Interaction between a bubbly flow and a moving obstacle

N. Chikhi, Sylviane Pascal-Ribot

Force on a transverse obstacle in a bubbly flow

N. Chikhi [a,*], S. Pascal-Ribot [a]

[a] *Commissariat à l'Energie Atomique DEN/DTN/SMTM/LMTR, CEA/Cadarache, 13108 Saint Paul Lez Durance, Cedex, France*

Abstract

In this paper, we derive an analytic expression for the force exerted by a bubbly flow on a transverse obstacle (sphere or cylinder). The bubbly fluid is described with the help of the Kogarko, Iordanski and Van Wijngaarden model. The flow is supposed to be potential. In the acoustic approximation, the wave equation, satisfied by the velocity potential, is derived. Coupled with the boundary conditions, the resolution of this equation leads to the force exerted on the obstacle. In a stationary flow, we generalize the d'Alembert paradox. When the obstacle is moved by harmonic oscillations, we find that the force is the sum of two terms, the added mass and the friction force. Finally, we apply the formula in the case of an air-water mixture.

Key words: Fluid-structure interaction, bubbly fluid, acoustic approximation

* Corresponding author. Tel:+33 442 25 40 36; Fax: +33 442 25 77 67
Email address: `nourdine.chikhi@cea.fr` (N. Chikhi).

1 Introduction

A fundamental understanding of the fluid-structure interactions in bubbly flows is essential for many applications in industry (nuclear, oil, oceanography...). For example, two-phase cross-flow exists in nuclear steam generators. It generates dynamic fluid forces which may induce structural vibrations and lead to component failures. Hence, the calculation of the force exerted by a two-phase flow on an obstacle is an important (and still open) problem. Several experimental studies deal with the two-phase flow induced vibrations and an overview on this topic can be found in [1]. They show, in particular, that the force exerted by two phase flows are more important than the force exerted by single phase flows. By contrast, there is a lack of theoretical approach for this problem.

The aim of this paper is to calculate the force exerted by a bubbly flow on an obstacle (sphere or cylinder) and to obtain an analytical expression of this force depending on the obstacle geometry and on the void fraction. To do this, it is useful to refer to the literature on single phase flows. The incompressible, potential, inviscid flows past an obstacle have been the subject of many theoretical investigations (see [2]). The main result is that the force exerted by the fluid on the obstacle is proportional and opposite to the obstacle acceleration (added mass effect). In [3], Lamb studied the force exerted by compressible flows of barotropic fluids on an obstacle such as a sphere or a cylinder (see section 3). In the case of the obstacle oscillations to and fro in a straight line, he calculated the force on the obstacle and showed that it was the sum of two terms: the added mass effect and a friction force due to the compressibility.

It is interesting to investigate the effects of the presence of bubbles in such a flow and this is what we have done here.

In this article, we consider an obstacle (sphere or cylinder) moving through an inviscid, infinite liquid containing gas bubbles. The bubbly liquid is described with the help of the Kogarko [4], Iordanski [5] and Van Wijngaarden [6] model (section 3). In this model, the bubbly liquid is a homogenous fluid in which any effect connected with the bubbly character is disregarded except for the compressibility. The fluid is supposed to be initially at rest. The amplitude of the obstacle motion is supposed to be small with respect to the obstacle size. Hence, the flow is potential (see [7]) and the flow perturbations can be considered as first order terms (acoustic approximation). We derive the equation satisfied by the velocity potential (section 4). This equation is studied when the obstacle oscillates to and fro in a straight line. The study of the obstacle oscillations is interesting because it corresponds to a physical situation in nuclear industry. We consider that the obstacle is a sphere in section 5 and a cylinder in section 6 . Then, we calculate an analytical expression of the force which, as in the case of compressible flows, is the sum of two terms, the added mass effect and a friction force. In the case of an air-water mixture, we discuss the force behavior with respect to the void fraction, the obstacle size and the oscillation frequency.

2 Position of the problem

Consider an obstacle with axial symmetry (sphere or cylinder, radius R) moving through an infinite bubbly liquid. The obstacle velocity is denoted $\overrightarrow{u(t)}$.

The fluid is initially at rest and the flow is supposed to be potential. The pressure, velocity and density fields in the fluid are respectively denoted $p(t, \overrightarrow{r})$, $\overrightarrow{v}(t, \overrightarrow{r})$ and $\rho(t, \overrightarrow{r})$ where t is the time coordinate and $\overrightarrow{r}$ is the position vector having for origin the center of the obstacle. The goal is to calculate the force exerted by the fluid on the obstacle.

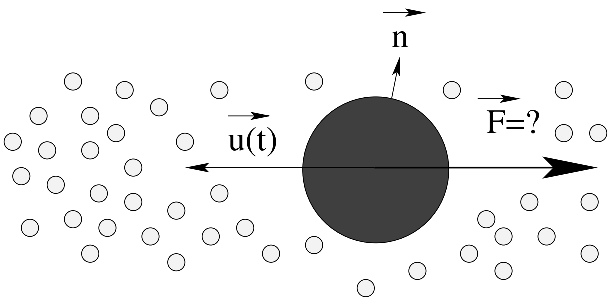

Fig. 1. Position of the problem

This force is expressed as:

$$\overrightarrow{F} = -\int\int_{\Gamma} p\overrightarrow{n}\, dS \tag{1}$$

where Γ is the obstacle surface and $\overrightarrow{n}$ is the unit vector normal to Γ. In order to calculate this force, we have to solve the motion equations in the bubbly fluids with respect to the boundary conditions at infinity

$$\overrightarrow{v}(t, \overrightarrow{r}) \to \overrightarrow{0}, |\overrightarrow{r}| \to \infty \tag{2}$$

and on the obstacle surface Γ

$$\overrightarrow{v}(t, \overrightarrow{r}) \cdot \overrightarrow{n} = \overrightarrow{u}(t) \cdot \overrightarrow{n}. \tag{3}$$

An equivalent problem would be to study the force on a fixed obstacle exerted by a bubbly flow with the boundary condition at infinity:

$$\overrightarrow{v}(t, \overrightarrow{r}) \to \overrightarrow{u}(t), |\overrightarrow{r}| \to \infty \tag{4}$$

and on the obstacle surface Γ

$$\overrightarrow{v}(t, \overrightarrow{r}) \cdot \overrightarrow{n} = \overrightarrow{0} \qquad (5)$$

For sake of simplicity, we will solve the problem with the boundary conditions (2) (3).

2.1 Incompressible, inviscid, potential flows

In this subsection, we remind the way to calculate the force exerted by an incompressible, inviscid, potential flow on an obstacle [2]. Although it concerns single phase flows, this result is a reference to discuss the force exerted by a bubbly flow. In the case of an incompressible and inviscid flow, the equations of motion are:

$$\rho_l \frac{d\overrightarrow{v}}{dt} = -\overrightarrow{\nabla} p \qquad (6)$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{v} = 0 \qquad (7)$$

where the liquid density ρ_l is constant. $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \overrightarrow{v} \cdot \overrightarrow{\nabla}$.

With the assumption of a potential flow $\overrightarrow{v} = \overrightarrow{\nabla}\varphi$, the velocity potential φ satisfies the Laplace equation

$$\Delta\varphi = 0 \qquad (8)$$

and the boundary conditions

$$\frac{\partial\varphi}{\partial r} \to 0, \quad |\overrightarrow{r}| \to \infty \qquad (9)$$

$$\frac{\partial\varphi}{\partial n}_{|\Gamma} = \overrightarrow{u} \cdot \overrightarrow{n} \qquad (10)$$

where $r = |\overrightarrow{r}|$. The resolution of the system (8) (9) (10) yields the force acting on the obstacle:

$$\overrightarrow{F} = -m_a \frac{d\overrightarrow{u}}{dt} \qquad (11)$$

where m_a is the added mass of the obstacle. It depends only on the geometry and the liquid density. For a sphere, m_a is defined by:

$$m_a = \frac{2}{3}\pi R^3 \rho_l \tag{12}$$

and corresponds to a half sphere full of liquid.

For a cylinder (height H), it becomes:

$$m_a = \pi R^2 H \rho_l \tag{13}$$

and corresponds to a cylinder full of liquid.

If the flow is stationary $\left(\frac{d\vec{u}}{dt} = \vec{0}\right)$, the force is equal to zero as stated by the d'Alembert paradox.

2.2 Compressible, inviscid, potential flows

Let us now consider the same problem for compressible flows of barotropic fluids. The equations of motion are:

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} p \tag{14}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{15}$$

$$p = f(\rho). \tag{16}$$

Few results exist in the non linear case. In [8], Finn and Gilbarg studied the compressible flows past an obstacle. In particular, they generalized the d'Alembert paradox in the case of subsonic flows.

In the acoustic approximation and with the assumption of a potential flow, the resolution of linearized equations leads to the so-called d'Alembert equation (satisfied by the velocity potential φ):

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = 0 \tag{17}$$

where the speed of sound c is defined by $c^2 = \dfrac{\partial p}{\partial \rho}$. Equation (17) coupled with the boundary conditions can be solved to calculate the force. For example, in [3], Lamb calculated the force exerted on an obstacle which oscillated (pulsation ω) to and fro in a straight line. He obtained (calculations are detailed in the sections 5 and 6) :

for a spherical obstacle,

$$\overrightarrow{F} = -\frac{4}{3}\pi\rho_l R^3 \frac{2 + k_c^2 R^2}{4 + k_c^4 R^4}\frac{d\overrightarrow{u}}{dt} - \frac{4}{3}\pi\rho_l R^3 \frac{k_c^3 R^3}{4 + k_c^4 R^4}\omega\overrightarrow{u} \tag{18}$$

for a cylindrical obstacle,

$$\overrightarrow{F} = \pi R^2 H\rho_l \mathcal{R}e\left[\frac{D_1(k_c R)}{k_c R D_1'(k_c R)}\right]\frac{d\overrightarrow{u}}{dt} - \pi R^2 H\rho_l \mathcal{I}m\left[\frac{D_1(k_c R)}{k_c R D_1'(k_c R)}\right]\omega\overrightarrow{u} \tag{19}$$

where D_1 is a Bessel function of the second kind and with $k_c = \dfrac{\omega}{c}$. $\mathcal{R}e$ (resp. $\mathcal{I}m$) is the real (resp. imaginary) part.

In each case, the force is the sum of two terms, the added mass effect and a friction force due to the compressibility. The expression for the force can be rewritten:

$$\overrightarrow{F} = -m_* T_a \frac{d\overrightarrow{u}}{dt} - m_* T_f \omega\overrightarrow{u} \tag{20}$$

The dimensionless coefficients T_a and T_f correspond respectively to the added mass effect and to the friction force. The mass m_* corresponds to $\dfrac{4}{3}\pi R^3\rho_l$ for a sphere and $\pi R^2 H\rho_l$ for a cylinder.

2.3 Potential flow for bubbly fluids

For single phase flows, the velocity potential is defined by $\overrightarrow{v} = \overrightarrow{\nabla}\varphi$ meaning that three unknowns are replaced by one. Then, the number of equations becomes greater than the number of unknowns either for the system (6) (7) (incompressible flows) or the system (14) (15) (16)(compressible flows). Such a system is said to be overdetermined. In general, such a system will have no solution. Nevertheless, here, both systems always admit a solution, due to their mathematical properties.

For bubbly fluids, the situation is different. The equations of motion are given by [10]:

$$\rho \frac{d\overrightarrow{v}}{dt} = -\overrightarrow{\nabla} p \tag{21}$$

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot (\rho \overrightarrow{v}) = 0 \tag{22}$$

$$p = f\left(\rho, \frac{d\rho}{dt}, \frac{d^2\rho}{dt^2}\right) \tag{23}$$

Gavrilyuk *et al* prove [9] that if the velocity potential is defined by $\overrightarrow{v} = \overrightarrow{\nabla}\varphi$, then the system (21)(22)(23) has no solution. They also propose the following definition

$$\overrightarrow{v} = \overrightarrow{\nabla}\varphi + \overrightarrow{g}\left(\rho, \frac{d\rho}{dt}\right) \tag{24}$$

Then, the system (21)(22)(23) admits a solution.

In the acoustic approximation (the equation of motion (21)(22)(23) are linearized), they show that the definition $\overrightarrow{v} = \overrightarrow{\nabla}\varphi$ leads to a compatible overdetermined system. For that reason, we will use it in section 4.

3 Bubbly fluids model

In this section, we present the model of Kogarko, Iordanski and Van Wijngaarden adopted in the present study. The validity of this model has been discussed in [11] and [12]. It can be used to describe bubbly fluids where the gas concentration is below 15%. The bubbles are supposed to be small and spherical. They pulse due to the effect of the compressibility of gas. The slip velocity between the gas and the liquid phases is assumed to be zero. It means that the viscosity is neglected everywhere in the liquid excepted in the region near the gas-liquid interface.

The bubbly fluid is described as an incompressible inviscid liquid containing $N(t, \overrightarrow{r})$ gas bubbles per unit volume. The bubble radius is denoted $R_b(t, \overrightarrow{r})$. This model is restricted to low void fractions $\alpha = \frac{4}{3}\pi N R_b^3$ so that the mixture can be described as a homogeneous medium with only one field for either the density $\rho(t, \overrightarrow{r})$, or the pressure $p(t, \overrightarrow{r})$ or the velocity $\overrightarrow{v}(t, \overrightarrow{r})$. The mixture density is defined by

$$\rho = (1 - \alpha)\rho_l + \alpha\rho_g \tag{25}$$

where ρ_l and ρ_g are respectively the liquid and gas densities.

Let us write the equations of motion of bubbly fluids. First, the mass and the momentum conservation laws can be expressed as:

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot (\rho \overrightarrow{v}) = 0 \tag{26}$$

$$\rho \frac{d\overrightarrow{v}}{dt} = -\overrightarrow{\nabla} p \tag{27}$$

The system is closed by an equation of state connecting the pressure and the density. In the case of compressible fluids, it is assumed that $p = f(\rho)$. This relation is no longer true for bubbly fluids. Actually, the pressure in bubbly fluids depends on the density and on its time derivatives: $p = f\left(\rho, \frac{d\rho}{dt}, \frac{d^2\rho}{dt^2}\right)$. For that reason, it is called sometimes "memory fluids".

The closure equation is obtained by considering a single bubble in a potential flow. It means that the interactions between bubbles are neglected. If both the liquid viscosity and the superficial tension are negligible, we get the Rayleigh-Lamb equation:

$$p_g - p = \rho_l \left(R_b \frac{d^2 R_b}{dt^2} + \frac{3}{2} \left(\frac{dR_b}{dt} \right)^2 \right), \tag{28}$$

where p_g is the gas pressure.

The bubble number is conserved:

$$\frac{\partial N}{\partial t} + \overrightarrow{\nabla} \cdot (N \overrightarrow{v}) = 0. \tag{29}$$

The mass concentration of gas C_g is supposed to be constant. It means that for a given mass of bubbly fluid, the mass of gas and the mass of liquid are also constant.

$$C_g = \frac{\alpha \rho_g}{\rho} = \text{constant} \tag{30}$$

Equations (26) and (29) suggest to introduce the constant number $n = \frac{N}{\rho}$ which represents the number of bubbles per unit mass. We can then write a direct relation between the mixture density and the bubble radius:

$$\frac{4}{3}\pi R_b^3 n = \frac{1}{\rho} - \frac{C_g}{\rho_l} \tag{31}$$

According to equation (28), we have $p = f\left(R_b, \frac{dR_b}{dt}, \frac{d^2 R_b}{dt^2}\right)$ so that

$$p = f\left(\rho, \frac{d\rho}{dt}, \frac{d^2\rho}{dt^2}\right) \tag{32}$$

To be complete, it remains to characterize the gas behaviour and, in particular, the thermodynamics. The gas evolution is supposed to be isentropic so that the pressure follows the Laplace equation:

$$p_g(R_b) = p_0 \left(\frac{R_{0b}}{R_b}\right)^{3\gamma}, \tag{33}$$

where R_{0b} is the mean bubble radius.

To summarize, following the present model, a bubbly fluid can be considered as a homogeneous medium with particular properties due to the definition of the pressure. The equations of motion can be written as:

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot (\rho \overrightarrow{v}) = 0 \tag{34}$$

$$\rho \frac{d\overrightarrow{v}}{dt} = -\overrightarrow{\nabla} p \tag{35}$$

$$p = f\left(\rho, \frac{d\rho}{dt}, \frac{d^2\rho}{dt^2}\right) \tag{36}$$

4 Wave equation in bubbly fluids

4.1 Acoustic approximation

Consider the motion of a body through a bubbly fluid initially at rest in the acoustic approximation. We also assume that the disturbance due to the obstacle presence is negligible so that the governing equations can be linearized. This involves that the amplitude of obstacle oscillations are small compared to its size. If the perturbations are denoted by prime, we get:

$$\overrightarrow{v} = \overrightarrow{v_0} + \overrightarrow{v}', \quad \text{with } \overrightarrow{v_0} = \overrightarrow{0} \tag{37}$$

$$\rho = \rho_0 + \rho', \quad \rho' << \rho_0 \tag{38}$$

$$p = p_0 + p', \quad p' << p_0 \tag{39}$$

$$R_b = R_{0b} + R'_b, \quad R'_b << R_{0b} \tag{40}$$

Then, we obtain the linearized system previously found by Gavrilyuk *et al* [9]:

$$\frac{\partial \rho'}{\partial t} + \overrightarrow{\nabla} \cdot (\rho_0 \overrightarrow{v}') = 0 \tag{41}$$

$$\rho_0 \frac{\partial \overrightarrow{v}'}{\partial t} = -\overrightarrow{\nabla} p' \tag{42}$$

$$p' = a^2 \rho' + b^2 \frac{\partial^2 \rho'}{\partial t^2} \tag{43}$$

with $b = \frac{R_{0b}}{\sqrt{\alpha}}$ [10] and a is the sound velocity in the bubbly fluids. Following the Woods formula, it depends on the gas and liquid densities and on the gas concentration:

$$a^2 = \frac{1}{\rho\left(\frac{\alpha\rho_g}{c_g^2} + \frac{(1-\alpha)\rho_l}{c_l^2}\right)} \tag{44}$$

where c_g and c_l are the sound velocity respectively in the gas and in the liquid. As we know, the Woods velocity a is lower than the speed of sound in single phase fluids (c_g, c_l). For example, at ambient temperature and pressure:

$$c_l = 1450\ m.s^{-1}, \quad c_g = 340\ m.s^{-1}, \quad \text{for } \alpha = 4\% \quad a = 20\ m.s^{-1} \tag{45}$$

4.2 Wave equation

The bubbly flow is supposed to be potential. As mentioned in section 2.3, the velocity potential is given by:

$$\overrightarrow{v} = \overrightarrow{v}' = \overrightarrow{\nabla}\varphi \tag{46}$$

Let us now introduce the velocity potential in equation (42):

$$\rho_0 \frac{\partial}{\partial t}\left(\overrightarrow{\nabla}\varphi\right) = -\overrightarrow{\nabla} p' \tag{47}$$

We can deduce the linearized Bernoulli relation:

$$p' + \rho_0 \frac{\partial \varphi}{\partial t} = 0 \tag{48}$$

Since the pressure and the potential are directly related, it is sufficient to determine the potential in order to calculate the force.

If the system (41) (42) (43) is rewritten in terms of p', ρ' and φ, we find:

$$\frac{\partial \rho'}{\partial t} = -\rho_0 \Delta \varphi \tag{49}$$

$$p' = a^2 \rho' + b^2 \frac{\partial^2 \rho'}{\partial t^2} \tag{50}$$

By suppressing the density and the pressure, we finally obtain the wave equation satisfied by the velocity potential:

$$\frac{\partial^2 \varphi}{\partial t} - a^2 \Delta \varphi = b^2 \Delta \frac{\partial^2 \varphi}{\partial t} \tag{51}$$

This equation is similar to the classical wave equation for inviscid compressible fluids since a is the speed of sound in the bubbly fluids. The righthand side is an additive term characterizing the wave dispersion due to the presence of bubbles. This original equation is useful when studying the wave propagations in bubbly fluids.

Equation (51) admits an infinite number of solutions. To provide the physical solution of our problem, we have to consider two boundary conditions. First, we suppose that the flow at infinity is not disturbed by the obstacle motion and the bubbly fluid is initially at rest, which gives:

$$\left| \frac{\partial \varphi}{\partial \overrightarrow{r}} \right| \to 0, \quad |\overrightarrow{r}| \to \infty \tag{52}$$

Second, we write the boundary condition at the obstacle surface:

$$\frac{\partial \varphi}{\partial n}_{|\Gamma} = \overrightarrow{u} \cdot \overrightarrow{n} \tag{53}$$

4.3 Stationary flow: the d'Alembert paradox

Let us suppose that the obstacle velocity $\overrightarrow{u}$ is constant. Then, the flow is stationary and every time derivative is equal to zero:

$$\frac{\partial \varphi}{\partial t} = 0 \tag{54}$$

The wave equation (51) becomes the Laplace equation:

$$\Delta \varphi = 0 \tag{55}$$

The force is found equal to zero so that the so-called d'Alembert paradox has been generalized for bubbly fluids in the acoustic approximation.

4.4 Non-stationary flow: harmonic oscillations

We consider now that the obstacle is oscillating along a fixed axis in the bubbly flow. The obstacle oscillations are characterized by a pulsation ω and a velocity $\overrightarrow{u}$:

$$\overrightarrow{u}(t) = \overrightarrow{u}_0 e^{i\omega t} \tag{56}$$

The time dependence of the perturbations can also be written as follows:

$$f(\overrightarrow{r}, t) = f(\overrightarrow{r}) e^{i\omega t} \tag{57}$$

Also equation (51) becomes:

$$-\omega^2 \varphi - a^2 \Delta \varphi = -b^2 \omega^2 \Delta \varphi \tag{58}$$

Using the Helmholtz form, we find

$$\Delta\varphi + k_b^2\varphi = 0 \tag{59}$$

where k_b is defined by the dispersion relation in bubbly fluids:

$$k_b^2 = \frac{\omega^2}{a^2 - b^2\omega^2} \tag{60}$$

In the following, two cases are considered. In section 5, the obstacle is a sphere and a cylinder in section 6. In each case, the Helmholtz equation (59), coupled with the boundary conditions (52) (53), is solved and the force exerted on the obstacle is calculated.

5 Application 1: force on a sphere

5.1 Analytical resolution

Consider a sphere moving parallel to a fixed axis (denoted z) with a velocity equal to:

$$\overrightarrow{u} = \overrightarrow{u}_0 e^{i\omega t}, \quad \text{with } \overrightarrow{u}_0 = u_0 \overrightarrow{z} \tag{61}$$

It is natural to refer the fluid motion in a spherical polar coordinate system (r, θ, ϕ) where the sphere center is the reference origin. The boundary condition (53) becomes:

$$\frac{\partial\varphi}{\partial r}_{|\Gamma} = u_0 \cos\theta \tag{62}$$

Several studies have been carried out on the Helmholtz equation (see Lamb [3]). The solutions are

$$f(r) = \frac{e^{ik_b r}}{r} \tag{63}$$

as well as its derivatives.

In our problem, the fluid motion is caused by the small oscillations of the obstacle. It is equivalent to a flow generated by a dipole in the center of the sphere (see Lamb). The solution takes the form:

$$\varphi(r) = C\frac{\partial}{\partial z}\left(\frac{e^{ik_b r}}{r}\right) = C\cos\theta\frac{d}{dr}\left(\frac{e^{ik_b r}}{r}\right) \tag{64}$$

with $z = r\cos\theta$.

The boundary condition (62) gives for $r = R$:

$$C = \frac{(2 - k_b^2R^2 - 2ik_bR)u_0R^3}{4 + k_b^4R^4} \tag{65}$$

Considering the symmetries of the problem, the force is:

$$\overrightarrow{F} = F\overrightarrow{z} \tag{66}$$

$$F = \int_0^\pi p\cos\theta.2\pi R^2\sin\theta d\theta \tag{67}$$

The pressure is given by the Bernoulli relation. At the obstacle surface, we get:

$$p = -i\omega\rho_0 C\cos\theta\frac{d}{dR}\left(\frac{e^{ik_bR}}{R}\right) \tag{68}$$

The integration over the sphere surface leads to:

$$\overrightarrow{F} = -\frac{4}{3}\pi\rho_0R^3\frac{2 + k_b^2R^2}{4 + k_b^4R^4}i\omega\overrightarrow{u} - \frac{4}{3}\pi\rho_0R^3\frac{k_b^3R^3}{4 + k_b^4R^4}\omega\overrightarrow{u} \tag{69}$$

or

$$\overrightarrow{F} = -\frac{4}{3}\pi\rho_0R^3\frac{2 + k_b^2R^2}{4 + k_b^4R^4}\frac{d\overrightarrow{u}}{dt} - \frac{4}{3}\pi\rho_0R^3\frac{k_b^3R^3}{4 + k_b^4R^4}\omega\overrightarrow{u} \tag{70}$$

Let us introduce $m_* = \frac{4}{3}\pi\rho_0R^3$ and two dimensionless numbers:

$$T_{abs} = \frac{2 + k_b^2R^2}{4 + k_b^4R^4} \tag{71}$$

$$T_{fbs} = \frac{k_b^3 R^3}{4 + k_b^4 R^4} \tag{72}$$

We get:

$$\overrightarrow{F} = -m_* T_{abs} \frac{d\overrightarrow{u}}{dt} - m_* T_{fbs} \omega \overrightarrow{u} \tag{73}$$

5.2 Discussion

The force exerted by the bubbly flow on the sphere is the sum of two terms. The first term, proportional and opposite to the obstacle acceleration, is the added mass of the obstacle. The second force is opposed to the obstacle motion, as it is proportional and opposite to the obstacle velocity: it is a friction force due to the compressibility of the gas bubbles.

This expression (70) is similar to the Lamb expression (18) for compressible flows (see section 2.2). The difference is in the definition of the wave number. For compressible flows, the wave number is $k_c^2 = \frac{\omega^2}{c^2}$ where c is the speed of sound in the medium. For the bubbly flows, we found $k_b^2 = \frac{\omega^2}{a^2 - \omega^2 b^2}$. Comparisons will be later performed.

Let us now introduce the dimensionless parameter $x = k_b R$. This parameter is used to study the behavior of the force and, in particular, the behavior of the coefficients T_{abs} and T_{fbs}.

It compares two lengths: the obstacle size and the wave length. If x tends to zero, the wave length is large compared with the obstacle radius. This corresponds to the incompressible flow limit.

$$T_{abs} \to \frac{1}{2}, \quad T_{fbs} \to 0 \tag{74}$$

$$\overrightarrow{F} \to -\frac{2}{3}\pi R^3 \rho_l \frac{d\overrightarrow{u}}{dt} \tag{75}$$

which is similar to the results of section 2.1.

The evolution of the coefficient T_{abs} with respect to the parameter x is described in figure 2. It passes through a maximum for $x_m \simeq 0.91$. The extreme values are $T_{abs}(0) = \frac{1}{2}$ and $T_{abs}(x_m) \simeq 0.6$. For $x > x_m$, T_{abs} decreases and $T_{abs}(x \to \infty) \to 0$.

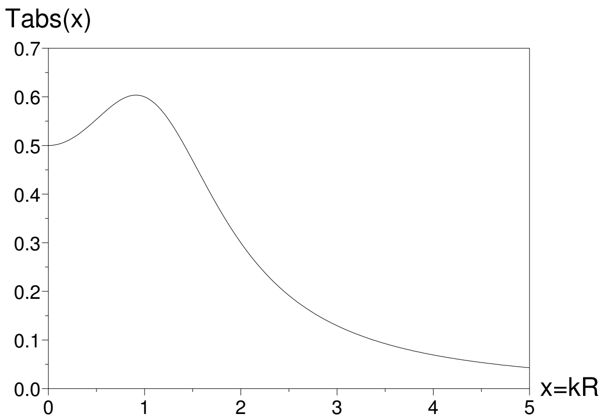

Fig. 2. The behavior of the added mass coefficient for a sphere: T_{abs}

The evolution of the coefficient T_{fbs} is described in figure 3: It passes through a maximum for $x'_m \simeq 1.86$. The extreme values are $T_{fbs}(0) = 0$ and $T_{fbs}(x'_m) \simeq 0.4$. For $x > x'_m$, T_{fbs} decreases and $T_{fbs}(x \to \infty) \to 0$

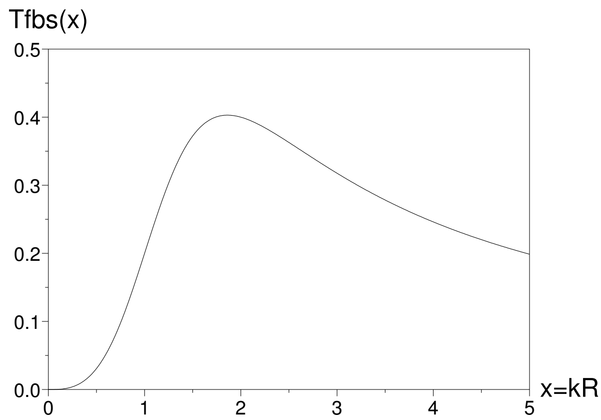

Fig. 3. The behavior of the friction coefficient for a sphere: T_{fbs}

Suppose now that the bubbly fluid is an air-water mixture with $0 < \alpha < 10\%$, $\rho_l = 1000\ kg.m^{-3}$, $\rho_g = 1.2\ kg.m^{-3}$, $R_{0b} = 0.001\ m$. The obstacle ($R = 1\ cm$) oscillates with the pulsation $\omega = 600\ rad.s^{-1}$ (i.e. the frequency is $f \simeq 100\ Hz$). In these conditions, the parameter x verifies $0.004 < x < 0.3 < x_m$

Then, in this range of values, the coefficients T_{abs} and T_{fbs} increase with x together with the force on the obstacle. To be more concrete, we will now study the evolution of the force with the void fraction, the obstacle radius and the oscillation pulsation. It is easy to see that $b^2\omega^2 \ll a^2$ and then $k_b \simeq \frac{\omega}{a}$. The speed of sound a decreases with the void fraction. Consequently, $x = k_b R$ increases with the void fraction, which means that the coefficients T_{abs} and T_{fbs}, and the force on the obstacle increase with the void fraction. In the same way, we see that x increases with R and ω, so that the force on the obstacle increases with the obstacle size and the oscillation pulsation.

6 Application 2: force on a cylinder

6.1 Analytical resolution

Consider now the 2D flow of a bubbly fluid around a cylinder. The wave equation and the boundary conditions remain the same in the 3D case (see section 5):

$$\Delta\varphi + k_b^2\varphi = 0 \tag{76}$$

with the boundary condition:

$$\frac{\partial\varphi}{\partial n}_{|\Gamma} = \overrightarrow{u} \cdot \overrightarrow{n} \tag{77}$$

It is now natural to use a cylindrical polar coordinate system, defined by (r, θ, z). Here, the problem is unchanged by translation along the z-axis. Then, the potential will not depend on z. The boundary condition (77) becomes:

$$\frac{\partial \varphi}{\partial r}_{|\Gamma} = u_0 \cos\theta \tag{78}$$

We seek a solution of the form:

$$\varphi = Af(r)\cos(m\theta) \tag{79}$$

Introducing this equation in the wave equation, we obtain:

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \left(k_b^2 - \frac{m^2}{r^2}\right) f = 0 \tag{80}$$

The solutions of this equation are the Bessel functions J_m and Y_m. As we work with complex notations, it is convenient to use the Bessel function of the second kind D_m defined by:

$$D_0(r) = -Y_0(r) - iJ_0(r) \tag{81}$$

$$D_m(r) = r^m \left(-\frac{d}{rdr}\right)^m D_0(r) \tag{82}$$

with the Mehler formula (see [3]):

$$J_0(r) = \frac{2}{\pi}\int_0^\infty \sin(r\cosh(u))du \tag{83}$$

$$Y_0(r) = \frac{2}{\pi}\int_0^\infty \cos(r\cosh(u))du \tag{84}$$

The expression of the boundary condition (78) suggests a solution of the following form:

$$\varphi(r) = AD_1(k_b r)\cos\theta \tag{85}$$

A is a constant which can be determinated with the help of (78):

$$A = \frac{u_0}{k_b D_1'(k_b R)} \tag{86}$$

Applying the Bernoulli relation, the pressure over the cylinder surface at $r = R$ can be expressed:

$$p = -i\omega\rho_0 u_0 \frac{D_1(k_b R)}{k_b D_1'(k_b R)} \cos\theta \tag{87}$$

The resulting force on the cylinder is

$$\overrightarrow{F} = \pi\rho_0 R^2 \frac{D_1(k_b R)}{k_b R D_1'(k_b R)} i\omega \overrightarrow{u} \tag{88}$$

or

$$\overrightarrow{F} = -m_* T_{abc} \frac{d\overrightarrow{u}}{dt} - m_* T_{fbc} \omega \overrightarrow{u} \tag{89}$$

with

$$T_{abc} = -\mathcal{R}e\left[\frac{D_1(k_b R)}{k_b R D_1'(k_b R)}\right], \quad T_{fbc} = \mathcal{I}m\left[\frac{D_1(k_b R)}{k_b R D_1'(k_b R)}\right] \tag{90}$$

6.2 Discussion

The results obtained with a cylinder are close to those obtained with a sphere. The force exerted by the bubbly flow on the cylinder is the sum of two terms, the added mass effect and a friction force.

If $x = k_b R$ tends to zero, the wave length is large compared with the obstacle radius.

$$T_{abc} \to 1, \quad T_{fbc} \to 0 \tag{91}$$

$$\overrightarrow{F} \to -\pi R^2 H \rho_l \frac{d\overrightarrow{u}}{dt} \tag{92}$$

It corresponds to the limit of incompressible flows and the above result is consistent with (11) (see section 2.1).

The behavior of the added mass coefficient T_{abc} is described in figure 4. It passes through a maximum for $x_{mc} \simeq 0.4$. The extremal values are $T_{abc}(0) = 1$

and $T_{abc}(x_{mc}) \simeq 1.13$. For $x > x_{mc}$, T_{abc} decreases and $T_{abc}(x \to \infty) \to 0$.

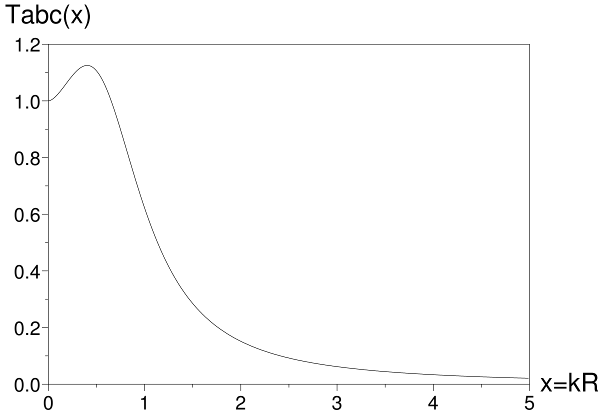

Fig. 4. The behavior of the added mass coefficient for a cylinder: T_{abc}

The behavior of the friction coefficient T_{fbc} is described in figure 5. It passes through a maximum for $x'_{mc} \simeq 1$. The extremal values are $T_{fbc}(0) = 1$ and $T_{fbc}(x'_{mc}) \simeq 0.74$. For $x > x'_{mc}$, T_{fbc} decreases and $T_{fbc}(x \to \infty) \to 0$.

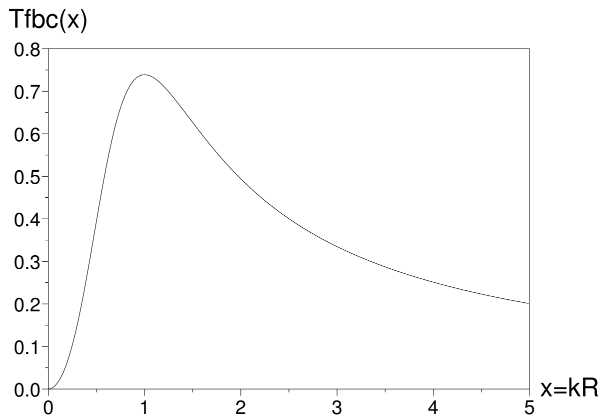

Fig. 5. The behavior of the friction coefficient for a cylinder: T_{fbc}

By analogy with application 1, we consider now that the bubbly fluid is an air-water mixture with $0 < \alpha < 10\%$, $\rho_l = 1000\ kg.m^{-3}$, $\rho_g = 1.2\ kg.m^{-3}$,$R_{0b} = 0.001m$. The obstacle ($R = 1\ cm$) oscillates with the pulsation $\omega = 600\ rad.s^{-1}$. In these conditions, the parameter x verifies $0.004 < x < 0.3 < x_{mc} < x'_{mc}$

Then, in this range of values, the coefficients T_{abc} and T_{fbc} increase with x together with the force on the obstacle. Using the argument of the section 5.2, it is straightforward to prove that the force on the cylinder increases with the void fraction, with the cylinder radius and with the oscillation frequency.

A number of experiments have shown that the force exerted by two phase flows is more important compared with single phase flows (see Pettigrew [14]). Thanks to a simplified model, the present approach contributes to an explanation of these observations.

7 Conclusion

In this article, a theoretical approach of the interaction between a bubbly flow and an obstacle is proposed. With the help of the model of Kogarko, Iordanski and Van Wijngaarden, we have calculated the force exerted by a bubbly flow on an obstacle (sphere or cylinder). In the acoustic approximation and with the assumption of a potential flow, we derived the "wave equation" satisfied by the velocity potential. By solving this equation coupled with the boundary conditions, the force has been calculated on both spherical and cylindrical obstacles.

For stationary flows, we have generalized the d'Alembert paradox showing that the force exerted on the obstacle is equal to zero. Considering harmonic oscillations of the obstacle (sphere or cylinder), an analytic expression of the force has been derived. It is the sum of two terms, the added mass effect and a friction force. In the limit of incompressible flows, the present results are consistent with the classical Euler equations. In the case of an air-water mixture, the influence of the void fraction, the obstacle size and the oscillation frequency have been investigated. We have showed, in particular, that the force increases with the void fraction meaning that the force exerted by a bubbly flow is more important that the force exerted by a single phase flow. Using a simplifed model, we have obtained a good qualitative behavior of the force. It is a first step in a domain where the need in theoretical studies is important. To go further and obtain quantitative results, numerical simulations are necessary due to the complexity of the problem.

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