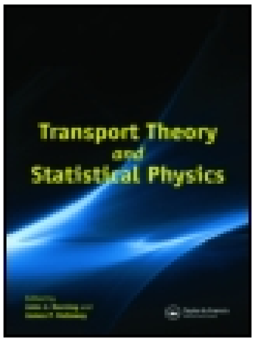

Transport Theory and Statistical Physics

ISSN: 0041-1450 (Print) 1532-2424 (Online) Journal homepage: https://www.tandfonline.com/loi/ltty20

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Richard Sanchez & Norman J. McCormick

Transport Theory and Statistical Physics, 37:236–263, 2008

ISSN: 0041-1450 print / 1532-2424 online
DOI: 10.1080/00411450802526301

ON THE UNIQUENESS OF THE INVERSE SOURCE PROBLEM FOR LINEAR PARTICLE TRANSPORT THEORY

RICHARD SANCHEZ

Commissariat à l'Energie Atomique, Direction de l'Energie Nucléaire, Service d'Etudes de Réacteurs et de Mathématiques Appliquées

NORMAN J. McCORMICK

University of Washington, Department of Mechanical Engineering, Seattle, Washington

Inverse source problems for time-independent linear transport with data from invasive and noninvasive detectors are analyzed. The former inverse problem is proven to have a unique solution, while for the latter we construct counterexamples that prove that the problem is underdetermined for the general case of anisotropic sources and prove that it has a unique solution for isotropic sources and scattering. Using duality we propose and analyze a general class of inverse source algorithms. The emphasis is on establishing new inversion techniques and in proving uniqueness or nonuniqueness as well as to find, when possible, ways to regularize the inverse source problem.

Keywords: Particle transport, Inverse problems, Radon transform, Nonradiating sources, Regularization

1. Introduction

We consider steady-state, linear particle transport in an open domain $D \subset \mathbb{R}^3$ with boundary ∂D of outward normal $\mathbf{n}_+$:

$$\begin{array}{ll} B\psi = S, & x \in X, \\ \psi = \psi_- + \beta\psi, & x \in \Gamma_-, \end{array} \qquad (1)$$

Received 10 October 2007; Accepted 7 May 2008.

One of the authors (RS) thanks Lahbib Bourhrara for helpful discussions and for suggesting the example in Eq. (2) and Mustapha Mokhtar-Kharroubi and Rémi Sentis for their advice.

Address correspondence to Richard Sanchez, Commissariat à l'Energie Atomique, Direction de l'Energie Nucléaire, Service d'Etudes de Réacteurs et de Mathématiques Appliquées, CEA de Saclay, DM2S/SERMA 91191 Gif-sur-Yvette cédex, France. E-mail: richard.sanchez@cea.fr

where $x = (\mathbf{r}, E, \mathbf{\Omega})$ is a point in phase space $X = \{x, \mathbf{r} \in D, E \in \mathcal{E}, \mathbf{\Omega} \in \mathcal{S}_2\}$, $\mathcal{E}$ is the energy domain, $\mathcal{S}_2$ the unit sphere, $\Gamma_- = \{x, \mathbf{r} \in \partial D, E \in \mathcal{E}, \mathbf{\Omega} \in \mathcal{S}_2, \mathbf{\Omega} \cdot \mathbf{n}_+ < 0\}$ is the incoming boundary of X, S and ψ_- are, respectively, the internal source and the incoming angular flux. The Boltzmann operator is

$$B = \mathbf{\Omega} \cdot \nabla + \Sigma - H,$$

with H the scattering operator

$$(H\psi)(x) = \int dE' \int d\mathbf{\Omega}' \Sigma_s(\mathbf{r}, E' \to E, \mathbf{\Omega}' \cdot \mathbf{\Omega}) \psi(\mathbf{r}, E', \mathbf{\Omega}').$$

Also, in (1) β stands for an albedo operator

$$(\beta\psi)(x) = \int_{\Gamma_+} d_b x' \beta(x' \to x) \psi(x'), \ \ x \in \Gamma_-,$$

where $\Gamma_+ = \{x, \mathbf{r} \in \partial D, E \in \mathcal{E}, \mathbf{\Omega} \in \mathcal{S}_2, \mathbf{\Omega} \cdot \mathbf{n}_+ > 0\}$ is the exiting boundary of X and $d_b x = |\mathbf{\Omega} \cdot \mathbf{n}_+| \, dS dE d\mathbf{\Omega}$. Henceforth we shall assume that problem (1) corresponds with a physically realizable situation. An equivalent statement is to say that (1) describes a subcritical problem, i.e., the associated homogeneous problem

$$\begin{aligned} B\psi &= 0, \ \ x \in X, \\ \psi &= 0, \ \ x \in \Gamma_-, \end{aligned}$$

admits the unique solution $\psi = 0$.

Inverse methods associated with linear transport problem (1) seek to recover cross section data Σ and Σ_s (Siewert,1979; Sanchez McCormick, 1981, 1982; Larsen, 1984, 1988; Choulli and Stefanov,1998), the albedo β (Barichello et al., 1997) or the source S (Yi et al.,1992) in terms of measurements of ψ and/or the boundary fluxes ψ_- and $\psi_+ = \psi|_{\Gamma_+}$. Early work is reviewed elsewhere (McCormick,1992). In this work we are interested in the inverse source problem for which cross section and albedo data are known, and one seeks to recover the internal source from flux measurements. Such inverse algorithms are based on differential operators or integral equations that are ill-conditioned and magnify measurement errors (Siewert,1993; Tao et al.,1994;

Sundman et al.,1998; Stephany et al.,2000; Chalhoub de Campos Velho,2003). This is aggravated by the fact that often the experimenter only has access to a partial set of the required data and the remaining data have to be reconstructed by approximate interpolation procedures. In this work we shall bypass the ill-conditioned aspects of inverse algorithms and discuss only the uniqueness of the inverse source problem.

We shall discuss invasive and noninvasive inverse problems. In the former, one has access to the interior flux ψ, while in the latter, only boundary fluxes can be measured. For the invasive inverse problem uniqueness is a consequence of the existence theorem for the solution of linear transport problem (1).

In the special case of the one-group problem (1) for transport at constant energy,

$$\Sigma_s(\mathbf{r}, E' \to E, \mathbf{\Omega}' \cdot \mathbf{\Omega}) = \delta(E - E')\Sigma_s(\mathbf{r}, E, \mathbf{\Omega}' \cdot \mathbf{\Omega}),$$

Larsen (1975) and then Zweifel (1999) have proved uniqueness for the noninvasive inverse problem in a semi-infinite slab but only for isotropic sources. In the presence of anisotropic sources, the noninvasive inverse problem can have more than one solution, as seen by considering the particular case of a three-dimensional medium with purely absorbing medium, $H = 0$, with interior source S. For this case we can use an exponential transformation to define a new source

$$S'(x_R) = e^{sg(R)\tau(x_R, x)} S(x), \; x, x_R \in X,$$

where $x_R = (\mathbf{r} - R\mathbf{\Omega}, E, \mathbf{\Omega})$, $sg(R)$ is the sign of R, $\tau(x_R, x) = \int_0^R \Sigma(x_{R'})\, dR'$ is the optical distance between x_R and x, and R may depend on x with the constraint $x_R \in X$. Clearly, the source $S'(x_R)$ produces the same exiting flux as the source $S(x)$ and therefore the noninvasive inverse source problem does not have a unique solution. A more interesting example that accounts for scattering can be derived via the exponential transformation $\psi(x) \to e^{f(\mathbf{r})}\psi'(x)$, where $f(\mathbf{r})$ is a function with bounded gradient such that $f|_{\partial D} = 0$. Using this form in Equation (1) gives the following equation for $\psi'(x)$:

$$\begin{aligned} B\psi' &= S', & x \in X, \\ \psi' &= \psi_- + \beta\psi', & x \in \Gamma_-, \end{aligned}$$

with the new source

$$S'(x) = e^{-f(\mathbf{r})}(S - \psi\boldsymbol{\Omega}\cdot\nabla f)(x). \tag{2}$$

Notice that the exponential transformation preserves the boundary fluxes; i.e., $\psi'|_\Gamma = .\psi|_\Gamma$. Therefore, both sources S and S' produce the same exiting flux. These examples do not contradict the result established by Larsen for S isotropic because S' is anisotropic, but it suggests the fact that the general noninvasive inverse problem can be degenerate.

Nonuniqueness for the inverse source problem is a well-established fact in the field of acoustics and electromagnetics (Bleistein and Kohen, 1977; Hoenders,1978; Arridge,1998) where it is associated with the existence of so-called nonradiating sources. These sources produce a field that vanishes outside a finite spatial domain. Recently Hoenders (1997) applied this idea to the transport equation to obtain a procedure to construct internal sources that produce a zero exiting flux, and he concluded nonuniqueness for the noninvasive inverse problem as well as the existence of nonradiating sources for linear transport. However, the sources defined by Hoenders' technique are not positive everywhere and his conclusions do not apply to physical (positive) sources. This caveat can be easily avoided, and in Sec. 2 we use Hoenders' procedure to show nonuniqueness for the general noninvasive inverse problem for positive anisotropic sources. Our construction is straightforward but, for completeness, in the Appendix we have revised the machinery of the adjoint Green's function used by Hoenders to demonstrate the existence of non positive, nonradiating sources.

In Sec. 2 we also consider the special case of isotropic sources and prove uniqueness for the noninvasive inverse problem for energy-dependent isotropic sources with isotropic scattering. Our proof consists on recasting the problem in the form of a three dimensional attenuated Radon transform (AtRT) and then using the proof of uniqueness for the inverse of the AtRT (Beylkin, 1984; Finch, 1986; Novikov, 2002a,2002b; Natterer, 2001; Bal, 2004). The section ends with an analysis of a regularization technique for the general source inverse problem and with a discussion of the relation between the inverse source problem and the so-called canonical inverse problem.

In Sec. 3 we briefly examine a family of invasive and noninvasive inverse source methods based on duality and give an explicit example for a homogeneous medium based on the use of Case's singular eigenfunctions. Conclusions are given in the last section where a number of open problems are proposed.

2. Analysis of the inverse source problem

2.1. *Invasive inverse source problem*

We consider transport problem (1) in an open domain D with a regular, piecewise C^1 boundary in the L^p functional setting, i.e. $S \in L^p$, $\psi_- \in L_-^p$ and $\psi \in W^p$, where, with the Lebesgue measures $dx = d\mathbf{r}dEd\mathbf{\Omega}$ and $d_b x = |\mathbf{\Omega} \cdot \mathbf{n}_+|\, dSdEd\mathbf{\Omega}$, $L^p = L^p(X, dx)$, $L_-^p = L^p(\Gamma_-, d_b x)$, and $W^p = \{f, f \in L^p, \mathbf{\Omega} \cdot \nabla f \in L^p, f|_{\Gamma_-} \in L_-^p\}$, $p \in [1, \infty)$. We note that functional space W^p also can be specified by replacing the condition $f|_{\Gamma_-} \in L_-^p$ by $f|_{\Gamma_+} \in L_+^p = L^p(\Gamma_+, d_b x)$ (Dautray and Lions, 1993).

Under general conditions on the cross section data, if problem (1) is subcritical then there is an isomorphism (Bourhrara, 2005; Dautray and Lions, 1993)

$$\begin{aligned} \pi : L^p \times L_-^p &\to W^p, \\ S, \psi_- &\mapsto \psi. \end{aligned}$$

If we consider an invasive inverse source problem, where measurements of the flux in the interior of the domain are made, then the above isomorphism shows that the inverse source problem has a unique solution explicitly given by $S = B\psi$. However, the numerical evaluation of derivatives with unavoidable measurement errors leads to a very ill-conditioned inverse algorithm. In the next section we discuss a better conditioned inverse method that also requires measurements of the boundary fluxes ψ_- and ψ_+, where $\psi_+ = \psi|_{\Gamma_+}$ is the flux exiting the domain.

Often the experimenter has access to only the boundary fluxes. The question arises as to whether such noninvasive source inverse methods have a unique solution. Clearly, the map π induces a linear continuous morphism

$$\begin{aligned} \pi_+ : L^p \times L^p_- &\to L^p_+, \\ S, \psi_- &\mapsto \psi_+. \end{aligned} \tag{3}$$

Unfortunately, the mapping π_+ proves to be only an epimorphism (Dautray,1985) and, therefore, the noninvasive inverse problem $\psi_-, \psi_+ \to S$ does not have a unique solution in general. Now we construct examples to prove that, indeed, the solution of this inverse problem is nonunique.

2.2. *Noninvasive inverse source problem (NISP)*

A necessary and sufficient condition for nonuniqueness is that the kernel of the application $S \to \psi_-, \psi_+$ contain non zero elements. Or, in other words, that there exist a nonradiating source $S_{nr} \neq 0$ such that the transport problem

$$\begin{aligned} B\psi_{nr} &= S_{nr}, \quad x \in X, \\ \psi_{nr} &= 0, \quad x \in \Gamma_-, \end{aligned} \tag{4}$$

admits a solution with zero exiting flux

$$\psi_{nr} = 0, \; x \in \Gamma_+. \tag{5}$$

To prove that the noninvasive inverse source problem is underdetermined for general anisotropic sources, we shall use a recent result in the literature (Hoenders, 1997; Pautz, 2001). The general idea is to construct an internal source S_{nr} that gives zero exiting flux. Let $D_{nr} \subseteq D$ be a bounded open subset, X_{nr} its associated phase space, and $\psi_{nr}(x) \in W^p$ a function with support X_{nr} such that $\psi_{nr}|_{\Gamma \cap D_{nr}} = 0$. Note that ψ_{nr} is absolutely continuous along trajectories, i.e., $\psi_{nr}(\mathbf{r} + t\mathbf{\Omega}, E, \mathbf{\Omega})$ is differentiable in t (Vladimirov, 1963). Clearly, ψ_{nr} is a solution of the transport equation with source $S_{nr} = B\psi_{nr}$ and zero boundary fluxes, $\psi_{nr}|_\Gamma = 0$. This proves that source S_{nr} is a nonradiating source: $(\psi_{nr})_+ = 0$.

A different way of constructing nonradiating sources is by taking the difference between two sources of the type in (2) for two different functions $f_1(\mathbf{r})$ and $f_2(\mathbf{r})$:

$$S_{nr} = e^{-f_1(\mathbf{r})}(S - \psi \mathbf{\Omega} \cdot \nabla f_1) - e^{-f_2(\mathbf{r})}(S - \psi \mathbf{\Omega} \cdot \nabla f_2),$$

where ψ is the flux produced by source S with zero incoming flux. The corresponding flux is $\psi_{nr}(x) = [e^{-f_1(\mathbf{r})} - e^{-f_2(\mathbf{r})}]\psi(x)$.

We conclude that any proof of existence of a unique solution for the NISP has to be based on some sort of regularization; i.e., a restriction on the type of sources considered.

A characterization of nonradiating sources arises from the observation that the flux produced by a physical (positive) source is never zero, at least for physically bounded cross sections, and therefore a nonradiating source S_{nr} cannot be overall positive. Indeed, consider the integral form of the transport equation

$$\psi_{nr}(x) = \psi_{nr}(x_R)e^{-\tau(x_R,x)} + \int_0^R dR' e^{-\tau(x_{R'},x)} q(x_{R'}), \qquad x, x_R \in X, \tag{6}$$

where $x_R = (\mathbf{r} - R\mathbf{\Omega}, E, \mathbf{\Omega})$, $\tau(x_R, x)$ is the optical distance between x_R and x, and

$$q = H\psi_{nr} + S_{nr}$$

is the local emission source. By using (6) with $x \in (\Gamma_{nr})_+$ and $x_R \in (\Gamma_{nr})_-$, where the flux is zero, we find that the integral along the back trajectory within X_{nr} must vanish:

$$\int_0^R dR' e^{-\tau(x_{R'},x)} q(x_{R'}) = 0.$$

Because in a subcritical domain the emission density q produced by a positive source is also positive, we conclude that the source S_{nr} must change sign in X_{nr}.

The lack of positivity is not an obstacle, however, for the construction of internal physical (positive) sources that will produce the same exiting flux ψ_+ on Γ_+. Consider (1) with a positive source S and let D_{nr} be a domain where the minimum value of the source is strictly positive, $\inf_{x \in X_{nr}} S(x) = S_{min} > 0$. In the manner described earlier we can construct an infinite number of sources S_{nr} with support D_{nr} and such that their resultant fluxes cancel on the boundary ∂D_{nr} and in the outside domain. Any of these sources can be scaled so that the overall source $S + S_{nr}$ is positive everywhere. By linearity the solution of (1) with source $S + S_{nr}$ is the sum of the solutions, $\psi + \psi_{nr}$, and, therefore, the exiting flux is the same for positive sources S and $S + S_{nr}$.

Nonuniqueness in the reconstruction of an anisotropic source is illustrated in a recent work by Bal (2004) who consider two-dimensional transport in a purely absorbing medium with an anisotropic source and seeks to determine the source in terms of flux boundary data. In this work he considers anisotropic sources with compact support that can be written as a Fourier series in φ,

$$S(\mathbf{r}, \varphi) = \sum_{k=-N}^{N} S_k(\mathbf{r}) e^{ik\varphi}, \; S_{-k} = \overline{S_k},$$

where $\mathbf{r} \in \mathbb{R}^2$ is the position in the plane and $\varphi \in [0, 2\pi)$ is the planar coordinate for the angular direction, and derives an expression for the inverse of the two-dimensional attenuated Radon transform. He then finds that the measured exiting fluxes provide only two independent equations for the determination of the source and derives an inverse algorithm for the reconstruction of the source

$$S_\omega(\mathbf{r}, \varphi) = S_0(\mathbf{r}) + S_1(\mathbf{r}) \cos(\varphi + \omega) \tag{7}$$

for given $\omega \in [0, 2\pi)$.

This work clearly illustrates that the exiting flux does not provide enough information for the determination of a general anisotropic source. More importantly, Bal demonstrate that there are only two independent equations that can be established from the known boundary data and that, therefore, only two functions $S_0(\mathbf{r})$ and $S_1(\mathbf{r})$ can be determined, as in (7). This is a consequence of the known redundancy of order two of the exiting flux data for the case of an isotropic source. Bal's proof for the uniqueness of the inverse of the AtRT is equivalent to proving that there are no nonradiating sources of the form (7).

1. Example for a slab

Practical constraints on the construction of the flux ψ_{nr} are that it must be bounded with bounded derivatives and that it must vanish on ∂D_{nr}. One can therefore consider factorized fluxes of the form $\psi_{nr}(x) = f(\mathbf{r})g(E)h(\mathbf{\Omega})$, where g and h are bounded and positive and the constraints are to be satisfied by f. As an example, let us consider transport in a one-dimensional slab $z \in (0, L)$. For $z \in D_{nr} = (a, b)$, $0 \leq a < b \leq L$, and with $x = (z, E, \mu)$ we define

$$\psi_{nr}(x) = (z - a)(b - z)g(E)h(\mu).$$

This is a positive flux in (a, b) that vanishes on its boundaries a and b. The corresponding source is obtained as

$$S_{nr}(x) = \theta_{D_{nr}}(z)\{\mu(a + b - 2z)g(E)h(\mu) \\ +(z - a)(b - z)[\Sigma(z, E)g(E)h(\mu) - K(E, \mu)]\},$$

where $K(E, \mu) = [H(gh)](E, \mu)$ and $\theta_{D_{nr}}$ is the characteristic function of D_{nr}: $\theta_{D_{nr}}(z) = 1$ for $z \in D_{nr}$, $\theta_{D_{nr}}(z) = 0$ otherwise. As expected, this source is non positive everywhere, as can be readily seen by realizing that the second contribution vanishes at the boundaries. For the particular case of an isotropic flux, $h(\mu) = 1$, $K(E) = \int dE'\Sigma_s(z, E' \to E)g(E')$. In this particular case, with an isotropic flux in domain D_{nr}, the source is linearly anisotropic.

2.3. *Noninvasive problem with isotropic sources*

The preceding proof of nonuniqueness applies to anisotropic sources. Larsen (1975) showed that a unique solution exists if the source is isotropic. This result was established for one group transport in a homogeneous, semi-infinite slab under the conditions that ψ_- and ψ_+ are Hölder-continuous and that S and ψ are exponentially bounded. Zweifel (1999) proved, for the same one-group problem with no internal sources, that the mapping $\psi_+ \to \psi_-$ is bijective, and this led to a different derivation of Larsen's result.

To our knowledge, however, at the present time the uniqueness result for the noninvasive isotropic source problem has not been extended to the general energy-dependent transport problem in a three dimensional domain. We turn now to a proof of uniqueness for the case of transport with isotropic scattering. In our proof we employ a uniqueness result from the field of emission tomography. We consider transport in a purely absorbing infinite medium with an isotropic source q

$$
\begin{aligned}
(\boldsymbol{\Omega} \cdot \nabla + \Sigma)\psi = q, \;\; x \in X, \\
\psi = 0, \;\; x \in \Gamma_-,
\end{aligned}
\tag{8}
$$

where $q \in L^2$, Σ is sufficiently smooth and both functions have compact support. In emission tomography the inverse source problem consist of determining q from the exiting boundary flux $\psi_+ = \psi|_{\Gamma_+}$. To this end one expresses the flux exiting the system as the three dimensional attenuated Radon transform (AtRT)

$$
(P_\Sigma q)(x) = \int_{\mathbb{R}} dR e^{-\tau(x_R, x_\infty)} q(x_R), \tag{9}
$$

where now $x \in TS^2 = \{(\mathbf{r}, \boldsymbol{\Omega}), \mathbf{r} \in R^3, \boldsymbol{\Omega} \in S^2, \mathbf{r} \cdot \boldsymbol{\Omega} = 0\}$ and $x_R = (\mathbf{r} + R\boldsymbol{\Omega}, \boldsymbol{\Omega})$.

The existence of the inverse of the AtRT has been proven under general conditions on q and Σ for the two dimensional AtRT (Beylkin,1984; Natterer, 2001; Novikov, 2002a, 2002b; Bal, 2004). This result applies to higher dimensions but with increasing redundancy of the data ψ_+. In particular, in three dimensions the source is uniquely reconstructed from the inverses of the two-dimensional AtRTs for all planes orthogonal to a fixed angular direction (Novikov, 2002; Bal Tamasan, 2006).

We now consider the energy-dependent transport problem in a three-dimensional domain D with isotropic scattering and source and zero incoming boundary flux, $\psi_- = 0$. We extend the domain to the entire space by preserving the smoothness and bounded support of Σ and by setting $q = 0$ in $\mathbb{R}^3 \backslash D$ so the extended transport problem can be written as in (8), where now $q = H\psi + S$ contains scattering and source. We note that q is isotropic and that the energy E appears as a parameter. From the existence of the inverse of the three-dimensional AtRT (Novikov,2002) we conclude that there is a unique isotropic q satisfying boundary data (9). It remains to obtain the source. By solving transport Equation (8) we obtain the flux ψ and from it the collision term $H\psi$ and, finally, the source $S = q - H\psi$. Since every step from ψ_+ to the final expression for S preserves uniqueness we have proved

Theorem 1. *The noninvasive inverse source problem for energy-dependent three-dimensional transport admits a unique solution for isotropic scattering and sources.*

In a recent work, Bal and Tamasan (2006) considered one-group transport with anisotropic scattering and used a Neumann series perturbation of the Novikov inversion formula for the AtRT to reconstruct an isotropic source from flux boundary data. The result is valid when the anisotropic component of the scattering is sufficiently small and, therefore, offers a constructive demonstration for theorem 1 in the one-group setting. We recall that Larsen (1975) has proved uniqueness for one-group transport in a semi-infinite slab with arbitrary anisotropy of scattering. Our conjecture is that Theorem 1 must be true for general anisotropy of scattering. In this respect, we note that the general transport source problem can be written as an equivalent problem in a purely absorbing medium, as in (8), with source

$$q = S + H\psi = (1 - HL^{-1})^{-1}S,$$

where H is the scattering operator and $L = \mathbf{\Omega} \cdot \nabla + \Sigma$ is the streaming operator. It seems plausible that the exiting angular flux, shown to have enough information to determine two functions $S_0(\mathbf{r})$ and $S_1(\mathbf{r})$ for the anisotropic source in (7), also can determine in a unique way the isotropic source $S(\mathbf{r})$ hidden in the source term q, regardless of the anisotropy of scattering.

In the next section we investigate the possibility of applying-Theorem 1 to the regularization of the inverse source problem for anisotropic sources and isotropic scattering.

2.4. Regularization of the noninvasive inverse source problem

Here we consider general anisotropic sources of the form $S(\mathbf{r}, E, \mathbf{\Omega})$. We introduce an equivalence relation $\mathcal{R}$ in the set of sources L^p for transport problem (4): $S\mathcal{R}S'$ iff S and S' have the same boundary flux $\psi_\Gamma = \{\psi_-, \psi_+\}$. Relation $\mathcal{R}$ defines a partition of the set of sources into equivalent classes. These classes are the inverse images by π_+ of the elements of $\pi_+(L^p) \subseteq L^p_+$, where π_+ denotes the epimorphism in (3). Hence, denoting by $\mathcal{R}[S]$ the class containing source S, $\mathcal{R}[0] = \pi_+^{-1}\{0\}$ is the set of all nonradiating sources and the difference between two sources in the same class is a nonradiating source. This implies that every class can be identified by a unique element in the class. It follows that the inverse source problem can be regularized if we are able to come up with a rule that identifies a single element in each class. In the case of isotropic scattering we just proved that a class cannot contain two different isotropic sources so we can formulate:

Proposition 1. *For isotropic scattering there is at most one isotropic source for each exiting flux* $\psi_+ \in L^p_+$.

This result defines a natural regularization for the general inverse problem that consists of determining the unique isotropic source satisfying the boundary data. However, this regularization is of limited utility because one has to be sure that the measured exiting flux is the result of an isotropic source. This raises the question of whether there are exiting distributions ψ_+ that are not the result of an isotropic source or, equivalently, if there are classes that do not contain an isotropic element. Unfortunately for the experimentalist the answer to this question is positive, as the following example shows.

Consider transport in a purely absorbing medium and let $\mathcal{R}[S_\omega]$ be the class containing a source as in (7) with $S_1 \neq 0$. Assume now that $\overline{S}_0(\mathbf{r})$ is an equivalent isotropic source. This implies that the problem with source $S_\omega - \overline{S}_0$ produces a zero exiting flux. The last source has the same structure as S_ω and, as shown by Bal (2004), the inverse problem with source (7) has a unique solution so we must have $\overline{S}_0(\mathbf{r}) = S_0(\mathbf{r})$ and $S_1(\mathbf{r}) = 0$. We conclude that $\mathcal{R}[S_\omega]$ does not contain an isotropic source. Bal found that from given exiting flux data there was a unique source of the form (7) for each $\omega \in [0, 2\pi)$, and therefore all the sources $S_{\omega'}$ obtained with Bal's inversion algorithm from the exiting flux produced by S_ω belong to the class $\mathcal{R}[S_\omega]$; however, this does not exhaust the elements in this class because we still can add nonradiating sources: $\mathcal{R}[S_\omega] = S_\omega + \mathcal{R}[0]$. We note, in particular, that the source $S_\omega - S_{\omega'}$ is nonradiating.

One could argue that the example is of limited value because it applies only to transport in a purely absorbing medium. To show that the example also holds in the more general context of energy-dependent transport with isotropic scattering we first generalize

Bal's uniqueness result to a class of weakly anisotropic sources. This class is implicitly defined by the set of functions

$$f(\varphi) = \sum_{k=1}^{N} (f_k e^{ik\varphi} + \overline{f_k}, e^{-ik\varphi}),\ f_1 \neq 0,\ \int_0^{2\pi} d\varphi f(\varphi) = 0 \quad (10)$$

with Fourier coefficients f_k such that the norm of operator K (defined bellow) is smaller than one.

Proposition 2. *Consider two dimensional transport in a purely absorbing medium with total cross section* $\Sigma(\mathbf{r})$ *and an anisotropic source of the form*

$$S(\mathbf{r}, \varphi) = S_0(\mathbf{r}) + S_1(\mathbf{r}) f(\varphi),\ \mathbf{r} \in \mathbb{R}^2, \varphi \in [0, 2\pi), \quad (11)$$

where Σ *is smooth and decaying exponentially at infinity,* S_0 *and* S_1 *have compact support and* $f(\varphi)$ *can be written as a finite Fourier series as in (10). Then, for weakly anisotropic* $f(\varphi)$, S_0 *and* S_1 *can be uniquely determined from the inversion of the AtRT* $P_\Sigma S$.

Proof. Our proof is based on the analysis in (Bal, 2004) that we briefly summarize hereafter. Following Bal we pose $z = x + iy$ and $\lambda = e^{ik\varphi}$ and write the transport equation for $\psi(z, \lambda) = \psi(\mathbf{r}, \varphi)$:

$$(\lambda\partial_z + \lambda^{-1}\partial_{\overline{z}} + \Sigma)\psi = S.$$

In this equation $\partial_z = (1/2)(\partial_x - i\partial_y)$, $\partial_{\overline{z}} = (1/2)(\partial_x + i\partial_y)$, $\Sigma(z)$ and $S(z, \lambda) = \sum_{k=-N}^{N} f_k(z)\lambda^k$ with $f_0(z) = S_0(z)$ and $f_k(z) = S_1(z) f_k$ for $k \neq 0$. By considering this equation for $\lambda \in \mathbb{C}$, one verifies that $\psi(z, \lambda)$ is sectionally analytic for $|\lambda| > 1$ and $0 < |\lambda| < 1$, but has a pole of finite multiplicity at $z = 0$ and is not necessarily of order $O(\lambda^{-1})$ at infinity. Next, by subtracting the diverging terms at $z = 0$ and $|z| = \infty$, one verifies that the jump of the modified function $\phi(z, \lambda)$ across $|\lambda| = 1$ is a function of ψ_+. Therefore, the reconstruction problem can be cast as a Riemann-Hilbert problem for $\phi(z, \lambda)$. Once this function is computed from the inverse of the Hilbert transform, the reconstruction formulas are obtained from the expression of $\psi(z, \lambda)$ in the vicinity of $\lambda = 0$. The technique used by Bal consists of expressing $\psi(z, \lambda)$ in terms of the Green's function, $G(z, \lambda) = [sgn(|\lambda| - 1)]/[\pi(\lambda\overline{z} - \lambda^{-1}z)]$, of operator $(\lambda\partial_z + \lambda^{-1}\partial_{\bar{z}})$ and expanding the resulting expression in powers of λ:

$$\psi(z,\lambda) = \begin{cases} \sum_{k=1}^{\infty} (\mathcal{H}_k S)(z,\lambda)\lambda^k, & |\lambda| < 1, \\ \sum_{k=1}^{\infty} (\overline{\mathcal{H}_k} S)(z,\lambda)\lambda^{-k}, & |\lambda| > 1. \end{cases}$$

Here the $\mathcal{H}_k$ are integral operators, related to Cauchy integrals, that satisfy

$$\sum_{k=1}^{\infty} \lambda^k (\lambda\partial_z + \lambda^{-1}\partial_{\bar{z}} + \Sigma)\mathcal{H}_k(z) = \mathcal{I},$$

where $\mathcal{I}$ is the identity operator, and, therefore, can be explicitly computed by equating like powers of λ in the above expression. In particular, for $k = 0$ we have $\partial_{\bar{z}}\mathcal{H}_1 = 1$. The final result is that there are only two independent reconstruction formulas (for details see Bal, 2004) that, for the source in (11), are :

$$\sum_{k=1}^{N} (\overline{f_k}\mathcal{H}_k - f_k\overline{\mathcal{H}_k}) S_1 = \varphi_0,$$
$$\mathcal{H}_1 S_0 + (\sum_{k=2}^{N+1} \overline{f_{k-1}}\mathcal{H}_k - \sum_{k=I}^{N-1} f_{k+1}\overline{\mathcal{H}_k}) S_1 = \varphi_1.$$

In these equations $\varphi_0(z)$ and $\varphi_1(z)$ depend on the measured exiting fluxes. We write the first equation as $(A + B)S_1 = \varphi_0$, where operator $A = \overline{f_1}\mathcal{H}_1 - f_1\overline{\mathcal{H}_1}$ has an inverse (Bal, 2004) and $B = \sum_{k=2}^{N} (\overline{f_k}\mathcal{H}_k - f_k\overline{\mathcal{H}_k})$ is bounded. This equation has a unique solution if the Fourier coefficients $\{f_k, k = 2, N\}$ are small enough so that the norm of operator $K = A^{-1}B$ is smaller than 1. Once $S_1(z)$ has been determined, we use the formula $\partial_{\bar{z}}\mathcal{H}_1 = \{1$ to obtain $S_0(\mathbf{r})$ from the second equation:

$$S_0 = \partial_{\bar{z}}\varphi_1 - \left[\partial_{\bar{z}} \left(\sum_{k=2}^{N+1} \overline{f_{k-1}}\mathcal{H}_k - \sum_{k=1}^{N-1} f_{k+1}\overline{\mathcal{H}_k} \right) \right] S_1.$$

□

Clearly, the class of weakly anisotropic sources contains the case $f(\varphi) = \cos(\varphi + \omega)$, $\omega \in [0, 2\pi)$ considered by Bal. We can then state

Proposition 3. *Under the conditions of Proposition 2 the noninvasive source problem for energy-dependent three dimensional transport with isotropic scattering and an anisotropic source of the form*

$$S(x) = S_0(\mathbf{r}, E) + S_1(\mathbf{r}, E)\, f(\mathbf{\Omega}),\ \mathbf{r} \in \mathbb{R}^3,\ E \in \mathcal{E},\ \mathbf{\Omega} \in \mathcal{S}_2, \tag{12}$$

admits a unique solution for $f(\mathbf{\Omega})$, $\int d\mathbf{\Omega}\, f(\mathbf{\Omega}) = 0$, *weakly anisotropic.*

Proof. We write the transport source problem with collisions as an equivalent problem $\mathcal{P}$ in a purely absorbing medium with source $q = S + H\psi$. For isotropic scattering $\overline{S}_0 = H\psi$ is isotropic and q is of the form (12). Let $\mathbf{n}$ be a unit vector and consider transport problem $\mathcal{P}$ on a plane σ orthogonal to $\mathbf{n}$. Note that the angular direction $\mathbf{\Omega}$ for particles streaming on σ can be described by a single coordinate $\varphi_\sigma \in [0, 2\pi)$ so $f(\mathbf{\Omega})|_\sigma = h_0 + h(\varphi_\sigma)$ with $\int_0^{2\pi} d\varphi_\sigma\, h(\varphi_\sigma) = 0$. Hence, the restriction of source (12) to σ can be written as in (11): $S(x)|_\sigma = S_{\sigma 0}(\mathbf{r}, E) + S_1(\mathbf{r}, E)\, h(\varphi_\sigma)$, where $S_{\sigma 0} = (\overline{S}_0 + S_0 + h_0 S_1)\big|_\sigma$. By Proposition 2, for weakly enough anisotropy this source can be uniquely reconstructed on σ by inversion of the two dimensional AtRT from the exiting flux data on σ. Thus, choosing $\mathbf{n}$ such that $f(\varphi_\sigma) = f(\mathbf{\Omega})|_\sigma \neq 0$, for every energy E the planar problem has a unique inverse solution $q|_\sigma$ with $(\overline{S}_0 + S_0)\big|_\sigma$ and $S_1|_\sigma$ uniquely determined. The reconstruction of the three dimensional sources $S_0 + \overline{S}_0$ and S_1 is achieved by applying this procedure to every plane orthogonal to $\mathbf{n}$. Note by $\psi_+[S_1 f]$ the exiting flux produced by source $S_1(\mathbf{r}, E)\, f(\mathbf{\Omega})$. Then, by theorem 1 there is a unique isotropic source S_0 that produces the exiting flux $\psi_+[S_0] = \psi_+[S] - \psi_+[S_1 f]$. □

In the above reconstruction one needs to write $f|_\sigma$ for $\mathbf{\Omega} \cdot \mathbf{n} = 0$ as a function $f(\varphi_\sigma)$ of the planar angular coordinate φ_σ. Let (θ, φ) and $(\theta, \varphi)_n$ be the polar and azimuthal coordinates of $\mathbf{\Omega}$ and $\mathbf{n}$ in the fixed system of coordinates (x, y, z) with unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. We define a local system of coordinates $(x, y, z)_\sigma$ for transport on plane σ by applying a rotation R such that $R\mathbf{e}_z = (\mathbf{e}_z)_\sigma = \mathbf{n}$. We choose $R = R_2 R_1$, where R_1 is the rotation of axis z and angle φ_n and R_2 is the rotation of axis $R_1\mathbf{e}_y$ and angle θ_n. By measuring the planar angular coordinate φ_σ by the angle between

$\boldsymbol{\Omega}$ and $R\mathbf{e}_x$ we have

$$\begin{aligned}\mu &= -\sqrt{1-\mu_n^2}\cos\varphi_\sigma,\\ \tan\varphi &= (\mu_n\tan\varphi_n + \tan\varphi_\sigma)/(\mu_n - \tan\varphi_n\tan\varphi_\sigma),\end{aligned} \tag{13}$$

where $\mu = \boldsymbol{\Omega}\cdot\mathbf{e}_z = \cos\theta$ and $\mu_n = \mathbf{n}\cdot\mathbf{e}_z = \cos\theta_n$.

From proposition 3 it follows that in the general setting of energy-dependent transport with isotropic scattering the class $\mathcal{R}[S]$ with S as in (12) and $S_1 \neq 0$ does not contain any isotropic source for $f(\boldsymbol{\Omega})$ weakly anisotropic. A simple example is an anisotropic source with $f(\boldsymbol{\Omega}) = \boldsymbol{\Omega}\cdot\mathbf{e}_z$ for which $f(\varphi_\sigma)$ behaves as $\cos\varphi_\sigma$, for $\mu_n \neq 0$. More general examples can be constructed by using the fact that the reconstruction can be done for any $\mathbf{n}$ such that $f(\boldsymbol{\Omega})|_\sigma \neq 0$. Assume an anisotropic factor of the form $f(\boldsymbol{\Omega}) = g(\mu)h(\varphi)$. For $\mu_n = \pm 1$ we have $f(\varphi_\sigma) = g(0)h(\varphi_\sigma \pm \varphi_n)$, then for any $g(\mu)$ such that $g(0) \neq 0$ and for any $h(\varphi)$ weakly anisotropic, in particular for $h(\varphi) = \cos\varphi$, the source in (12) can be reconstructed in a unique way. Also, for $\mu_n = 0$ we have $f(\varphi_\sigma) = g(-\cos\varphi_\sigma)h(\pi/2 \pm \varphi_n)$ and the same conclusion applies for any $h(\varphi)$ such that $h(\pi/2 \pm \varphi_n) \neq 0$ and for $g(\mu) = \mu$.

The implication of Proposition 3 is that, with no a priori knowledge of the type of source producing the radiation field the experimentalist can only use the flux boundary data to try to determine an isotropic source. If the source is found then the problem has an infinite number of solutions of the form $S + S_{nr}$, where S_{nr} is a nonradiating source. If the problem does not admit an isotropic solution then one has to look for an anisotropic one. However, the ill-conditioning due to the inverse algorithm and to the errors in the measured data can mar the boundary between the two situations and one may find an isotropic solution for a problem that, with exact measurements, does not accept one.

2.5. *Relation with the canonical inverse problem*

Zweifel (1999) has defined the canonical inverse problem as finding the entering flux from the exiting flux for a transport problem without sources,

$$\begin{aligned} B\psi &= 0, \quad x \in X,\\ \psi &= \psi_-, \quad x \in \Gamma_-,\end{aligned} \tag{14}$$

and has demonstrated uniqueness for the inverse map $\psi_- \to \psi_+$ for one-group transport in a semi-infinite slab.

Here we analyze the relation between the canonical and the source inverse problems. We introduce the mapping

$$\begin{aligned} \mathrm{i} : L^p_- &\to L^p, \\ \psi_- &\mapsto S \end{aligned} \tag{15}$$

that associates to ψ_- its first collision source $S = \mathrm{i}\psi_- = HT\psi_-$. Here H is the scattering operator and $T\psi_-$ is the uncollided flux

$$(T\psi_-)(x) = e^{-\tau(x_-,x)}\psi_-(x_-),$$

where x_- is the entering point for the trajectory through x. The solution of (14) can be written as the sum of the uncollided and the collided fluxes, $\psi = T\psi_- + \psi_{col}$, where the latter is solution of the transport equation with first collided source $\mathrm{i}\psi_-$ and zero incoming flux:

$$\begin{aligned} B\psi_{col} &= \mathrm{i}\psi_-, \quad x \in X, \\ \psi_{col} &= 0, \qquad x \in \Gamma_-. \end{aligned}$$

In a semi-infinite slab geometry no particle entering the domain can leave it without undergoing collisions and, therefore, ψ_- and its first collision source $\mathrm{i}\psi_-$ give the same exiting flux. For this geometry, mapping (15) associates to each ψ_- a unique source $\mathrm{i}\psi_-$ that is equivalent to ψ_- in the sense that it produces the same exiting flux. As an aside comment we note that $\mathrm{i}\psi_-$ is not the only source with this property: any source of the form $\mathrm{i}\psi_- + S_{nr}$, where S_{nr} is a nonradiating source, will give the same exiting flux. For isotropic scattering source $\mathrm{i}\psi_-$ is isotropic and theorem 1 shows that $\mathrm{i}\psi_-$ is the only isotropic source that gives the same exiting flux than ψ_-, which proves the equivalence between the canonical and the isotropic source inverse problems for energy dependent transport in a semi-infinite slab with isotropic scattering.

Regarding the uniqueness of the canonical inverse problem in three dimensional geometry, it is obvious that uniqueness exists for transport in a purely absorbing medium. Whether this result extends to the case with scattering remains an open problem.

3. Inverse source methods based on duality

3.1. General case

A general formulation for the inverse source problem can be obtained from duality (Sanchez,1998). The duality relation for the transport equation is a consequence of Green's formula

$$(f, Bg) = (B^* f, g) + < f, g >, \quad f \in L^{p'}, g \in L^p, \tag{16}$$

where $B^* = -\mathbf{\Omega} \cdot \nabla + \Sigma - H^*$ is the formal adjoint of B, H^* is similar to H but with kernel $\Sigma_s^*(\mathbf{r}, E' \to E, \mathbf{\Omega}' \cdot \mathbf{\Omega}) = \Sigma_s(\mathbf{r}, E \to E', \mathbf{\Omega}' \cdot \mathbf{\Omega})$, p' is the conjugate index of p,

$$(f, g) = \int_X dx(fg)(x) \tag{17}$$

is the volume scalar product and

$$< f, g >= \int_\Gamma \mathbf{\Omega} \cdot \mathbf{dS} dE d\mathbf{\Omega}(fg)(x) \tag{18}$$

is the associated surface scalar product. The latter can be written as the difference between the contributions on the exiting and entering boundaries, $< f, g >=< f, g >_+ - < f, g >_-$ with the scalar products

$$< f, g >_\pm= \int_{\Gamma\pm} d_b x(fg)(x).$$

To the direct source problem (1) we associate the adjoint problem

$$\begin{aligned} B^* \psi^* &= S^*, \qquad x \in X, \\ \psi^* &= \psi_+^* + \beta^* \psi^*, \ x \in \Gamma_+ \end{aligned} \tag{19}$$

with $S^* \in L^{p'}$, $\psi_+^* \in L_+^{p'}$, and $\psi^* \in W^{p'}$. From duality formula (16) with $f \to \psi^*$ and $g \to \psi$ we write

$$(\psi^*, S) = (S^*, \psi) + < \psi^*, \psi > . \tag{20}$$

The last equation provides an inverse source method that requires volume and surface measurements of the flux. (An elementary example of this was constructed for slab geometry by Siewert (1978)). The source can be obtained from (20) by looking for a solution in a finite subspace $\mathcal{F}$. A formal approach consists of introducing a partition of the geometric domain $D = \cup D_i$ and looking for a piecewise source in $\mathcal{F} = \oplus \mathcal{F}_i$, where $\mathcal{F}_i = \{f(\mathbf{r})g(E)h(\boldsymbol{\Omega}),\ f \in P_N,\ g \in \mathcal{E}_G,\ h \in Q_K\}$. Typically, P_N is a finite polynomial space with support D_i, $\mathcal{E}_G$ is a piecewise constant (multigroup) approximation, and Q_K is the subspace generated by the spherical harmonics of degree $\leq K$. The resulting numerical inverse algorithm reads

$$\sum_{i,n,m,k} S_{i,nmk}(\psi^*,\ f_{i,n}g_{i,m}h_{i,k}) = (S^*,\psi) + <\psi^*,\psi> .$$

In order to have a determined system of equations for the $S_{i,nmk}$, one would have to compute as many independent adjoint fluxes as there are source coefficients. However, the construction of the auxiliary adjoint pairs (ψ^*, S^*) for use in (20) can be done by postulating any suitable functions ψ^* and computing the corresponding sources and boundary values as $S^* = B^*\psi^*$ and $\psi^*|_\Gamma$. These adjoint fluxes also can be defined in terms of factorized functions, $\psi^*(x) = f(\mathbf{r})g(E)h(\boldsymbol{\Omega})$, and even with discontinuous f's. In the latter case, the discontinuities will be accounted for by introducing delta-like sources in (20). Better yet, to avoid the discontinuities, one may follow the ideas discussed in Section 2 and construct fluxes $(\psi^*)'$ with supports in regions that are more accessible to measurements. This choice would, of course, give information for the source only in the supports of the $(\psi^*)'$. Finally, note that if the adjoint problems are solved with the adjoint albedo, i.e. $<\beta^* f, g>_+ = <f, \beta g>_-$, then $<\psi^*,\psi> = <\psi^*_+,\psi>_+ - <\psi^*,\psi_->_-$ and, therefore, the albedo does not enter the inverse source algorithm.

A different approach to source reconstruction for invasive inverse methods has been suggested by Mokhtar-Kharroubi (1997); in this work, the author introduces suitable measures in velocity space and particular constraints in the physical data and the source to prove that the source can be explicitly obtained from two moments of the angular flux (Mokhtar-Kharroubi,1997; Mokhtar-Kharroubi and Zeghal, 2000).

In practice, inside measurements can prove to be difficult or impossible to obtain, such as for planetary atmospheres and most biological applications. In order to obtain a noninvasive source inverse method, the adjoint problem in (19) has to have zero source, $S^* = 0$. For this case (20) yields the inverse method

$$(\psi^*, S) \ = \ < \psi^*, \psi >, \tag{21}$$

which only requires measurement of the entering and exiting fluxes at the surface of the domain. However, as discussed in Section 2, this inverse method does not have a unique solution for anisotropic sources. This means that in the presence of unknown anisotropic sources one has to revert to inverse method (20) and, therefore, detailed flux measurements will have to be carried inside the domain. However, for isotropic scattering or anisotropic scattering in one-group theory the problem admits a unique isotropic solution.

3.2. Use of singular eigenfunctions

Often in practical applications of radiation problems one may neglect the energy dependence and consider a one-group problem. For an isotropic medium this implies that $\mathcal{R}B^* = B\mathcal{R}$ and $\mathcal{R}\beta^* = \beta\mathcal{R}$, where $\mathcal{R}$ is the operator that inverts the angular direction, $(\mathcal{R}f)(\mathbf{\Omega}) = f(-\mathbf{\Omega})$. Hence, the solution of adjoint problem (19) can be written as $\psi^* = \mathcal{R}\widetilde{\psi}$, where $\widetilde{\psi}$ is the solution of a related direct problem:

$$\begin{array}{rl} B\widetilde{\psi} = \widetilde{S}, & x \in X, \\ \widetilde{\psi} = \widetilde{\psi}_- + \beta\widetilde{\psi}, & x \in \Gamma_- \end{array} \tag{22}$$

with $\widetilde{S} = \mathcal{R}S^*$ and $\widetilde{\psi}_- = \mathcal{R}\psi^*_-$.

Furthermore, if the medium is homogeneous, then one can use Caseology (Case and Zweifel, 1967) to obtain analytical solutions for this problem. For simplicity we consider here the case of slab geometry for the noninvasive inverse algorithm (21) for which $\widetilde{S} = \mathcal{R}S^* = 0$. Then (22) admits homogeneous solutions of the form

$$\psi_\nu(x) = e^{-\tau/\nu}\phi_\nu(\mu), \ \ \nu \in \sigma, \tag{23}$$

where $\phi_\nu(\mu)$ are Case's singular eigenfunctions normalized to $\int_{-1}^{1} \phi_\nu(\mu)\, d\mu = 1$. The spectrum σ contains a continuum $[-1, 1]$ for which the ϕ_ν's are distributions and a finite discrete spectrum σ_d with regular eigenfunctions. Here we have measured the spatial location in optical distance units so that $x = (\tau, \mu)$ and $B = \mu\partial_\tau + 1 - H$.

Singular eigenfunctions have been extensively discussed in the literature; e.g., Case and Zweifel (1967) or McCormick and Kuščer (1973). Here we limit ourselves to mention that the ϕ_ν's form a complete orthogonal base for Hölder continuous functions in $[-1, 1]$ and satisfy the full-range orthogonality relation

$$[\phi_\nu, \phi_{\nu'}] = \int_{-1}^{1} \mu\, d\mu\, (\phi_\nu \phi_{\nu'})(\mu) = N(\nu)\delta(\nu - \nu'),$$

where $N(\nu)$ is the square of the norm of ϕ_ν and the delta function is to be understood as defined over σ.

In the present case it is more expedient to directly use the homogeneous solution $\psi^* = \mathcal{R}\widetilde{\psi}$, with $\widetilde{\psi}$ as in (23), in (21) and disregard any boundary conditions constraint. Then the noninvasive inverse source method (21) gives the "distributed-source law"

$$(e^{\tau/\nu}\phi_\nu, S) = < e^{\tau/\nu}\phi_\nu, \psi >, \ \ \nu \in \sigma, \tag{24}$$

where we have used the property $\mathcal{R}\phi_\nu = \phi_{-\nu}$. A particular case is that of a purely absorbing medium for which the eigenvalue spectrum consists only of $\sigma = [-1, 1]$ and $\phi_\nu = \delta(\nu - \mu)$, $-1 \le \nu \le 1$. In this case S can be viewed as a blackbody source and (24) is an equation sometimes solved in temperature inversion computations for atmospheres (Liou, 2002).

Another interesting special case of (24) arises for a Dirac delta source, $S = S_\delta(\tau)\delta(\tau - \tau_0)$. With $\tau_0^+ = \tau_0 + \varepsilon$ and $\tau_0^- = \tau_0 - \varepsilon$, we now have

$$e^{\tau_0/\nu} < \phi_\nu, [\psi - S_\delta(\tau_0) + O(\varepsilon)] >= 0.$$

Because the term in brackets depends only on μ and the ϕ_ν are complete, for $\varepsilon \to 0$ we obtain the "localized-source law" (Case Zweifel, 1967)

$$< \psi >= \mu^{-1} S_\delta(\tau_0).$$

Equation (24) also provides a quick way of proving that the solution of the inverse source problem for a half-space $\tau \geq 0$ has a unique isotropic solution. For such a source, only the spectrum $\nu \leq 0$ may be used to avoid $e^{\tau/\nu}$ for $\tau \to \infty$, so (24) becomes

$$\int_0^\infty d\tau S(\tau) e^{\tau/\nu} = -\int_{-1}^1 \mu d\mu \psi(0, \mu) \phi_\nu(\mu), \quad \nu \leq 0.$$

Substitution of $s = -1/\nu$ shows that $S(\tau)$ is

$$S(\tau) = -\mathcal{L}^{-1} \left[\int_{-1}^1 \mu d\mu \psi(0, \mu) \phi_{-1/s}(\mu) \right], \quad s > 0,$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform. From this point on, the proof of Larsen (1975) can be followed to prove uniqueness of $S(\tau)$.

Finally, with regard to the case of the invasive inverse transport problem in (20), one could construct pairs of solutions $(\widetilde{\psi}, \widetilde{S})$ of the form

$$\begin{array}{l} \widetilde{\psi}(x) = A(\tau) \psi_\nu(x) \\ \widetilde{S}(x) = (\partial_\tau A)(\tau) \mu \psi_\nu(x) \end{array}, \quad \nu \in \sigma,$$

where $\widetilde{S}(x) = (\partial_\tau A)(\tau) \mu \psi_\nu(x)$ is any differentiable function, and extract the boundary fluxes by taking traces at the boundaries.

4. Conclusions

Inverse source problems in linear transport theory can be classified as invasive and noninvasive, depending on whether they require interior flux measurements or only measurements of the boundary fluxes. Uniqueness for the invasive case follows from the existence theorem for the direct transport problem. This is not the case, however, for the noninvasive method. In this work we have used a constructive technique given by Hoenders (1997) to construct counterexamples of positive sources that result in the same exiting flux, thus proving that the noninvasive inverse problem has a nonunique solution for the case of general anisotropic sources. The proof is based on the construction of a source that produces a zero exiting flux.

For the case of isotropic scattering we have proved that such sources not only must change of sign but they also have to be anisotropic. Thanks to the last property we have been able to prove uniqueness for the noninvasive problem for isotropic sources and scattering with no constraints in the dimensionality of the geometry or energy domains. In contrast, the previous proofs of uniqueness (Larsen, 1975; Zweifel, 1999) that were restricted to one-group transport in a semi-infinite slab were based on the use of an analytical technique to recover the source from boundary data.

An open problem is to prove or disprove this result for energy-dependent, three-dimensional geometry and general anisotropy of scattering. Note that this is equivalent to a proof that a nonradiating source necessarily must be anisotropic. The flux resulting from a nonradiating source vanishes outside some domain D_{nr} and this gives a means to analyze the local behavior of the flux in the vicinity of the boundary of the domain. The flux ψ_{nr} created by a nonradiating source S_{nr} obeys (4) and (5). Let $\mathbf{\Omega}$ be an angular direction exiting the boundary ∂D_{nr} at x. Because ψ_{nr} is differentiable a.e.w. along particle trajectories, the value of the source at the interior location $x_\varepsilon = (\mathbf{r} - \varepsilon\mathbf{\Omega}, E, \mathbf{\Omega})$, $\varepsilon > 0$, is given by

$$S_{nr}(x_\varepsilon) = [-\partial_\varepsilon \psi_{nr} + (\Sigma - H)\psi_{nr}](x_\varepsilon).$$

As $\varepsilon \to 0$ the collision term goes to zero and the term $\partial_\varepsilon \psi_{nr}$ becomes dominant so $S_{nr}(x_\varepsilon) \sim -(\partial_\varepsilon \psi_{nr})(x_\varepsilon)$. By changing the sign of ε this argument applies also to an entering trajectory. We conclude that near the boundary $S_{nr} \sim \mathbf{\Omega} \cdot \nabla \psi_{nr}$. Because the flux behaves linearly near the boundary along any trajectory one may conclude that S_{nr} is anisotropic or that both the source and the gradient of the flux vanish on the boundary ∂D_{nr}. This argument is neither rigorous, especially on the detail of how fast $(H\psi_{nr})(x_\varepsilon)$

goes to zero as $\varepsilon \to 0$, nor conclusive but hints to the fact that under some general conditions a nonradiating source should be anisotropic. We believe, however, that the proof has to come from global analysis, that is, from an extension of the construction of the inverse of the AtRT to general anisotropy of scattering. Indeed, the AtRT reconstruction of an isotropic source with isotropic scattering uses only the exiting fluxes with directions orthogonal to a fixed unit vector $\mathbf{n}(\mu, \phi) = \mu\mathbf{e}_z + \sqrt{1-\mu^2}(\cos\phi\mathbf{e}_x + \sin\phi\mathbf{e}_y)$. Therefore, one could exploit the redundancy of the data for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi)$ in order to extend the result to general anisotropic scattering.

We have introduced an equivalence relation between sources: two sources are equivalent if they are indistinguishable by external flux measurements. This equivalence relation gives a partition of the set of sources into classes, where every class can be defined by a unique element of the class. This could have led to a regularization technique for the inverse source problem that would have consisted of determining the isotropic source in the class. Unfortunately, as we prove with examples, there exist anisotropic sources that are not equivalent to any isotropic source.

The relation between the inverse source problem and the canonical inverse problem, which consists of constructing the exiting flux in terms of the entering one for a sourceless problem, also has been analyzed. We have considered energy-dependent transport with isotropic scattering in a semi-infinite slab and shown that for every entering flux ψ_- there is a unique isotropic source that produces the same exiting flux, thus extending Zweifel's result for one-group transport to the energy-dependent case. A problem that remains to be solved is whether a comparable result holds for general 3D geometry. With $T\psi_-$ and $\mathrm{i}\psi_-$, respectively, the uncollided flux and the first collision source produced by the entering flux ψ_-, the question is whether the mapping

$$\begin{gathered} \gamma : L^p_- \to L^p_+, \\ \psi_- \mapsto T\psi_- + \psi_+[\mathrm{i}\psi_-] \end{gathered}$$

is one-to-one. This is equivalent to prove that there is no $\psi_- \neq 0$ that produces a zero exiting flux without internal sources.

Finally, we have analyzed a general class of invasive and noninvasive inverse source algorithms based on the transport duality relation. With invasive methods it is even possible to construct an inverse algorithm that targets the source structure in a given region and that requires only measurements of the flux within the region. Although such a procedure may be better conditioned, the method is equivalent to the brutal reconstruction $S = B\psi$. For the case of noninvasive inverse methods we have shown how the use of Case's eigenfunctions can be applied to define an inverse algorithm to recover the structure of internal isotropic sources.

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Appendix

Existence of nonradiating sources based on duality

We consider transport Equation (1) in the W^p functional setting. In the spirit of Sec. 2, let $D' \subset D$ be an interior domain and $\psi' \in W^p(X')$ be a function that is positive, bounded with bounded gradient, and such that $\psi'\big|_{\Gamma'} = 0$. We extend ψ' to $W^p(X)$ by posing $\psi'(x) = 0$ for $x \in X''$, where X'' is the complementary of X' in X. We shall prove that the extended function is a solution of the direct transport problem in X, which implies that the flux exiting and entering D is zero.

Consider the adjoint equation for the adjoint Green's function $G_y^*(x) = G^*(y \to x)$:

$$\begin{aligned} B^* G_y^* &= \delta_y, \quad x \in X, \\ G_y^* &= 0, \quad x \in \Gamma_+, \end{aligned} \tag{25}$$

where $\delta_y(x) = \delta(x - y)$ is the unit Dirac mass at $y \in X$.

For any measurable regular subset Z of phase space X we define the scalar product

$$(f, g)_Z = \int_Z dx'(fg)(x')$$

and from the definition of the adjoint we obtain for $f \in L^{p'}$ and $g \in L^p$:

$$(f, Bg)_Z + < f, g >_{\Gamma(Z)_-} = (B^* f, g)_Z + < f, g >_{\Gamma(Z)_+}. \tag{26}$$

In this equation $\Gamma(Z)_\pm$ are the incoming ($-$) and outgoing ($+$) boundaries of Z and

$$< f, g >_{\Gamma(Z)_\pm} = \int_{\Gamma(Z)_\pm} d_b x' (fg)(x').$$

In particular, by using complementarity relation (26) with $g = \psi'$ and $f = G_y^*$, the solutions of the direct and the adjoint transport Equations (1) and (25), we obtain

$$(G_y^*, S)_Z + < G_y^*, \psi' >_{\Gamma(Z)_-} = \theta_Z(y)\psi'(y) + < G_y^*, \psi' >_{\Gamma(Z)_+},$$

where θ_Z is the characteristic function of Z: $\theta_Z(x) = 1$ for $x \in Z$, $\theta_Z(x) = 0$ otherwise.

Finally, we apply the preceding duality relation to $Z = X''$. By noticing that $\Gamma''_- = \Gamma'_+ \cup \Gamma_-$ and $\Gamma''_+ = \Gamma'_- \cup \Gamma_+$ and by accounting for the boundary condition for G_y^* and for the fact that $\psi' = 0$ on $\partial D'$ and $S = 0$ in X'' we obtain

$$\theta_{X''}(y)\psi'(y) = < G_y^*, \psi' >_{\Gamma_-} .$$

For $y \in X'$ we find that $< G_y^*, \psi' >_{\Gamma_-} = 0$. Then for $y \in X''$ we get $\psi' = 0$ in X'' and, therefore, on Γ_+.