Residence times of branching diffusion processes

Eric Dumonteil, Alain Mazzolo

PHYSICAL REVIEW E **94**, 012131 (2016)

Residence times of branching diffusion processes

E. Dumonteil[1] and A. Mazzolo[2,*]
[1]*IRSN, Nuclear Safety Division, 31 Avenue de la Division Leclerc, 92260 Fontenay-aux-Roses, France*
[2]*Den-Service d'études des réacteurs et de mathématiques appliquées (SERMA), CEA, Université Paris-Saclay, F-91191 Gif-sur-Yvette, France*
(Received 22 April 2016; published 21 July 2016)

The residence time of a branching Brownian process is the amount of time that the mother particle and all its descendants spend inside a domain. Using the Feynman-Kac formalism, we derive the residence-time equation as well as the equations for its moments for a branching diffusion process with an arbitrary number of descendants. This general approach is illustrated with simple examples in free space and in confined geometries where explicit formulas for the moments are obtained within the long time limit. In particular, we study in detail the influence of the branching mechanism on those moments. The present approach can also be applied to investigate other additive functionals of branching Brownian process.

DOI: 10.1103/PhysRevE.94.012131

I. INTRODUCTION

Branching diffusion processes are commonly used for studying the dynamics of populations and chemical reactions where the stochastic spatial displacement is coupled with creation and annihilation processes [1]. Relevant examples where such a mechanism plays a fundamental role included the spread of epidemics [2], the dynamics of bacterial colonies [3], mutation-propagation of genes [4], or the evolution of the population of neutrons in a nuclear reactor containing fissile material [5], just to name a few. Among the quantities of interest to characterize stochastic processes, the residence time (RT) [6], i.e., the amount of time that a particle spends inside a subdomain, plays an important role, since many physical phenomena strongly depend on how long diffusing particles remain in a zone of interest. For instance, the total numbers of photons emitted by a dye molecule diffusing within a laser spot is proportional to its residence time [7]. In reactor physics, measuring the residence time that neutrons spend inside a detector gives access to the neutron flux and other quantities of interest [8].

This subject is far from being limited to the physics world and in the field of mathematics, the residence time, usually called occupation times, has also been extensively studied since the seminal work of Lévy and his arcsine law giving the residence time of a Brownian particle on an infinite line [9]. In particular, Kac developed a formalism (based on Feynman path integrals) for deriving the residence time distribution when the underlying stochastic process is a pure Brownian motion [10–12]. Since then, The Feynman-Kac formalism has been successfully applied to various kinds of Brownian functionals (for instance, see Ref. [13] for a recent review) as well as for other Markovian continuous-time processes [14] and non-Markovian processes [15,16].

However, despite the considerable literature on the subject, very little is known regarding the RT of branching diffusion processes. Indeed, except the article of Cox and Griffeath on the occupation times for critical binary branching Brownian motions [17], most works on spatial branching random walks concern either discrete quantities (the number of particles and related quantities like the survival probability [18,19] and their spatial distribution [20]) or the RT for exponentially distributed random walks [21–23]. The purpose of this article is to fill this gap. It is organized as follows: In Sec. II after specifying some properties of branching diffusion processes we derive a general approach for the RT equations and its moments in a subdomain based on the Feynman-Kac formalism. Analytical solutions are then carried out in Sec. III for several spherical geometries in the long time limit. In Sec. IV an ergodic property is established in confined geometries. Finally, conclusions and perspectives are presented in Sec. V.

II. BRANCHING DIFFUSION PROCESS

Basically, a branching diffusion process (or branching Brownian motion) combines two classical objects: a Galton-Watson process (a pure reproductive process) and a Brownian motion (a pure random motion process) where each particle of the Galton-Watson process performs Brownian motion independently of any other [24,25]. More precisely, we consider a particle initially located at some position $\mathbf{x_0}$ at time $t = 0$. This particle obeys a regular d-dimensional Brownian motion with a diffusion coefficient D independent of the location (i.e., a homogeneous medium). At rate β it undergoes a Galton-Watson reproduction event: the particle disappears and is replaced by a random number i of identical and independent descendants whose number follows a discrete probability law p_i. We assume that each descendant thus created behaves as the parent particle and evolves independently of the other individuals. For such a branching diffusion process, $\mathbf{X}(t)$, evolving in a domain Ω we wish to calculate the time spent by the initial particle and all its descendants in a subdomain $V \subset \Omega$ up to an observation time t. This quantity known as residence time is formally defined in term of a stochastic integral,

$$\tau_V(t) = \int_0^t \mathbb{1}_V(\mathbf{X}(u))du, \qquad (1)$$

where $\mathbb{1}_V(\mathbf{x})$ is the indicator function of the domain V, which equals 1 if $\mathbf{x} \in V$ and 0 otherwise. A schematic representation is given in Fig. 1. In order to obtain the residence time equation,

*alain.mazzolo@cea.fr

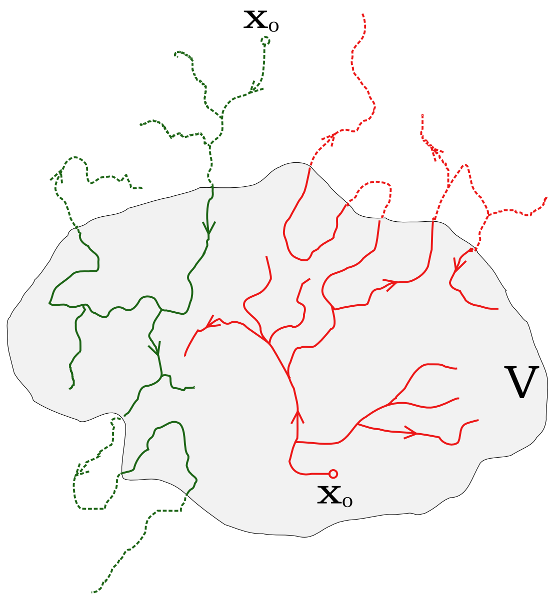

FIG. 1. A schematic representation of two branching diffusion processes up to some time t, one starting outside the domain V (green) and one starting inside the domain (red). By diffusion, the particles can re-enter in V an arbitrary number of times and eventually escape to infinity. Furthermore, due to the branching mechanism, they can be absorbed or created anywhere. Only paths drawn with a continuous line contribute to the residence time in V.

the moment generating function is introduced,

$$Q_t(s,\mathbf{x_0}) = \mathbb{E}\left[\exp\left\{-s\int_0^t \mathbb{1}_V(\mathbf{X}(\mathbf{u}))d\mathbf{u}\right\}\middle|\mathbf{X}(\mathbf{0}) = \mathbf{x_0}\right], \quad (2)$$

or with the simplified notation

$$Q_t(s,\mathbf{x_0}) = \mathbb{E}[e^{-s\tau_V(t)}|\mathbf{x_0}]. \quad (3)$$

In Eqs. (2) and (3) the expectation is performed over all possible Brownian paths $\mathbf{X}(t)$ that start at $\mathbf{x_0}$ at time $t = 0$ and propagate up to time t. By noticing $p(\tau_V(t),\mathbf{x_0})$ the probability density function (pdf) of the RT, $Q_t(s,\mathbf{x_0})$, is expressed as

$$Q_t(s,\mathbf{x_0}) = \int_0^\infty e^{-s\tau_V(t)} p(\tau_V(t),\mathbf{x_0}) d\tau_V, \quad (4)$$

so that $Q_t(s,\mathbf{x_0})$ is also the Laplace transform of the pdf $\tau_V(t)$.

To obtain an equation for $Q_t(s,\mathbf{x_0})$ the standard procedure is given by the Feynman-Kac approach. To this aim, we consider a branching Brownian motion $\mathbf{X}(t)$ whose trajectory (by trajectory we understand the trajectory of the mother particle and those of all its descendants) starts at $\mathbf{x_0}$ (at time $t = 0$) and is observed up to a time $t + \Delta t$. The interval $[0,t + \Delta t]$ is split into two parts: a first small interval $[0,\Delta t]$, where the process evolves from the initial position $\mathbf{x_0}$, and a second interval $[\Delta t,t + \Delta t]$, in which the process reaches $\mathbf{X}(t + \Delta t)$ at the final observation time. The branching Brownian process evolves according to the fundamental law: during any infinitesimal time Δt the probability of a reproductive event is $\beta\Delta t$ and with complementary probability $1 - \beta\Delta t$ the particle keeps diffusing. If a diffusion occurs during the infinitesimal time $[0,\Delta t]$, then the particle moves to the random position $\mathbf{x_0} + \Delta\mathbf{x}$, otherwise the particle undergoes a reproductive event while remaining in $\mathbf{x_0}$ [37]. By decomposing the two mutually exclusive events (diffusion and branching), and thanks to the Markov property of the branching Brownian process [25], Eq. (2) becomes

$$\begin{aligned} Q_{t+\Delta t}(s,\mathbf{x_0}) = {}& (1 - \beta\Delta t)e^{-s\mathbb{1}_V(\mathbf{x_0})\Delta t} \\ & \times \langle \mathbb{E}[e^{-s\int_{\Delta t}^{t+\Delta t}\mathbb{1}_V(\mathbf{X}(\mathbf{u}))d\mathbf{u}}|\mathbf{x_0} + \Delta\mathbf{x}]\rangle_{\Delta\mathbf{x}} \\ & + \beta\Delta t e^{-s\mathbb{1}_V(\mathbf{x_0})\Delta t} \\ & \times (p_0 + p_1\mathbb{E}[e^{-s\int_{\Delta t}^{t+\Delta t}\mathbb{1}_V(\mathbf{X}(\mathbf{u}))d\mathbf{u}}|\mathbf{x_0}] \\ & + p_2\mathbb{E}[e^{-s\int_{\Delta t}^{t+\Delta t}\mathbb{1}_V(\mathbf{X}(\mathbf{u}))d\mathbf{u}}|\mathbf{x_0}]^2 + \dots), \end{aligned} \quad (5)$$

where the notation $\langle\rangle_{\Delta\mathbf{x}}$ means that the average is performed over all random realizations of $\Delta\mathbf{x}$. In the preceding equation, terms due to the branching mechanism can be explained as follows. If no-particle is re-emitted, an event occurring with the probability p_0, the only contribution to $Q_{t+\Delta t}(s,\mathbf{x_0})$ is the nonrandom quantity $e^{-s\mathbb{1}_V(\mathbf{x_0})\Delta t}$ weighted by its probability of occurrence $\beta\Delta t$. If a single particle is re-emitted (with probability p_1), then during $[\Delta t,t + \Delta t]$, the remaining trajectory's contribution to the moment generating function is simply $\mathbb{E}[e^{-s\int_{\Delta t}^{t+\Delta t}\mathbb{1}_V(\mathbf{X}(\mathbf{u}))d\mathbf{u}}|\mathbf{x_0}] = Q_t(s,\mathbf{x_0})$ weighted again by its probability of occurrence $\beta\Delta t$. If two particles emerge from a reproduction event (with probability p_2), then the remaining contribution of two trajectories [says $\mathbf{X_1}(t)$ and $\mathbf{X_2}(t)$] during $[\Delta t,t + \Delta t]$ to the moment generating function is given by

$$\mathbb{E}[e^{-s\int_{\Delta t}^{t+\Delta t}\mathbb{1}_V(\mathbf{X_1}(\mathbf{u}))d\mathbf{u}} \times e^{-s\int_{\Delta t}^{t+\Delta t}\mathbb{1}_V(\mathbf{X_2}(\mathbf{u}))d\mathbf{u}}|\mathbf{x_0}] = \mathbf{Q_t^2}(\mathbf{s},\mathbf{x_0}), \quad (6)$$

where in Eq. (6) we used the fact that both particles evolve independently. This rule applies to higher-order contributions as well, due to three and more particles born during the small interval time Δt, each event weighted by the probability p_i. Furthermore, by introducing the offspring generating function of the branching mechanism,

$$G[z] = p_0 = p_1 z + p_2 z^2 + \dots = \sum_i p_i z^i, \quad (7)$$

Eq. (5) rewrites,

$$\begin{aligned} Q_{t+\Delta t}(s,\mathbf{x_0}) = {}& (1 - \beta\Delta t)e^{-s\mathbb{1}_V(\mathbf{x_0})\Delta t} \\ & \times \langle \mathbb{E}[e^{-s\int_{\Delta t}^{t+\Delta t}\mathbb{1}_V(\mathbf{X}(\mathbf{u}))d\mathbf{u}}|\mathbf{x_0} + \Delta\mathbf{x}]\rangle_{\Delta\mathbf{x}} \\ & + \beta\Delta t e^{-s\mathbb{1}_V(\mathbf{x_0})\Delta t} G[Q_t(s,\mathbf{x_0})]. \end{aligned} \quad (8)$$

The last step consists of developing the preceding equation for small Δt. Since for a pure Brownian motion $\langle\Delta\mathbf{x}\rangle_{\Delta\mathbf{x}} = 0$ and $\langle(\Delta\mathbf{x})^2\rangle_{\Delta\mathbf{x}} = 2D\Delta t$, this procedure yields to

$$\begin{aligned} \frac{\partial Q_t(s,\mathbf{x_0})}{\partial t} = {}& D\Delta_{x_0} Q_t(s,\mathbf{x_0}) - \beta Q_t(s,\mathbf{x_0}) \\ & - s\mathbb{1}_V(\mathbf{x_0})Q_t(s,\mathbf{x_0}) + \beta G[Q_t(s,\mathbf{x_0})], \end{aligned} \quad (9)$$

where Δ_{x_0} is the d-dimensional Laplacian in the initial variable $\mathbf{x_0}$.

When $\beta = 0$ (in the absence of branching), Eq. (9) reduces to the usual backward Fokker-Planck equation for the moment generating function of the standard Brownian motion, and for certain simple geometrical configurations, Eq. (9) can be solved by Laplace transform technique [13]. However, once the branching mechanism is turned on with at least one of the

$p_i \neq 0, i \geqslant 2$, Eq. (9) becomes nonlinear (through the G term) and is extremely difficult to solve.

Equation (9) is close to the residence time equation obtained for Pearson branching random walks with exponentially distributed jump length that model the behavior of fission chain in a water-moderated reactor [5]. Indeed, for such random walks, a similar equation holds for the number of neutrons generated in a subvolume V (the so-called Pál-Bell equation [5]) and for the total traveled length in V [21].

III. MOMENT OF THE RESIDENCE TIME

To circumvent the difficulty related to the resolution of Eq. (9), a somewhat simpler approach to the analysis of the RT is provided by the moment equations. Two quantities then play a particularly important role to characterize the RT: the mean residence time (MRT) and the variance of the RT [7,26]. From the definition of $Q_t(s,\mathbf{x_0})$, the kth moment of the RT follows immediately [27],

$$\mathbb{E}\left[\tau_V^k(t)\middle|\mathbf{x_0}\right] = (-1)^k \left.\frac{\partial^k Q_t(s,\mathbf{x_0})}{\partial s^k}\right|_{s=0}. \tag{10}$$

Applying this formula to Eq. (9) and using Faà di Bruno's formula for the nth derivative of the composition $f[g(x)]$,

$$\frac{d^n}{dx^n} f[g(x)] = \sum_{j=1}^{n} f^{(j)}[g(x)]\mathcal{B}_{n,j}[g'(x),\dots,g^{(n-j+1)}(x)], \tag{11}$$

where $\mathcal{B}_{n,j}[x_1,\dots,x_{n-j+1}]$ are the Bell's polynomials, gives a set of recursive partial differential equations for the moments of the RT (to simplify the notation, the index x_0 is removed from the definition of the moments),

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{E}\left[\tau_V^n(t)\right] &= D\Delta_{x_0}\mathbb{E}\left[\tau_V^n(t)\right] + n\mathbb{1}_V(\mathbf{x_0})\mathbb{E}\left[\tau_V^{n-1}(t)\right]\\ &\quad + \beta(\nu_1 - 1)\mathbb{E}\left[\tau_V^n(t)\right]\\ &\quad + \beta\sum_{j=2}^{n}\nu_j\mathcal{B}_{n,j}\left\{\mathbb{E}[\tau_V(t)],\dots,\mathbb{E}\left[\tau_V^{n-j+1}(t)\right]\right\}.\end{aligned} \tag{12}$$

In Eq. (12), $\nu_1 = \frac{d}{dz}G[z]\big|_{z=1} = \sum_i ip_i$ is the mean number of secondary particles per reproduction event, and the quantities ν_j are the falling factorial moments of the number of descendants per reproduction event,

$$\nu_j = \left.\frac{d^j G[z]}{dz^j}\right|_{z=1} = \sum_i i(i-1)\dots(i-j+1)p_i. \tag{13}$$

This set of equations, Eqs. (12), for the moments of the RT is linear since Bell's polynomials are at most of order $k-1$. Therefore, all the nonlinearity arising from these polynomials can be considered as a source term and Eqs. (12) can be solved recursively. We will adopt this approach to obtain the MRT and the variance of the RT for simple geometries in the next paragraphs [38]. Recalling that the first few Bell's polynomials read

$$\mathcal{B}_{1,1}[x_1] = x_1 \quad \mathcal{B}_{2,1}[x_1,x_2] = x_2 \quad \mathcal{B}_{2,2}[x_1,x_2] = x_1^2, \tag{14}$$

the equations for the first two moments are

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{E}[\tau_V(t)] &= D\Delta_{x_0}\mathbb{E}[\tau_V(t)] + \mathbb{1}_V(\mathbf{x_0})\\ &\quad + \beta(\nu_1 - 1)\mathbb{E}[\tau_V(t)]\\ \frac{\partial}{\partial t}\mathbb{E}\left[\tau_V^2(t)\right] &= D\Delta_{x_0}\mathbb{E}\left[\tau_V^2(t)\right] + 2\mathbb{1}_V(\mathbf{x_0})\mathbb{E}[\tau_V(t)]\\ &\quad + \beta(\nu_1 - 1)\mathbb{E}\left[\tau_V^2(t)\right] + \beta\nu_2(\mathbb{E}[\tau_V(t)])^2.\end{aligned} \tag{15}$$

The previous equations therefore generalize to branching diffusion processes, the equations for the MRT and for the second moment of the RT of a standard Brownian motion. Remark that in the absence of branching mechanism ($\beta = 0$), Eqs. (15) reduce, as expected, to those of the first two moments of the RT for a standard Brownian motion [26].

A. Asymptotic behavior: the long time limit

In an infinite medium, the behavior of the branching diffusion process at long times strongly depends on the spatial dimension. More precisely, as $t \to \infty$ the moments of the RT diverge when the random walk is recurrent (which appends for $d \leqslant 2$ in the case of pure Brownian motion). Besides, for $\nu_1 \leqslant 1$ a pure Galton-Watson process goes to extinction with probability 1. In this paragraph, we will assume that $d > 2$ and $\nu_1 \leqslant 1$. In such cases, the limit $\tau_n(\mathbf{x_0}) = \lim_{t\to\infty}\mathbb{E}[\tau_V^n(t)]$ exists and $\partial\mathbb{E}[\tau_V^n(t)]/\partial t \to 0$ as $t \to \infty$. Under such conditions, Eq. (12) becomes

$$\begin{aligned}D\Delta_{x_0}\tau_n(\mathbf{x_0}) &= -\,n\mathbb{1}_V(\mathbf{x_0})\tau_{n-1}(\mathbf{x_0}) + \beta(1-\nu_1)\tau_n(\mathbf{x_0})\\ &\quad - \beta\sum_{j=2}^{n}\nu_j\mathcal{B}_{n,j}[\tau_1(\mathbf{x_0}),\dots,\tau_{n-j+1}(\mathbf{x_0})],\end{aligned} \tag{16}$$

which for the first two moments simplifies to

$$\begin{aligned}\Delta_{x_0}\tau_1(\mathbf{x_0}) &= -\,\mathbb{1}_V(\mathbf{x_0})/D + k^2\tau_1(\mathbf{x_0}),\\ \Delta_{x_0}\tau_2(\mathbf{x_0}) &= -\,2\mathbb{1}_V(\mathbf{x_0})\tau_1(\mathbf{x_0})/D + k^2\tau_2(\mathbf{x_0})\\ &\quad - \beta_2[\tau_1(\mathbf{x_0})]^2,\end{aligned} \tag{17}$$

where we have set $k^2 \equiv \beta(1-\nu_1)/D$ and $\beta_2 \equiv \beta\nu_2/D$.

We now illustrate our method by solving these equations within a ball B_d of radius R (centered at the origin) for an infinite observation time. In this case, solutions are amenable to explicit expressions. To this end, we follow the route described in Ref. [26] for a pure Brownian motion.

For spherical domains, only the radial part of the Laplacian,

$$\Delta_r = \frac{d-1}{r}\frac{d}{dr} + \frac{d^2}{dr^2}, \tag{18}$$

where $r = |\mathbf{x_0}|$ is the norm of the vector $\mathbf{x_0}$, contributes to the solution. For the MRT, we thus have to solve the following equations:

$$\Delta_r \tau_1(r) = -1/D + k^2 \tau_1(r) \quad \text{for} \quad r < R$$
$$\Delta_r \tau_1(r) = k^2 \tau_1(r) \quad \text{for} \quad r > R, \tag{19}$$

with the appropriate boundary conditions.

(i) Continuity of the solution $\tau_1(r)$ at $r = R$.

(ii) Continuity of the derivative $d\tau_1(r)/dr$ at $r = R$. This property comes from the structure of Eqs. (19): if $d\tau_1(r)/dr$ were not continuous at $r = R$ (i.e., a jump occurs at this distance), then the Laplacian would have a discontinuity of higher order. However, the right-hand side of Eqs. (19) contains only continuous functions. Consequently, $d\tau_1(r)/dr$ must be continuous at $r = R$.

(iii) Vanishing MRT when the particle starts infinitely far away: $\tau_1(r) \to 0$ as $r \to \infty$. This property comes from the original Brownian particle and remains valid for its descendants since they are also born infinitely far away.

(iv) The derivative $d\tau_1(r)/dr \to 0$ at $r = 0$. This last condition comes from the following point: consider an arbitrary direction $\vec{u}(r) = \vec{r}/r$. From the spherical shape of the domain, $\tau_1(r)$ is an even function along this direction and therefore its derivative is an odd function. Furthermore, $d\tau_1(r)/dr$ is continuous (from the second condition). These two properties imply that $\lim_{r\to 0} d\tau_1(r)/dr = 0$.

Contrary to the first three conditions the last one depends on the shape of the domain and is valid for spherical symmetric geometries only.

The previous arguments can be extended to all $\tau_n(r)$ since the right-hand side of Eqs. (16) is a combination of continuous functions.

In three dimensions, the solution of Eqs. (19) with the four boundary conditions previously enumerated is given by

$$\tau_{1<}(r) = \frac{1}{Dk^2}\left[1 - (1 + kR)e^{-kR}\frac{\sinh(kr)}{kr}\right],$$
$$\tau_{1>}(r) = \frac{1}{Dk^2}[kR\cosh(kR) - \sinh(kR)]\frac{e^{-kr}}{kr}, \tag{20}$$

where the subscript $<$ denotes the solution for $r \leqslant R$ and $>$ the solution for $r \geqslant R$. In another context, similar equations to those of Eqs. (20) were reported in Refs. [28,29]. Putting these solutions in Eqs. (17) allows us to obtain formally the expressions for second moment of the RT. These analytical solutions, quite cumbersome, are reported in Appendix A. Figure 2 shows the behavior of the RT and that of the variance for different set of parameters. In three-dimensional free space, for subcritical system, one can observe that the branching mechanism has a limited influence on the variance of the RT.

In the following, rather than working with the long exact analytical expression, we look at the limit when β is small so that the branching mechanism can be treated as a perturbation. Under this approximation, the above expressions become

$$\tau_{1<}(r) \simeq \frac{3R^2 - r^2}{6D} - \frac{R^3}{3D}k + o(k^2),$$
$$\tau_{1>}(r) \simeq \frac{R^3}{3Dr} - \frac{R^3}{3D}k + o(k^2). \tag{21}$$

Note that the first terms in the right-hand side of Eqs. (21) correspond to the MRT for Brownian particles without branching [26]. Putting these expressions in Eq. (17) leads to

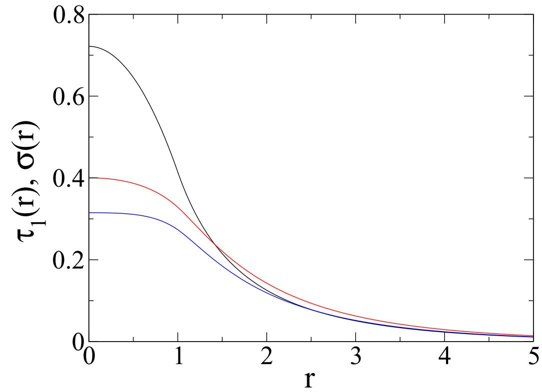

FIG. 2. Three dimensions: Mean residence time and variance inside a sphere of radius unity for different parameters ν_2 with $D = 1/2$ and $\beta = 1/2$. Each case has the same mean number of descendants $\nu_1 = 0.75$. Black line, $\tau_1(r)$; blue line, $\sigma(r)$ with $\nu_2 = 0$; red line, $\sigma(r)$ with $\nu_2 = 0.25$. Whatever the starting point $\mathbf{x_0}$, the branching mechanism has the effect of increasing the variance (as expected), but its effect remains, however, relatively moderate.

$$\tau_{2<}(r) \simeq \frac{(5R^2 - r^2)^2}{60D^2} - \frac{(27R^2 - 5r^2)R^3}{45D^2}k + o(k^2),$$
$$\tau_{2>}(r) \simeq \frac{4R^5}{15D^2 r} - \frac{2(5R + 6r)R^5}{45D^2 r}k + o(k^2). \tag{22}$$

Again, the first terms in the right-hand side of Eqs. (22) correspond to the second moment of the RT for Brownian particles without branching [26].

The development in small β shows that the second moment of the RT and therefore the variance do not depend of ν_2. Thus, at first approximation, the fine structure of the branching mechanism can be ignored and the branching diffusion process behaves as a *diffusion-absorption* process [30].

B. Residence time of reflected branching diffusions

In the preceding paragraph we studied the RT of branching diffusion in a infinite medium. However, the motion of diffusing particles (neutrons, species) is often restricted by a geometrical confinement. This geometrical constraint causes significant changes of the transport [31] and affects the clustering of particles as it was recently reported [32]. To illustrate our method we consider a one-dimensional branching diffusion process evolving in a segment $\Omega = [-R_2, R_2]$, with reflecting boundary conditions at $\pm R_2$. We also consider two "concentric" regions $R_I = [-R_1, R_1]$ and $R_{II} = [-R_2, -R_1] \cup [R_1, R_2]$ with $R_1 < R_2$ as shown on Fig. 3. Besides the geometrical constraints, we assume that the mean number of descendants $\nu_1 < 1$ so that the system is again subcritical. For such a system, the extinction is certain [24] and all the moments of the RT remain finite, even in the long time limit. Therefore, the approach developed in the preceding paragraphs can be safely applied to obtain recursively the moments of the RT. As previously, our discussion is limited to the first two moments of the RT. These moments are studied in both regions I and II. For this one-dimensional system, in the long time limit, within

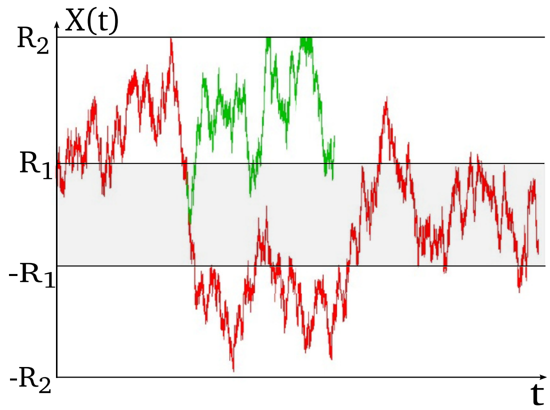

FIG. 3. An example of realization of the one-dimensional branching diffusion process in the segment $[-R_2;R_2]$ with reflecting boundary conditions on the borders. Only paths between $[-R_1;R_1]$ contribute to the RT of Region I.

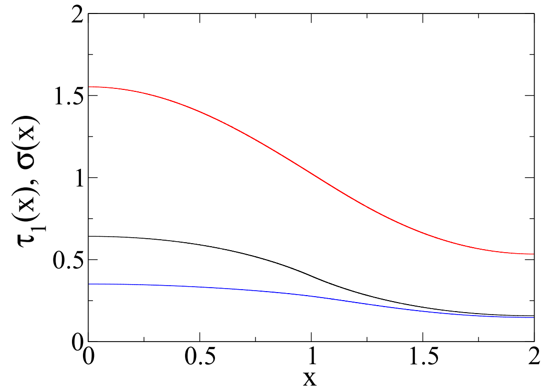

FIG. 4. Residence time of reflected branching diffusions: mean and variance in the region I for different set of parameters ν_2 with $D = 1/2$ and $\beta = 5$. Each case has the same mean number of descendants $\nu_1 = 0.75$. Black line: $\tau_1(x)$; Blue line: $\sigma(x)$ with $\nu_2 = 0$; Red line: $\sigma(x)$ with $\nu_2 = 0.25$. Due to the branching mechanism, when $\nu_2 \neq 0$, the variance of the process increases drastically.

region I, Eqs. (17) become

$$\begin{aligned}\frac{d^2\tau_1(x)}{dx^2} &= -\frac{1}{D}\times\mathbb{1}_{[-R_1,R_1]}(x) + k^2\tau_1(x),\\ \frac{d^2\tau_2(x)}{dx^2} &= -\frac{2}{D}\times\mathbb{1}_{[-R_1,R_1]}(x)\tau_1(x)\\ &\quad + k^2\tau_2(x) - \beta_2[\tau_1(x)]^2.\end{aligned} \tag{23}$$

Boundary conditions (i), (ii), and (iv) are unchanged. The boundary condition (iii) at infinity is now replaced by a Neumann reflective boundary condition $d\tau_1(\pm R_2)/dx = 0$. The same conditions hold for $\tau_2(x)$. With such boundary conditions, the unique solution of the differential system Eq. (23) is given by

$$\tau_{1R_I}(x\in R_I) = \frac{1}{Dk^2}\left\{1 - \frac{\sinh[k(R_2-R_1)]}{\sinh(kR_2)}\cosh(kx)\right\} \tag{24}$$

$$\tau_{1R_I}(x\in R_{II}) = \frac{1}{Dk^2}\frac{\sinh(kR_1)}{\sinh(kR_2)}\cosh[k(R_2-x)].$$

In the above equations, $x \in R_I$ indicates that the starting position belong to the region I. The variance $\sigma_{R_I}(x) = \tau_{2R_I}(x) - \tau_{1R_I}(x)^2$ in the region R_I, again rather lengthy, is left in Appendix B. Figure 4 is a plot of the analytical solutions for different set of parameters. Because of the confined geometry and one-dimensionality of the system, the variance exhibits a strong dependence on the branching mechanism. Indeed, the variance increases considerably, even with a small number of reproductive event.

Note that $\tau = \tau_{R_I}(x) + \tau_{R_{II}}(x)$, the total time spent by the particle inside the reflecting domain, is independent of the position of the particle. Moreover, it satisfies $d^2\tau/dx^2 = -1/D + k^2\tau$ whose obvious solution $\tau = 1/Dk^2 = 1/\beta(1-\nu_1)$ corresponds to the (trivial) mean lifetime (before absorption) of a particle in an infinite medium killed with a constant absorption rate $\beta(1-\nu_1)$ per unit time. Expression of the MRT in region II, $\tau_{R_{II}}(x) = 1/Dk^2 - \tau_{R_I}(x)$ follows immediately.

IV. AN ERGODIC PROPERTY

So far, we considered that the process began at a fixed position $\mathbf{x_0}$. From now on, we assume that the position of the initial particle is uniformly distributed in the volume Ω, corresponding to the equilibrium distribution. The total time spent τ_{R_I} in R_I by the uniform normalized distribution $dx/(2R_2)$ is given by

$$\begin{aligned}\tau_{R_I} &= \int_{-R_1}^{R_1}\frac{dx}{2R_2}\tau_{1R_I}(x\in R_I) + \int_{R_1}^{R_2}\frac{dx}{2R_2}\tau_{1R_I}(x\in R_{II})\\ &\quad + \int_{-R_2}^{-R_1}\frac{dx}{2R_2}\tau_{1R_I}(x\in R_{II}) = \frac{R_1}{Dk^2R_2} = \tau\frac{2R_1}{2R_2}.\end{aligned} \tag{25}$$

The total time spent in R_I is thus proportional to the MRT in the whole domain multiplied by the ratio of the volume of observation over the total volume, reflecting an ergodic-like property. This rather intuitive property (as long as the branching random walk is unbiased) is indeed very general since it does not depend of the shape of the different volumes and holds in any dimension. A simple proof is given by considering a domain $\Omega \in \mathbb{R}^n$ with reflective condition on its boundary and a subdomain $V \subset \Omega$. For such geometries, the MRT in the subdomain V satisfies

$$\begin{aligned}\Delta\tau_V(\boldsymbol{r}) &= -1/D + k^2\tau_V(\boldsymbol{r}) \quad \text{for} \quad \boldsymbol{r}\in V,\\ \Delta\tau_{V'}(\boldsymbol{r}) &= k^2\tau_{V'}(\boldsymbol{r}) \quad \text{for} \quad \boldsymbol{r}\in V' = \Omega\setminus V,\end{aligned} \tag{26}$$

with boundary conditions between the two regions V and V',

$$\begin{aligned}\tau_V(\boldsymbol{r})|_{\boldsymbol{r}\in\partial V} &= \tau_{V'}(\boldsymbol{r})|_{\boldsymbol{r}\in\partial V},\\ \nabla\tau_V(\boldsymbol{r})|_{\boldsymbol{r}\in\partial V} &= \nabla\tau_{V'}(\boldsymbol{r})|_{\boldsymbol{r}\in\partial V},\end{aligned} \tag{27}$$

and with reflective boundary conditions on $\partial\Omega$,

$$\boldsymbol{n}(\boldsymbol{r})\cdot\nabla\tau(\boldsymbol{r})|_{\boldsymbol{r}\in\partial\Omega} = 0, \tag{28}$$

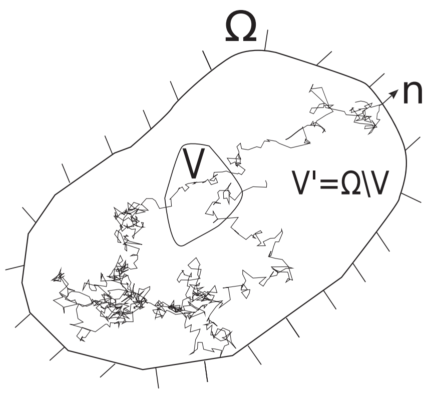

FIG. 5. Residence time of a particle inside a subvolume $V \subset \Omega$. The whole domain Ω has reflecting boundary conditions on its surface.

$\boldsymbol{n}(\boldsymbol{r})$ being the unit normal vector (pointing outward) to the surface $\partial\Omega$ as shown on Fig. 5. For an initial uniform distributed density normalized to unity $d\boldsymbol{r}/\Omega$, the total time spent τ_V in V is given by

$$\tau_V = \in \tau_V \frac{d\boldsymbol{r}}{\Omega}\,\tau_V(\boldsymbol{r}) + \int_{V'} \frac{d\boldsymbol{r}}{\Omega}\,\tau_{V'}(\boldsymbol{r})$$

Replacing Eqs. (26) in the preceding expression and applying the Gauss divergence theorem yields

$$\tau_V = \frac{V}{\Omega D k^2} + \frac{1}{\Omega k^2}\left[\int_{\partial V} \nabla\tau_V(\boldsymbol{r})\cdot\boldsymbol{n}dS + \int_{\partial V'} \nabla\tau_{V'}(\boldsymbol{r})\cdot\boldsymbol{n}dS\right]. \quad (29)$$

The last integral splits into two parts,

$$\int_{\partial V'} \nabla\tau_{V'}(\boldsymbol{r})\cdot\boldsymbol{n}dS = -\int_{\partial V} \nabla\tau_{V'}(\boldsymbol{r})\cdot\boldsymbol{n}dS + \int_{\partial\Omega} \nabla\tau_{V'}(\boldsymbol{r})\cdot\boldsymbol{n}dS. \quad (30)$$

The minus sign arises since the normal vector at the frontier between V' and V has an orientation opposite to that between V and V'. By condition Eq. (28) the last integral equals zero and thanks to Eq. (27) the remaining integral cancels the first one in Eq. (29).The MRT becomes

$$\tau_V = \frac{V}{Dk^2\Omega} = \tau\frac{V}{\Omega}. \quad (31)$$

As announced, the total time spent in a subvolume is proportional to the time spent in the whole domain multiplied by the ratio of the subdomain volume over the volume of the domain, reflecting an ergodic-like property. Some similar ergodic properties were also recently observed regarding the total time spent and the number of collisions in a subdomain for Pearson random walks in confined geometries [33,34].

V. CONCLUSION

Based on the Feynman-Kac formalism we derived the residence time equation for a branching diffusion process with an arbitrary number of descendants. Solving this nonlinear equation established within Laplace space, and then reversing it to obtain the density of probability of the residence time is currently out of reach. Rather than searching the entire distribution, we derived the equations for the moments, and focused on the first two, namely, the mean residence time and its variance. These equations are linear and can be solved recursively, as in the pure Brownian case [26]. We gave examples of analytical solutions for simple spherical geometries showing the crucial role played by the branching mechanism, in particular for reflected branching Brownian motion in the subcritical regime. Furthermore, for subcritical branching diffusion processes evolving in confined geometries when the initial particles distribution is at equilibrium, we showed that the system has an ergodic-like property once the observation time is long enough to reach the extinction.

When the branching mechanism is reduced to a mere absorption, the general approach presented allows us to study *diffusion-absorption* processes when particles disappear at a constant rate. A relevant phenomena for analyzing certain diffusion reactions in solution, such as energy transfer and fluorescence quenching [35].

The present approach may be also useful to investigate other additive functionals of branching Brownian motion and the residence time of branching particles diffusing in a potential.

APPENDIX A: VARIANCE OF THE RESIDENCE TIME IN THREE-DIMENSIONAL GEOMETRIES

In this Appendix we give the analytical expression of the variance of the RT inside a sphere of radius R in an infinite medium. Hypothesis are those of Sec. III: the system is subcritical and the observation time goes to infinity (again the subscript $<$ denotes the solution for $r \leqslant R$ and $>$ that for $r \geqslant R$). Inserting Eqs. (20) in Eqs. (17) leads to the second moment of the RT from which variances $\sigma_<(r) = \tau_{2<}(r) - \tau_{1<}(r)^2$ and $\sigma_>(r) = \tau_{2>}(r) - \tau_{1>}(r)^2$ follow immediately. Analytical results are obtained by *mathematica 9.0*,

$$\begin{aligned}\sigma_<(r) = &-\frac{e^{-6kR-4kr}}{4D^2k^6}\{2(kR+1)^2e^{4k(R+r)} - (kR+1)^2e^{2k(2R+r)} - (kR+1)^2(e^{4kR+6kr}) + kr(kR+1)e^{3k(R+r)}\\ &+ 4k^2r^2e^{6kR+4kr} - kr(kR+1)e^{3kR+5kr} + kr(kR+1)e^{5k(R+r)}[2k(r-R)-1] + kr(kR+1)e^{5kR+3kr}[2k(R+r)+1]\}\\ &+ \frac{\beta_2}{8D^2k^7r}[(1-kR)[2kR+e^{2kR}(kR-1)+2]2\sinh(kr)\,\mathrm{Ei}(-3kR)\\ &+ e^{-k(3R+r)}(8kre^{k(3R+r)} + 2(kR+1)\{e^{2k(R+r)}(-2kR+2kr-3) + e^{2kR}[2k(R+r)+3] - e^{2kr} + 1\}\end{aligned}$$

$$+ e^{kR}(kR+1)^2\{-e^{2kr}[2\,\mathrm{Ei}(-kR) - \mathrm{Ei}(kR) - 2\,\mathrm{Ei}(-kr) + \mathrm{Ei}(-3kr) + \mathrm{Ei}(kr)] + 2\,\mathrm{Ei}(-kR)$$
$$- \mathrm{Ei}(kR) + \mathrm{Ei}(-kr) - 2\,\mathrm{Ei}(kr) + \mathrm{Ei}(3kr)\})],$$

$$\sigma_>(r) = \frac{e^{-2kr}}{4D^2k^6r^2}\{kr[4e^{2kR}(kR+1)^2 + kR + e^{4kR}(3kR-5)+1]e^{k(r-3R)} - 4[kR\cosh(kR) - \sinh(kR)]^2\}$$
$$+ \beta_2\frac{e^{-2kr}}{8D^2k^7r}((kR-1)[2kR + e^{2kR}(kR-1)+2]e^{kr}\,\mathrm{Ei}(-3kR) - (kR+1)^2[3\,\mathrm{Ei}(kR) - \mathrm{Ei}(3kR)]e^{k(r-2R)}$$
$$- 4e^{kr}[e^{2kr}\,\mathrm{Ei}(-3kr) - \mathrm{Ei}(-kr)][\sinh(kR) - kR\cosh(kR)]^2 + 4e^{k(r-R)}[(kR+1)(2kR+1)$$
$$- 2\sinh(2kR) + (kR-1)\cosh(2kR)] + e^{kr}\,\mathrm{Ei}(-kR)\{-2k^2R^2 - [kR(3kR+2)+3]\sinh(2kR)$$
$$+ [kR(kR+6)+1]\cosh(2kR) + 2\}), \tag{A1}$$

where Ei is the exponential integral function [36]. In the absence of reproductive event, i.e., when $\beta_2 = 0$, the preceding equations become much simpler and correspond to the important case of *diffusion-absorption* processes [30].

APPENDIX B: VARIANCE OF THE RT IN ONE-DIMENSIONAL CONFINED GEOMETRIES

With the technique briefly outlined in Appendix A, the variance in region I, $\sigma_{R_I}(x) = \tau_{2R_I}(x) - \tau_{1R_I}(x)^2$, is obtained after straightforward but lengthy calculations. The variance $\sigma_{R_{II}}(x) = \tau_{2R_{II}}(x) - \tau_{1R_{II}}(x)^2$ in the region R_{II}, not reported here, can be obtained in the same way.

$$\sigma_{R_I}(x\in R_I) = \frac{1}{2D^2k^4}[2 + \mathrm{csch}(kR_2)(\cosh(kx)\sinh(k(R_1-R_2)\{1 + 2kR_1\coth(kR_2) + \mathrm{csch}(kR_2)\sinh[k(2R_1-R_2)]\}$$
$$- 2\cosh(kx)^2\,\mathrm{csch}(kR_2)\sinh[k(R_2-R_1)]^2 - 2kx\sinh[k(R_1-R_2)]\sinh(kx))]$$
$$+ \frac{\beta\nu_2}{6D^3k^6}[6 - \mathrm{csch}(kR_2)\sinh[k(R_1-R_2)]([-3+\cosh(2kx)]\,\mathrm{csch}(kR_2)\sinh[k(R_1-R_2)]$$
$$+ \cosh(kx)\{-7 - 6kR_1\coth(kR_2) + \mathrm{csch}(kR_2)\sinh[k(2R_1-R_2)]\} + 6kx\sinh(kx))],$$

$$\sigma_{R_I}(x\in R_{II}) = \frac{1}{8D^2k^4}\,\mathrm{csch}(kR_2)^2(-4\cosh[k(R_1-x)] + \cosh[k(3R_1-x)] + 3\cosh[k(R_1+2R_2-x)]$$
$$+ 3\cosh[k(R_1+x)] - 4\cosh[k(R_1-2R_2+x)] + \cosh[k(3R_1-2R_2+x)]$$
$$- 8\cosh[k(R_2-x)]^2\sinh(kR_1)^2 + 4kR_1\{\sinh[k(R_1-x)] + \sinh[k(R_1-2R_2+x)]\})$$
$$+ \frac{\beta\nu_2}{24D^3k^6}\,\mathrm{csch}(kR_2)^2[6\cosh(2kR_1) - 4\cosh[k(R_1-x)] - \cosh[k(3R_1-x)] + 2\cosh[2k(R_2-x)]$$
$$- \cosh[2k(R_1+R_2-x)] + 5\cosh[k(R_1+2R_2-x)] - \cosh[2k(R_1-R_2+x)] - 4\cosh[k(R_1-2R_2+x)]$$
$$- \cosh[k(3R_1-2R_2+x)] + 5\cosh[k(R_1+x)] - 6(1 - 2kR_1\{\sinh[k(R_1-x)] - \sinh[k(R_1-2R_2+x)]\})]. \tag{B1}$$

The variance involves only elementary functions. As in the three-dimensional case, in the absence of reproductive event, the above equations simplify considerably and allow us to study *diffusion-absorption* processes [30].

[1] A. S. Mikhailov, Phys. Rep. **184**, 307 (1989).
[2] N. T. J. Bailey, *The Mathematical Theory of Infectious Diseases and its Applications* (Griffin, London, 1957).
[3] P. Jagers, *Branching Processes with Biological Applications* (John Wiley & Sons, London 1975).
[4] S. Sawyer and J. Fleischman, Proc. Natl. Acad. Sci. USA **76**, 872 (1979).
[5] I. Pázsit and L. Pál, *Neutron Fluctuations: A Treatise on the Physics of Branching Processes* (Elsevier, Oxford, 2008).
[6] N. Agmon, J. Chem. Phys. **81**, 3644 (1984).
[7] N. Agmon, J. Phys. Chem. A **115**, 5838 (2011).
[8] I. Lux and L. Koblinger, *Monte Carlo Particle Transport Methods: Neutron and Photon Calculations* (CRC Press, Boca Raton, 1991).
[9] W. Feller, *An Introduction to Probability Theory and Its Applications*, 3rd ed. (Wiley, New York, 1970).
[10] M. Kac, Trans. Amer. Math. Soc. **65**, 1 (1949).
[11] M. Kac, *Probability and Related Topics in Physical Sciences*, Lectures in applied mathematics (Wiley, New York, 1957).
[12] D. A. Darling and M. Kac, Trans. Amer. Math. Soc. **84**, 444 (1957).
[13] S. N. Majumdar, Curr. Sci. **89**, 2076 (2005).

[14] A. Zoia, E. Dumonteil, and A. Mazzolo, Phys. Rev. E **84**, 061130 (2011); **85**, 011132 (2012).
[15] E. Barkai, J. Stat. Phys. **123**, 883 (2006).
[16] L. Turgeman, S. Carmi, and E. Barkai, Phys. Rev. Lett. **103**, 190201 (2009).
[17] J. T. Cox and D. Griffeath, Ann. Prob. **13**, 1108 (1985).
[18] Y. Iwasa and E. Teramoto, J. Math. Biol. **19**, 109 (1984).
[19] B. A. Sevast'yanov, Theory Prob. Appli. **2**, 111 (1958); **3**, 253 (1961).
[20] K. Ramola, S. N. Majumdar, and G. Schehr, Phys. Rev. Lett. **112**, 210602 (2014); Phys. Rev. E **91**, 042131 (2015).
[21] A. Zoia, E. Dumonteil, A. Mazzolo, and S. Mohamed, J. Phys. A: Math. Theor. **45**, 425002 (2012).
[22] C. de Mulatier, A. Mazzolo, and A. Zoia, Europhys. Lett. **107**, 30001 (2014).
[23] A. Zoia, E. Dumonteil, and A. Mazzolo, Phys. Rev. E **84**, 021139 (2011).
[24] T. E. Harris, *The Theory of Branching Processes* (Springer, Berlin, 1963).
[25] K. B. Athreya and P. Ney, *Branching Processes*, Grundlehren Series (Springer-Verlag, New York, 1972).
[26] N. Agmon, Chem. Phys. Lett. **497**, 184 (2010).
[27] A. M. Berezhkovskii, V. Zaloj, and N. Agmon, Phys. Rev. E **57**, 3937 (1998).
[28] S. S. Khokhlova and N. Agmon, J. Chem. Phys. **137**, 184103 (2012).
[29] S. S. Khokhlova and N. Agmon, Bull. Korean Chem. Soc. **33**, 1020 (2012).
[30] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, New York, 2001).
[31] D. S. Grebenkov, Phys. Rev. E **76**, 041139 (2007).
[32] A. Zoia, E. Dumonteil, A. Mazzolo, C. de Mulatier, and A. Rosso, Phys. Rev. E **90**, 042118 (2014).
[33] A. Zoia, E. Dumonteil, and A. Mazzolo, Europhys. Lett. **100**, 40002 (2012).
[34] O. Bénichou, M. Coppey, M. Moreau, P. H. Suet, and R. Voituriez, Europhys. Lett. **70**, 42 (2005).
[35] A. Szabo, J. Phys. Chem. **93**, 6929 (1989).
[36] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed., edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[37] One can also consider that the branching occurs after a diffusion (with the same probability $\beta \Delta t$) in which case Eq. (5) must be modified to take into account of the random displacement $\Delta \mathbf{x}$ before the splitting. However, these modifications (which consist in replacing $\mathbf{x_0}$ by $\mathbf{x_0} + \Delta \mathbf{x}$ in Eq. (5) are of second-order and after having averaged on the random positions $\Delta \mathbf{x}$, the final equation Eq. (9) remains unchanged.
[38] Since Eq. (12), and consequently Eqs. (15), have the general form $\frac{\partial}{\partial t} f(\mathbf{x},t|\mathbf{x_0}) = \mathcal{L}^*_{x_0} f(\mathbf{x},t|\mathbf{x_0}) + a(\mathbf{x},t,\mathbf{x_0})$ where $\mathcal{L}^*_{x_0} f(\mathbf{x},t|\mathbf{x_0}) = D(\Delta_{x_0} - k^2)$ is the backward operator ($k^2 = \beta(1-\nu_1)/D$) and where $a(\mathbf{x},t,\mathbf{x_0})$ is some known function, they admit a solution in terms of the Green function for diffusion $p(\mathbf{x},t|\mathbf{x_0})$, solution of $\frac{\partial}{\partial t} p(\mathbf{x},t|\mathbf{x_0}) = \mathcal{L}^*_{x_0} p(\mathbf{x},t|\mathbf{x_0})$ with the appropriate boundary conditions. In a recent article [32], we developed this approach for studying the clustering of branching Brownian motion (see appendix A in [32] for a full discussion). However, for the present work, the Green function method appears more complicated than the direct approach exposed in [26] (i.e. solving recursively the partial differential equations for the residence time and higher moments), especially in the case of confined geometry where the Green function can only be expressed as a series of eigenfunctions [32].