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Abstract

We consider a model for thermal contact through a diathermal interface between two macroscopic bodies at different temperatures: an Ising spin chain with nearest neighbor interactions is endowed with a Glauber dynamics with different temperatures and kinetic parameters on alternating sites. The inhomogeneity of the kinetic parameter is a novelty with respect to the model of Ref.[1] and we exhibit its influence upon the stationary non equilibrium values of the two-spin correlations at any distance. By mapping to the dynamics of spin domain walls and using free fermion techniques, we determine the scaled generating function for the cumulants of the exchanged heat amounts per unit of time in the long time limit.

Keywords : exact analytical results, thermal contact, Glauber spin dynamics, current fluctuations

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1 Introduction

Thermal contact between two macroscopic bodies initially at different temperatures corresponds to a situation where the heat transfer between the two bodies is ensured by a thin diathermal interface. The latter may be an immaterial interface between two solids or a diathermal wall between two fluids. Over a time window during which the two macroscopic bodies have negligible energy variations, they behave as thermostats with constant thermodynamic temperatures, while the interface is a mesoscopic system with traceable configurations. After a long enough time inside the considered time window the interface tends to a stationary non equilibrium state where the instantaneous heat current which it receives from a thermostat has a non-vanishing mean value. Even if the interface is described by a model where its degrees of freedom obey a (deterministic or stochastic) microscopic dynamics, there is no general framework, such as Gibbs equilibrium ensemble theory, which would allow to determine the probability distribution of the interface configurations and the corresponding mean instantaneous heat current. Therefore it is most valuable to exhibit solvable models which would shed some light into the dependence of the heat instantaneous current upon the model parameters and the temperatures of the two thermostats.

Such a solvable model has been introduced by Racz and Zia in 1994 [1]. They consider a chain of classical spins with periodic boundary conditions and interacting with a nearest-neighbor ferromagnetic (Ising) interaction. They endow it with a Glauber stochastic dynamics with single-spin flips at a time and such that the spins on odd and even lattice sites are flipped by thermostats at two different temperatures. In our view, the model may be seen as a zig-zag shaped chain inside a very thin strip between two half-planes occupied by two different thermostats, and where the odd (even) sites are located on the left (right) side of the strip. The stationary two-spin correlations at any distance as well as higher order spin correlations have been extensively studied in Refs.[1, 2, 3, 4].

We point out that the latter model indeed satisfies two requirements needed for a correct description of a thermal contact between two macroscopic bodies during a transient time window where their temperature can be considered as constant. First the contact must be mediated by changes in the internal energy of the spin interface. Second, if the energies of the macroscopic bodies were kept tracked of, the transition rates for both the interface configurations and these two energies would obey the detailed balance with the microcanonical equilibrium probability distribution for these variables of the whole system. Then, in the infinite time limit the two bodies and the interface would be at the same temperature; in a time window where the energy variations of the macroscopic bodies are negligible, the transition rates for the interface depend only on the temperatures of the macroscopic bodies and they obey the local detailed balance [5, 6][1]. In the case of a spin interface the two corresponding requirements become: 1) a transition between two spin configurations involves only one thermostat; 2) the transition rate involving a given thermostat obey the detailed balance at the same temperature. For the present two-temperature Ising chain the first requirement is obviously fulfilled[2], and the simplest transition rates which fulfill the second requirement are those chosen by Glauber in the case of an Ising chain in contact with a unique thermostat[8]. (We recall that Glauber looked for single-spin flip dynamics such that in the infinite time limit the spin chain indeed relaxes to the canonical equilibrium state at the thermostat temperature and then he chose the simplest transition rates.)

On the other hand, the quantities of interest for exchange processes which have been considered over the last three decades, both theoretically and experimentally, are amounts of microscopically conserved entities (matter, energy, ...), which are exchanged over a very long time (in the case of thermal contact the corresponding quantities are the heat amounts received by the interface from each thermostat during a fixed long time interval). They have been focused on because the scaled generating function for their cumulants per unit of time in the long time limit as well as its Laplace transform, the large deviation function of the corresponding time-integrated current, have been

[1] In Ref.[5] the terminology "generalized" detailed balance is used.

[2] We notice that the first requirement is not satisfied by the Ising chain model of Ref.[7], where two thermostats at different temperatures act on every spin : the latter model does not describe a situation of thermal contact.

shown to obey generic symmetry relations, the so-called fluctuation relations. The latter relations are derived from properties of the system dynamics and they are a milestone of the stochastic thermodynamics theory (For a review see Ref.[9].)

In this context it is also interesting to have at hand solvable models where the statistics of the time-integrated currents can be calculated. For instance, such a model has been exhibited in the case where the heat transfer between two thermostats is ensured by a wire. The energy quanta are represented as particles whose stochastic dynamics is a Symmetric Simple Exclusion Process (SSEP) with adequate boundary conditions: particles with hard cores hop on the sites of a one-dimensional lattice with the same hopping rate in both directions, but have different in-coming and out-going rates at the two lattice ends in contact with two particle reservoirs with different chemical potentials. Then all cumulants of the heat coming out of one thermostat per unit of time can be calculated in the long time limit [10, 11] .

In the present paper we consider a generalized version of the diathermal interface model of Ref.[1]: the two Glauber dynamics which flip the spins on sites with odd or even indices respectively have not only different temperatures but also different kinetic parameters (see Sec.2). Our aim is to calculate the scaled generating function of the joint cumulants per unit of time for the heats flowing out of the two thermostats in the long time limit.

The difference between the kinetic constants is relevant for two macroscopic bodies made with different materials. Some of the corresponding kinetic effects have been previously investigated in our study of a very simple model for a one-dimensional interface between two half-planes occupied by bodies at two different temperatures : a model of independent two-spin pairs where the left-side (right-side) spin in each pair is flipped only by the thermostat on the same side according to a Glauber dynamics. The whole statistics for the spin configurations and the heat amounts exchanged by every pair with the thermostats have been calculated explicitly [12].

The method which we use to obtain the scaled generating function for the heat cumulants in the present model is the following. From its definition this function can be obtained as the largest eigenvalue of the modified Markov matrix which rules the evolution of the joint probability for the system configurations and the heat amounts $Q_{\rm o}$ and $Q_{\rm e}$ received on each (odd or even) sublattice.

In order to study energy exchanges more conveniently, instead of considering the spin configurations, we rather formulate the problem in terms of the position configurations for the domain walls, which sit on the dual lattice. In other words we consider the well-known lattice gas representation on the *dual* lattice ('antiparallel adjacent spin pair' $\leftrightarrow$ 'particle' and 'parallel adjacent spin pair' $\leftrightarrow$ 'hole') where a particle is in fact a domain wall. The mapping of Glauber spin dynamics to the particle dynamics then includes the possibility for the creation and annihilation of adjacent particle pairs, which correspond to the injection or the loss of energy in the chain respectively[3]. Our model with two temperatures and two kinetic parameters is mapped to a reaction-diffusion system with two different creation (annihilation) rates as well as two different hopping rates on spatially alternating sites[4].

The crucial point is that, in a suitable basis for the representation of the configurations of domain wall positions, the Markov matrix for the evolution of the configuration probability involves only products of two operators : the Markov matrix is mapped to a free fermion Hamiltonian [14]. Moreover the mapping can be readily generalized for the modified Markov matrix which rules the evolution of the joint probability for a configuration of domain wall positions and the amounts of heat $Q_{\rm o}$ and $Q_{\rm e}$. The latter matrix can be diagonalized by using free fermions techniques: Jordan-Wigner transformation and antiperiodic Fourier transform. Thus we obtain a block diagonal matrix, made of 4×4 blocks, which can be straighforwardly diagonalized by introducing four pseudo-fermion operators. (In the case of two pseudo-fermion operators and the associated

[3]The latter correspondance has been used for instance in Ref.[13] for the calculation of the scaled generating function of the energy injected in an Ising spin ring through the random flips of one spin while all other spins evolve according to a Glauber dynamics at zero temperature which dissipates energy along the ring.

[4]In Ref.[4] the mapping has been used in the reverse sense in order to study the relaxation towards the stationary state for the reaction-diffusion system from results obtained for the Ising spin chain dynamics with two temperatures but a unique kinetic parameter.

Bogoliubov-like transformation, see for instance Ref. [15, 16].) The largest eigenvalue of the modified Markov matrix is obtained by filling up all pseudo-fermion states associated with an eigenvalue with a positive real part.

The paper is organized as follows. The model is defined in Sec.2 and the relaxation time for the mean global magnetization on each sublattice is calculated. Moreover by solving a hierarchy of equations for the stationary global two-spin correlations at any distance (see Appendix A), we determine the mean instantaneous energy current which flows from each thermostat into the spin lattice in the stationary state. The mapping to the dynamics of domain walls on the dual lattice and the associated modified matrix is derived in Sec.3. Its eigenvalues are determined in Sec.4 by free fermion techniques. The cumulants are obtained in Sec.5 and various physical regimes are discussed. In conclusion we summarize some finite-size effects and their possible cancellation in heat cumulants.

2 Model

2.1 Description of a thermalization process

We consider a one-dimensional lattice with a finite even number of sites $L = 2N$, where each site j is occupied by a classical spin s_j ($s_j = \pm 1$, $j = 1, \ldots, 2N$) with periodic boundary condition $s_{j+2N} = s_j$. Spins interact via the Ising ferromagnetic nearest-neighbor interaction with coupling $K > 0$: the energy of a spin configuration $\mathbf{s}$ is

$$\mathcal{E}(\mathbf{s}) = -K \sum_{j=1}^{2N} s_j s_{j+1}. \tag{2.1}$$

When the spin at site j is flipped, the energy variation of the Ising chain is equal to

$$\Delta\mathcal{E}(s_j \to -s_j) = s_j \frac{s_{j-1} + s_{j+1}}{2} \Delta E, \tag{2.2}$$

with $\Delta E = 4K$. This variation can take the values $+\Delta E$, 0, or $-\Delta E$,

The model is endowed with a stochastic dynamics where the spin flips at odd (even) sites are due to energy exchanges with a macroscopic body at temperature $T_{\rm o}$ ($T_{\rm e}$) in the course of a thermalization process of the two macroscopic bodies. The transition rates must obey local detailed balance [5, 6]: the transition rates $w(s_j \to -s_j)$ and $w(-s_j \to s_j)$ for two reversed flips of the spin at site j, while all other spins are kept fixed, must obey the ratio

$$\frac{w(s_j \to -s_j)}{w(-s_j \to s_j)} = \mathrm{e}^{-\beta_j \Delta\mathcal{E}(s_j \to -s_j)}, \tag{2.3}$$

where β_j is the inverse temperature of the thermostat acting on site j : $\beta_j = 1/(k_B T_j)$ where k_B is Boltzmann constant and T_j is equal either to $T_{\rm o}$ or $T_{\rm e}$, depending on the parity of j. As shown by Glauber [8] in the case of a unique temperature, the simplest transition rates which obey the constraint (2.3) read

$$w(s_j \to -s_j) = \frac{\nu_j}{2} \left[1 - \gamma_j \frac{s_j \left(s_{j-1} + s_{j+1} \right)}{2} \right], \tag{2.4}$$

while $w(-s_j \to s_j)$ is given by the latter expression where s_j is replaced by $-s_j$. In (2.4) γ_j is the thermodynamic parameter at site j,

$$\gamma_j = \tanh\left(\frac{\beta_j \Delta E}{2} \right) \tag{2.5}$$

where $\gamma_j = \gamma_{\rm o}$ or $\gamma_{\rm e}$, depending on the parity of the site index, and ν_j is the kinetic parameter ν_j at site j. The latter is not determined by the local detailed balance; it can be interpreted as the

mean frequency at which the macroscopic body tries to flip the spin at site j. Therefore we set $\nu_j = \nu_{\rm o}$ or $\nu_{\rm e}$, depending on the parity of the site index (note that when the two kinetic parameters are equal the present model coincides with that of Ref.[1]). Since the time scale is arbitrary, it is convenient to introduce the dimensionless kinetic parameters $\overline{\nu}_{\rm a}$, with a = o or e, defined as

$$\overline{\nu}_{\rm a} = \frac{\nu_{\rm a}}{\nu_{\rm o} + \nu_{\rm e}}. \tag{2.6}$$

They satisfy the relation $\overline{\nu}_{\rm o} + \overline{\nu}_{\rm e} = 1$ and, apart from the arbitrary time scale, the model has only three independent parameters, $\gamma_{\rm o}$, $\gamma_{\rm e}$ and $\overline{\nu}_{\rm o}$.

The probability $P(\mathbf{s};t)$ for the system to be in spin configuration $\mathbf{s}$ at time t evolves according to the master equation

$$\frac{{\rm d}P(\mathbf{s};t)}{{\rm d}t} = \sum_j^{2N} w(-s_j \to s_j) P(\mathbf{s}_j;t) - \left(\sum_j^{2N} w(s_j \to -s_j) \right) P(\mathbf{s};t) \tag{2.7}$$

where $\mathbf{s}_j$ denotes the spin configuration obtained from $\mathbf{s}$ by changing s_j into $-s_j$. The number of configurations is finite, and the transition rates allow the system to evolve from any configuration to any other one after a suitable succession of transitions. Therefore there is a unique stationary solution of the master equation. In the following we focus on the mean values of global quantities and denote $\langle \cdots \rangle$ and $\langle \cdots \rangle_{\rm st}$ the expectation values calculated with the time-dependent probability $P(\mathbf{s};t)$ and the stationary probability $P_{\rm st}(\mathbf{s})$ respectively.

2.2 Relaxation of the mean global sublattice magnetizations

The transition rates are invariant under a global flip of the spins, so that a configuration and the corresponding one where all spins are flipped have the same probability in the stationary state. As a result all stationary correlations for an odd number of spins vanish identically; in particular $\langle s_j \rangle_{\rm st} = 0$. As a consequence the mean values of the global magnetizations on the two sublattices, $M_{\rm o} = \sum_{n=1}^{N} s_{2n-1}$ and $M_{\rm e} = \sum_{n=1}^{N} s_{2n}$ respectively, vanish in the stationary state,

$$\langle M_{\rm o} \rangle_{\rm st} = \langle M_{\rm e} \rangle_{\rm st} = 0. \tag{2.8}$$

The relaxation of the mean values of global sublattice magnetizations is readily studied. As in the case of the homogeneous spin chain considered by Glauber [8], the evolution equation for the mean value of the spin at site j reads

$$\frac{{\rm d}\langle s_j \rangle}{{\rm d}t} = -2 \langle s_j w(s_j \to -s_j) \rangle. \tag{2.9}$$

According to the expression of the transition rates (2.4)

$$\frac{{\rm d}\langle s_j \rangle}{{\rm d}t} = -\nu_j \left[\langle s_j \rangle - \gamma_j \frac{\langle s_{j-1} \rangle + \langle s_{j+1} \rangle}{2} \right]. \tag{2.10}$$

Then the coupled evolutions of the magnetizations on the two sublattices read

$$\begin{aligned} \frac{{\rm d}\langle M_{\rm o} \rangle}{{\rm d}t} &= -\nu_{\rm o} \left[\langle M_{\rm o} \rangle - \gamma_{\rm o} \langle M_{\rm e} \rangle \right] \\ \frac{{\rm d}\langle M_{\rm e} \rangle}{{\rm d}t} &= -\nu_{\rm e} \left[\langle M_{\rm e} \rangle - \gamma_{\rm e} \langle M_{\rm o} \rangle \right]. \end{aligned} \tag{2.11}$$

From these equations we retrieve that both mean magnetizations vanish in the stationary state, as predicted by symmetry arguments. The matrix associated with this system of linear equations has two strictly negative eigenvalues $\frac{1}{2}(\nu_{\rm o} + \nu_{\rm e}) \left[-1 \pm \sqrt{(\overline{\nu}_{\rm o} - \overline{\nu}_{\rm e})^2 + 4 \overline{\nu}_{\rm o} \overline{\nu}_{\rm e} \gamma_{\rm o} \gamma_{\rm e}} \right]$, each of which is associated with a couple of right and left eigenvectors (the eigenvalues are negative because

$(\overline{\nu}_{\rm o}-\overline{\nu}_{\rm e})^2+4\overline{\nu}_{\rm o}\overline{\nu}_{\rm e}\gamma_{\rm o}\gamma_{\rm e}=1-4\overline{\nu}_{\rm o}\overline{\nu}_{\rm e}(1-\gamma_{\rm o}\gamma_{\rm e})<1$). For generic values of the initial magnetizations $M_{\rm o}$ and $M_{\rm e}$, the inverse relaxation time $1/t_{\rm rel}$ to their stationary value is given by the opposite of the negative eigenvalue with the smallest modulus, and the relaxation time $t_{\rm rel}$ reads

$$t_{\rm rel}=\frac{2}{\nu_{\rm o}+\nu_{\rm e}}\left[1-\sqrt{(\overline{\nu}_{\rm o}-\overline{\nu}_{\rm e})^2+4\overline{\nu}_{\rm o}\overline{\nu}_{\rm e}\gamma_{\rm o}\gamma_{\rm e}}\right]^{-1}. \tag{2.12}$$

2.3 Mean global heat current in the stationary state

The mean instantaneous heat current $\langle j_k\rangle$ received by the spin chain at site k from the thermostat at temperature T_k is equal to the expectation value of the variation of the chain energy when the spin s_k is flipped times the transition rate for the flip. According to the expressions for the energy variation (2.2) and for the transition rates (2.4), the mean instantaneous current reads

$$\langle j_k\rangle=K\nu_k\left[-\gamma_k-\gamma_k\langle s_{k-1}s_{k+1}\rangle+\langle s_{k-1}s_k\rangle+\langle s_k s_{k+1}\rangle\right]. \tag{2.13}$$

Therefore the stationary mean value of the global heat current coming from the thermostat acting on spins at even sites, namely $J_{\rm e}=\sum_{n=1}^{N}j_{2n}$, is determined as

$$\langle J_{\rm e}\rangle_{\rm st}=NK\nu_{\rm e}\left[-\gamma_{\rm e}-\gamma_{\rm e}D_2^{\rm oo}+D_1^{\rm oe}+D_1^{\rm eo}\right] \tag{2.14}$$

with the following definitions : $D_2^{\rm oo}$ is the average over the sublattice of odd sites of the stationary correlation between two spins separated by two sites,

$$D_2^{\rm oo}=\frac{1}{N}\sum_{n=1}^{N}\langle s_{2n-1}s_{2n+1}\rangle_{\rm st}, \tag{2.15}$$

$D_1^{\rm oe}=(1/N)\sum_{n=1}^{N}\langle s_{2n-1}s_{2n}\rangle_{\rm st}$ and $D_1^{\rm eo}$ has an analogous definition. Similarly the stationary mean value of the global heat current coming from the thermostat acting on spins at odd sites, $J_{\rm o}=\sum_{n=1}^{N}j_{2n-1}$, reads

$$\langle J_{\rm o}\rangle_{\rm st}=NK\nu_{\rm o}\left[-\gamma_{\rm o}-\gamma_{\rm o}D_2^{\rm ee}+D_1^{\rm oe}+D_1^{\rm eo}\right] \tag{2.16}$$

with $D_2^{\rm ee}=(1/N)\sum_{n=1}^{N}\langle s_{2n}s_{2n+2}\rangle_{\rm st}$.

The values of the stationary global two-spin correlations $D_2^{\rm oo}$, $D_2^{\rm ee}$, $D_1^{\rm oe}$ and $D_1^{\rm eo}$ can be determined from a hierarchy of equations for similar quantities with two spins at any distance on the lattice. Details are given in Appendix A with the results (A.11), (A.12), (A.19), and (A.20) for any distance between spins. From the latter results we get

$$D_2^{\rm ee}=\frac{\gamma\eta_-}{\gamma_{\rm o}}\,\frac{1+\eta_-^{N-2}}{1+\eta_-^{N}} \tag{2.17}$$

with

$$\gamma=\overline{\nu}_{\rm o}\gamma_{\rm o}+\overline{\nu}_{\rm e}\gamma_{\rm e}, \tag{2.18}$$

where the dimensionless kinetic parameters have been defined in (2.6), and η_-, with $0<\eta_-<1$, is defined by

$$\sqrt{\eta_-}=\frac{1-\sqrt{1-\gamma_{\rm o}\gamma_{\rm e}}}{\sqrt{\gamma_{\rm o}\gamma_{\rm e}}}. \tag{2.19}$$

Moreover $D_2^{\rm oo}=(\gamma_{\rm o}/\gamma_{\rm e})D_2^{\rm ee}$, while

$$D_1^{\rm oe}=D_1^{\rm eo}=\frac{\gamma\sqrt{\eta_-}}{\sqrt{\gamma_{\rm o}\gamma_{\rm e}}}\,\frac{1+\eta_-^{N-1}}{1+\eta_-^{N}}. \tag{2.20}$$

We notice that the dependence upon the kinetic parameters $\nu_{\rm o}$ and $\nu_{\rm e}$ occurs only through the parameter γ defined in (2.18).

Eventually the stationary mean value of the global heat current received on even sites can be calculated from expression (2.14) and relation (A.18); we get

$$\langle J_{\rm e}\rangle_{\rm st} = NK\frac{\nu_{\rm o}\nu_{\rm e}}{\nu_{\rm o}+\nu_{\rm e}}(\gamma_{\rm o}-\gamma_{\rm e}). \tag{2.21}$$

Similarly the stationary mean value of the global heat current received on odd sites can be obtained from the expression (2.16) ; it proves to be opposite to that on even sites, $\langle J_{\rm o}\rangle_{\rm st} = -\langle J_{\rm e}\rangle_{\rm st}$, as it should in the stationary state where the mean energy of the chain is constant.

We point out the following remarkable property : though $D_2^{\rm ee}$, $D_2^{\rm oo}$, $D_1^{\rm oe}$ and $D_1^{\rm eo}$ involve finite size corrections (see (2.17) and (2.20)), these corrections cancel one another in the value of the mean global current $\langle J_{\rm e}\rangle_{\rm st}$: $\langle J_{\rm e}\rangle_{\rm st}$ is exactly proportional to the size $L = 2N$ of the ring. Moreover it happens to be equal to L times the mean current received by a spin in the independent pair model of Ref.[12].

3 Mapping to a reaction-diffusion system with pair creation-annihilation

To prepare the study of the heat amounts exchanged with the thermostats we consider a mapping to another model for which the evolution operator is quadratic.

3.1 Domain wall system

When two spins on neighboring sites are antiparallel, one may consider that there is a domain wall between them, whereas there is no domain wall when they are parallel. The domain walls sit on the edges of the initial lattice. Labeling each edge by its mid-point, one gets another lattice which we call the dual lattice in what follows. The edge $(j-1, j)$ and the corresponding site on the dual lattice are labeled by j. If s_{j-1} and s_j are antiparallel, $s_{j-1}s_j = -1$, then the occupation number by a domain wall at site j on the dual lattice is $n_j = 1$, whereas if s_{j-1} and s_j are parallel $n_j = 0$. Thus the correspondance reads

$$n_j = \frac{1 - s_{j-1}s_j}{2}. \tag{3.1}$$

On a ring the number of domain walls is even and $\sum_{j=1}^{L} n_j$ is even.

As a result a spin configuration can be characterized either by the set $\mathbf{s} = \{s_1, \cdots s_L\}$ of spin configurations or by the knowledge of the value of s_1 and the set of the positions of the domains walls, namely the set of occupations numbers $\mathbf{n} = \{n_1, \cdots, n_L\}$. The energy of the system can be expressed solely in terms of domain walls as

$$\mathcal{E}(\mathbf{n}) = -2NK + 2K\sum_{j=1}^{2N} n_j. \tag{3.2}$$

3.2 Quantum mechanics notations

In the following we use the quantum mechanics notations, as commonly done in the literature. Then a column vector is denoted as a "ket", $|\dots\rangle$ and a row vector is denoted as a "bra" $\langle\dots|$. The configuration of occupation numbers by domain walls $\mathbf{n} = \{n_1, \cdots, n_L\}$ is represented as a tensor product

$$|\mathbf{n}\rangle = \otimes_{j=1}^{L} |n_j\rangle, \tag{3.3}$$

where $|n_j\rangle$ is a two-component column vector. The convention used for kets associated to vacant and occupied states is

$$\begin{pmatrix}1\\0\end{pmatrix}_j = |n_j = 0\rangle \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix}_j = |n_j = 1\rangle. \tag{3.4}$$

This convention is the standard choice of basis in the condensed matter literature on quantum spin chains. With the representation (3.3)-(3.4) the row-column product $\big(\mathbf{n}'|\mathbf{n}\big)$ takes the form $\big(\mathbf{n}'|\mathbf{n}\big) = \prod_{j=1}^{L} \delta_{n'_j,n_j}$. Therefore the probability of the domain wall configuration $\mathbf{n}$ at time t, $P(\mathbf{n};t)$, can be represented as a row-column (scalar) product $P(\mathbf{n};t) = \big(\mathbf{n}|P_t\big)$, where $|P_t\big)$ is the column vector defined as

$$|P_t\big) = \sum_{\mathbf{n}} P(\mathbf{n};t)|\mathbf{n}\big). \tag{3.5}$$

With the latter definitions the master equation for the stochastic evolution of the probability $P(\mathbf{n};t)$, which takes the generic form written in (2.7) in the case of $P(\mathbf{s};t)$, can be represented as the evolution of the column vector $|P_t\big)$ under the Markov matrix $\mathbb{M}$

$$\frac{\mathrm{d}|P_t\big)}{\mathrm{d}t} = \mathbb{M}|P_t\big) \tag{3.6}$$

with

$$\begin{aligned} \big(\mathbf{n}'|\mathbb{M}|\mathbf{n}\big) &= w(\mathbf{n}\to\mathbf{n}') \quad \text{if} \quad \mathbf{n}'\neq\mathbf{n} \\ \big(\mathbf{n}|\mathbb{M}|\mathbf{n}\big) &= -\sum_{\mathbf{n}'\neq\mathbf{n}} w(\mathbf{n}\to\mathbf{n}'), \end{aligned} \tag{3.7}$$

where $w(\mathbf{n}\to\mathbf{n}')$ denotes the transition rate from configuration $\mathbf{n}$ to configuration $\mathbf{n}'$.

3.3 Markov matrix for the model

For the present model of domain walls the matrix elements of $\mathbb{M}$ can be expressed in terms of Pauli matrices. Indeed the operator for the occupation number at site j reads

$$\widehat{n}_j = \tfrac{1}{2}\left(\mathbb{1}_j - \sigma_j^z\right) \tag{3.8}$$

where $\mathbb{1}_j$ denotes the identity 2×2 matrix at site j, and $\sigma_j^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_j$; the operator which changes the occupation number at site j is $\sigma_j^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_j$. By inspection of the transition rates for the spin configurations $w(\mathbf{s}\to\mathbf{s}')$ given by (2.4), every transition rate for the occupation numbers by domain walls, $w(\mathbf{n}\to\mathbf{n}')$, can be written as a matrix element $\big(\mathbf{n}'|\mathbb{W}|\mathbf{n}\big)$, where

- for a hop of a domain wall from site j to site $j+1$

$$\mathbb{W} = \frac{\nu_j}{2}\sigma_j^x\sigma_{j+1}^x\widehat{n}_j\left(\mathbb{1}_j - \widehat{n}_{j+1}\right) \tag{3.9}$$

- for a hop of a domain wall from site $j+1$ to site j

$$\mathbb{W} = \frac{\nu_j}{2}\sigma_j^x\sigma_{j+1}^x\left(\mathbb{1}_j - \widehat{n}_j\right)\widehat{n}_{j+1} \tag{3.10}$$

- for the annihilation of two domain walls at sites j and $j+1$

$$\mathbb{W} = \frac{\nu_j}{2}(1+\gamma_j)\sigma_j^x\sigma_{j+1}^x\widehat{n}_j\widehat{n}_{j+1} \tag{3.11}$$

- for the creation of two domain walls at sites j and $j+1$

$$\mathbb{W} = \frac{\nu_j}{2}(1-\gamma_j)\sigma_j^x\sigma_{j+1}^x\left(\mathbb{1}_j - \widehat{n}_j\right)\left(\mathbb{1}_j - \widehat{n}_{j+1}\right). \tag{3.12}$$

The latter expressions can be written in a more compact form by using the spin-$\frac{1}{2}$ ladder operators $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_j$ and $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j$. They are such that $\sigma^x\widehat{n} = \sigma^+$ and $\sigma^x(\mathbb{1}-\widehat{n}) = \sigma^-$. With the convention (3.4) σ_j^+ annihilate a domain wall at site j, while σ_j^- creates a domain wall at site j.

Eventually the Markov matrix $\mathbb{M}$ derived from the master equation for the evolution of the probability of spin configurations (2.7) reads

$$\mathbb{M} = \frac{\nu_1+\nu_2}{2} \tag{3.13}$$
$$\times \left[-(1-\gamma)N\mathbb{1} - \gamma \sum_{j=1}^{2N} \widehat{n}_j + \sum_{j=1}^{2N} \overline{\nu}_j \left[\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + (1+\gamma_j)\sigma_j^+ \sigma_{j+1}^+ + (1-\gamma_j)\sigma_j^- \sigma_{j+1}^- \right] \right],$$

where $\mathbb{1}$ denotes the identity $2N \times 2N$ matrix and γ has been defined in (2.18). The advantage of the domain wall representation with respect to the spin representation is that the Markov matrix is quadratic in terms of operators acting on different sites instead of involving three operators acting on different sites (for the latter case see for instance Ref.[17, 18]).

4 Eigenvalues of the modified Markov matrix

4.1 Modified Markov matrix

We are interested in the joint cumulants per unit of time for the heat amounts $Q_{\rm o}$ and $Q_{\rm e}$ which are received by the chain from the thermostat acting on spins at odd and even sites during a time t in the long time limit. The corresponding scaled generating function is

$$g_{2N}(\lambda_{\rm o}, \lambda_{\rm e}; t) = \lim_{t\to\infty} \frac{1}{t} \ln \langle {\rm e}^{\lambda_{\rm o} Q_{\rm o} + \lambda_{\rm e} Q_{\rm e}} \rangle, \tag{4.1}$$

where $\lambda_{\rm o}$ and $\lambda_{\rm e}$ are real parameters. In fact an evolution equation can be written for the probability $P(\mathbf{n}, Q_{\rm o}, Q_{\rm e}; t)$ for the system to be in configuration $\mathbf{n}$ at time t and to have received heat amounts $Q_{\rm o}$ and $Q_{\rm e}$ between times 0 and t. Therefore the expectation value in the definition (4.1) can be expressed in terms of the discrete Laplace transform of $P(\mathbf{n}, Q_{\rm o}, Q_{\rm e}; t)$, and then $g_{2N}(\lambda_{\rm o}, \lambda_{\rm e}; t)$ reads

$$g_{2N}(\lambda_{\rm e}, \lambda_{\rm o}; t) = \lim_{t\to\infty} \frac{1}{t} \ln \sum_{\mathbf{n}} \widehat{P}(\mathbf{n}, \lambda_{\rm o}, \lambda_{\rm e}; t) \tag{4.2}$$

with

$$\widehat{P}(\mathbf{n}, \lambda_{\rm o}, \lambda_{\rm e}; t) = \sum_{\{Q_{\rm o}, Q_{\rm e}\}} {\rm e}^{\lambda_{\rm o} Q_{\rm o} + \lambda_{\rm e} Q_{\rm e}} P(\mathbf{n}, Q_{\rm o}, Q_{\rm e}; t). \tag{4.3}$$

By inspection of the transition rates (2.4) and according to the correspondence (3.1), when the spin at site j is flipped under the action of the thermostat at temperature T_j, the variation of Q_j is equal to $+\Delta E$ if a pair of domain walls is created at sites j and $j+1$, $-\Delta E$ if a pair of domain walls is annihilated at these sites, 0 if a domain wall jumps either from j to $j+1$ or from $j+1$ to j. As a consequence, with a definition for $|\widehat{P}_t(\lambda_{\rm o}, \lambda_{\rm e})\big)$ analogous to that for $|P_t\big)$ given in (3.5), namely $\big(\mathbf{n}|\widehat{P}_t(\lambda_{\rm o}, \lambda_{\rm e})\big) = \widehat{P}(\mathbf{n}, \lambda_{\rm o}, \lambda_{\rm e}; t)$, we get the evolution equation

$$\frac{{\rm d}|\widehat{P}_t(\lambda_{\rm o}, \lambda_{\rm e})\big)}{{\rm d}t} = \widehat{\mathbb{M}}(\lambda_{\rm o}, \lambda_{\rm e}) |\widehat{P}_t(\lambda_{\rm o}, \lambda_{\rm e})\big), \tag{4.4}$$

where the so-called modified Markov matrix $\widehat{\mathbb{M}}(\lambda_{\rm o}, \lambda_{\rm e})$ reads

$$\frac{2}{\nu_1+\nu_2} \widehat{\mathbb{M}}(\lambda_{\rm o}, \lambda_{\rm e}) = -(1-\gamma)N\mathbb{1} - \gamma \sum_{j=1}^{2N} \widehat{n}_j + \sum_{j=1}^{2N} \overline{\nu}_j \left[\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + b_j \sigma_j^+ \sigma_{j+1}^+ + c_j \sigma_j^- \sigma_{j+1}^- \right]. \tag{4.5}$$

The coefficient b_j (c_j) is equal either to $b_{\rm o}$ or $b_{\rm e}$ ($c_{\rm o}$ or $c_{\rm e}$) according to the parity of j; with the notation a = o or e

$$b_{\rm a} = (1+\gamma_{\rm a})\, {\rm e}^{-\lambda_{\rm a} \Delta E} \tag{4.6}$$

and

$$c_{\rm a} = (1-\gamma_{\rm a})\,{\rm e}^{\lambda_{\rm a}\Delta E}. \tag{4.7}$$

According to the evolution equation (4.4) the Laplace transform $\widehat{P}(\mathbf{n},\lambda_{\rm o},\lambda_{\rm e};t)$ is equal to $\big(\mathbf{n}|\exp[\widehat{\mathbb{M}}(\lambda_{\rm o},\lambda_{\rm e})t]|\widehat{P}_{t=0}(\lambda_{\rm o},\lambda_{\rm e})\big)$. Thus the scaled generating function $g_{2N}(\lambda_{\rm o},\lambda_{\rm e};t)$ given by (4.2) is equal to the largest eigenvalue of the matrix $\widehat{\mathbb{M}}(\lambda_{\rm o},\lambda_{\rm e})$ which rules the evolution of $|\widehat{P}_t(\lambda_{\rm o},\lambda_{\rm e})\big)$.

4.2 Jordan-Wigner transformation

In order to find the eigenvalues of the modified Markov matrix $\widehat{\mathbb{M}}(\lambda_{\rm o},\lambda_{\rm e})$ given by (4.5) we take advantage of its structure analogous to a free fermion Hamiltonian and we introduce the following Jordan-Wigner transformation [19]

$$f_j^\dagger = \left(\prod_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^- \quad \text{and} \quad f_j = \left(\prod_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^+. \tag{4.8}$$

The operator $f_j^\dagger$ is indeed the adjoint of f_j, because $(\sigma^z)^\dagger = \sigma^z$ and $(\sigma^-)^\dagger = \sigma^+$. Operators σ acting on different sites commute, whereas σ_j^z anticommutes with σ_j^+ and σ_j^-; moreover $(\sigma_j^+)^2 = 0$ and $(\sigma_j^-)^2 = 0$. Therefore the operators f_j and $f_j^\dagger$ obey the fermionic anticommutation relations

$$\{f_j, f_{j'}\} = 0 \quad \{f_j^\dagger, f_{j'}^\dagger\} = 0 \quad \{f_j, f_{j'}^\dagger\} = \delta_{j,j'}. \tag{4.9}$$

The occupation number of site j by a domain wall, given by (3.8), also reads $\widehat{n}_j = \sigma_j^-\sigma_j^+ = f_j^\dagger f_j$. The expression (4.5) of the modified matrix $\widehat{\mathbb{M}}(\lambda_{\rm o},\lambda_{\rm e})$ is rewritten in terms of fermionic operators as

$$\begin{aligned}\frac{2}{\nu_1+\nu_2}\widehat{\mathbb{M}}(\lambda_{\rm o},\lambda_{\rm e}) &= -(1-\gamma)N\mathbb{1} - \gamma\sum_{j=1}^{2N} f_j^\dagger f_j + \sum_{j=1}^{2N-1}\overline{\nu}_j\left[f_j^\dagger f_{j+1} - f_j f_{j+1}^\dagger + c_j f_j^\dagger f_{j+1}^\dagger - b_j f_j f_{j+1}\right] \\ &- \overline{\nu}_{\rm e}(-1)^{\mathcal{N}_{\rm f}}\left[f_{2N}^\dagger f_1 - f_{2N} f_1^\dagger + c_{\rm e} f_{2N}^\dagger f_1^\dagger - b_{\rm e} f_{2N} f_1\right]\end{aligned} \tag{4.10}$$

where $\mathcal{N}_{\rm f} = \sum_{j=1}^{2N} f_j^\dagger f_j$ is the total number of fermions.

Since the spin system is on a ring, there can be only an even number of domain walls in the system. As noticed above, the operator for the occupation number by a domain wall $\widehat{n}_j$ coincides with the number of fermions at site j, $f_j^\dagger f_j$. Therefore we have to consider the restriction $\widehat{\mathbb{M}}_+(\lambda_{\rm o},\lambda_{\rm e})$ of $\widehat{\mathbb{M}}(\lambda_{\rm o},\lambda_{\rm e})$ to the sector with an even number of fermions. According to (4.10) the expression of this rectriction is invariant by translation along the ring if the fermionic operators are chosen to satisfy the antiperiodic boundary conditions

$$f_{2N+1} = -f_1 \quad \text{and} \quad f_{2N+1}^\dagger = -f_1^\dagger. \tag{4.11}$$

Then the restriction reads

$$\frac{2}{\nu_1+\nu_2}\widehat{\mathbb{M}}_+(\lambda_{\rm o},\lambda_{\rm e}) = -(1-\gamma)N\mathbb{1} - \gamma\sum_{j=1}^{2N} f_j^\dagger f_j + \sum_{j=1}^{2N}\overline{\nu}_j\left[f_j^\dagger f_{j+1} - f_j f_{j+1}^\dagger + c_j f_j^\dagger f_{j+1}^\dagger - b_j f_j f_{j+1}\right]. \tag{4.12}$$

4.3 Antiperiodic Fourier transform

The next step to the diagonalization is to rewrite the fermionic operators as antiperiodic Fourier transforms which satisfy the antiperiodic boundary conditions (4.11) The wave numbers are of the form $q = (2k+1)\pi/(2N)$ and we work with a complete family of representatives in the set

$$\mathcal{B}(2N) = \{q = (2k+1)\frac{\pi}{2N}, k = -N, -N+1, \cdots, -1, 0, 1, \cdots N-1\}, \tag{4.13}$$

namely

$$\mathcal{B}(2N) = \{-\pi + \frac{\pi}{2N}, -\pi + \frac{3\pi}{2N}, \cdots - \frac{\pi}{2N}, \frac{\pi}{2N}, \cdots \pi - \frac{\pi}{2N}\}. \tag{4.14}$$

The operator f_j can be written as the antiperiodic Fourier transform

$$f_j = \frac{1}{\sqrt{2N}} \sum_{q \in \mathcal{B}(2N)} \mathrm{e}^{\imath qj} \eta_q \tag{4.15}$$

in terms of the wave fermions

$$\eta_q = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} \mathrm{e}^{-\imath qj} f_j. \tag{4.16}$$

Going from (4.15) to (4.16) relies on the identity

$$\sum_{j=1}^{2N} \mathrm{e}^{\imath(q-q')j} = 2N \, \mathbb{1}_{q-q' \equiv 0(2\pi)}, \tag{4.17}$$

where $\mathbb{1}_{q-q' \equiv 0(2\pi)} = 1$ if $q - q'$ is equal to 0 modulo 2π and $\mathbb{1}_{q-q' \equiv 0(2\pi)} = 0$ otherwise. All q's in $\mathcal{B}(2N)$ satisfy $\mathrm{e}^{\imath q 2N} = -1$, and subsequently f_j does obey the antiperiodic boundary conditions (4.11). The adjoint operator $f_j^\dagger$ reads

$$f_j^\dagger = \frac{1}{\sqrt{2N}} \sum_{q \in \mathcal{B}(2N)} \mathrm{e}^{-\imath qj} \eta_q^\dagger. \tag{4.18}$$

These representations are inserted in the expression (4.12). In the term $\sum_{j=1}^{2N} f_j^\dagger f_j$ there occurs a summation over all sites of the ring and one uses the identity (4.17). In the other summations one has to distinguish the two sublattices ; for instance one has to consider the sum $\sum_{n=1}^{N} f_{2n}^\dagger f_{2n+1}^\dagger$ and then one uses the identity

$$\sum_{n=1}^{N} \mathrm{e}^{\imath(q+q')2n} = N \, \mathbb{1}_{2(q+q') \equiv 0(2\pi)} = N \, \mathbb{1}_{q+q' \equiv 0(\pi)}. \tag{4.19}$$

According to definition (4.13), the set $\mathcal{B}(2N)$ does not contain the value 0 and the solution of $q + q' = 0$ corresponds to two distinct values q and $-q$.

In order to simplify the following discussion, we assume from now on that N *is even.* Then $\mathcal{B}(2N)$ does not contain $\pi/2$ and all values q and $\pi - q$, are also distinct. After a symmetrization over the values q and $\pi - q$ the matrix $\widehat{\mathbb{M}}_+(\lambda_\mathrm{o}, \lambda_\mathrm{e})$ appears as a sum of contributions each of which involves only the operators associated with the wave numbers q, $\pi - q$, $-q$ and $-(q - \pi)$. Let us introduce the first quadrant in the set $\mathcal{B}$ defined as

$$\mathcal{QB}(2N) = \{q = (2k+1)\frac{\pi}{2N}, k = 0, 1, \cdots (N/2) - 1\} = \{\frac{\pi}{2N}, \frac{3\pi}{2N} \cdots \frac{\pi}{2} - \frac{\pi}{2N}\}. \tag{4.20}$$

Then the expression (4.12) for $\widehat{\mathbb{M}}_+(\lambda_\mathrm{o}, \lambda_\mathrm{e})$ can be rewritten as

$$\frac{2}{\nu_1 + \nu_2} \widehat{\mathbb{M}}_+(\lambda_\mathrm{o}, \lambda_\mathrm{e}) = -N\mathbb{1} + \sum_{q \in \mathcal{QB}} \left[V_q^\dagger\right]^T \mathbb{A}_q(\lambda_\mathrm{o}, \lambda_\mathrm{e}) \, V_q, \tag{4.21}$$

where V_q is the column vector

$$V_q = \begin{pmatrix} \eta_q \\ \eta_{q-\pi} \\ \eta^\dagger_{-q} \\ \eta^\dagger_{\pi-q} \end{pmatrix}, \tag{4.22}$$

$\left[V_q^\dagger\right]^T$ denotes the transposed row vector corresponding to the column vector $V_q^\dagger$ built with the adjoints of the components of V_q, and

$$\mathbb{A}_q(\lambda_\mathrm{o}, \lambda_\mathrm{e}) = \begin{pmatrix} -\gamma + \cos q & \imath a_q \sin q & \imath c'_q \sin q & c_q \cos q \\ -\imath a_q \sin q & -\gamma - \cos q & -c_q \cos q & -\imath c'_q \sin q \\ -\imath b'_q \sin q & -b_q \cos q & \gamma - \cos q & -\imath a_q \sin q \\ b_q \cos q & \imath b'_q \sin q & \imath a_q \sin q & \gamma + \cos q \end{pmatrix}, \tag{4.23}$$

with

$$\begin{aligned} a_q &= \overline{\nu}_\mathrm{o} - \overline{\nu}_\mathrm{e} \\ b_q &= \overline{\nu}_\mathrm{o} b_\mathrm{o} - \overline{\nu}_\mathrm{e} b_\mathrm{e} \\ b'_q &= \overline{\nu}_\mathrm{o} b_\mathrm{o} + \overline{\nu}_\mathrm{e} b_\mathrm{e} \\ c_q &= \overline{\nu}_\mathrm{o} c_\mathrm{o} - \overline{\nu}_\mathrm{e} c_\mathrm{e} \\ c'_q &= \overline{\nu}_\mathrm{o} c_\mathrm{o} + \overline{\nu}_\mathrm{e} c_\mathrm{e}, \end{aligned} \tag{4.24}$$

where the b_a's and the c_a's are defined in (4.6) and (4.7).

4.4 Diagonalization of $\mathbb{A}_q(\lambda_\mathrm{o}, \lambda_\mathrm{e})$

The characteristic polynomial of $\mathbb{A}_q$, $\det(\mathbb{A}_q - \alpha \mathbb{1}_q)$, proves to be a second order polynomial in α^2, with a constant which is a squared quantity,

$$\det(\mathbb{A}_q - \alpha \mathbb{1}_q) = \alpha^4 - 2D\alpha^2 + F^2. \tag{4.25}$$

Moreover both coefficients D and F^2 depend on the parameters λ_o and λ_e only through the difference

$$\overline{\lambda} = (\lambda_\mathrm{e} - \lambda_\mathrm{o})\,\Delta E. \tag{4.26}$$

They read

$$D = 1 + (\overline{\nu}_\mathrm{o} - \overline{\nu}_\mathrm{e})^2 + \overline{\nu}_\mathrm{o}\overline{\nu}_\mathrm{e}\left[4\gamma_\mathrm{o}\gamma_\mathrm{e}\cos^2 q + (1 - 2\cos^2 q)\theta(\overline{\lambda})\right] \tag{4.27}$$

and

$$F = \overline{\nu}_\mathrm{o}\overline{\nu}_\mathrm{e}\left[4(1 - \gamma_\mathrm{o}\gamma_\mathrm{e}\cos^2 q) + \theta(\overline{\lambda})\right], \tag{4.28}$$

where the function $\theta(\overline{\lambda})$ vanishes when $\overline{\lambda}$ is set to zero :

$$\theta(\overline{\lambda}) = 2\left[(1 - \gamma_\mathrm{o}\gamma_\mathrm{e})(\cosh\overline{\lambda} - 1) + (\gamma_\mathrm{o} - \gamma_\mathrm{e})\sinh\overline{\lambda}\right]. \tag{4.29}$$

The coefficient F can be rewritten as

$$F = \overline{\nu}_\mathrm{o}\overline{\nu}_\mathrm{e}\left[2 + 2\gamma_\mathrm{o}\gamma_\mathrm{e}(1 - 2\cos^2 q) + (1 - \gamma_\mathrm{o}\gamma_\mathrm{e})\cosh\overline{\lambda} + (\gamma_\mathrm{o} - \gamma_\mathrm{e})\sinh\overline{\lambda}\right], \tag{4.30}$$

and the property $(1 - \gamma_\mathrm{o}\gamma_\mathrm{e}) > |\gamma_\mathrm{o} - \gamma_\mathrm{e}|$ for $\gamma_\mathrm{o} < 1$ and $\gamma_\mathrm{e} < 1$ ensures that $F > 0$. The squared roots of the characteristic polynomial are

$$\alpha^2 = D \pm \sqrt{D^2 - F^2} = \left(\sqrt{\frac{D+F}{2}} \pm \sqrt{\frac{D-F}{2}}\right)^2. \tag{4.31}$$

where $\sqrt{\cdots}$ denotes a possibly complex square root. Let us introduce the notations $R_\pm(q, \overline{\lambda}) = \frac{1}{2}(D \pm F)$, namely

$$R_+(q, \overline{\lambda}) = 1 + \overline{\nu}_\mathrm{o}\overline{\nu}_\mathrm{e}\theta(\overline{\lambda})\sin^2 q \tag{4.32}$$

$$R_-(q,\lambda) = (\overline{\nu}_{\rm o} - \overline{\nu}_{\rm e})^2 + \overline{\nu}_{\rm o}\overline{\nu}_{\rm e}\left[4\gamma_{\rm o}\gamma_{\rm e} - \theta(\overline{\lambda})\right]\cos^2 q. \tag{4.33}$$

We notice that, since F is positive, $R_+ > R_-$.

Note that $1 + \overline{\nu}_{\rm o}\overline{\nu}_{\rm e}\theta(\overline{\lambda}) > 0$ because of the definition (4.29) of $\theta(\overline{\lambda})$ and the identities $1 - 2\overline{\nu}_{\rm o}\overline{\nu}_{\rm e}(1-\gamma_{\rm o}\gamma_{\rm e}) > 0$ and $(1-\gamma_{\rm o}\gamma_{\rm e}) > |\gamma_{\rm o} - \gamma_{\rm e}|$ for $\gamma_{\rm o} < 1$ and $\gamma_{\rm e} < 1$. According to (4.32), R_+ can be rewritten as $R_+ = 1 - \sin^2 q + \left(1 + \overline{\nu}_{\rm o}\overline{\nu}_{\rm e}\theta(\overline{\lambda})\right)\sin^2 q$ and we conclude that $R_+ > 0$.

On the other hand, by virtue of definitions (2.5) and (4.29),

$$4\gamma_{\rm o}\gamma_{\rm e} - \theta(\overline{\lambda}) = 2\frac{\cosh\left((\beta_{\rm e}+\beta_{\rm o})\Delta E/2\right) - \cosh\left(\overline{\lambda} - (\beta_{\rm e}-\beta_{\rm o})\Delta E/2\right)}{\cosh\left(\beta_{\rm e}\Delta E/2\right)\cosh\left(\beta_{\rm o}\Delta E/2\right)}, \tag{4.34}$$

so that $4\gamma_{\rm o}\gamma_{\rm e} - \theta(\overline{\lambda}) > 0$ only if $-\beta_{\rm o}\Delta E < \overline{\lambda} < \beta_{\rm e}\Delta E$, and we infer from (4.33) that R_- can take both signs.

Eventually the four eigenvalues of $\mathbb{A}_q(\lambda_{\rm o},\lambda_{\rm e})$ are

$$\alpha_1 = \sqrt{R_+} + \sqrt{R_-} \qquad \alpha_2 = \sqrt{R_+} - \sqrt{R_-} \qquad \alpha_3 = -\alpha_2 \qquad \alpha_4 = -\alpha_1, \tag{4.35}$$

In these expressions $\sqrt{R_+}$ denotes the usual positive square root of the positive number R_+, whereas $\sqrt{R_-}$ is either real positive or purely imaginary depending on the sign of R_-, i.e. on the values of $\overline{\lambda}$ and q. As noticed previously $R_+ > R_-$, and in the case where $\sqrt{R_-}$ is real, all α_k's, with $k = 1,\ldots,4$, are real and $\alpha_1 > \alpha_2 > 0 > \alpha_3 > \alpha_4$.

4.5 Largest eigenvalue of the modified matrix $\widehat{\mathbb{M}}_+(\lambda_{\rm o},\lambda_{\rm e})$

For the sake of conciseness we omit all dependences upon $\lambda_{\rm o}$ and $\lambda_{\rm e}$ in the present section. We denote by $\mathbb{D}_q$ the diagonal matrix built with the eigenvalues $\alpha_1(q),\ldots,\alpha_4(q)$ of the matrix $\mathbb{A}_q(\lambda_{\rm o},\lambda_{\rm e})$. The matrix $\mathbb{A}_q$ reads

$$\mathbb{A}_q = \mathbb{P}_q\mathbb{D}_q\mathbb{P}_q^{-1}, \tag{4.36}$$

where the $k^{\rm th}$ column of $\mathbb{P}_q$ is made with the components of a (column) right eigenvector of $\mathbb{A}_q$ associated with the eigenvalue $\alpha_k(q)$, and $\mathbb{P}_q^{-1}$ is the inverse matrix of $\mathbb{P}_q$. Let $\xi_k(q)$ denote the $k^{\rm th}$ component of the column vector $\mathbb{P}_q^{-1}V_q$, while $\xi_k^\star(q)$ denotes the $k^{\rm th}$ component of the row vector $\left[V_q^\dagger\right]^T\mathbb{P}_q$, where V_q and $\left[V_q^\dagger\right]^T$ are defined in (4.22). With these definitions, the relation (4.36) implies that

$$\left[V_q^\dagger\right]^T\mathbb{A}_qV_q = \sum_{k=1}^{4}\alpha_k(q)\,\xi_k^\star(q)\,\xi_k(q). \tag{4.37}$$

The operators ξ_k and $\xi_k^\star$ obey the anticommutation rules $\{\xi_k,\xi_{k'}\} = 0$, $\{\xi_k^\star,\xi_{k'}^\star\} = 0$ and $\{\xi_k,\xi_{k'}^\star\} = \delta_{k,k'}$. However, since $\mathbb{A}_q$ is not hermitian (for the usual scalar product), $\mathbb{P}_q$ is not unitary and the operator $\xi_k^\star$ is not the adjoint of ξ_k.

Nevertheless the anticommutation rules are enough to ensure that the spectrum of the operator $\xi_k^\star\xi_k$ is the set $\{0,1\}$. Then, according to the expressions (4.35) of the α_k's, the value of the right-hand side of (4.37) with the largest real part is equal to the sum of two eigenvalues and proves to be real positive,

$$\alpha_1 + \alpha_2 = 2\sqrt{R_+(q)} > 0. \tag{4.38}$$

Eventually, according to (4.21), the largest eigenvalue of $\widehat{\mathbb{M}}_+(\lambda_{\rm o},\lambda_{\rm e})$ is

$$\frac{\nu_{\rm o}+\nu_{\rm e}}{2}\left[-N + 2\sum_{q\in\mathcal{QB}}\sqrt{R_+(q)}\right], \tag{4.39}$$

where $R_+(q)$ is given by (4.32).

We notice that if $\lambda_{\rm o} = \lambda_{\rm e} = 0$, the modified Markov matrix $\widehat{\mathbb{M}}(\lambda_{\rm o},\lambda_{\rm e})$ becomes the usual Markov matrix for the evolution of $P(\mathbf{n};t)$; since $\theta(0) = 0$, we retrieve that the largest eigenvalue of the Markov matrix is 0. Moreover the eigenvalue closest to 0 is obtained by setting $\xi_3^\star(q)\xi_3(q)$ equal to 1 for $q = 0$, and we retrieve the value (2.12) for the inverse relaxation time of the magnetizations on the two sublattices.

5 Cumulants of heat amounts per unit of time in the long time limit

5.1 Scaled generating function for joint cumulants

According to the remark at the end of section 4.1, the scaled generating function for the joint cumulants of $Q_{\rm o}$ and $Q_{\rm e}$ coincides with the largest eigenvalue of $\widehat{\mathbb{M}}_+(\lambda_{\rm o},\lambda_{\rm e})$. By virtue of (4.39) it reads

$$g_{2N}(\lambda_{\rm o},\lambda_{\rm e}) = \frac{\nu_{\rm o}+\nu_{\rm e}}{2}\left[-N+2\sum_{q\in\mathcal{QB}}\sqrt{1+\overline{\nu}_{\rm o}\overline{\nu}_{\rm e}\theta(\overline{\lambda})\sin^2 q}\right], \tag{5.1}$$

where $\mathcal{QB}(2N)$ is defined in (4.20) and $\theta(\overline{\lambda})$ is given in (4.29). The joint cumulants per unit of time in the long time limit are determined from the relation

$$\lim_{t\to\infty}\frac{1}{t}\langle Q_{\rm e}^p Q_{\rm o}^{p'}\rangle_c = \left.\frac{\partial^{p+p'} g_{2N}(\lambda_{\rm e},\lambda_{\rm o};t)}{\partial\lambda_{\rm e}^p\partial\lambda_{\rm o}^{p'}}\right|_{\lambda_{\rm e}=\lambda_{\rm o}=0}, \tag{5.2}$$

where the index c refers to the truncation of the mean value $\langle Q_{\rm e}^p Q_{\rm o}^{p'}\rangle$ involved in the definition of the cumulant. The fact that g_{2N} depends only on the difference $\lambda_{\rm e}-\lambda_{\rm o}$ entails the properties

$$\lim_{t\to\infty}\frac{1}{t}\langle Q_{\rm e}^p Q_{\rm o}^{p'}\rangle_c = (-1)^{p'}\lim_{t\to\infty}\frac{1}{t}\langle Q_{\rm e}^{p+p'}\rangle_c \tag{5.3}$$

and, in particular,

$$\lim_{t\to\infty}\frac{1}{t}\langle Q_{\rm o}^p\rangle_c = (-1)^p\lim_{t\to\infty}\frac{1}{t}\langle Q_{\rm e}^p\rangle_c. \tag{5.4}$$

These properties are linked to the fact that the interface energy can take only a finite number of values whereas the cumulants have no upper bounds in the infinite time limit.

For the sake of completeness we point out that, according to (4.34), $\theta(\overline{\lambda})$ depends on $\overline{\lambda}$ through the function $\cosh\big(\overline{\lambda}-(\beta_{\rm e}-\beta_{\rm o})\Delta E/2\big)$; therefore the scaled generating function satisfies the symmetry $g_{2N}(\lambda_{\rm o},\lambda_{\rm e}) = g_{2N}(\beta_{\rm o}-\lambda_{\rm o},\beta_{\rm e}-\lambda_{\rm e})$, which is in fact a consequence of the local detailed balance (2.3). Since $g_{2N}(\lambda_{\rm o},\lambda_{\rm e})$ depends only on the difference $\lambda_{\rm e}-\lambda_{\rm o}$, this entails a symmetry for the scaled generating function for the cumulants of $Q_{\rm e}$, $g_{2N}^{\rm e}(\lambda_{\rm e}) = g_{2N}(0,\lambda_{\rm e})$, namely the symmetry $g_{2N}^{\rm e}(\lambda_{\rm e}) = g_{2N}^{\rm e}(\beta_{\rm e}-\beta_{\rm o}-\lambda_{\rm e})$. Then the corresponding large deviation function for the time-integrated current $\mathcal{J}_{\rm e} = Q_{\rm e}/t$, which can be obtained as the Legendre-Fenchel transform of $g_{2N}^{\rm e}(\lambda_{\rm e})$, obeys the fluctuation relation $f(\mathcal{J}_{\rm e}) - f(-\mathcal{J}_{\rm e}) = (\beta_{\rm o}-\beta_{\rm e})\mathcal{J}_{\rm e}$.

5.2 Cumulants for heat amount $Q_{\rm e}$

We give the explicit expressions for the first four cumulants of $Q_{\rm e}$ per unit of time. They are determined as $\partial^n g_{2N}(0,\lambda_{\rm e})/\partial\lambda_{\rm e}^n|_{\lambda_{\rm e}=0} = \Delta E^n \times \partial^n g_{2N}(0,\lambda_{\rm e})/\partial\overline{\lambda}^n|_{\overline{\lambda}=0}$, where $\overline{\lambda} = (\lambda_{\rm e}-\lambda_{\rm o})\,\Delta E$ and $\Delta E = 4K$ is the energy gap in the chain energy. The function $\theta(\overline{\lambda})$ defined in (4.29) can be rewritten as

$$\overline{\nu}_{\rm o}\overline{\nu}_{\rm e}\theta(\overline{\lambda}) = 2A[\cosh\overline{\lambda}-1] + 2B\sinh\overline{\lambda} \tag{5.5}$$

where the parameters A and B are those introduced in Ref.[12] for a model with only two spins, namely

$$A = \overline{\nu}_{\rm o}\overline{\nu}_{\rm e}(1-\gamma_{\rm o}\gamma_{\rm e}) \tag{5.6}$$

$$B = \overline{\nu}_{\rm o}\overline{\nu}_{\rm e}(\gamma_{\rm o}-\gamma_{\rm e}). \tag{5.7}$$

Then, according to (5.1), the first four cumulants of $Q_{\rm e}$ read

$$\begin{aligned}
\lim_{t\to\infty}\frac{\langle Q_{\rm e}\rangle}{(\nu_{\rm e}+\nu_{\rm o})t} &= \frac{N}{2}BS_2\times\Delta E \qquad (5.8)\\
\lim_{t\to\infty}\frac{\langle Q_{\rm e}^2\rangle_c}{(\nu_{\rm e}+\nu_{\rm o})t} &= \frac{N}{2}\left[AS_2-B^2S_4\right]\times\Delta E^2\\
\lim_{t\to\infty}\frac{\langle Q_{\rm e}^3\rangle_c}{(\nu_{\rm e}+\nu_{\rm o})t} &= \frac{N}{2}B\left[S_2-3AS_4+3B^2S_6\right]\times\Delta E^3\\
\lim_{t\to\infty}\frac{\langle Q_{\rm e}^4\rangle_c}{(\nu_{\rm e}+\nu_{\rm o})t} &= \frac{N}{2}\left[AS_2-(3A^2+4B^2)S_4+18AB^2S_6-15B^4S_8\right]\times\Delta E^4,
\end{aligned}$$

where

$$S_{2n}(N)=\frac{2}{N}\sum_{k=0}^{(N/2)-1}\sin^{2n}\left(\frac{(2k+1)\pi}{2N}\right). \qquad (5.9)$$

The structure of the cumulants in terms of the coefficients A and B is similar to the structure found for the two-spin model of Ref.[12] as well as for the ring of Ref.[7]; indeed, in the three cases the scaled cumulant generating function depends on $\lambda_{\rm e}$ only through the same function $\theta(\lambda_{\rm e}\Delta E)$.

The coefficients $S_{2n}(N)$ (with N even) can be calculated explicitly. In the special case $N=2$, a direct calculation leads to $S_{2n}(2)=(1/2)^n$. For any $N\geq 2$, by extending the summation up to N, rewriting $\sin u=[{\rm e}^{\imath u}-{\rm e}^{-\imath u}]/(2\imath)$, using the binomial formula and an identity similar to (4.17), we get that

$$\text{if } n<N \quad S_{2n}(N)=\left(\frac{1}{2}\right)^{2n}\frac{[2n]!}{(n!)^2}=W_{2n}, \qquad (5.10)$$

where W_{2n} denotes the normalized Wallis integral, $W_{2n}=(2/\pi)\int_0^{\pi/2}(\sin q)^{2n}{\rm d}q$. The first four values of the latter integrals are

$$W_2=\frac{1}{2}, \qquad W_4=\frac{3}{8}, \qquad W_6=\frac{5}{16}, \qquad \text{and} \quad W_8=\frac{35}{128}. \qquad (5.11)$$

For n larger than N, a finite-size correction arises. For instance,

$$\text{if } N\leq n<2N \quad S_{2n}(N)=W_{2n}\left[1-2\frac{[n!]^2}{(n-N)!(n+N)!}\right]. \qquad (5.12)$$

From the study of the $S_{2n}(N)$ we get that all cumulants of order n smaller than N, the number of sites connected to a given thermostat, are strictly proportional to the size $L=2N$ of the chain. Finite-size corrections appear only in cumulants of order $n\geq N$.

In particular, $S_2(N)=1/2$ for all N, by virtue of (5.10)-(5.11) valid for all $N\geq 2$ with N even. Therefore the first cumulant $\lim_{t\to+\infty}\langle Q_{\rm e}\rangle/t$ given in (5.8) does coincide with the expression (2.21) of the mean global instantaneous current $\langle J_{\rm e}\rangle_{\rm st}$ in the stationary state. We have already pointed out that the latter mean global current contains no finite-size corrections. If $N>4$ the first four cumulants are given by (5.8) where the $S_{2n}(N)$ are to be replaced by the corresponding W_{2n} given in (5.11).

Eventually the dependences upon the thermodynamic parameters, $\gamma_{\rm o}$ and $\gamma_{\rm e}$, and the kinetic parameters, $\nu_{\rm o}$ and $\nu_{\rm e}$, arise only through the coefficients A and B. This is in contrast with what happens for another Ising chain model, where both thermostats act on every spin [7]: for this model the dependence upon the combination γ of the thermodynamic and kinetic parameters defined in (2.18) also arises in the coefficients $\Sigma_n(N,\gamma)$ which replace the coefficients $S_n(N)$ in the expressions (5.8) for the cumulants.

5.3 Various physical regimes

According to the last remark of the previous section, the discussion of the various physical regimes is the same as that performed in the case of the two-spin model of Ref.[12].

At equilibrium $\gamma_{\rm o} = \gamma_{\rm e}$ and, according to (4.29), $\theta(\overline{\lambda}) = 2(1-\gamma_{\rm e}^2)(\cosh\overline{\lambda} - 1)$. Therefore only cumulants of even order do not vanish. According to (5.8) the first two cumulants of even order read

$$\begin{aligned} \lim_{t\to\infty} \frac{\langle Q_{\rm e}^2\rangle_c}{(\nu_{\rm e}+\nu_{\rm o})t} &= \frac{1}{4}\nu_{\rm o}\nu_{\rm e}\left(1-\gamma_{\rm e}^2\right) \times N\Delta E^2 \\ \lim_{t\to\infty} \frac{\langle Q_{\rm e}^4\rangle_c}{(\nu_{\rm e}+\nu_{\rm o})t} &= \frac{1}{4}\nu_{\rm o}\nu_{\rm e}\left(1-\gamma_{\rm e}^2\right)\left[1-\frac{9}{4}\nu_{\rm o}\nu_{\rm e}\left(1-\gamma_{\rm e}^2\right)\right] \times N\Delta E^4. \end{aligned} \tag{5.13}$$

The probability distribution of $Q_{\rm e}$ is not a Gaussian, since all cumulants of even order have non-vanishing values.

When a thermostat has a kinetic parameter far larger than the other one, the scaled generating function becomes proportional to $\theta(\overline{\lambda})$,

$$g_{2N}(\overline{\lambda}) = \frac{1}{8}N\nu_{\rm s}\theta(\overline{\lambda}), \tag{5.14}$$

where $\nu_{\rm s}$ is the kinetic parameter of the slower thermostat. As a consequence $g_{2N}(\overline{\lambda})$ coincides with the scaled generating function of a continuous-time random walk, because $\theta(\overline{\lambda})$ can be rewritten as

$$\theta(\overline{\lambda}) = 2\left[p_+ {\rm e}^{\overline{\lambda}} + p_- {\rm e}^{-\overline{\lambda}} - (p_+ + p_-)\right] \tag{5.15}$$

with the probabilities $p_+ = (1+\gamma_{\rm o})(1-\gamma_{\rm e})/2$ and $p_- = (1-\gamma_{\rm o})(1+\gamma_{\rm e})/2$. As a consequence all cumulants of even (odd) order are equal to the same value when they are measured in unit of ΔE: for all $p \geq 1$

$$\begin{aligned} \lim_{t\to\infty} \frac{\langle Q_{\rm e}^{2p-1}\rangle_c}{t\,\Delta E^{2p-1}} &= \frac{1}{4}\nu_{\rm s}(\gamma_{\rm o}-\gamma_{\rm e})\,N \\ \lim_{t\to\infty} \frac{\langle Q_{\rm e}^{2p}\rangle_c}{t\,\Delta E^{2p}} &= \frac{1}{4}\nu_{\rm s}(1-\gamma_{\rm o}\gamma_{\rm e})\,N. \end{aligned} \tag{5.16}$$

In the same kinetic regime, if one thermostat is at zero temperature, for instance $\gamma_{\rm o} = 1$, then the scaled generating function coincides with that of a continuous-time Poisson process, because

$$\theta(\overline{\lambda}) = 2(1-\gamma_{\rm e})\left[{\rm e}^{\overline{\lambda}} - 1\right]. \tag{5.17}$$

As a consequence all cumulants of $Q_{\rm e}$ in unit of ΔE are equal to the same value: for all $p \geq 1$

$$\lim_{t\to\infty} \frac{\langle Q_{\rm e}^{p}\rangle_c}{t\,\Delta E^{p}} = \frac{1}{4}\nu_{\rm s}(1-\gamma_{\rm e})\,N. \tag{5.18}$$

6 Conclusion

In the present paper we have investigated the heat currents in an Ising spin ring where alternating spins are coupled to two macroscopic bodies at different temperatures and with different kinetic parameters. The stationary mean values of the global two-spin correlations at any distance have been calculated. The dependence upon the kinetic parameters arises only through the linear combination γ defined in (2.18)[5]. The finite-size corrections in the global two-spin correlations disappear in the mean instantaneous global heat current coming out of one thermostat.

The scaled generating function of the joint cumulants per unit of time for the heat amounts exchanged with the two thermostats over a long time have been calculated exactly. At leading

[5] When the kinetic parameters are set equal, our result are compatible with those of Ref.[2].

order in the ring size they prove to be proportional to the ring size, as it is the case for the model where two thermostats act on the same site [7]. Moreover, if the order of the cumulant is lower than the number of spins connected to one thermostat, the finite-size corrections again disappear exactly, and the cumulant is strictly proportional to the ring size.

We notice that the proportionality to the ring size at leading order in the size has already been observed for the cumulants of two other kinds of cumulative quantities when the system is homogeneous (only one temperature and one kinetic parameter) and seen as a Simple Exclusion Process with pair creation and annihilation [20]: the two cumulative quantities are the difference between the numbers of domain wall jumps in the clockwise and anticlockwise directions respectively, and the number of pair annihilations. This is in contrast with the case of the purely diffusive Simple Exclusion Process on a ring, where the cumulants for the difference between the numbers of jumps in the two directions and the cumulants for the total number of jumps are proportional to powers of the ring size which increase with the order of the cumulants [21].

An interesting open problem is the calculation of the heat cumulants in another solvable model for thermal contact : two joined semi-infinite Ising chains coupled to thermostats at two different temperatures [22, 23]. Then the mean global current which flows from one thermostat to the other through the junction between the two half-chains is obtained by summing the mean currents received by all spins in a semi-infinite Ising chain. The intrinsic inhomogeneity of these currents would have to be dealt with by specific methods.

A Two-spin correlations

By analogy with the homogeneous case [8] the evolution equation for the two-spin correlations reads

$$\frac{\mathrm{d}\langle s_j s_k\rangle}{\mathrm{d}t} = -2\langle s_j s_k \left[w(s_j \to -s_j) + w(s_k \to -s_k)\right]\rangle. \tag{A.1}$$

Inserting the expression (2.4) for the transition rates we get

$$\frac{\mathrm{d}\langle s_j s_k\rangle}{\mathrm{d}t} = -(\nu_j+\nu_k)\langle s_j s_k\rangle + \tfrac{1}{2}\nu_j\gamma_j\left[\langle s_{j-1}s_k\rangle + \langle s_{j+1}s_k\rangle\right] + \tfrac{1}{2}\nu_k\gamma_k\left[\langle s_j s_{k-1}\rangle + \langle s_j s_{k+1}\rangle\right]. \tag{A.2}$$

The latter equations imply that D_2^{ee}, D_2^{oo}, D_1^{oe} and D_1^{eo} involved in the expressions (2.14) and (2.16) for the mean global currents $\langle J_{\mathrm{e}}\rangle_{\mathrm{st}}$ and $\langle J_{\mathrm{o}}\rangle_{\mathrm{st}}$ are to be determined from a hierarchy of equations for the two-spin quantities

$$D_{2p}^{\mathrm{oo}} = \frac{1}{N}\sum_{n=1}^{N}\langle s_{2n-1}s_{2n-1+2p}\rangle_{\mathrm{st}}, \tag{A.3}$$

$$D_{2p+1}^{\mathrm{oe}} = \frac{1}{N}\sum_{k=1}^{N}\langle s_{2n-1}s_{2n+2p}\rangle_{\mathrm{st}} \tag{A.4}$$

with an analogous definition for D_{2p+1}^{eo}, and

$$D_{2p}^{\mathrm{ee}} = \frac{1}{N}\sum_{n=1}^{N}\langle s_{2n}s_{2n+2p}\rangle_{\mathrm{st}}. \tag{A.5}$$

We notice that if the initial probability distribution for the spin configurations is translationally invariant, this property is conserved by the evolution under the transition rates and the stationary two-spin correlation $\langle s_k s_{k+p}\rangle_{\mathrm{st}}$ depends only on the difference p between the site indices; then it is equal to one of the D's defined in (A.3), (A.4) and (A.5). From the latter definitions we get the boundary conditions

$$D_0^{\mathrm{ee}} = 1 \quad \text{and} \quad D_0^{\mathrm{oo}} = 1. \tag{A.6}$$

From the evolution equation for the spin correlations (A.2) we get that, in the stationary state where the D's are defined, for $2 \leq 2p \leq 2(N-1)$

$$\begin{aligned} 0 &= -4\overline{\nu}_{\rm e} D^{\rm ee}_{2p} + \overline{\nu}_{\rm e}\gamma_{\rm e}\left[D^{\rm oe}_{2p+1} + D^{\rm oe}_{2p-1} + D^{\rm eo}_{2p-1} + D^{\rm eo}_{2p+1}\right] \qquad \text{(A.7)} \\ 0 &= -4\overline{\nu}_{\rm o} D^{\rm oo}_{2p} + \overline{\nu}_{\rm o}\gamma_{\rm o}\left[D^{\rm eo}_{2p+1} + D^{\rm eo}_{2p-1} + D^{\rm oe}_{2p-1} + D^{\rm oe}_{2p+1}\right], \qquad \text{(A.8)} \end{aligned}$$

while for $1 \leq 2p+1 \leq 2N-1$

$$\begin{aligned} 0 &= -2D^{\rm oe}_{2p+1} + \overline{\nu}_{\rm o}\gamma_{\rm o}\left[D^{\rm ee}_{2p+2} + D^{\rm ee}_{2p}\right] + \overline{\nu}_{\rm e}\gamma_{\rm e}\left[D^{\rm oo}_{2p} + D^{\rm oo}_{2p+2}\right] \qquad \text{(A.9)} \\ 0 &= -2D^{\rm eo}_{2p+1} + \overline{\nu}_{\rm o}\gamma_{\rm o}\left[D^{\rm ee}_{2p+2} + D^{\rm ee}_{2p}\right] + \overline{\nu}_{\rm e}\gamma_{\rm e}\left[D^{\rm oo}_{2p} + D^{\rm oo}_{2p+2}\right]. \qquad \text{(A.10)} \end{aligned}$$

Comparison of equations (A.7) and (A.8) leads to the relation valid for $2 \leq 2p \leq 2(N-1)$

$$D^{\rm oo}_{2p} = \frac{\gamma_{\rm o}}{\gamma_{\rm e}} D^{\rm ee}_{2p}, \qquad \text{(A.11)}$$

while comparison of equations (A.9) and (A.10) leads to the relation valid for $1 \leq 2p+1 \leq 2N-1$

$$D^{\rm eo}_{2p+1} = D^{\rm oe}_{2p+1}. \qquad \text{(A.12)}$$

By taking into account these relations in (A.7) and (A.9) we have to solve the coupled equations for the $D^{\rm ee}$'s and the $D^{\rm oe}$'s

$$\begin{aligned} -2D^{\rm ee}_{2p} + \gamma_{\rm e}\left[D^{\rm oe}_{2p-1} + D^{\rm oe}_{2p+1}\right] &= 0 \quad \text{for} \quad 2 \leq 2p \leq 2(N-1) \qquad \text{(A.13)} \\ -2D^{\rm oe}_{2p+1} + \gamma_{\rm o}\left[D^{\rm ee}_{2p} + D^{\rm ee}_{2p+2}\right] &= 0 \quad \text{for} \quad 2 \leq 2p \leq 2(N-2), \end{aligned}$$

where the second equation is to be supplemented by the extra boundary conditions for $p=0$ and $p=N-1$ respectively. The latter conditions are derived from (A.9) and (A.6),

$$\begin{aligned} -2D^{\rm oe}_{1} + \gamma_{\rm o} D^{\rm ee}_{2} + \gamma &= 0 \qquad \text{(A.14)} \\ -2D^{\rm oe}_{2N-1} + \gamma_{\rm o} D^{\rm ee}_{2(N-1)} + \gamma &= 0, \end{aligned}$$

where γ is defined in (2.18). The equations (A.13) allow to determine recursively $D^{\rm oe}_3, D^{\rm ee}_4, \ldots, D^{\rm oe}_{2N-1}$ from a given set $(D^{\rm oe}_1, D^{\rm ee}_2)$, and then the boundary conditions (A.14) determine the values of $D^{\rm oe}_1$ and $D^{\rm ee}_2$.

The recursive equations (A.13) are linear and their generic solution, which depends on the two parameters $D^{\rm oe}_1$ and $D^{\rm ee}_2$, can be looked for as linear combinations of two linearly independent solutions. Because of the invariance of these equations under the translation over two sites, one can look for independent solutions which are also eigenfunctions of the translation operator on each sublattice, namely solutions of the form $f^{\rm ee}_{2p+2} = \eta f^{\rm ee}_{2p}$ and $f^{\rm oe}_{2p+3} = \eta f^{\rm oe}_{2p+1}$. These solutions can be written as

$$\begin{aligned} f^{\rm ee}_{2p} &= \eta^{p-1} a \qquad \text{(A.15)} \\ f^{\rm oe}_{2p+1} &= \eta^{p} b. \end{aligned}$$

By inserting the latter expressions into the recursive equations (A.13) one gets two coupled linear equations for a and b. The latter do not vanish if η is equal to one of the two values

$$\eta_\pm = \frac{2 - \gamma_{\rm o}\gamma_{\rm e} \pm 2\sqrt{1-\gamma_{\rm o}\gamma_{\rm e}}}{\gamma_{\rm o}\gamma_{\rm e}} = \left[\frac{1 \pm \sqrt{1-\gamma_{\rm o}\gamma_{\rm e}}}{\sqrt{\gamma_{\rm o}\gamma_{\rm e}}}\right]^2. \qquad \text{(A.16)}$$

The model is defined for $0 < \gamma_{\rm o} < 1$ and $0 < \gamma_{\rm e} < 1$, so that $\eta_+ \neq \eta_-$. Then in the two solutions of (A.15) with η_+ and η_- respectively, $b_\pm = \frac{1}{2}\gamma_{\rm o}[1+\eta_\pm^{-1}]a_\pm$. Then the generic solution of (A.13) reads

$$\begin{aligned} D^{\rm ee}_{2p} &= a_+\eta_+^{p-1} + a_-\eta_-^{p-1} \qquad \text{(A.17)} \\ D^{\rm oo}_{2p} &= \tfrac{1}{2}\gamma_{\rm o}(1+\eta_+^{-1})a_+\eta_+^{p} + \tfrac{1}{2}\gamma_{\rm o}(1+\eta_-^{-1})a_-\eta_-^{p}, \end{aligned}$$

In fact the solution can be written only in terms of η_- (with $0 < \eta < 1$) by using the relation $\eta_+\eta_- = 1$. Then the boundary conditions (A.14) determine the values of a_+ and a_-. After straightforward calculations and use of the relation

$$\sqrt{\eta_-} + \frac{1}{\sqrt{\eta_-}} = \frac{2}{\sqrt{\gamma_o \gamma_e}}, \tag{A.18}$$

we get that for $2 \leq 2p \leq 2(N-1)$

$$D^{ee}_{2p} = \frac{\gamma}{\gamma_o} \frac{1}{1+\eta_-^N} \left[\eta_-^p + \eta_-^{N-p} \right], \tag{A.19}$$

and for $1 \leq 2p+1 \leq 2N-1$

$$D^{oe}_{2p+1} = \frac{\gamma}{\sqrt{\gamma_o \gamma_e}} \frac{1}{1+\eta_-^N} \left[\eta_-^{p+1/2} + \eta_-^{N-p-1/2} \right]. \tag{A.20}$$

The latter formulae are compatible with the expressions of the stationary two-spin correlations determined in Ref.[2] in the case where $\nu_o = \nu_e$.

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