I, I')]^{1+\omega_a}} \sigma_I^{[g]a} \sigma_{I'}^{[g]a} \quad (86)$$

The energy of Eq. 81 can be rewritten as the energy of this renormalized Hamiltonian $\mathcal{H}^{[g]}$

$$\mathcal{E}^{[G]} = \mathcal{E}^{[g]} \equiv \text{Tr}_{\{g\}}(\mathcal{H}^{[g]} \rho^{[g]}) = E^{[g]} - \sum_{I=(i_1, i_2, \dots, i_g)} h_I^{[g]} m_I^{[g]} - \sum_{I=(i_1, i_2, \dots, i_g) < I'=(i'_1, i'_2, \dots, i'_g)} \sum_{a=x,y,z} \frac{J_{I,I'}^{[g]a}}{[r(I, I')]^{1+\omega_a}} C_{I,I'}^{[g]a} \quad (87)$$

in terms of the magnetizations $m_I^{[g]}$ and correlations $C_{I,I'}^{[g]a=x,y,z}$ of the generation g .

C. Renormalization rules for the parameters of the Hamiltonian

The renormalization rules for the parameters of the renormalized Hamiltonian of Eq. 86 can be derived via the application of the ascending superoperator (Eq. 85). Here to stress the duality with the recursions for the magnetizations and correlations derived in the previous section, it will be more instructive the use instead the identification

between the energy computed at generation g with Eq. 87 and the energy computed at generation $(g + 1)$

$$\begin{aligned} \mathcal{E}^{[G]} = \mathcal{E}^{[g+1]} &= E^{[g+1]} - \sum_{I=(i_1, i_2, \dots, i_g)} \sum_{i_{g+1}} h_{I i_{g+1}}^{[g+1]} m_{I i_{g+1}}^{[g+1]} - \sum_{I=(i_1, i_2, \dots, i_g)} \sum_{a=x, y, z} J_{I_1, I_2}^{[g+1]a} C_{I_1, I_2}^{[g+1]a} \\ &- \sum_{I=(i_1, i_2, \dots, i_g) < I'=(i'_1, i'_2, \dots, i'_g)} \sum_{i_{g+1}} \sum_{i'_{g+1}} \sum_{a=x, y, z} \frac{J_{I i_{g+1}, I' i'_{g+1}}^{[g+1]a}}{[2r(I, I')]^{1+\omega_a}} C_{I i_{g+1}, I' i'_{g+1}}^{[g+1]a} \end{aligned} \quad (88)$$

Plugging the recursions for the magnetization $m_{I i_{g+1}}^{[g+1]}$ (Eq 64) and for the correlations in the same block $C_{I_1, I_2}^{[g+1]a}$ (Eq 65) or in two different blocks (Eqs 68, 77 and 70) into Eq. 88

$$\begin{aligned} \mathcal{E}^{[g+1]} = E^{[g+1]} &- \sum_{I=(i_1, i_2, \dots, i_g)} \left[\sum_{i_{g+1}} h_{I i_{g+1}}^{[g+1]} (\lambda_{i_{g+1}}^{[g, I]0} + \lambda_{i_{g+1}}^{[g, I]z} m_I^{[g]}) + \sum_{I=(i_1, \dots, i_g)} J_{I_1, I_2}^{[g+1]z} m_I^{[g]} \right] \\ &- \sum_{I=(i_1, \dots, i_g)} \left[J_{I_1, I_2}^{[g+1]x} (\lambda_1^{[g, I]x} \lambda_2^{[g, I]x} - \lambda_1^{[g, I]y} \lambda_2^{[g, I]y}) m_I^{[g]} + J_{I_1, I_2}^{[g+1]y} (\lambda_1^{[g, I]y} \lambda_2^{[g, I]y} - \lambda_1^{[g, I]x} \lambda_2^{[g, I]x}) m_I^{[g]} \right] \\ &- \sum_{I=(i_1, \dots, i_g) < I'=(i'_1, i'_2, \dots, i'_g)} \sum_{i_{g+1}} \sum_{i'_{g+1}} \left[\frac{J_{I i_{g+1}, I' i'_{g+1}}^{[g+1]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i_{g+1}}^{[g, I]x} \lambda_{i'_{g+1}}^{[g, I']x} C_{I, I'}^{[g]x} + \frac{J_{I i_{g+1}, I' i'_{g+1}}^{[g+1]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i_{g+1}}^{[g, I]y} \lambda_{i'_{g+1}}^{[g, I']y} C_{I, I'}^{[g]y} \right] \\ &- \sum_{I=(i_1, \dots, i_g) < I'=(i'_1, \dots, i'_g)} \sum_{i_{g+1}} \sum_{i'_{g+1}} \frac{J_{I i_{g+1}, I' i'_{g+1}}^{[g+1]z}}{[2r(I, I')]^{1+\omega_z}} (\lambda_{i_{g+1}}^{[g, I]0} \lambda_{i'_{g+1}}^{[g, I']0} + \lambda_{i_{g+1}}^{[g, I]z} \lambda_{i'_{g+1}}^{[g, I']z} m_I^{[g]} + \lambda_{i_{g+1}}^{[g, I]0} \lambda_{i'_{g+1}}^{[g, I']z} m_{I'}^{[g]} + \lambda_{i_{g+1}}^{[g, I]z} \lambda_{i'_{g+1}}^{[g, I']z} C_{I, I'}^{[g]z}) \end{aligned} \quad (89)$$

one obtains via the identification with Eq. 87 the following renormalization rules.

The renormalized couplings $J_{I, I'}^{[g]a}$ are simply given by linear combinations of the four corresponding couplings between their children $(I1, I2)$ and $(I'1, I'2)$ of generation $(g + 1)$

$$J_{I, I'}^{[g]a} = \sum_{i_{g+1}=1, 2} \sum_{i'_{g+1}=1, 2} \frac{\lambda_{i_{g+1}}^{[g, I]a} \lambda_{i'_{g+1}}^{[g, I']a}}{2^{1+\omega_a}} J_{I i_{g+1}, I' i'_{g+1}}^{[g+1]a} \quad (90)$$

As a consequence, if some coupling component $a = x, y, z$ is not present in the initial Hamiltonian, it will not be generated via renormalization.

The renormalized field $h_I^{[g]}$ involves local terms coming from the two fields $h_{I_1}^{[g+1]}$ and $h_{I_2}^{[g+1]}$ of its children and the three couplings between its two children $J_{I_1, I_2}^{[g+1]a=x, y, z}$, but also long-ranged terms coming from all the z-couplings between one child $(I1)$ or $(I2)$ with other children from other blocks $I' \neq I$:

$$\begin{aligned} h_I^{[g]} &= h_{I_1}^{[g+1]} \lambda_1^{[g, I]z} + h_{I_2}^{[g+1]} \lambda_2^{[g, I]z} + J_{I_1, I_2}^{[g+1]z} - J_{I_1, I_2}^{[g+1]x} \lambda_1^{[g, I]y} \lambda_2^{[g, I]y} - J_{I_1, I_2}^{[g+1]y} \lambda_1^{[g, I]x} \lambda_2^{[g, I]x} \\ &+ \sum_{I'=(i'_1, i'_2, \dots, i'_g) > I} \sum_{i_{g+1}} \sum_{i'_{g+1}} \frac{J_{I i_{g+1}, I' i'_{g+1}}^{[g+1]z}}{[2r(I, I')]^{1+\omega_z}} \lambda_{i_{g+1}}^{[g, I]z} \lambda_{i'_{g+1}}^{[g, I']0} + \sum_{I'=(i'_1, i'_2, \dots, i'_g) < I} \sum_{i_{g+1}} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I i_{g+1}}^{[g+1]z}}{[2r(I, I')]^{1+\omega_z}} \lambda_{i'_{g+1}}^{[g, I']0} \lambda_{i_{g+1}}^{[g, I]z} \end{aligned} \quad (91)$$

As a consequence, even if the fields are not present in the initial Hamiltonian, they will be generated via renormalization. In addition, the presence of z-couplings $J_{I, I'}^{[G]z}$ in the initial Hamiltonian leads to qualitatively different RG rules with non-local contributions, while if the z couplings vanish $J_{I, I'}^{[G]z} = 0$, the RG rule for the field $h_I^{[g]}$ is fully local and only involves the fields and couplings of its two children.

Finally, the renormalization of the constant contribution involves the random fields, the x-couplings and the y-couplings between the two spins $(I1, I2)$ of the blocks, as well as the z-couplings between spins $(I1, I2)$ and $(I'1, I'2)$ belonging to different blocks $I < I'$

$$\begin{aligned} E^{[g]} &= E^{[g+1]} - \sum_{I=(i_1, i_2, \dots, i_g)} \left(h_{I_1}^{[g+1]} \lambda_1^{[g, I]0} + h_{I_2}^{[g+1]} \lambda_2^{[g, I]0} + J_{I_1, I_2}^{[g+1]x} \lambda_1^{[g, I]x} \lambda_2^{[g, I]x} + J_{I_1, I_2}^{[g+1]y} \lambda_1^{[g, I]y} \lambda_2^{[g, I]y} \right) \\ &- \sum_{I=(i_1, i_2, \dots, i_g) < I'=(i'_1, i'_2, \dots, i'_g)} \sum_{i_{g+1}=1, 2} \sum_{i'_{g+1}=1, 2} \frac{J_{I i_{g+1}, I' i'_{g+1}}^{[g+1]z}}{[2r(I, I')]^{1+\omega_z}} \lambda_{i_{g+1}}^{[g, I]0} \lambda_{i'_{g+1}}^{[g, I']0} \end{aligned} \quad (92)$$

So again the presence of z-couplings $J_{\cdot\cdot}^{[G]z}$ in the initial Hamiltonian leads to qualitatively different RG rules with non-local contributions.

D. How the energy of the Tree-Tensor-State depends on the parameters of generation g

We have seen above how the energy of the Tree-Tensor-State can be computed at any generation $g = G, G-1, \dots, 0$

$$\mathcal{E}^{[G]} = \mathcal{E}^{[g]} \quad (93)$$

At generation $g = G$, the energy of Eq. 82 involves the fields and couplings of the initial Hamiltonian $\mathcal{H}^{[G]}$, while the whole dependence on the parameters of the Tree-Tensor-State is contained in the magnetization $m_I^{[G]}$ and correlations $C_{I,I'}^{[G]a}$ for the $N_G = 2^G$ spins of generation G . At generation $g = 0$ where there is a single spin left with magnetization $m^{[0]} = 1$, the energy involves instead the constant term $E^{[0]}$ and the renormalized field $h^{[0]}$ obtained at the end of the renormalization procedure for the Hamiltonian

$$\mathcal{E}^{[G]} = \mathcal{E}^{[0]} = E^{[0]} - h^{[0]} \quad (94)$$

So here the whole dependence on the parameters of the Tree-Tensor-State is contained in the renormalized parameters of the Hamiltonian at generation $g = 0$.

At any intermediate generation $g = 1, \dots, G-1$, the dependence on the parameters of the Tree-Tensor-State of the energy $\mathcal{E}^{[g]}$ of Eq. 87 is divided in two parts : the magnetizations $m_I^{[g]}$ and the correlations $C_{I,I'}^{[g]a}$ of generation g only involve the Tree-Tensor-State parameters of smaller generations $g' = 0, \dots, g-1$, while the renormalized parameters $(E^{[g]}, h_I^{[g]}, \mathcal{J}_{I,I'}^{[g]a})$ of the Hamiltonian of generation g only involve the Tree-Tensor-State parameters of bigger generations $g' = g, \dots, G-1$.

As a consequence, the dependence of the energy with respect to the parameters of generation g can be seen in Eq. 89 via the scaling factors $\lambda_{i_{g+1}=1,2}^{[g,I]a=0,x,y,z}$ of generation g , while the dependence with respect to smaller generations $g' = 0, \dots, g-1$ is contained in the magnetizations $m_I^{[g]}$ and the correlations $C_{I,I'}^{[g]a}$ of generation g , and the dependence with respect to bigger generations $g' = g+1, \dots, G-1$ is contained in the renormalized parameters of the Hamiltonian of generation $(g+1)$.

Since the general case with the the z-couplings leads to somewhat heavy expressions for the disordered models described by inhomogeneous Tree-Tensor-States, it is more instructive to focus now on the simpler models without z-couplings, while we will return to the general case with the three type of couplings $a = x, y, z$ in the next sections concerning pure models.

E. Optimization of the parameters of the Tree-Tensor-State for the case without z-couplings

For the case without z-couplings, Eq. 89 yields the following optimization equation with respect to the angle $\phi^{[g,I]}$ of Eqs 30

$$\begin{aligned} 0 = & \frac{\partial \mathcal{E}^{[g+1]}}{\partial \phi^{[g,I]}} = -h_{I_1}^{[g+1]} \left(\frac{\partial \lambda_1^{[g,I]0}}{\partial \phi^{[g,I]}} + \frac{\partial \lambda_1^{[g,I]z}}{\partial \phi^{[g,I]}} m_I^{[g]} \right) - h_{I_2}^{[g+1]} \left(\frac{\partial \lambda_2^{[g,I]0}}{\partial \phi^{[g,I]}} + \frac{\partial \lambda_2^{[g,I]z}}{\partial \phi^{[g,I]}} m_I^{[g]} \right) \\ & - J_{I_1, I_2}^{[g+1]x} \left(\frac{\partial (\lambda_1^{[g,I]x} \lambda_2^{[g,I]x})}{\partial \phi^{[g,I]}} - \frac{\partial (\lambda_1^{[g,I]y} \lambda_2^{[g,I]y})}{\partial \phi^{[g,I]}} m_I^{[g]} \right) - J_{I_1, I_2}^{[g+1]y} \left(\frac{\partial (\lambda_1^{[g,I]y} \lambda_2^{[g,I]y})}{\partial \phi^{[g,I]}} - \frac{\partial (\lambda_1^{[g,I]x} \lambda_2^{[g,I]x})}{\partial \phi^{[g,I]}} m_I^{[g]} \right) \\ & - \sum_{i_{g+1}} \frac{\partial \lambda_{i_{g+1}}^{[g,I]x}}{\partial \phi^{[g,I]}} \left[\sum_{I' > I} \sum_{i'_{g+1}} \frac{J_{I_{g+1}, I' i'_{g+1}}^{[g+1]x} C_{I, I'}^{[g]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i'_{g+1}}^{[g, I']x} + \sum_{I' < I} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I_{g+1}}^{[g+1]x} C_{I', I}^{[g]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i'_{g+1}}^{[g, I']x} \right] \\ & - \sum_{i_{g+1}} \frac{\partial \lambda_{i_{g+1}}^{[g,I]y}}{\partial \phi^{[g,I]}} \left[\sum_{I' > I} \sum_{i'_{g+1}} \frac{J_{I_{g+1}, I' i'_{g+1}}^{[g+1]y} C_{I, I'}^{[g]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i'_{g+1}}^{[g, I']y} + \sum_{I' < I} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I_{g+1}}^{[g+1]y} C_{I', I}^{[g]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i'_{g+1}}^{[g, I']y} \right] \end{aligned} \quad (95)$$

and the analogous equation with respect to the angle $\tilde{\phi}^{[g,I]}$.

Using the explicit forms of the scaling factors $\lambda_{i_{g+1}=1,2}^{[g,I]a=0,x,y,z}$ of Eqs 35 39 45 48, one obtains the optimization equation with respect to the angle $\phi^{[g,I]}$

$$\begin{aligned}
0 = \frac{\partial \mathcal{E}^{[g+1]}}{\partial \phi^{[g,I]}} &= -h_{I_1}^{[g+1]} \left(c^{[g,I]} \tilde{c}^{[g,I]} - s^{[g,I]} \tilde{s}^{[g,I]} m_I^{[g]} \right) - h_{I_2}^{[g+1]} \left(-s^{[g,I]} \tilde{s}^{[g,I]} + c^{[g,I]} \tilde{c}^{[g,I]} m_I^{[g]} \right) \\
&+ J_{I_1, I_2}^{[g+1]x} \left(s^{[g,I]} \tilde{c}^{[g,I]} + c^{[g,I]} \tilde{s}^{[g,I]} m_I^{[g]} \right) - J_{I_1, I_2}^{[g+1]y} \left(\tilde{s}^{[g,I]} c^{[g,I]} + s^{[g,I]} \tilde{c}^{[g,I]} m_I^{[g]} \right) \\
&+ s^{[g,I]} \left[\sum_{I' > I} \sum_{i'_{g+1}} \frac{J_{I_1, I' i'_{g+1}}^{[g+1]x} C_{I, I'}^{[g]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i'_{g+1}}^{[g, I']x} + \sum_{I' < I} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I_1}^{[g+1]x} C_{I', I}^{[g]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i'_{g+1}}^{[g, I']x} \right] \\
&- c^{[g,I]} \left[\sum_{I' > I} \sum_{i'_{g+1}} \frac{J_{I_2, I' i'_{g+1}}^{[g+1]y} C_{I, I'}^{[g]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i'_{g+1}}^{[g, I']y} + \sum_{I' < I} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I_2}^{[g+1]y} C_{I', I}^{[g]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i'_{g+1}}^{[g, I']y} \right]
\end{aligned} \tag{96}$$

and the optimization equation with respect to the angle $\tilde{\phi}^{[g,I]}$

$$\begin{aligned}
0 = \frac{\partial \mathcal{E}^{[g+1]}}{\partial \tilde{\phi}^{[g,I]}} &= -h_{I_1}^{[g+1]} \left(-s^{[g,I]} \tilde{s}^{[g,I]} + c^{[g,I]} \tilde{c}^{[g,I]} m_I^{[g]} \right) - h_{I_2}^{[g+1]} \left(c^{[g,I]} \tilde{c}^{[g,I]} - s^{[g,I]} \tilde{s}^{[g,I]} m_I^{[g]} \right) \\
&+ J_{I_1, I_2}^{[g+1]x} \left(c^{[g,I]} \tilde{s}^{[g,I]} + s^{[g,I]} \tilde{c}^{[g,I]} m_I^{[g]} \right) - J_{I_1, I_2}^{[g+1]y} \left(\tilde{c}^{[g,I]} s^{[g,I]} + c^{[g,I]} \tilde{s}^{[g,I]} m_I^{[g]} \right) \\
&+ \tilde{s}^{[g,I]} \left[\sum_{I' > I} \sum_{i'_{g+1}} \frac{J_{I_2, I' i'_{g+1}}^{[g+1]x} C_{I, I'}^{[g]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i'_{g+1}}^{[g, I']x} + \sum_{I' < I} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I_2}^{[g+1]x} C_{I', I}^{[g]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i'_{g+1}}^{[g, I']x} \right] \\
&- \tilde{c}^{[g,I]} \left[\sum_{I' > I} \sum_{i'_{g+1}} \frac{J_{I_1, I' i'_{g+1}}^{[g+1]y} C_{I, I'}^{[g]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i'_{g+1}}^{[g, I']y} + \sum_{I' < I} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I_1}^{[g+1]y} C_{I', I}^{[g]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i'_{g+1}}^{[g, I']y} \right]
\end{aligned} \tag{97}$$

It is simpler to replace these two equations by their sum

$$\begin{aligned}
0 &= (1 + m_I^{[g]}) \left[-(h_{I_1}^{[g+1]} + h_{I_2}^{[g+1]}) (c^{[g,I]} \tilde{c}^{[g,I]} - s^{[g,I]} \tilde{s}^{[g,I]}) + (J_{I_1, I_2}^{[g+1]x} - J_{I_1, I_2}^{[g+1]y}) (s^{[g,I]} \tilde{c}^{[g,I]} + c^{[g,I]} \tilde{s}^{[g,I]}) \right] \\
&+ (s^{[g,I]} + \tilde{s}^{[g,I]}) \left[\sum_{I' > I} \sum_{i'_{g+1}} \frac{J_{I_1, I' i'_{g+1}}^{[g+1]x} C_{I, I'}^{[g]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i'_{g+1}}^{[g, I']x} + \sum_{I' < I} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I_1}^{[g+1]x} C_{I', I}^{[g]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i'_{g+1}}^{[g, I']x} \right] \\
&- (c^{[g,I]} + \tilde{c}^{[g,I]}) \left[\sum_{I' > I} \sum_{i'_{g+1}} \frac{J_{I_2, I' i'_{g+1}}^{[g+1]y} C_{I, I'}^{[g]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i'_{g+1}}^{[g, I']y} + \sum_{I' < I} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I_2}^{[g+1]y} C_{I', I}^{[g]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i'_{g+1}}^{[g, I']y} \right]
\end{aligned} \tag{98}$$

and by their difference

$$\begin{aligned}
0 &= (1 - m_I^{[g]}) \left[-(h_{I_1}^{[g+1]} - h_{I_2}^{[g+1]}) (c^{[g,I]} \tilde{c}^{[g,I]} + s^{[g,I]} \tilde{s}^{[g,I]}) + (J_{I_1, I_2}^{[g+1]x} + J_{I_1, I_2}^{[g+1]y}) (s^{[g,I]} \tilde{c}^{[g,I]} - c^{[g,I]} \tilde{s}^{[g,I]}) \right] \\
&+ (s^{[g,I]} - \tilde{s}^{[g,I]}) \left[\sum_{I' > I} \sum_{i'_{g+1}} \frac{J_{I_1, I' i'_{g+1}}^{[g+1]x} C_{I, I'}^{[g]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i'_{g+1}}^{[g, I']x} + \sum_{I' < I} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I_1}^{[g+1]x} C_{I', I}^{[g]x}}{[2r(I, I')]^{1+\omega_x}} \lambda_{i'_{g+1}}^{[g, I']x} \right] \\
&- (c^{[g,I]} - \tilde{c}^{[g,I]}) \left[\sum_{I' > I} \sum_{i'_{g+1}} \frac{J_{I_2, I' i'_{g+1}}^{[g+1]y} C_{I, I'}^{[g]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i'_{g+1}}^{[g, I']y} + \sum_{I' < I} \sum_{i'_{g+1}} \frac{J_{I' i'_{g+1}, I_2}^{[g+1]y} C_{I', I}^{[g]y}}{[2r(I, I')]^{1+\omega_y}} \lambda_{i'_{g+1}}^{[g, I']y} \right]
\end{aligned} \tag{99}$$

It is now clear that the usual block-spin RG rules based on the diagonalization of the intra-Hamiltonian in each block (see Appendix A) are recovered if one neglects the correlations $C_{I, I'}^{[g]a=x,y} \rightarrow 0$ that correspond to the future RG steps : then Eq 98 simplifies into

$$\begin{aligned}
0 &= -(h_{I_1}^{[g+1]} + h_{I_2}^{[g+1]}) (c^{[g,I]} \tilde{c}^{[g,I]} - s^{[g,I]} \tilde{s}^{[g,I]}) + (J_{I_1, I_2}^{[g+1]x} - J_{I_1, I_2}^{[g+1]y}) (s^{[g,I]} \tilde{c}^{[g,I]} + c^{[g,I]} \tilde{s}^{[g,I]}) \\
&= -(h_{I_1}^{[g+1]} + h_{I_2}^{[g+1]}) \cos(\phi^{[g,I]} + \tilde{\phi}^{[g,I]}) + (J_{I_1, I_2}^{[g+1]x} - J_{I_1, I_2}^{[g+1]y}) \sin(\phi^{[g,I]} + \tilde{\phi}^{[g,I]})
\end{aligned} \tag{100}$$

for the sum of the two angles (Eq 30)

$$\phi^{[g,I]} + \tilde{\phi}^{[g,I]} = \frac{\pi}{2} - 2\theta^{[g,I]+} \quad (101)$$

in agreement with Eq. A5 of the Appendix. Similarly, Eq. 99 simplifies into

$$\begin{aligned} 0 &= -(h_{I_1}^{[g+1]} - h_{I_2}^{[g+1]}) \left(c^{[g,I]} \tilde{c}^{[g,I]} + s^{[g,I]} \tilde{s}^{[g,I]} \right) + \left(J_{I_1, I_2}^{[g+1]x} + J_{I_1, I_2}^{[g+1]y} \right) \left(s^{[g,I]} \tilde{c}^{[g,I]} - c^{[g,I]} \tilde{s}^{[g,I]} \right) \\ &= -(h_{I_1}^{[g+1]} - h_{I_2}^{[g+1]}) \cos(\phi^{[g,I]} - \tilde{\phi}^{[g,I]}) + \left(J_{I_1, I_2}^{[g+1]x} + J_{I_1, I_2}^{[g+1]y} \right) \sin(\phi^{[g,I]} - \tilde{\phi}^{[g,I]}) \end{aligned} \quad (102)$$

for the difference of the two angles (Eq 30)

$$\phi^{[g,I]} - \tilde{\phi}^{[g,I]} = \frac{\pi}{2} - 2\theta^{[g,I]-} \quad (103)$$

in agreement with Eq. A9 of the Appendix.

In summary, with respect to the block-spin RG rules of Eqs 100 and 102 based on the diagonalization of the renormalized intra-Hamiltonian in each block that contains the isometries of bigger generations $g' = g + 1, \dots, G - 1$, the variational optimization of the whole Tree-Tensor-State of Eqs 98 and 99 requires to take into account the magnetizations $m_I^{[g]}$ and the correlations $C_{I,I'}^{[g]a=x,y}$ that contain the isometries of smaller generations $g' = 0, \dots, g - 1$.

VI. HOMOGENEOUS TREE-TENSOR-STATES FOR THE PURE DYSON MODELS

Up to now, we have considered inhomogeneous Tree-Tensor-States for disordered spin chains. In this section, we turn to the case of pure Dyson models, where their supplementary symmetries need to be taken into account in the Tree-Tensor description.

A. Supplementary symmetries of the pure Dyson models

When the fields h_I and the couplings $J_{I,I'}^a$ of the Dyson Hamiltonian (Eqs 12 13) are uniform

$$\begin{aligned} h_I &= h \\ J_{I,I'}^a &= J^a \end{aligned} \quad (104)$$

one needs to take into account two supplementary symmetries for the choice of the isometries $w^{[g,I]}$ of Eq. 16. The first symmetry concerns the equivalence between the various branches of the tree, so that the two angles $\theta^{[g,I]\pm}$ of Eqs 19 and 20 will only depend on the generation g but not on the position $I = (i_1, \dots, i_g)$ anymore

$$\theta^{[g,I]\pm} = \theta^{[g]\pm} \quad (105)$$

The second symmetry concerns the equivalence of the two children of a given ancestor. In the parity sector $P = +$, the ket of Eq. 19 is symmetric with respect to the two children for any value of the angle $\theta^{[g],+}$. However in the parity sector $P = -$, the ket of Eq. 20 is symmetric with respect to the two children only for the value

$$\theta^{[g],-} = \frac{\pi}{4} \quad (106)$$

As a consequence, the number of parameters of Eq. 53 needed for the inhomogeneous Tree-Tensor-States corresponding to disordered models is now reduced to the number G of generations for the homogeneous Tree-Tensor-States corresponding to pure models

$$\mathcal{N}_{\text{pure}}^{\text{Parameters}} = \sum_{g=0}^{G-1} 1 = G \quad (107)$$

The two symmetries of Eq. 105 and 106 yields that the two angles of Eq. 30 now coincide and do not depend on the position I anymore

$$\phi^{[g]} = \tilde{\phi}^{[g]} = \frac{\pi}{4} - \theta^{[g]+} \quad (108)$$

so the coefficients of Eq. 31 reduce to

$$\begin{aligned} c^{[g]} &\equiv \cos\left(\phi^{[g]}\right) \equiv \tilde{c}^{[g]} \\ s^{[g]} &\equiv \sin\left(\phi^{[g]}\right) \equiv \tilde{s}^{[g]} \end{aligned} \quad (109)$$

and the scaling factors of Eqs 35 39 45 48 simplify into

$$\begin{aligned} \lambda^{[g]x} &\equiv c^{[g]} \\ \lambda^{[g]y} &\equiv s^{[g]} \\ \lambda^{[g]z} &\equiv c^{[g]}s^{[g]} \\ \lambda^{[g]0} &\equiv c^{[g]}s^{[g]} \end{aligned} \quad (110)$$

B. Explicit solution for the one-point magnetizations

The magnetization now only depends on the generation g . The recursion of Eq. 64 simplifies into

$$m^{[g+1]} = c^{[g]}s^{[g]}(1 + m^{[g]}) \quad (111)$$

and the solution of Eq. 74 reduces to

$$\begin{aligned} m^{[g]} &= \sum_{g''=0}^{g-1} \left(\prod_{g'=g''}^{g-1} c^{[g']}s^{[g']} \right) + \left(\prod_{g'=0}^{g-1} c^{[g']}s^{[g']} \right) \\ &= c^{[g-1]}s^{[g-1]} + c^{[g-1]}s^{[g-1]}c^{[g-2]}s^{[g-2]} + \dots + \prod_{g'=1}^{g-1} c^{[g']}s^{[g']} + 2 \prod_{g'=0}^{g-1} c^{[g']}s^{[g']} \end{aligned} \quad (112)$$

C. Explicit solutions for the two-point correlations

The two-point correlations between two sites of generation G now only depend on the generation $g = G - k - 1$ of their Last Common Ancestor i.e. on their corresponding distance $r_k = 2^k$ on the tree. Eqs 65 give the values of the correlations at distance $r_0 = 2^0 = 1$ as a function of the magnetization given in Eq. 112

$$\begin{aligned} C_{(r_0=1)}^{[g+1]x} &= (c^{[g]})^2 - (s^{[g]})^2 m^{[g]} \\ C_{(r_0=1)}^{[g+1]y} &= (s^{[g]})^2 - (c^{[g]})^2 m^{[g]} \\ C_{(r_0=1)}^{[g+1]z} &= m^{[g]} \end{aligned} \quad (113)$$

while Eqs 68, 69 71 correspond to the following recursions for $k \geq 0$

$$\begin{aligned} C_{(2r_k=2^{k+1})}^{[g+1]x} &= (c^{[g]})^2 C_{(r_k=2^k)}^{[g]x} \\ C_{(2r_k=2^{k+1})}^{[g+1]y} &= (s^{[g]})^2 C_{(r_k=2^k)}^{[g]y} \\ \left(C_{(2r_k=2^{k+1})}^{[g+1]z} - (m^{[g+1]})^2 \right) &= (c^{[g]}s^{[g]})^2 \left(C_{(r_k=2^k)}^{[g]z} - (m^{[g]})^2 \right) \end{aligned} \quad (114)$$

The solutions of Eqs 75 77 and 79 reduce to

$$\begin{aligned} C_{(r_k=2^k)}^{[G]x} &= \left(\prod_{g'=G-k}^{G-1} (c^{[g']})^2 \right) \left[(c^{[G-k-1]})^2 - (s^{[G-k-1]})^2 m^{[G-k-1]} \right] \\ C_{(r_k=2^k)}^{[G]y} &= \left(\prod_{g'=G-k}^{G-1} (s^{[g']})^2 \right) \left[(s^{[G-k-1]})^2 - (c^{[G-k-1]})^2 m^{[G-k-1]} \right] \\ \left[C_{(r_k=2^k)}^{[G]z} - (m^{[G]})^2 \right] &= \left(\prod_{g'=G-k}^{G-1} (c^{[g']}s^{[g']})^2 \right) \left[m^{[G-k-1]} - (c^{[G-k-1]}s^{[G-k-1]})^2 (1 + m^{[G-k-1]})^2 \right] \end{aligned} \quad (115)$$

D. Energy of the homogeneous Tree-Tensor-State and optimization of its G parameters

For the pure Dyson model of Eq. 15 where the magnetization $m^{[G]}$ depends only on the generation and where the correlation depends only on the generation G and on the distance $r_k = 2^k$, the energy of the Tree-Tensor-State of Eq. 82 becomes

$$\mathcal{E}^{[G]} = -2^G h m^{[G]} - 2^{G-1} \sum_{k=0}^{G-1} \sum_{a=x,y,z} \frac{J^a}{2^{k\omega_a}} C_{(r_k=2^k)}^{[G]a} \quad (116)$$

while the equivalent computation of the energy at any generation g (Eq 87) reads similarly

$$\mathcal{E}^{[g]} = E^{[g]} - 2^g h^{[g]} m^{[g]} - 2^{g-1} \sum_{k=0}^{g-1} \sum_{a=x,y,z} \frac{J^{[g]a}}{2^{k\omega_a}} C_{(r_k=2^k)}^{[g]a} \quad (117)$$

in terms of the parameters $(E^{[g]}, h^{[g]}, J^{[g]a})$ of the renormalized Hamiltonian.

The dependence with respect to the parameter of the generation g can be obtained from the energy computed at generation $(g+1)$ when the magnetization and the correlations of generation $(g+1)$ are written in terms of the magnetizations and the correlations of generation g via the recursions of Eqs 111, 113 and 114

$$\begin{aligned} \mathcal{E}^{[g+1]} &= E^{[g+1]} - 2^{g+1} h^{[g+1]} m^{[g+1]} - 2^g \sum_{a=x,y,z} J^{[g+1]a} C_{(r_0=1)}^{[g+1]a} - 2^g \sum_{k=0}^{g-1} \sum_{a=x,y,z} \frac{J^{[g+1]a}}{2^{\omega_a} 2^{k\omega_a}} C_{(r_{k+1}=2^{k+1})}^{[g+1]a} \quad (118) \\ &= E^{[g+1]} - 2^{g+1} h^{[g+1]} c^{[g]} s^{[g]} (1 + m^{[g]}) - 2^g \left[J^{[g+1]x} \left((c^{[g]})^2 - (s^{[g]})^2 m^{[g]} \right) + J^{[g+1]y} \left((s^{[g]})^2 - (c^{[g]})^2 m^{[g]} \right) + J^{[g+1]z} m^{[g]} \right] \\ &\quad - 2^{g-1} \sum_{k=0}^{g-1} \left[\frac{2^{1-\omega_x} (c^{[g]})^2 J^{[g+1]x}}{2^{k\omega_x}} C_{(r_k=2^k)}^{[g]x} + \frac{2^{1-\omega_y} (s^{[g]})^2 J^{[g+1]y}}{2^{k\omega_y}} C_{(r_k=2^k)}^{[g]y} + \frac{2^{1-\omega_z} (c^{[g]} s^{[g]})^2 J^{[g+1]z}}{2^{k\omega_z}} \left(1 + 2m^{[g]} + C_{(r_k=2^k)}^{[g]z} \right) \right] \end{aligned}$$

The identification with Eq. 87 yields the renormalization rules for the couplings (instead of Eq. 90)

$$\begin{aligned} J^{[g]x} &= 2^{1-\omega_x} (c^{[g]})^2 J^{[g+1]x} \\ J^{[g]y} &= 2^{1-\omega_y} (s^{[g]})^2 J^{[g+1]y} \\ J^{[g]z} &= 2^{1-\omega_z} (c^{[g]} s^{[g]})^2 J^{[g+1]z} \end{aligned} \quad (119)$$

for the field (instead of Eq. 91)

$$h^{[g]} = 2c^{[g]} s^{[g]} h^{[g+1]} - (s^{[g]})^2 J^{[g+1]x} - (c^{[g]})^2 J^{[g+1]y} + J^{[g+1]z} \left[1 + 2^{1-\omega_z} (c^{[g]} s^{[g]})^2 \sum_{k=0}^{g-1} \frac{1}{2^{k\omega_z}} \right] \quad (120)$$

and for the constant term (instead of Eq. 92)

$$E^{[g]} = E^{[g+1]} - 2^g \left[2c^{[g]} s^{[g]} h^{[g+1]} + (c^{[g]})^2 J^{[g+1]x} + (s^{[g]})^2 J^{[g+1]y} + 2^{-\omega_z} (c^{[g]} s^{[g]})^2 J^{[g+1]z} \sum_{k=0}^{g-1} \frac{1}{2^{k\omega_z}} \right] \quad (121)$$

Eq 118 also gives the explicit dependence of the energy with respect to the angle $\phi^{[g]}$ associated to the generation g

$$\begin{aligned} 2^{-g} \mathcal{E}^{[g+1]} &= 2^{-g} E^{[g+1]} - h^{[g+1]} (1 + m^{[g]}) \sin(2\phi^{[g]}) - J^{[g+1]z} m^{[g]} \\ &\quad - J^{[g+1]x} \left(\frac{(1 - m^{[g]}) + (1 + m^{[g]}) \cos(2\phi^{[g]})}{2} \right) - J^{[g+1]y} \left(\frac{(1 - m^{[g]}) - (1 + m^{[g]}) \cos(2\phi^{[g]})}{2} \right) \\ &\quad - 2^{-1-\omega_x} [1 + \cos(2\phi^{[g]})] J^{[g+1]x} \sum_{k=0}^{g-1} \frac{C_{(r_k=2^k)}^{[g]x}}{2^{k\omega_x}} - 2^{-1-\omega_y} [1 - \cos(2\phi^{[g]})] J^{[g+1]y} \sum_{k=0}^{g-1} \frac{C_{(r_k=2^k)}^{[g]y}}{2^{k\omega_y}} \\ &\quad - 2^{-2-\omega_z} \sin^2(2\phi^{[g]}) J^{[g+1]z} \sum_{k=0}^{g-1} \left[\frac{(1 + 2m^{[g]} + C_{(r_k=2^k)}^{[g]z})}{2^{k\omega_z}} \right] \end{aligned}$$

The optimization equation with respect to the angle $\phi^{[g]}$ reads

$$\begin{aligned}
0 = \frac{\partial(2^{-g}\mathcal{E}^{[g+1]})}{\partial\phi^{[g]}} &= (1 + m^{[g]}) \left[-2h^{[g+1]} \cos(2\phi^{[g]}) + (J^{[g+1]x} - J^{[g+1]y}) \sin(2\phi^{[g]}) \right] \\
&+ \sin(2\phi^{[g]}) \left[2^{-\omega_x} J^{[g+1]x} \sum_{k=0}^{g-1} \frac{C_{(r_k=2^k)}^{[g]x}}{2^{k\omega_x}} - 2^{-\omega_y} J^{[g+1]y} \sum_{k=0}^{g-1} \frac{C_{(r_k=2^k)}^{[g]y}}{2^{k\omega_y}} \right] \\
&- \cos(2\phi^{[g]}) \sin(2\phi^{[g]}) 2^{-1-\omega_z} J^{[g+1]z} \sum_{k=0}^{g-1} \left[\frac{\left(1 + 2m^{[g]} + C_{(r_k=2^k)}^{[g]z}\right)}{2^{k\omega_z}} \right]
\end{aligned} \tag{122}$$

If one neglects the contributions of the second and third lines, the first line allows to recover the usual criterion based on the diagonalization of the intra-Hamiltonian in each block (Eq A5) for the angle (Eq 108)

$$2\phi^{[g]} = \frac{\pi}{2} - 2\theta^{[g]+} \tag{123}$$

VII. SCALE-INVARIANT TREE-TENSOR-STATES FOR THE CRITICAL PURE DYSON MODELS

In this section, we focus on the possible critical points of pure Dyson models, where the corresponding homogeneous Tree-Tensor-State of the last section becomes in addition scale invariant.

A. Supplementary symmetry : scale invariance

At the critical points of the pure Dyson models discussed in the previous section, the scale invariance means that the isometries do not even depend on the generation g anymore, so that the only remaining parameter is the angle θ^+ or the angle

$$\phi = \frac{\pi}{4} - \theta^+ \tag{124}$$

so the parameters $c^{[g]}$ and $s^{[g]}$ of the previous section do not depend of g anymore

$$\begin{aligned}
c &\equiv \cos(\phi) \\
s &\equiv \sin(\phi)
\end{aligned} \tag{125}$$

B. Explicit solutions for the one-point magnetizations

The magnetization of Eq. 112 reduces to

$$m^{[g]} = \frac{cs}{1 - cs} + (cs)^g \left(\frac{1 - 2cs}{1 - cs} \right) \tag{126}$$

The dependence with respect to the generation g comes only from the finite size and from the initial condition $m^{[0]} = +1$ at generation $g = 0$. In the thermodynamic limit $g \rightarrow +\infty$, the influence of this initial condition disappears and the asymptotic magnetization is simply

$$m^{[\infty]} = \frac{cs}{1 - cs} \tag{127}$$

C. Explicit solutions for the two-point correlations

The two-point correlations of Eq. 115 simplify into

$$\begin{aligned}
C_{(r_k=2^k)}^{[G]x} &= (c^2)^k \left(c^2 - s^2 m^{[G-k-1]} \right) \\
C_{(r_k=2^k)}^{[G]y} &= (s^2)^k \left(s^2 - c^2 m^{[G-k-1]} \right) \\
\left[C_{(r_k=2^k)}^{[G]z} - (m^{[G]})^2 \right] &= (c^2 s^2)^k \left[m^{[G-k-1]} - (cs)^2 \left(1 + m^{[G-k-1]} \right)^2 \right]
\end{aligned} \tag{128}$$

Again the dependence with respect to the generation G comes only from the finite size via the magnetization $m^{[G-k-1]}$ of the Last Common Ancestor. In the thermodynamic limit $G \rightarrow +\infty$ where the asymptotic magnetization is given by Eq. 127, the two-point-correlations become simple power-laws with respect to the distance $r_k = 2^k$

$$\begin{aligned}
C_{(r_k=2^k)}^{[\infty]x} &= (c^2)^k \left(c^2 - s^2 m^{[\infty]} \right) = (c^2)^k \frac{c(c-s)}{1-cs} \equiv \frac{A_x}{r_k^{2\Delta_x}} \\
C_{(r_k=2^k)}^{[\infty]y} &= (s^2)^k \left(s^2 - c^2 m^{[\infty]} \right) = (s^2)^k \frac{s(s-c)}{1-cs} \equiv \frac{A_y}{r_k^{2\Delta_y}} \\
\left[C_{(r_k=2^k)}^{[\infty]z} - (m^{[\infty]})^2 \right] &= (c^2 s^2)^k \left[m^{[\infty]} - (cs)^2 \left(1 + m^{[\infty]} \right)^2 \right] = (c^2 s^2)^k \frac{cs(1-2cs)}{(1-cs)^2} \equiv \frac{A_z}{r_k^{2\Delta_z}}
\end{aligned} \tag{129}$$

where the scaling dimensions Δ_a that govern the power-law decay with respect to the distance $r_k = 2^k$

$$\begin{aligned}
\Delta_x &= -\frac{\ln |c|}{\ln 2} \\
\Delta_y &= -\frac{\ln |s|}{\ln 2} \\
\Delta_z &= -\frac{\ln |cs|}{\ln 2}
\end{aligned} \tag{130}$$

and the amplitudes

$$\begin{aligned}
A_x &= \frac{c(c-s)}{1-cs} \\
A_y &= \frac{s(s-c)}{1-cs} \\
A_z &= \frac{cs(1-2cs)}{(1-cs)^2}
\end{aligned} \tag{131}$$

depend only on the angle ϕ .

D. Scale-invariance of the renormalized Hamiltonian with the dynamical exponent z

The renormalization rules for the couplings (Eqs 119) become

$$\begin{aligned}
J^{[g]x} &= 2^{1-\omega_x} c^2 J^{[g+1]x} \\
J^{[g]y} &= 2^{1-\omega_y} s^2 J^{[g+1]y} \\
J^{[g]z} &= 2^{1-\omega_z} (cs)^2 J^{[g+1]z}
\end{aligned} \tag{132}$$

while the renormalization rule for the field (Eq 120) reads (when the thermodynamic limit is taken in the last sum)

$$h^{[g]} = 2csh^{[g+1]} - s^2 J^{[g+1]x} - c^2 J^{[g+1]y} + J^{[g+1]z} \left[1 + 2^{1-\omega_z} (cs)^2 \sum_{k=0}^{\infty} \frac{1}{2^{k\omega_z}} \right] \tag{133}$$

At the critical point, the field and the couplings that do not vanish in the renormalized scale-invariant Hamiltonian should all have the same scaling dimension given by the dynamical exponent z

$$\begin{aligned} J^{[g]a} &\simeq 2^{-z} J^{[g+1]a} \\ h^{[g]} &\simeq 2^{-z} h^{[g+1]} \end{aligned} \quad (134)$$

As a consequence, the ratios $K^{[g]a} \equiv \frac{J^{[g]a}}{h^{[g]}}$ associated to the couplings surviving in the scale-invariant renormalized Hamiltonian should take fixed point values independent of the generation g

$$K^{[g]a} = \frac{J^{[g]a}}{h^{[g]}} = K^{[g+1]a} \equiv K^a \quad (135)$$

The optimization equation of Eq. 122 can be then rewritten in the thermodynamic limit $g \rightarrow +\infty$ as

$$\begin{aligned} 0 &= (1 + m^{[\infty]}) [-2 \cos(2\phi) + (K^x - K^y) \sin(2\phi)] + \sin(2\phi) \left[2^{-\omega_x} K^x \sum_{k=0}^{\infty} \frac{C_{(r_k=2^k)}^{[\infty]x}}{2^{k\omega_x}} - 2^{-\omega_y} K^y \sum_{k=0}^{\infty} \frac{C_{(r_k=2^k)}^{[\infty]y}}{2^{k\omega_y}} \right] \\ &\quad - \cos(2\phi) \sin(2\phi) 2^{-1-\omega_z} K^z \sum_{k=0}^{\infty} \left[\frac{(1 + m^{[\infty]})^2 + \left(C_{(r_k=2^k)}^{[\infty]z} - (m^{[\infty]})^2 \right)}{2^{k\omega_z}} \right] \end{aligned} \quad (136)$$

where the magnetization $m^{[\infty]}$ of Eq. 127 and the correlations of Eq. 129 only depend on the angle ϕ .

E. Critical points of the pure Dyson quantum Ising model ($K^x \neq 0$ and $K^y = 0 = K^z$)

Let us consider the critical points where the y-couplings and the z-couplings vanish in the scale-invariant renormalized Hamiltonian. This will occur either if the y-couplings and the z-couplings already vanish in the initial condition, i.e. if the initial condition corresponds to the pure Dyson quantum Ising model, or if their scaling dimensions in Eq. 132 make the two ratios $\frac{J^{[g]a=y,z}}{J^{[g]x}}$ converge towards zero via renormalization. Then the scale invariance with the dynamical exponent z of Eq. 134 yields the two conditions from Eq. 132 and 133

$$\begin{aligned} 2^{-z} &= \frac{J^{[g]x}}{J^{[g+1]x}} = 2^{1-\omega_x} c^2 \\ 2^{-z} &= \frac{h^{[g]}}{h^{[g+1]}} = 2cs - s^2 K^x \end{aligned} \quad (137)$$

while the optimization equation of Eq. 136 gives the constraint

$$0 = -2 \cos(2\phi) + K^x \sin(2\phi) \left[1 + \frac{2^{-\omega_x}}{(1 + m^{[\infty]})} \sum_{k=0}^{\infty} \frac{C_{(r_k=2^k)}^{[\infty]x}}{2^{k\omega_x}} \right] \quad (138)$$

If one neglects the correlations $C_{(r_k=2^k)}^{[\infty]x} \rightarrow 0$ in this optimization equation, one recovers Eq. A5

$$0 = -2 \cos(2\phi) + K^x \sin(2\phi) \quad (139)$$

based on diagonalization of the intra-Hamiltonian in each block (see Appendix A), and the properties of the corresponding critical point have been discussed in Refs [42, 43] as a function of the power-law exponent ω_x (called σ in Refs [42, 43]).

When the correlations $C_{(r_k=2^k)}^{[\infty]x}$ are not neglected, the line of critical points is parametrized by the four variables (ϕ, z, ω_x, K^x) related by the three equations (the two Eqs of 137 and Eq 138).

F. Critical points where $K^x \neq 0$ and $K^y \neq 0$ while $K^z = 0$

Let us now consider the case where both the x-coupling $K^x \neq 0$ and the y-coupling $K^y \neq 0$ survive in the renormalized scale invariant Hamiltonian, while the z coupling vanishes $K^z = 0$.

Then the scale invariance with the dynamical exponent z of Eq. 134 yields the following three conditions from Eq. 132 and 133

$$\begin{aligned} 2^{-z} &= \frac{J^{[g]x}}{J^{[g+1]x}} = 2^{1-\omega_x} c^2 \\ 2^{-z} &= \frac{J^{[g]y}}{J^{[g+1]y}} = 2^{1-\omega_y} s^2 \\ 2^{-z} &= \frac{h^{[g]}}{h^{[g+1]}} = 2cs - s^2 K^x - c^2 K^y \end{aligned} \quad (140)$$

while the optimization equation of Eq. 136 gives the constraint

$$0 = (1 + m^{[\infty]}) [-2 \cos(2\phi) + (K^x - K^y) \sin(2\phi)] + \sin(2\phi) \left[2^{-\omega_x} K^x \sum_{k=0}^{\infty} \frac{C_{(r_k=2^k)}^{[\infty]x}}{2^{k\omega_x}} - 2^{-\omega_y} K^y \sum_{k=0}^{\infty} \frac{C_{(r_k=2^k)}^{[\infty]y}}{2^{k\omega_y}} \right] \quad (141)$$

As a consequence of the two first equations of Eq. 137, the angle ϕ is now completely fixed by the difference between the exponents ω_x and ω_y

$$\tan^2(\phi) = \frac{s^2}{c^2} = 2^{\omega_y - \omega_x} \quad (142)$$

Then z is fixed by the two first equations of Eq. 137, then the fixed-point values $K^{x,y}$ for the x-coupling and the y-coupling are given by the solutions of the two remaining equations, namely the third equation of 137 and Eq. 141.

VIII. CONCLUSIONS

In this paper, we have analyzed the simplest Tree-Tensor-States (TTS) respecting the Parity and the Time-Reversal symmetries in order to describe the ground states of Long-Ranged Quantum Spin Chains with or without disorder.

We have first focused on inhomogeneous TTS for disordered Long-Ranged spin-chains. Explicit formulas have been given for the one-point and two-point reduced density matrices as parametrized by the magnetizations and the two-point correlations. We have then analyzed how the total energy of the TTS depend on each parameter of the TTS in order to obtain the optimization equations and to compare them with the traditional block-spin renormalization procedure based on the diagonalization of some intra-block renormalized Hamiltonian.

We have then considered the pure Long-Ranged spin-chains in order to include the supplementary symmetries in the TTS description, both for the off-critical region where the homogeneous TTS is made of isometries that only depend on the generation, and for critical points where the homogeneous TTS becomes scale invariant with isometries that do not depend on the generation anymore.

Further work is needed to investigate whether the variational optimization with respect to parameters can be also written explicitly for other types of Tensor-States based on different entanglement architectures.

Appendix A: Comparison with the isometries determined by the intra-block Hamiltonians

In this Appendix, we recall the usual block-spin RG rules based on the diagonalization of the intra-Hamiltonian in each block in order to compare with the variational optimization of the isometries discussed in the text. The renormalized intra-Hamiltonian associated to the block of the two children ($I1, I2$) of generation ($g+1$) having the same ancestor I at generation g reads

$$H_{I1, I2}^{intra} = -h_{I1}^{[g+1]} \sigma_{I1}^{[g+1]z} - h_{I2}^{[g+1]} \sigma_{I2}^{[g+1]z} - J_{I1, I2}^{[g+1]z} \sigma_{I1}^{[g+1]z} \sigma_{I2}^{[g+1]z} - J_{I1, I2}^{[g+1]x} \sigma_{I1}^{[g+1]x} \sigma_{I2}^{[g+1]x} - J_{I1, I2}^{[g+1]y} \sigma_{I1}^{[g+1]y} \sigma_{I2}^{[g+1]y} \quad (A1)$$

1. Diagonalization in the parity sector $P = +$

In the parity sector $\sigma_{I1}^{[g+1]z} \sigma_{I2}^{[g+1]z} = +$, the diagonalization of the Hamiltonian of Eq. A1

$$\begin{aligned} H_{I1, I2}^{intra} |++\rangle &= -(h_{I1}^{[g+1]} + h_{I2}^{[g+1]} + J_{I1, I2}^{[g+1]z}) |++\rangle - (J_{I1, I2}^{[g+1]x} - J_{I1, I2}^{[g+1]y}) |--\rangle \\ H_{I1, I2}^{intra} |--\rangle &= -(J_{I1, I2}^{[g+1]x} - J_{I1, I2}^{[g+1]y}) |++\rangle + (h_{I1}^{[g+1]} + h_{I2}^{[g+1]} - J_{I1, I2}^{[g+1]z}) |--\rangle \end{aligned} \quad (A2)$$

leads to the two eigenvalues

$$e_{\pm}^{[P=+]} = -J_{I_1, I_2}^{[g+1]z} \pm \sqrt{(h_{I_1}^{[g+1]} + h_{I_2}^{[g+1]})^2 + (J_{I_1, I_2}^{[g+1]x} - J_{I_1, I_2}^{[g+1]y})^2} \quad (\text{A3})$$

The eigenvector associated to the lowest eigenvalue $e_{-}^{[P=+]}$ is the kept state $|\psi_{I_1, I_2}^{[g+1]+}\rangle$ of Eq. 19

$$|\psi_{I_1, I_2}^{[g+1]+}\rangle = \cos(\theta^{[g, I]^+}) |++\rangle + \sin(\theta^{[g, I]^+}) |--\rangle \quad (\text{A4})$$

where the angle $\theta^{[g, I]^+}$ is fixed by the parameters of the renormalized intra-Hamiltonian

$$\begin{aligned} \cos(2\theta^{[g, I]^+}) &= \frac{h_{I_1}^{[g+1]} + h_{I_2}^{[g+1]}}{\sqrt{(h_{I_1}^{[g+1]} + h_{I_2}^{[g+1]})^2 + (J_{I_1, I_2}^{[g+1]x} - J_{I_1, I_2}^{[g+1]y})^2}} \\ \sin(2\theta^{[g, I]^+}) &= \frac{J_{I_1, I_2}^{[g+1]x} - J_{I_1, I_2}^{[g+1]y}}{\sqrt{(h_{I_1}^{[g+1]} + h_{I_2}^{[g+1]})^2 + (J_{I_1, I_2}^{[g+1]x} - J_{I_1, I_2}^{[g+1]y})^2}} \end{aligned} \quad (\text{A5})$$

2. Diagonalization in the parity sector $P = -$

In the parity sector $\sigma_{I_1}^{[g+1]z} \sigma_{I_2}^{[g+1]z} = -$, the diagonalization of the Hamiltonian of Eq. A1

$$\begin{aligned} H_{I_1, I_2}^{intra} |+-\rangle &= (-h_{I_1}^{[g+1]} + h_{I_2}^{[g+1]} + J_{I_1, I_2}^{[g+1]z}) |+-\rangle - (J_{I_1, I_2}^{[g+1]x} + J_{I_1, I_2}^{[g+1]y}) |-+\rangle \\ H_{I_1, I_2}^{intra} |-+\rangle &= -(J_{I_1, I_2}^{[g+1]x} + J_{I_1, I_2}^{[g+1]y}) |+-\rangle + (h_{I_1}^{[g+1]} - h_{I_2}^{[g+1]} + J_{I_1, I_2}^{[g+1]z}) |-+\rangle \end{aligned} \quad (\text{A6})$$

leads to the two eigenvalues

$$e_{\pm}^{[P=-]} = J_{I_1, I_2}^{[g+1]z} \pm \sqrt{(h_{I_1}^{[g+1]} - h_{I_2}^{[g+1]})^2 + (J_{I_1, I_2}^{[g+1]x} + J_{I_1, I_2}^{[g+1]y})^2} \quad (\text{A7})$$

The eigenvector associated to the lowest eigenvalue $e_{-}^{[P=-]}$ is the kept state $|\psi_{I_1, I_2}^{[g+1]-}\rangle$ of Eq. 20

$$|\psi_{I_1, I_2}^{[g+1]-}\rangle = \cos(\theta^{[g, I]^-}) |+-\rangle + \sin(\theta^{[g, I]^-}) |-+\rangle \quad (\text{A8})$$

where the angle $\theta^{[g, I]^-}$ is fixed by the parameters of the renormalized intra-Hamiltonian

$$\begin{aligned} \cos(2\theta^{[g, I]^-}) &= \frac{h_{I_1}^{[g+1]} - h_{I_2}^{[g+1]}}{\sqrt{(h_{I_1}^{[g+1]} - h_{I_2}^{[g+1]})^2 + (J_{I_1, I_2}^{[g+1]x} + J_{I_1, I_2}^{[g+1]y})^2}} \\ \sin(2\theta^{[g, I]^-}) &= \frac{J_{I_1, I_2}^{[g+1]x} + J_{I_1, I_2}^{[g+1]y}}{\sqrt{(h_{I_1}^{[g+1]} - h_{I_2}^{[g+1]})^2 + (J_{I_1, I_2}^{[g+1]x} + J_{I_1, I_2}^{[g+1]y})^2}} \end{aligned} \quad (\text{A9})$$

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