Enhancing Financial Portfolio Robustness with an Objective Based on -Neighborhoods

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Financial portfolio optimization is a challenging task. One of the major difficulties is managing the uncertainty arising from different aspects of the process. This paper suggests a solution based on ϵ-neighborhoods that, combined with a time-stamped resampling mechanism, increases the robustness of the solutions. The approach is tested on four of the most popular evolutionary multiobjective algorithms over a long period of time. This results in a significant enhancement in the reliability of the estimated efficient frontier.

Keywords: Portfolio optimization; robustness; multiobjective optimization.

1. Introduction

The selection of appropriate mix of assets in financial portfolios is one of the key problems faced by money managers. The search for good strategies that consistently

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lead to the best combinations to meet the needs of investor has driven an extremely large amount of research. This line of investigation is still very active and involves both researchers and practitioners from different disciplines.

Since the seminal work of Markowitz,[1] the problem of portfolio optimization has been mainly aimed at the identification of sets of assets that balance two key characteristics, risk and return. While the initial efforts to tackle this financial multiobjective problem[2] relied mostly on classic quadratic programming, the use of heuristics[3] is getting progressively more traction due to their flexibility when it comes to modeling additional objectives[4] and real-world constraints.[5] Among them, Multiobjective Evolutionary Algorithms (MOEAs)[6] seem to be specially well suited for the task.

Regardless of the optimization approach chosen, the process of constructing good portfolios is subject to uncertainty at different levels.[7,8] Among them, we will mention two important ones: the quality of the estimates for key optimization parameters,[9] and the uncertainty coming from the implementation of the trading strategy chosen to build the portfolio. The efforts to keep problems like these under control has led to the subfield of robust financial portfolio optimization.[10] Within it, the bulk of the literature is focused in the first one, so we intend to make a contribution that tackles the second one and is compatible with optimization strategies based on MOEAs.

When asset managers provide orders to the trading desk to buy and sell towards a target allocation, there are implementation costs that relate to the timing of the order. It is normal that the execution is not immediate due to liquidity constraints, the need to hide intentions, etc. If the execution of the instructions is not simultaneous, the result is likely to be intermediate portfolios that are very similar to the initial or the target portfolio, but slightly different. If those small differences in the solution space translate into large divergences in the risk-return profile, it is likely that the decision maker considers them undesirable (and vice-versa). This paper shows how adding an additional objective based on ϵ-neighborhoods allows the improvement in this aspect, making the solutions more robust, while, at the same time, it either has very little cost, or adds value in terms of the conception of robustness based on sensitivity to deviation in the values of the parameters. In addition to this, the approach is also compatible with a wide range of MOEAs.

In order to illustrate these features, we will show the performance of this approach combined with another one that controls for errors in parameter estimation on a large set of historical data using four different core MOEAs. The sample was already used to present the resampled strategy, time-stamped resampling, as a stand-alone method,[11] and those results will be partially used as benchmark. The core algorithms that will be tested are among the most widely studied MOEAs in the domain, specifically Nondominated Sorting Genetic Algorithm II (NSGA-II), Strength Pareto Evolutionary Algorithm 2 (SPEA2), Speed-Constrained Multiobjective PSO algorithm (SMPSO), and Generalized Differential Evolution 3 (GDE3). NSGA-II[12] is one of the most popular MOEAS. This MOEA, together with SPEA2,[13] the second algorithm that we use, has been reported in the literature[14] to offer good performance in portfolio optimization. We will complete the set with a differential evolution strategy, GDE3,[15] and a multiobjective algorithm based on particle swarm optimization.[16]

The rest of the paper is organized as follows. First, we review recent literature on MOEAs in portfolio optimization. Then, we make a formal introduction to the financial portfolio optimization problem and describe in detail our approach, outlined above, to obtain robust solutions. This will be followed by a section devoted to the experimental validation. Finally, there will be a section dedicated to summary and conclusions.

2. Related Literature Overview

The literature exploring applications of MOEAs on portfolio optimization is ample. Innovation on algorithms[17] and modeling has resulted in a constant buildup of evidence supporting the case for their use in the domain. Besides the references covered in the survey by Metaxiotis and Liagkouras,[18] among recent contributions, we could mention another one by the same authors[19] that presents a new version of the polynomial mutation operation applicable to the cardinality constrained portfolio optimization problem. Lwim *et al.*[20] studied the performance of a learning-guided multiobjective evolutionary algorithm, MODEwAwL, versus four well-known MOEAs for portfolio optimization with four real-world constraints.

More recently, Babaei *et al.*[21] proposed a multiobjective particle swarm optimization (MOPSO) on a multiobjective mixed integer programming formulation of the problem, and Yue *et al.*[22] introduced an evolutionary algorithm for multiobjective fuzzy portfolio selection aimed at maintaining the diversity of portfolios.

The interest in robust portfolio optimization using MOEAs is more recent and, as we mentioned before, has been specially focused on ways to control uncertainty related to the true values of key parameters. Within the Markowitz framework, the output of the optimization process relies heavily on the quality of the estimates for the expected asset returns and their variance–covariance matrices. Unfortunately, major predictability issues for financial asset returns often result expected optimal solutions that often turn out to lie far from the real ones.

Some authors have done some descriptive work regarding the sensitivity of NSGA-II to the presence of outliers in the dataset[23] but, when it comes to searching for potential solutions, the main strategies put forward to by researchers using evolutionary computation adapt the idea of resampling. Resampling is a strategy found in the financial literature for some time.[24] The core idea is generating sets of solutions for a number of scenarios, and then aggregating the output to obtain a set of portfolios that is not specialized on a single forecast for the returns and the variance–covariance matrix. The aggregation solutions for different sets parameter values could either be simple or based on clustering algorithms, as Idzorek suggests.[25]

The mentioned approach traditionally combines sets of portfolios obtained using quadratic programming, but the idea of testing different scenarios has been adapted to evolutionary computation. In this case, the strategies affect the way the algorithms operate to include the resampling process in the optimization loop. This way, the solution is found in a single go. Hassan and Clack[26] used this idea to create trading strategies based on genetic programming, but we have to wait until García *et al.*[27] to find an application tailored specifically to the single-period portfolio optimization problem. These authors use multiple scenarios to compute the fitness of intermediate solutions using SPEA2. This strategy was subsequently improved adding a time-stamped mechanism, and generalized to other MOEAs.[11]

The problem of limiting exposure to deviations from the target portfolio due to trading strategies is an open aspect of the robust portfolio optimization problem. Given that we know in advance that the portfolios resulting from either a regular or even a resampled optimization process, like those already mentioned, carry a degree of uncertainty, we suggest prioritizing those solutions that are less sensitive to minor specification errors. The process samples the space around any candidate portfolio and favors those that, in addition to offering good relative risk and return, are located in stable sections of the solution space.

As it will be discussed in detail later, the driver of the suggested strategy is a stability indicator based on ϵ-neighborhoods. This component captures dispersion in terms of the two mentioned basic objectives resulting from the application of small perturbations to intermediate solutions. This technique is often regarded as a way to control problems that may arise from slight reading errors from sensors, small deviations in expected completion times in scheduling tasks, robust design, etc. Its generic nature makes it adaptable to many different contexts. The basic idea is controlling the impact of unexpected deviations in the solution space. In order to do that, the algorithm defines a neighborhood around the potential solution, samples it and analyzes the structure of samples in the objective space.

Within the framework of robust optimization,[28] the use of ϵ-neighborhoods to enhance the reliability of the solutions provided by the evolutionary algorithms is not new. Deb and Gupta[29] discuss their application in their general discussion of robustness in multiobjective optimization, and Gaspar-Cunha and Covas[30] go deeper into the specifics of this particular approach. Having said that, to the best of our knowledge, the application to limit implementation risks in portfolio optimization, such as the one described in the sections that follow, is novel.

3. Financial Portfolio Optimization Problem

Financial portfolios can be defined as a collection of investments or assets held by an institution or a private individual. The Modern Portfolio Theory was originated in the article published by Harry Markowitz, in 1952.[1] It explains how to use the diversification to optimize the Portfolio. In general, the portfolio optimization problem is the choice of an optimum set of assets to include in the portfolio and the distribution of investor's wealth among them. Markowitz[31] assumed that solving the problem requires the simultaneous satisfaction of maximizing the expected portfolio return $E(R_p)$ and minimizing the portfolio risk. Despite the fact that there are different ways to model risk,[32] this work will follow the canonical variance metric σ_p^2. Therefore, we will be solving a multiobjective optimization problem with two output objective functions that can be formally defined as:

- Minimize the *risk* (variance) of the portfolio:

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}. \tag{1}$$

- Maximize the *expected return* of the portfolio:

$$E(R_p) = \sum_{i=1}^{n} w_i \mu_i. \tag{2}$$

- Subject to:

$$\sum_{i=1}^{n} w_i = 1, \tag{3}$$

$$0 \leq w_i \leq 1; \quad i = 1, \ldots, n, \tag{4}$$

where n is the number of available assets, μ_i the expected return of the asset i, σ_{ij} the covariance between asset i and j, and w_i are the decision variables giving the composition of the portfolio. The constraints referenced in Eqs. (3) and (4) require the full investment of funds and prevent the investor from shorting any asset, respectively. In a quantitative way, the risk is represented with the standard deviation σ_p.

The solution to the problem should also consider some real world constraints such as

- Cardinality constraint: it is possible to define the maximum $C_{\max}$ and minimum $C_{\min}$ number of assets in which it is possible to invest ($w_i \neq 0$):

$$C_{\min} \leq \Sigma(w_i \neq 0) \leq C_{\max} \tag{5}$$

- Values limit constraint: each weight w_i must have a value in the interval [$\lim_{inf}$, $\lim_{sup}$], where

$$0.0 \leq \lim_{inf} \leq w_i \leq \lim_{sup} \leq 1.0. \tag{6}$$

All of these equations are solved by a set of points that constitute the efficient frontier of the problem. The points will define a curve in the *risk-return* space that represents, out of all possible portfolios, the subset that has the minimum amount of risk given a certain expected return (and viceversa).

4. Robust Approach

As we mentioned in the introduction, portfolio optimization faces a major challenge: the process depends on forecasts for two sets of parameters that are very likely to be inaccurate. The virtual impossibility of getting perfect estimates for future returns and the variance–covariance matrix results in a large potential for major deviations both between the expected and actual behavior of selected portfolios, and the real efficient frontier.

In this paper, we introduce a way to mitigate the problem, an extended formulation of the mean-variance model that includes a third stability objective based on ϵ-neighborhoods. There are previous efforts based on the perturbation of the mentioned key parameters like time-stamped resampling (R+T)[11] which have proved to be effective. That approach generates sets of likely scenarios and filters out those portfolios that are too sensitive to deviations in expected returns and variance–covariance matrix. The algorithm also considers the age of candidate portfolios, and favors those solutions that have performed better for a longer time, the time-stamped component of the approach. The strategy that we suggest, perfectly complementary, focuses its attention on a different aspect. In this case, the key is the structure of the landscape. The core idea is biasing the search towards areas of the solution space where small perturbations in the candidate solution do not lead to major differences in terms of risk and return.

The mechanism used to foster the robustness of the solution relies on a previous idea already found on robust optimization in other domains, ϵ-neighborhoods.[29] Each portfolio p will be assessed based on the traditional features, return $E(R_p)$ and risk σ_p^2, plus a new one Rob_p. The computation of the last figure requires the generation of H neighboring portfolios, $p_1, \ldots, p_H$, which are created using the Latin hypercube (LH) approach.[33] LH works by dividing the domain of each asset of p (around $[-\epsilon, \epsilon]$) into H equal grids, thus dividing the ϵ-neighborhood into H^n hyperboxes. The greater the value of ϵ, larger the size of the sampled neighborhood. That is, ϵ determines the scale of the perturbation made to the composition portfolios. Once the hyperboxes are selected, a random point within each hyperbox is chosen and used to compute the return $E(R_{p_i})$ and risk $\sigma_{p_i}^2$ of portfolio p_i. The average Mahalanobis distance[34] between p and p_i is considered as a new objective function to be minimized, since these distances measures the dispersion of the returns and risks of the set of portfolios, i.e., the larger the Mahalanobis distance, the larger the dispersion, and thus, the more sensitive is the solution to small perturbations. The choice of Mahalanobis distance is justified by some desirable properties like independence of scale or the fact that it takes into account correlation structures to evaluate similarity. This reduces the differences in risk and return to a single number. More formally, the new objective function is

$$\mathrm{Rob}_p = \frac{1}{H} \sum_{i=1}^{H} md(p, p_i), \tag{7}$$

where $md(p, p_i)$ is the Mahalanobis distance in terms of risk and return between portfolios p and p_i. If we define the pair $(E(R_{p_i}), \sigma^2_{p_i})$ as $\bar{x}$, $(E(R_p), \sigma^2_p)$ as $\bar{y}$, and Σ as the variance-covariance matrix, the distance would be computed using the expression

$$md(\bar{x}, \bar{y}) = \sqrt{(\bar{x} - \bar{y})^T \Sigma^{-1} (\bar{x} - \bar{y})}. \quad (8)$$

As we just mentioned, high values of Rob_p indicate of low stability in the neighborhood of p and, therefore, are something to be avoided. That is the reason why the new objective should be minimized.

The fronts resulting from the experimentation consist of portfolios that are not dominated according to the three objectives driving the evolution process. That means that the process results in surface that could be used by the decision maker to choose the right portfolio for his needs.

Given that these three-objective solutions are not readily comparable with the ones resulting from the standard two-objective formulation, we make a projection to the risk-return space. The sets of portfolios will be split in subsets according to the new objective, that we will refer to a ϵ-sensitivity, that we just described. This will allow us to analyze the connection between the values of the third objective, and the sensitivity to uncertainty regarding the real values of the optimization parameters.

The process is as follows: at the end of the execution, the final solution set is ranked by the ϵ-sensitivity objective, Rob_p, and split in three sets. The first of these, the third of the individuals with the lowest values, will be classified as "High stability". The second will be labeled "Medium stability" and finally the last one, the set of portfolios located in the least stable portion of the solution space, will be considered "Low stability". This is illustrated in Fig.1, where we represent a set of

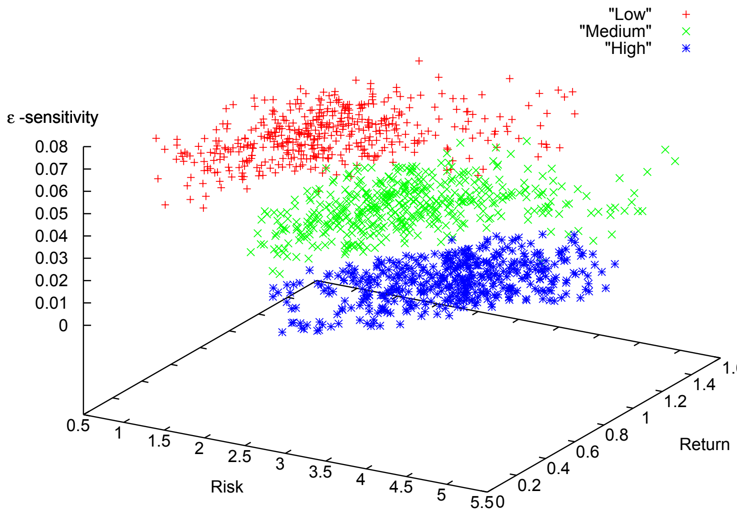

Fig. 1. Solutions grouped by ϵ-sensitivity objective.

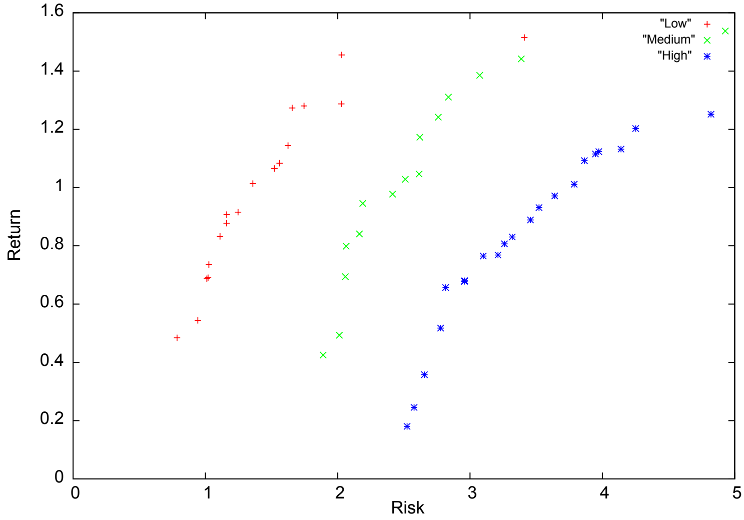

Fig. 2. Dominant solutions by ϵ-sensitivity objective.

portfolios broken in the mentioned categories according to the dispersion metric obtained sampling its neighborhood.

The processing stage ends with the identification of nondominated portfolios in terms of risk and return objectives. As we observe in Fig. 2, once we discard dominated solutions, we obtain three subfronts whose quality can be evaluated at a later stage and compared to the solution provided by any other optimization method.

It is worth noting that Figs. 1 and 2 only intend to illustrate the process. The actual experimental results that we describe in the next section produce charts where the three fronts are so close, that they almost overlap. Only by enlarging significantly small sections of the front we could observe the depicted structure. The second difference that we should remark is that, even though the three fronts cover a similar range of risk-return combinations, medium and high sensitivity fronts tend to concentrate more solutions in the low-risk (variance) section of the efficient frontier. This is specially the case in the latter.

5. Experimentation

In this section, we report the experimentation perform made to test the approach described in the previous section. Given that the model extension is compatible with both complementary solutions R+T and a wide range of MOEAs, we performed a set of experiments designed to assess the contribution of the third objective under different setups on a real-world problem. The problem chosen is robust portfolio optimization for strategic asset allocation. In this case, the investor requires robust approximations to the efficient frontier resulting from the combination of eight investment categories represented by a broad indices. The details regarding the setup will be provided in sections devoted to the introduction of the dataset, core algorithms, adaptations to the problem and parametrization and finally, evaluation metrics.

5.1. *Dataset*

The experiments were performed using a sample of 240 monthly returns for eight broad financial indexes representing eight asset classes. As we mentioned in the introduction, this sample was already used in Ref. 11. The series of monthly returns covers the time period from January 1990 to December 2009 and the source for the data is commercial provider *Datastream*. The list of indexes is provided in Table 1.

In order to make sure that the algorithms face a wide range of historical situations and, therefore, that we get proper validation for the approach, we test it in 120 different single-period market scenarios based on real data. In order to create these single-period portfolio optimization problem instances, we will use a sliding window. It is important to emphasize that this use differs from the usual single testing ground for dynamic multiperiod approaches. In this paper, the rolling window is just a tool to create in systematic way 120 different single-period scenarios that happen to be consecutive in time.

The size of the window is set to $n = 120$, return periods that correspond to 10 years of data. This means that the algorithm will rely on data from t_1 to t_n to identify the best possible allocations for the period t_{n+1}, that is, t_1 to t_n will be used for training and the clean data from t_{n+1} for testing. Each time, the 10-year window will move one month, 120 times in total to cover the date interval from 31/01/1990 to 31/12/2009.

Due to the stochastic nature of the algorithms, they will be run 100 times per window. This means that, for each window, we will obtain 100 solutions sets per algorithm.

5.2. *Algorithms*

The nature of the strategies designed to improve robustness is very flexible. It is compatible with many different metaheuristics. We briefly describe the four used in

Table 1. Data sets.

Name	Code
Frank Russell 1000 Growth	FRUS1GR
Frank Russell 1000 Value	FRUS1VA
Frank Russell 2000 Growth	FRUS2GR
Frank Russell 2000 Value	FRUS2VA
S&P GSCI Commodity Total Return	GSCITOT
MSCI EAFE	MSEAFEL
BOFA ML CORP MSTR ($)	MLCORPM
BOFA ML US TRSY /AGCY MSTRAAA($)	MLUSALM

Algorithm 5.1 Template of a multiobjective metaheuristic

```
1: S(0) ← GenerateInitialSolutions() // S can contain only a solution
2: Evaluation(S)
3: A(0) ← Update(A(0), S(0))
4: t ← 0
5: while not StoppingCriterion( ) do
6:     t ← t + 1
7:     S(t) ← Variation(A(t − 1), S(t − 1))
8:     Evaluate(S(t))
9:     A(t) ← Update(A(t), S(t))
10: end while
11: Output: A
```

this study, namely NSGA-II, SPEA2, SMPSO, and GDE3. They all are either population-based metaheuristics,[35] i.e., they operate on a set of solutions at every iteration, or include an external archive for storing the nondominated solutions found during the search, or both. A general template for a multiobjective metaheuristic is displayed in Algorithm 5.1. The general operation of these algorithms begins by generating the initial solutions, S (usually in a fully random manner), and updating the set of nondominated solutions found in this first sampling, A (lines 1–3). Then, the search loop starts. It lies in stochastically varying the solutions included in S and A, and generating a (hopefully improved) new set of solutions (line 7) from which those that are nondominated are retrieved (line 9). The matching of this general scheme on the four algorithms used in this work is briefly presented below, after introducing the encoding of the tentative solutions they manipulate, i.e., the portfolios. For a detailed description of the algorithms, interested readers are referred to the references provided for each one.

5.2.1. *NSGA-II*

The NSGA-II, was proposed by Deb *et al.*[12] It is a genetic algorithm based on generating a new population from the original one by applying the typical genetic operators (selection, crossover, and mutation); then, the individuals in the new and old population are sorted according to their rank, and the best solutions are chosen to create a new population. In case of having to select some individuals with the same rank, a density estimation based on measuring the crowding distance to the surrounding individuals belonging to the same rank is used to get the most promising solutions. From Algorithm 5.1, S and A are considered to be one single set $P = S \cup A$ so that, at each iteration, the nondominated solutions found are used to generate new solutions within the evolutionary loop.

5.2.2. *SPEA2*

The SPEA2 was proposed by Zitzler *et al.* in Ref. 13. In this algorithm, each individual has a fitness value that is the sum of its strength raw fitness plus a density estimation. SPEA2 fits perfectly in the general template of Algorithm 5.1, having a population of solutions plus an external archive. That is, the algorithm applies the selection, crossover, and mutation operators with solutions from S to fill the archive A of individuals; then, the nondominated individuals of both the original population and the archive are copied into a new population. If the number of nondominated individuals is greater than the population size, a truncation operator based on calculating the distances to the kth nearest neighbor is used. This way, the individuals having the minimum distance to any other individual are chosen.

5.2.3. *SMPSO*

SMPSO algorithm is a particle swarm optimization algorithm for solving MOPs.[16] From a high level of abstraction, in a PSO algorithm, a set (swarm) of candidate solutions (particles) to the problem navigates through the search space of an optimization problem. This navigation takes place attending to a velocity equation, which rules the way how particles change their position. Among the factors that govern the velocity equation, two of them can be highlighted: the current position of the particle and the best positions visited so far, also referred as leaders. Usually, the best position visited by a particle (local leader) and the best particle visited by any particle in the swarm (global leader) are considered. The main innovation of SMPSO is the incorporation of a constraining mechanism already applied in mono-objective PSO algorithms, which modulate the speed at which particles fly.[36] SMPSO uses an external archive to store the nondominated solutions, so it fits into the template described in Algorithm 5.1.

5.2.4. *GDE3*

The GDE3[15] is an improved version of the GDE algorithm. It starts with a population of random solutions, which becomes the current population. At each generation, an offspring population is created using the differential evolution operators; then, the current population for the next generation is updated using the solutions of both, the offspring and the current population. Before proceeding to the next generation, the size of the population is reduced using nondominated sorting and a pruning technique aimed at diversity preservation, in a similar way as NSGA-II, although the pruning used in GDE3 modifies the crowding distance of NSGA-II in order to solve some of its drawbacks when dealing with problems having more than two objectives. Therefore, the matching with Algorithm 5.1 is the same as NSGA-II: S and A are merged into one single set.

5.3. *Algorithm Adaptation and Parametrization*

All the algorithms evaluated in this work use the same encoding for the tentative solutions that they are iteratively improving. These solutions are encoded as arrays of floating point numbers, w_i, that represent the percentage of investment of the asset i, that is, a portfolio. As long as the solutions must fulfill a set of hard constraints (Eqs. (3)–(6), a repair operator has been used to deal with unfeasibility.[11] Whenever a new solution is generated, updated by the search operators, or sampled within the ϵ-neighborhood, its constraints are evaluated and it undergoes repair if needed.

In order to have a fair comparison among all the algorithms to be performed, they all are required to evaluate 150,000 tentative solutions and to obtain an approximation to the Pareto optimal set limited to 500 nondominated points. Recall that, for every sliding windows, each algorithm is run 100 times in order to provide the results with statistical confidence. Regarding the third objective, the neighborhood of each portfolio is sampled with 1,000 small perturbations, with $\epsilon = 0.001$. The detailed settings for each of the four algorithms are included in Table 2. We want to clarify two relevant points. On the one hand, we have not paid attention to the particular parametrization of the recombination and mutation rates for each algorithm as we have used the standard values given in the seminal works in which they are presented. Such a thorough analysis is out of the scope of this work because we want to focus the attention to the effect of the robust mechanism devised. On the other hand, we want to underline again that the comparison is fair in terms of both the numerical performance (i.e., the size of the sampling in the search space) and the maximum size of the approximated fronts (i.e., no algorithm is given more chance to cover regions of the Pareto front by using nondominated sets of unbounded size). The parameters for the R+T component of the algorithm are the same that were used in the cited reference paper.

Table 2. Parametrization of the algorithms. L is the individual length.

Parametrization used in NSGA-II and SPEA2	
Population Size	500 individuals
Selection of Parents	binary tournament + binary tournament
Recombination	uniform
Mutation	polynomial, $p_m = 1.0/L$
Parametrization used in SMPSO	
Swarm Size	500 individuals
Leaders Size	500 individuals
Mutation	polynomial, $p_m = 1.0/L$
Parametrization used in GDE3	
Population Size	500 individuals
Recombination	Differential Evolution, $CR = 0.1$, $F = 0.5$

5.4. *Evaluation metrics*

The evaluation of the solutions requires quality indicators. The most widely used in multiobjective optimization, such as Hypervolume (HV) and Spread,[37] are not appropriate to measure the quality of solutions from a robustness point of view. These are very useful to identify evenly distributed and strong dominant fronts but, unfortunately, they do not capture divergence between expected results and actual results. For this reason, we use some other metrics to measure the robustness of the solutions. We will consider four: Estimation Error, Stability, Unrealized Returns, and Extreme Risk, introduced in Ref. 11, that we succinctly describe below.

The *Estimation Error* (EE) considers the average Mahalanobis distance[34] between the expected risk and return for every portfolio in the efficient frontier and the actual risk and return *a posteriori*, once the real values of the parameters are observed. A small distance would imply that the expected behavior of the optimized portfolios is close to the real one observed *a posteriori*.

The *Stability* (ST) metric measures the average difference between the expected scenario and 500 feasible scenarios generated using nonparametric bootstrap. A larger set of scenarios is likely to result on a more accurate approximation to the potential distribution of parameters so that high values of this metric would represent higher sensitivity to likely scenarios and lower reliability.

Unrealized Returns (UR) evaluates, for every portfolio, the difference between its realized return and the maximum potential return for that risk level. This means that the higher the values of this metric, the larger the unrealized potential returns. Hence, a low value for this indicator would be considered something positive.

Extreme Risk (ER) metric measures portfolio sensitivity to worst-case scenarios. This indicator is especially important for portfolio managers that are concerned about potentially extreme deviations in the expected risks and returns. Under these circumstances, changes in the forecasted parameters could result in large deviations that would make the behavior of the selected portfolios very unreliable. The concept is very related to Value at Risk but, instead of focusing in losses, it captures unexpected portfolio behaviors balancing both changes in portfolio risk and return.

These are defined as the 1% resampled scenarios from a 500 sample with the highest average Mahalanobis distance between the risk/return of the portfolios evaluated on these parameters and the expected risk/return of the same portfolios. The higher the metric, the lower the robustness.

5.5. *Results*

In this section, we present the experimental results based on the data and procedures previously discussed. The approach was tested on the four MOEAs already introduced (SPEA2, NSGA-II, GDE3 and SMPSO) and the robustness of the solutions was assessed using EE, ST, ER and UR. For each algorithm, we compared the results of the canonical version with four alternatives: a time-stamped resampled version, and the three reliability levels arising from the extension of the previous version with the new objective based on ϵ-neighborhoods described in Sec. 4. For each metric and configuration we provide a set of standard descriptive statistics. Specifically, the average, median and variance over 100 runs are reported in the tables that follow. In addition to that, the differences in the metrics with respect to the basic versions of the algorithms have been formally tested for statistical significance using the Wilcoxon test due to their lack of normality.

The sensitivity of the results to the values of ϵ can be inferred from the comprehensive results reported in Appendix A. As it can be observed, the choice of parameter does not seem to play a major role. Despite testing the values of ϵ that differ by orders of magnitude, the impact is seldom beyond 2% regardless of the robustness quality indicator.

Conversely, the situation is very different when we study the importance of genetic operators. Appendix B shows the combined effect of replacing uniform crossover in the genetic algorithms with Simulated Binary Crossover (SBX) and Blend Alpha Crossover (BLX), together with the use of uniform mutation as an alternative to polynomial. As for SMPSO, the implications of using uniform mutation versus polynomial mutation are also reported. While the choice of operators affects the results of all of these algorithms, the genetic algorithms, specially SPEA2, show the largest dispersion. ST is, by far, the most influenced quality indicator, while UR is at the opposite side of the spectrum. In terms of operators, depending on the circumstance either uniform crossover or SBX offer the best results over BLX. Polynomial mutation, however, beats uniform mutation across the board.

The estimation error, the indicator that shows the discrepancy between the expected behavior of the selected portfolios and reality, is very variable depending on the underlying MOEA. As we can see in Table 3, out to the four basic algorithms, GD3 provides the best starting point. Adding the time-stamped resampling makes a major contribution to the results across the board, and we obtain the minimum value with SMPSO. The largest overall gain is achieved with SPEA2, but it is largely explained by the high baseline of the canonical algorithm, which performed the worst for EE. This algorithm, together with NSGA-II also seems to offer the lowest consistency as the variance of the results tends to be much higher for this alternative than the others. The best mean value for the metric is obtained by the high stability subset of SMPSO. Here, the addition of the ϵ-sensitivity objective results in additional mean improvement over the time-stamped resampled version of 13.5% in global terms, or 43% in relative terms.

All the median differences between the baseline values of the metric for the standard algorithms and the robust versions but one are significant at 1%. The exception is the comparison between the low stability solution for SMPSO and the standard setup. In that case the test could not reject equality at 5%.

Table 3. Estimation error indicator results.

EE	Average	Median	Variance	Av. imp. (%)
NSGAII	2.2613	1.8071	3.9781	
NSGAII R+T	1.2216	0.6357	2.5894	45.98
NSGAII High	1.4099	0.6717	4.3607	37.65
NSGAII Medium	1.2878	0.5192	3.8437	43.05
NSGAII Low	1.4934	0.7328	3.5194	33.96
SPEA2	2.5196	1.8162	5.1751	
SPEA2 R+T	1.1016	0.5270	2.4888	56.28
SPEA2 High	1.1595	0.4096	3.9791	53.98
SPEA2 Medium	1.1438	0.4172	3.8321	54.60
SPEA2 Low	1.2503	0.4826	3.9048	50.38
SMPSO	1.4939	1.2324	1.7256	
SMPSO R+T	1.0234	0.7044	0.9181	31.49
SMPSO High	0.8211	0.5162	1.1562	45.04
SMPSO Medium	1.0785	0.7184	1.1759	27.80
SMPSO Low	1.5171	1.1020	1.9934	−1.55
GDE3	1.4807	1.1839	1.6799	
GDE3 R+T	1.1001	0.6798	1.3447	25.71
GDE3 High	1.0406	0.6387	1.5822	29.72
GDE3 Medium	1.1947	0.7302	1.6907	19.31
GDE3 Low	1.4308	0.9410	2.2181	3.37

In terms of stability, SPEA2 is the core algorithm that both obtains the best values for the metric and the largest average improvement. Surprisingly, the improvement in the metric of 76.70% is not achieved by the high stability subset of portfolios, but the medium one. Having said that, Table 4 shows that the addition of the ϵ-neighborhood objective to the resampled version of SPEA2 improves the performance for the three fronts. In this case, the largest contribution of the new objective is lower than in the previous case, 6.7% in global terms or 8.7% in relative terms. The previous observation holds also for the median results. Even though the sets of portfolios identified by GDE3 show higher stability for the basic configuration, it does not profit as much as the rest from the inclusion of the resampling and the third objective. This pattern is also mirrored by the multi-objective PSO.

Once again, the only exception to the statistical significance of the differences at 1% between the robust versions and the baseline, is the solution set SMPSO Low. In this occasion, the equality null hypothesis is rejected at the 5% conventional level.

Table 5 reports the metric that captures the sensitivity of the solution to a set of worst-case scenarios, where the deviations of risk and return are especially large (the Extreme Risk indicator). In these cases, the introduction of the new approach results in a clear improvement across basic algorithms. Having said that, the contribution as

Table 4. Stability indicator results.

ST	Average	Median	Variance	Av. imp. (%)
NSGAII	6.5078	6.1306	13.0883	
NSGAII R+T	2.6528	2.4189	2.3038	59.24
NSGAII High	2.5756	2.0301	4.0390	60.42
NSGAII Medium	2.3863	1.9149	3.0975	63.33
NSGAII Low	3.2434	2.8207	4.2727	50.16
SPEA2	7.4009	7.0380	20.8347	
SPEA2 R+T	2.2201	1.8669	2.2534	70.00
SPEA2 High	1.7629	1.6965	0.9544	76.18
SPEA2 Medium	1.7243	1.5875	0.9450	76.70
SPEA2 Low	2.0391	1.7974	1.5289	72.45
SMPSO	5.2076	4.8454	7.0554	
SMPSO R+T	3.2402	2.8408	3.7394	37.78
SMPSO High	2.2425	2.0529	1.6246	56.94
SMPSO Medium	3.7007	3.1776	5.7124	28.94
SMPSO Low	5.1728	4.4513	10.1250	0.67
GDE3	5.1506	4.7763	6.9177	
GDE3 R+T	3.3691	3.0426	3.9137	34.59
GDE3 High	3.0536	2.6485	5.3222	40.71
GDE3 Medium	3.8238	3.1992	6.0094	25.76
GDE3 Low	4.8841	4.0340	9.9212	5.17

Table 5. Extreme risk indicator results.

ER	Average	Median	Variance	Av. imp. (%)
NSGAII	3.2148	2.8235	3.8975	
NSGAII R+T	1.7640	1.1865	2.6789	45.13
NSGAII High	2.0830	1.2043	5.1970	35.20
NSGAII Medium	1.8719	1.1469	3.7793	41.77
NSGAII Low	2.2642	1.5608	3.9371	29.57
SPEA2	3.5315	3.0472	4.9711	
SPEA2 R+T	1.5991	1.0266	2.5787	54.72
SPEA2 High	1.6308	0.8174	4.1723	53.82
SPEA2 Medium	1.6519	0.8571	3.8335	53.22
SPEA2 Low	1.8729	1.0022	4.3176	46.97
SMPSO	2.2303	1.9779	1.8379	
SMPSO R+T	1.4872	1.2325	1.0194	33.32
SMPSO High	1.3114	0.9860	1.1583	41.20
SMPSO Medium	1.7062	1.3493	1.4805	23.50
SMPSO Low	2.2328	1.9063	2.3973	−0.11
GDE3	2.2279	1.9722	1.8898	
GDE3 R+T	1.6970	1.2665	1.6804	23.83
GDE3 High	1.6236	1.2107	1.8473	27.12
GDE3 Medium	1.8660	1.4252	2.1637	16.24
GDE3 Low	2.1845	1.7783	2.7579	1.95

a percentage over the time-stamped resampling tends to be negligible. The exception is SMPSO, for which the highest stability portion of the new third objective shows a sizeable improvement of 41.20% versus 33.32%. This combination is also the one that gets the highest global results. The results, however, are poor for the genetic algorithms. Surprisingly, even though the results outperform the baseline, the new approach underperforms time-stamped resampling regardless of the tertile chosen. The outcome of the statistical testing is exactly the same one that we obtained in the previous metric. The differences for all the robust versions but SMPSO Low, are significant at 1%. The exception requires relaxing the criterion and setting it at 5%.

The introduction of the third objective also has an impact on the divergence between the selected set of portfolios and the observed efficient frontier. For the sake of these comparisons, the efficient frontier was determined using the individual fronts provided by the standard nonrobust versions of the core MOEAs using the ex-post parameters. All the fronts were combined into a single set of portfolios, and the nondominated subset was used as reference solution.

In all cases, as it is apparent in Table 6, the addition of strategies to increase robustness reduces the sum of money left of the table. The specific amount varies with the basic algorithm, but we obtained the best results with SMPSO. The multiobjective PSO also turns out to be the most consistent alternative among algorithm runs, as we can see in the variances. Once again, the lowest third of the

Table 6. Unrealized returns results.

UR	Average	Median	Variance	Av. imp. (%)
NSGAII	3.4788	2.7252	7.9201	
NSGAII R+T	2.5265	1.9556	4.2799	27.37
NSGAII High	2.5901	2.0154	6.4126	25.55
NSGAII Medium	2.4786	1.8850	5.4828	28.75
NSGAII Low	2.7654	2.1186	5.8123	20.51
SPEA2	3.5859	2.7953	8.6577	
SPEA2 R+T	2.3506	1.7922	4.1225	34.45
SPEA2 High	2.2843	1.7146	5.3432	36.30
SPEA2 Medium	2.2655	1.6740	5.1199	36.82
SPEA2 Low	2.4537	1.8015	5.5949	31.58
SMPSO	2.9953	2.4891	5.3718	
SMPSO R+T	2.2702	1.7245	3.0980	24.21
SMPSO High	1.8212	1.4248	2.1816	39.20
SMPSO Medium	2.2934	1.8028	3.4946	23.43
SMPSO Low	2.8357	2.2378	5.2096	5.33
GDE3	3.3427	2.8714	6.0005	
GDE3 R+T	2.6886	2.1756	4.6265	19.57
GDE3 High	2.4340	1.9484	3.5869	27.19
GDE3 Medium	2.7611	2.2202	4.8470	17.40
GDE3 Low	3.1077	2.5057	5.9904	7.03

set in terms of ϵ-sensitivity (high stability front) gets the largest reduction for the metric. Compared to the basic version of SMPSO which, we should bear in mind, is the algorithm with the best starting point, it gets a 39.20% drop. As the lower the value for UR, the better, the data shows an important contribution of the new objective to the enhancement of the robustness of the solution. In this case, the global improvement over R+T was 15%, 38.2% in relative terms. The differences observed for this metric between the standard MOEAs and the robust versions were all significant at 1%.

As we have seen, the results of the experimental process suggest that the addition of the third objective based on ϵ-neighborhoods improves the robustness of the solutions. It is therefore a reliable mechanism for attaining such a goal that can be adopted in any MOEA. While the magnitude of the gain varies depending on the metric and core algorithm chosen, there is a clear pattern that suggests that the lower the value of the third objective, the more robust is the solution. Regardless of the robustness indicator chosen, the best performance was always achieved extending the time-stamped resampled version of the algorithm with the third ϵ-sensitivity objective, and selecting the portfolios in either the lowest or the mid third in terms of third new variable. Unsurprisingly, the portfolios in the highest third in terms of the new objective, low stability fronts, tend to perform worse. Most of the time they drag the robustness provided by R+T significantly.

A visual inspection of the resulting approximated Pareto fronts in terms of risk and return shows a truly interesting conclusion. The inclusion of the ϵ-sensitivity as a new objective has enabled the algorithms to explore portions of the solution space that the R+T versions cannot reach. Indeed, Fig. 3 displays the 50%-attainment

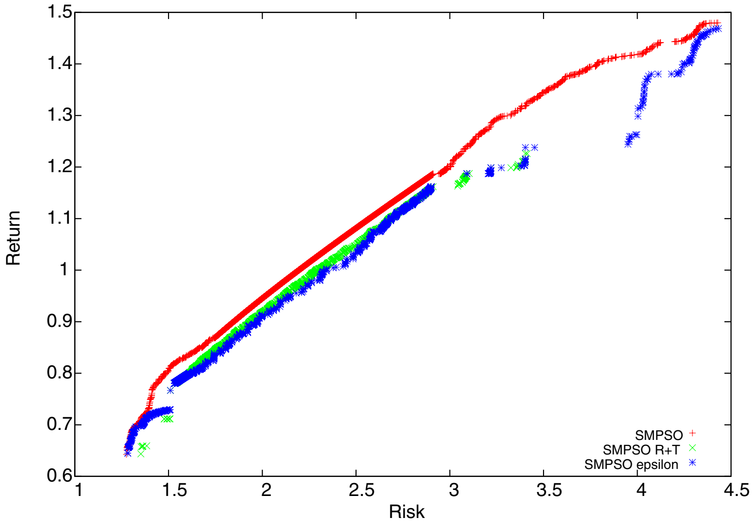

Fig. 3. Risk versus returns attainment surfaces for the three versions of SMPSO.

function[38] (it can be considered as the "median" approximated Front) for the three versions of SMPSO: the core algorithm, and those with the R+T approach, and the ϵ-sensitivity approach. It can be seen that the core algorithm is able to cover the wider portion of the search space, but the robust metrics in the tables above clearly point out that the portfolios reached are the worst in terms of robustness.

The R+T version, which shows a rather similar value for these metrics, does support the robustness by avoiding reaching portfolios with high risk, and also high return (located on the upper right part of the figure), that tend to be specially sensitive to uncertainty regarding the predicted returns and the variance–covariance matrix.

This idea of the existence of differentiated areas in terms of robustness along the Pareto front is also supported by the analysis of the traditional metrics considered in Appendices A and B. There we can see the sensitivity of HV and spread metrics to the choice of ϵ and operators. The results are in line with the ones we reported for the other indicators. While we observe that the sensitivity of HV and spread to ϵ are very limited, the selection of operators is important, specially for the former.

As we already discussed, there is an inverse relationship between robustness and HV. This makes sense since, as we mentioned before, robust solutions are a subset of the potential ones. The more robust is the subset, the smaller the range of alternatives in terms of risk and return and the smaller HV. This is consistent with the findings that, for NSGAII and SPEA2, uniform crossover and SBX provide lower values than BLX, and that polynomial mutation tends to offer higher results than the uniform alternative. Regarding SMPSO, the influence of the mutation operator is very limited. The impact of these choices on spread is not that clear. The indicator is mostly driven by the core algorithm. The evidence of the influence of the mutation operator is mixed and, regarding crossover, the only consistent pattern that we identified is that SBX tends to be slightly better.

To summarize, the results mentioned suggest that this approach provides not only advantages in term of sensitivity to the trading strategy used to build the selected portfolio, but also offers the decision maker portfolio choices of with good risk/return profiles that are at least as robust as the R+T strategy.

6. Summary and Conclusions

In this paper, we introduced a method to enhance the robustness of financial portfolios. The approach relies on the extension of the basic mean-variance standard formulation, with a third objective. This objective models portfolio implementation risk through the structure of the landscape surrounding a candidate portfolio and favors solutions located in reliable areas.

We define these as sections of the solution space where similar portfolios are also close in the objective space, that is, they offer similar risks and rewards. The rationale for this is the idea that, in these cases, temporary deviations from the desired solutions caused by trading strategies, are less likely to result in a major unexpected behavior. This reduces the exposure and increases the robustness of solutions.

As we mentioned, the method focuses its attention on the structure of the landscape, biasing the search towards areas of the solution space where small differences in the structure of portfolios do not result in major disparity in risk and return. This represents a major difference versus alternatives like time-stamped resampling (R+T), which are focused on the sensitivity of the solutions to perturbation in the two key parameters of the optimization, expected returns and the variance–covariance matrix. In other words, these approaches target two different intermediate objectives in their search for robustness that are not only compatible, but complementary, and can be implemented at the same time.

The implementation of this strategy is based on ϵ-neighborhoods together with the use of Mahalanobis distance to create a dispersion metric that should be minimized. The output of the method is a three-dimensional surface that could be used by the decision maker to select the portfolio that suits his needs better.

In order to assess the performance of the solutions, once the optimization is done, the resulting set of portfolios is broken into three sets of equal size according to the new variable. The selection of the nondominated solutions in terms of risk and return results on three subfronts with different degrees of stability, the best of which is likely to consist of robust solutions.

The approach is both compatible with the use of very different evolutionary multiobjective optimization methods, and alternatives like the one already mentioned. For this reason, the experimental setup consisted in four core popular MOEAs (NSGA II, SPEA2, SMPSO and GDE3), as baseline, plus the time-stamped resampling working in tandem with R+T versions of all of them. These alternatives were tested on historic financial data using a sliding window approach, and they were compared according to four different robustness metrics.

Even though the best results differ in their basic multiobjective algorithm depending on the metric, they were mostly achieved by a combination of R+T and the high stability subset of the ϵ-sensitivity objective. The combined effect resulted in major improvements for the robustness metrics. The contribution of the new objective to this result was generally substantial. Unsurprisingly, those portfolios with low values for the stability objective resulted in a significant drag to R+T in terms of increasing the reliability of the solutions.

Regarding the algorithm setup, the sensitivity of the results to neighborhood size, controlled by the parameter ϵ, depends on the core algorithm chosen. Having said that, it does not seem to play a major role in this particular context. However, it is not the case when we consider changes in evolutionary operators. The choice of the crossover and the mutation operators severely affects the robustness of the solutions. The best combinations include either uniform or SBX as crossover operators with polynomial mutation. The most affected algorithm is SPEA2 and the most sensitive performance indicator is ST.

Since solutions represent a subset of the range of alternatives, we observe an inverse relationship between robustness an HV. For this reason, the pattern of results in terms of this indicator basically mirrors the one described for robustness. Regarding spread, its connection with the different parameters and operators seem to be much weaker, and is driven by the choice of the core algorithm.

There are a number of potential ways to extend this work. Among them, we could mention comprehensive scalability studies; the exploration of different perturbation strategies or increasing the range basic algorithms to be tested.

In conclusion, the experimental evidence obtained by testing the approach over a very long period on real data supports the idea that it contributes to identifying robust portfolios. Given its flexibility, the low sensitivity of results to its only parameter, and its compatibility with different core algorithms and other robustness enhancing strategies, we think it is an option to be considered by both practitioners and researchers working with MOEAs in this domain.

Acknowledgments

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Appendix A. Sensitivity of Results to ϵ

Table A.1. Sensitivity to ϵ of the estimation error indicator.

EE	ϵ	Average	Median	Variance	Av. imp. (%)
NSGAII		2.2613	1.8071	3.9781	
NSGAII R+T		1.2216	0.6357	2.5894	45.98
NSGAII High	0.01	1.4160	0.6662	4.4399	37.38
NSGAII Medium		1.3058	0.5213	3.9894	42.26
NSGAII Low		1.5023	0.7374	3.6167	33.57
NSGAII High	0.001	1.4099	0.6717	4.3607	37.65
NSGAII Medium		1.2878	0.5192	3.8437	43.05
NSGAII Low		1.4934	0.7328	3.5194	33.96
NSGAII High	0.0001	1.4324	0.6847	4.3981	36.65
NSGAII Medium		1.3045	0.5151	3.9954	42.31
NSGAII Low		1.5135	0.7324	3.6646	33.07
NSGAII High	0.00001	1.4214	0.6661	4.4190	37.14
NSGAII Medium		1.3026	0.5091	3.9540	42.40
NSGAII Low		1.5044	0.7102	3.6359	33.47
SPEA2		2.5196	1.8162	5.1751	
SPEA2 R+T		1.1016	0.5270	2.4888	56.28

Table A.1. (*Continued*)

EE	ϵ	Average	Median	Variance	Av. imp. (%)
SPEA2 High	0.01	1.1699	0.4355	4.0348	53.57
SPEA2 Medium		1.1485	0.4245	3.9249	54.42
SPEA2 Low		1.2720	0.4907	4.1532	49.52
SPEA2 High	0.001	1.1595	0.4096	3.9791	53.98
SPEA2 Medium		1.1438	0.4172	3.8321	54.60
SPEA2 Low		1.2503	0.4826	3.9048	50.38
SPEA2 High	0.0001	1.1548	0.4195	3.9423	54.17
SPEA2 Medium		1.1518	0.4261	3.8796	54.29
SPEA2 Low		1.2366	0.4877	3.8586	50.92
SPEA2 High	0.00001	1.1473	0.4170	3.8978	54.46
SPEA2 Medium		1.2510	0.4871	3.9619	54.46
SPEA2 Low		1.1474	0.4191	3.9399	50.35
SMPSO		1.4939	1.2324	1.7256	
SMPSO R+T		1.0234	0.7044	0.9181	31.49
SMPSO High	0.01	0.8334	0.5243	1.2212	44.21
SMPSO Medium		1.0864	0.7259	1.1941	27.28
SMPSO Low		1.5237	1.1091	1.9836	−2.00
SMPSO High	0.001	0.8211	0.5162	1.1562	45.04
SMPSO Medium		1.0785	0.7184	1.1759	27.80
SMPSO Low		1.5171	1.1020	1.9934	−1.55
SMPSO High	0.0001	0.8224	0.5163	1.1420	44.95
SMPSO Medium		1.0903	0.7370	1.2037	27.02
SMPSO Low		1.5437	1.1318	2.0695	−3.33
SMPSO High	0.00001	0.8264	0.5213	1.1749	44.68
SMPSO Medium		1.0815	0.7427	1.1570	27.61
SMPSO Low		1.5181	1.1121	1.9642	−1.62
GDE3		1.4807	1.1839	1.6799	
GDE3 R+T		1.1001	0.6798	1.3447	25.71
GDE3 High	0.01	1.0567	0.6538	1.6219	28.63
GDE3 Medium		1.2213	0.7590	1.7732	17.52
GDE3 Low		1.4567	0.9770	2.1729	1.62
GDE3 High	0.001	1.0406	0.6387	1.5822	29.72
GDE3 Medium		1.1947	0.7302	1.6907	19.31
GDE3 Low		1.4308	0.9410	2.2181	3.37
GDE3 High	0.0001	1.0415	0.6462	1.5620	29.66
GDE3 Medium		1.2038	0.7398	1.7029	18.70
GDE3 Low		1.4453	0.9648	2.0794	2.39
GDE3 High	0.00001	1.0542	0.6511	1.5695	28.80
GDE3 Medium		1.2138	0.7445	1.7917	18.03
GDE3 Low		1.4287	0.9538	2.1284	3.51

Table A.2. Sensitivity to ϵ of the stability indicator.

ST	ϵ	Average	Median	Variance	Av. imp. (%)
NSGAII		6.5078	6.1306	13.0883	
NSGAII R+T		2.6528	2.4189	2.3038	59.24
NSGAII High	0.01	2.5566	2.0398	3.8033	60.71
NSGAII Medium		2.4027	1.9133	3.1669	63.08
NSGAII Low		3.2265	2.8121	4.2622	50.42
NSGAII High	0.001	2.5756	2.0301	4.0390	60.42
NSGAII Medium		2.3863	1.9149	3.0975	63.33
NSGAII Low		3.2434	2.8207	4.2727	50.16
NSGAII High	0.0001	2.5746	2.0487	3.6795	60.44
NSGAII Medium		2.3878	1.9169	2.8363	63.31
NSGAII Low		3.2344	2.8007	4.1223	50.30
NSGAII High	0.00001	2.5679	2.0328	3.8883	50.14
NSGAII Medium		2.3893	1.9168	2.9934	63.29
NSGAII Low		3.2184	2.7829	4.1869	50.55
SPEA2		7.4009	7.0380	20.8347	
SPEA2 R+T		2.2201	1.8669	2.2534	70.00
SPEA2 High	0.01	1.7877	1.7189	0.9625	75.84
SPEA2 Medium		1.7290	1.5949	0.9446	76.64
SPEA2 Low		2.0624	1.8056	1.5423	72.13
SPEA2 High	0.001	1.7629	1.6965	0.9544	76.18
SPEA2 Medium		1.7243	1.5875	0.9450	76.70
SPEA2 Low		2.0391	1.7974	1.5289	72.45
SPEA2 High	0.0001	1.7511	1.6950	0.8967	76.34
SPEA2 Medium		1.7153	1.5890	0.8871	76.82
SPEA2 Low		2.0128	1.7881	1.3832	72.80
SPEA2 High	0.00001	1.7615	1.7030	0.9200	76.20
SPEA2 Medium		1.7199	1.5922	0.8919	76.76
SPEA2 Low		2.0440	1.7972	1.4515	72.38
SMPSO		5.2076	4.8454	7.0554	
SMPSO R+T		3.2402	2.8408	3.7394	37.78
SMPSO High	0.01	2.2559	2.0676	1.6274	56.68
SMPSO Medium		3.7002	3.1791	5.6781	28.95
SMPSO Low		5.1610	4.4768	9.7711	0.89
SMPSO High	0.001	2.2425	2.0529	1.6246	56.94
SMPSO Medium		3.7007	3.1776	5.7124	28.94
SMPSO Low		5.1728	4.4513	10.1250	0.67
SMPSO High	0.0001	2.2381	2.0517	1.5802	57.02
SMPSO Medium		3.6960	3.2029	5.5885	29.03
SMPSO Low		5.1913	4.4998	10.0789	0.31
SMPSO High	0.00001	2.2226	2.0391	1.5911	57.32
SMPSO Medium		3.6761	3.1820	5.5142	29.41
SMPSO Low		5.1592	4.4767	10.0110	0.93
GDE3		5.1506	4.7763	6.9177	
GDE3 R+T		3.3691	3.0426	3.9137	34.59
GDE3 High	0.01	3.1418	2.7035	5.7365	39.00
GDE3 Medium		3.9129	3.2678	6.4121	24.03
GDE3 Low		4.9855	4.1357	10.3618	3.21

Table A.2. (*Continued*)

ST	ϵ	Average	Median	Variance	Av. imp. (%)
GDE3 High	0.001	3.0536	2.6485	5.3222	40.71
GDE3 Medium		3.8238	3.1992	6.0094	25.76
GDE3 Low		4.8841	4.0340	9.9212	5.17
GDE3 High	0.0001	3.0822	2.6843	5.1351	40.16
GDE3 Medium		3.8555	3.2606	6.0403	25.14
GDE3 Low		4.8853	4.0893	9.7035	5.15
GDE3 High	0.00001	3.0984	2.6715	5.5051	39.84
GDE3 Medium		3.8488	3.2365	6.0192	25.28
GDE3 Low		4.8480	4.0590	9.7989	5.88

Table A.3. Sensitivity to ϵ of the extreme risk indicator.

ER	ϵ	Average	Median	Variance	Av. imp. (%)
NSGAII		3.2148	2.8235	3.8975	
NSGAII R+T		1.7640	1.1865	2.6789	45.13
NSGAII High	0.01	2.0821	1.2042	5.2588	35.23
NSGAII Medium		1.8916	1.1513	3.8595	41.16
NSGAII Low		2.2715	1.5507	4.0121	29.34
NSGAII High	0.001	2.0830	1.2043	5.1970	35.20
NSGAII Medium		1.8719	1.1469	3.7793	41.77
NSGAII Low		2.2642	1.5608	3.9371	29.57
NSGAII High	0.0001	2.1075	1.2257	5.2513	34.44
NSGAII Medium		1.8880	1.1491	3.8730	41.27
NSGAII Low		2.2835	1.5775	4.0353	28.97
NSGAII High	0.00001	2.0871	1.2038	5.2419	35.08
NSGAII Medium		1.8884	1.1359	3.8793	41.26
NSGAII Low		2.2720	1.5603	4.0169	29.33
SPEA2		3.5315	3.0472	4.9711	
SPEA2 R+T		1.5991	1.0266	2.5787	54.72
SPEA2 High	0.01	1.6560	0.8300	4.2195	53.11
SPEA2 Medium		1.6658	0.8852	3.9287	52.83
SPEA2 Low		1.9061	1.0162	4.5500	46.03
SPEA2 High	0.001	1.6308	0.8174	4.1723	53.82
SPEA2 Medium		1.6519	0.8571	3.8335	53.22
SPEA2 Low		1.8729	1.0022	4.3176	46.97
SPEA2 High	0.0001	1.6217	0.8175	4.1016	54.08
SPEA2 Medium		1.6585	0.8677	3.8694	53.04
SPEA2 Low		1.8566	1.0011	4.2722	47.43
SPEA2 High	0.00001	1.6175	0.8172	4.0528	54.20
SPEA2 Medium		1.6593	0.8650	3.9331	53.02
SPEA2 Low		1.8801	1.0028	4.4195	46.76
SMPSO		2.2303	1.9779	1.8379	
SMPSO R+T		1.4872	1.2325	1.0194	33.32

Table A.3. (*Continued*)

ER	ϵ	Average	Median	Variance	Av. imp. (%)
SMPSO High	0.01	1.3263	0.9891	1.2227	40.53
SMPSO Medium		1.7204	1.3656	1.5185	22.86
SMPSO Low		2.2431	1.9261	2.4140	−0.57
SMPSO High	0.001	1.3114	0.9860	1.1583	41.20
SMPSO Medium		1.7062	1.3493	1.4805	23.50
SMPSO Low		2.2328	1.9063	2.3973	−0.11
SMPSO High	0.0001	1.3129	0.9969	1.1344	41.13
SMPSO Medium		1.7145	1.3583	1.4992	23.13
SMPSO Low		2.2615	1.9571	2.4844	−1.40
SMPSO High	0.00001	1.3137	0.9847	1.1708	41.10
SMPSO Medium		1.7061	1.3499	1.4779	23.50
SMPSO Low		2.2355	1.9213	2.4019	−0.23
GDE3		2.2279	1.9722	1.8898	
GDE3 R+T		1.6970	1.2665	1.6804	23.83
GDE3 High	0.01	1.6523	1.2325	1.8790	25.84
GDE3 Medium		1.9049	1.4598	2.2477	14.50
GDE3 Low		2.2192	1.8683	2.7076	0.39
GDE3 High	0.001	1.6236	1.2107	1.8473	27.12
GDE3 Medium		1.8660	1.4252	2.1637	16.24
GDE3 Low		2.1845	1.7783	2.7579	1.95
GDE3 High	0.0001	1.6336	1.2240	1.8797	26.68
GDE3 Medium		1.8776	1.4500	2.1692	15.72
GDE3 Low		2.1986	1.8330	2.6659	1.32
GDE3 High	0.00001	1.6542	1.2306	1.9386	25.75
GDE3 Medium		2.1729	1.8013	2.6769	2.47
GDE3 Low		1.8897	1.4315	2.2555	15.18

Table A.4. Sensitivity to ϵ of the unrealized returns indicator.

UR	ϵ	Average	Median	Variance	Av. imp. (%)
NSGAII		3.4788	2.7252	7.9201	
NSGAII R+T		2.5265	1.9556	4.2799	27.37
NSGAII High	0.01	2.5961	2.0124	6.5110	25.37
NSGAII Medium		2.4815	1.8862	5.5478	28.67
NSGAII Low		2.7626	2.1029	5.9501	20.59
NSGAII High	0.001	2.5901	2.0154	6.4126	25.55
NSGAII Medium		2.4786	1.8850	5.4828	28.75
NSGAII Low		2.7654	2.1186	5.8123	20.51
NSGAII High	0.0001	2.6264	2.0312	6.5937	24.50
NSGAII Medium		2.5045	1.8831	5.6675	28.01
NSGAII Low		2.7834	2.1199	6.0204	19.99
NSGAII High	0.00001	2.6198	2.0223	6.6731	24.69
NSGAII Medium		2.5071	1.8763	5.8220	27.93
NSGAII Low		2.7746	2.1233	6.0407	20.24
SPEA2		3.5859	2.7953	8.6577	
SPEA2 R+T		2.3506	1.7922	4.1225	34.45

Table A.4. (*Continued*)

UR	ϵ	Average	Median	Variance	Av. imp. (%)
SPEA2 High	0.01	2.2857	1.7652	5.0552	36.26
SPEA2 Medium		2.2563	1.6774	4.9483	37.08
SPEA2 Low		2.4650	1.8161	5.6578	31.26
SPEA2 High	0.001	2.2843	1.7146	5.3432	36.30
SPEA2 Medium		2.2655	1.6740	5.1199	36.82
SPEA2 Low		2.4537	1.8015	5.5949	31.58
SPEA2 High	0.0001	2.2720	1.7110	5.1008	36.64
SPEA2 Medium		2.2530	1.6674	5.0165	37.17
SPEA2 Low		2.4323	1.8021	5.4562	32.17
SPEA2 High	0.00001	2.2778	1.7108	5.1167	36.48
SPEA2 Medium		2.2602	1.6713	4.9607	36.97
SPEA2 Low		2.4576	1.8134	5.6130	31.46
SMPSO		2.9953	2.4891	5.3718	
SMPSO R+T		2.2702	1.7245	3.0980	24.21
SMPSO High	0.01	1.8383	1.4231	2.2844	38.63
SMPSO Medium		2.2866	1.7815	3.5252	23.66
SMPSO Low		2.8306	2.2418	5.1108	5.50
SMPSO High	0.001	1.8212	1.4248	2.1816	39.20
SMPSO Medium		2.2934	1.8028	3.4946	23.43
SMPSO Low		2.8357	2.2378	5.2096	5.33
SMPSO High	0.0001	1.8305	1.4297	2.1990	38.89
SMPSO Medium		2.3008	1.7994	3.4689	23.19
SMPSO Low		2.8586	2.2653	5.2400	4.56
SMPSO High	0.00001	1.8146	1.4214	2.1751	39.42
SMPSO Medium		2.2843	1.7924	3.4498	23.74
SMPSO Low		2.8289	2.2380	5.1331	5.56
GDE3		3.3427	2.8714	6.0005	
GDE3 R+T		2.6886	2.1756	4.6265	19.57
GDE3 High	0.01	2.4671	1.9619	3.7148	26.20
GDE3 Medium		2.7911	2.2465	4.7696	16.50
GDE3 Low		3.1406	2.5447	5.9849	6.05
GDE3 High	0.001	2.4340	1.9484	3.5869	27.19
GDE3 Medium		2.7611	2.2202	4.8470	17.40
GDE3 Low		3.1077	2.5057	5.9904	7.03
GDE3 High	0.0001	2.4702	1.9898	3.7410	26.10
GDE3 Medium		2.7874	2.2484	4.7802	16.61
GDE3 Low		3.1242	2.5336	5.9004	6.54
GDE3 High	0.00001	2.4734	2.0024	3.7607	26.01
GDE3 Medium		2.7995	2.2467	4.9233	16.25
GDE3 Low		3.1107	2.5129	6.0031	6.94

Table A.5. Sensitivity to ϵ of HV.

HV	ϵ	Average	Median	Variance	Av. imp. (%)
NSGAII		0.6743	0.6534	0.0034	
NSGAII R+T		0.5703	0.6066	0.0148	−15.42
NSGAII High	0.01	0.2888	0.2763	0.0252	−57.17
NSGAII Medium		0.3715	0.3787	0.0335	−44.90
NSGAII Low		0.5181	0.5666	0.0208	−23.17
NSGAII High	0.001	0.2924	0.2771	0.0255	−56.64
NSGAII Medium		0.3763	0.3861	0.0326	−44.20
NSGAII Low		0.5223	0.5677	0.0192	−22.54
NSGAII High	0.0001	0.2910	0.2734	0.0260	−56.84
NSGAII Medium		0.3775	0.3889	0.0321	−44.02
NSGAII Low		0.5209	0.5675	0.0197	−22.75
NSGAII High	0.00001	0.2920	0.2736	0.0264	−56.69
NSGAII Medium		0.3761	0.3863	0.0326	−44.22
NSGAII Low		0.5192	0.5657	0.0196	−23.00
SPEA2		0.6746	0.6538	0.0034	
SPEA2 R+T		0.5230	0.5845	0.0213	−22.48
SPEA2 High	0.01	0.2370	0.1860	0.0308	−64.87
SPEA2 Medium		0.2587	0.2089	0.0344	−61.65
SPEA2 Low		0.4038	0.4302	0.0259	−40.15
SPEA2 High	0.001	0.2404	0.1984	0.0307	−64.36
SPEA2 Medium		0.2578	0.2074	0.0341	−61.79
SPEA2 Low		0.4025	0.4276	0.0260	−40.34
SPEA2 High	0.0001	0.2397	0.1932	0.0310	−64.47
SPEA2 Medium		0.2577	0.2093	0.0334	−61.81
SPEA2 Low		0.4006	0.4240	0.0262	−40.62
SPEA2 High	0.00001	0.2415	0.1931	0.0315	−64.20
SPEA2 Medium		0.2601	0.2117	0.0337	−61.45
SPEA2 Low		0.4026	0.4273	0.0258	−40.32
SMPSO		0.5895	0.5781	0.0047	
SMPSO R+T		0.4058	0.4323	0.0133	−31.16
SMPSO High	0.01	0.3153	0.3186	0.0159	−46.51
SMPSO Medium		0.3738	0.3994	0.0155	−36.60
SMPSO Low		0.3840	0.4190	0.0161	−34.86
SMPSO High	0.001	0.3668	0.3849	0.0154	−37.77
SMPSO Medium		0.3989	0.4217	0.0134	−32.33
SMPSO Low		0.3981	0.4262	0.0146	−32.47
SMPSO High	0.0001	0.3165	0.3221	0.0160	−46.32
SMPSO Medium		0.3754	0.4007	0.0156	−36.32
SMPSO Low		0.3831	0.4181	0.0163	−35.02
SMPSO High	0.00001	0.3153	0.3194	0.0160	−46.52
SMPSO Medium		0.3752	0.4022	0.0157	−36.35
SMPSO Low		0.3827	0.4178	0.0166	−35.08
GDE3		0.5935	0.5815	0.0045	
GDE3 R+T		0.4564	0.4931	0.0168	−23.09
GDE3 High	0.01	0.3824	0.4234	0.0241	−35.57
GDE3 Medium		0.3820	0.4106	0.0154	−35.64
GDE3 Low		0.3864	0.4136	0.0155	−34.90

Table A.5. (*Continued*)

HV	ϵ	Average	Median	Variance	Av. imp. (%)
GDE3 High	0.001	0.3875	0.4236	0.0231	−34.71
GDE3 Medium		0.3881	0.4134	0.0145	−34.61
GDE3 Low		0.3908	0.4145	0.0148	−34.15
GDE3 High	0.0001	0.3782	0.4151	0.0236	−36.28
GDE3 Medium		0.3757	0.4037	0.0151	−36.70
GDE3 Low		0.3828	0.4109	0.0149	−35.50
GDE3 High	0.000001	0.3802	0.4199	0.0235	−35.94
GDE3 Medium		0.3771	0.4057	0.0149	−36.46
GDE3 Low		0.3818	0.4114	0.0151	−35.66

Table A.6. Sensitivity to ϵ of spread.

Spread	ϵ	Average	Median	Variance	Av. imp. (%)
NSGAII		0.5907	0.5689	0.0066	
NSGAII R+T		1.0742	1.0675	0.0361	−81.86
NSGAII High	0.01	1.2028	1.0620	0.0608	−103.62
NSGAII Medium		1.0946	1.0233	0.0358	−85.32
NSGAII Low		0.9838	0.9610	0.0365	−66.56
NSGAII High	0.001	1.2004	1.0596	0.0601	−103.22
NSGAII Medium		1.0859	1.0204	0.0343	−83.85
NSGAII Low		0.9776	0.9520	0.0354	−65.50
NSGAII High	0.0001	1.1986	1.0532	0.0614	−102.92
NSGAII Medium		1.0869	1.0210	0.0341	−84.00
NSGAII Low		0.9817	0.9572	0.0365	−66.19
NSGAII High	0.00001	1.1921	1.0494	0.0593	−101.81
NSGAII Medium		1.0819	1.0197	0.0333	−83.16
NSGAII Low		0.9819	0.9546	0.0367	−66.24
SPEA2		0.3984	0.3825	0.0106	
SPEA2 R+T		1.2193	1.1868	0.0562	−206.01
SPEA2 High	0.01	1.0891	1.0196	0.0267	−173.33
SPEA2 Medium		1.0431	1.0120	0.0115	−161.79
SPEA2 Low		1.0587	1.0493	0.0199	−165.72
SPEA2 High	0.001	1.0898	1.0225	0.0259	−173.52
SPEA2 Medium		1.0411	1.0132	0.0106	−161.29
SPEA2 Low		1.0603	1.0538	0.0194	−166.11
SPEA2 High	0.0001	1.0866	1.0209	0.0257	−172.72
SPEA2 Medium		1.0388	1.0123	0.0101	−160.71
SPEA2 Low		1.0611	1.0507	0.0195	−166.32
SPEA2 High	0.00001	1.0851	1.0202	0.0258	−172.33
SPEA2 Medium		1.0390	1.0126	0.0100	−160.76
SPEA2 Low		1.0593	1.0514	0.0198	−165.85
SMPSO		1.2391	1.2422	0.0674	
SMPSO R+T		0.9624	0.9674	0.0162	22.33

Table A.6. (*Continued*)

Spread	ϵ	Average	Median	Variance	Av. imp. (%)
SMPSO High	0.01	1.0062	0.9944	0.0247	18.80
SMPSO Medium		0.9473	0.9552	0.0116	23.55
SMPSO Low		0.8926	0.8939	0.0082	27.97
SMPSO High	0.001	1.0336	1.0000	0.0386	16.59
SMPSO Medium		0.9481	0.9525	0.0139	23.48
SMPSO Low		0.8888	0.8945	0.0087	28.27
SMPSO High	0.0001	0.9963	0.9852	0.0256	19.60
SMPSO Medium		0.9396	0.9453	0.0121	24.17
SMPSO Low		0.8908	0.8934	0.0087	28.11
SMPSO High	0.00001	0.9964	0.9847	0.0249	19.59
SMPSO Medium		0.9398	0.9459	0.0115	24.15
SMPSO Low		0.8910	0.8940	0.0088	28.09
GDE3		1.2858	1.3171	0.1042	
GDE3 R+T		0.9986	0.9981	0.0338	22.34
GDE3 High	0.01	1.1482	1.1102	0.0689	10.71
GDE3 Medium		0.9592	0.9757	0.0177	25.40
GDE3 Low		0.9311	0.9357	0.0097	27.58
GDE3 High	0.001	1.1295	1.0868	0.0689	12.16
GDE3 Medium		0.9557	0.9738	0.0178	25.68
GDE3 Low		0.9332	0.9399	0.0121	27.42
GDE3 High	0.0001	1.1317	1.0935	0.0646	11.99
GDE3 Medium		0.9561	0.9748	0.0163	25.64
GDE3 Low		0.9405	0.9447	0.0104	26.86
GDE3 High	0.00001	1.1260	1.0864	0.0652	12.43
GDE3 Medium		0.9544	0.9734	0.0168	25.78
GDE3 Low		0.9386	0.9437	0.0098	27.00

Appendix B. Sensitivity of Results to Evolutionary Operators

Table B.1. Sensitivity of estimation error to evolutionary operators.

Crossover: SBX $\eta_c = 20$, BLX $\alpha = 0.5$, Uniform Prob. = 0.5
Mutation: Uniform Perturb. = 1, Polynomial $\eta_m = 20$.

EE	Cross. Op.	Mut. Op.	Average	Median	Variance	Av. imp. (%)
NSGAII High	Uniform	Polynomial	1.4099	0.6717	4.3607	
NSGAII Medium			1.2878	0.5192	3.8437	
NSGAII Low			1.4934	0.7328	3.5194	
NSGAII High	Uniform	Uniform	1.5144	1.0320	2.2401	−7.42
NSGAII Medium			1.4321	0.8662	2.5313	−11.21
NSGAII Low			1.6267	1.0907	2.5183	−8.93
NSGAII High	SBX	Polynomial	1.0899	0.4971	2.3081	22.70
NSGAII Medium			1.0928	0.4723	2.4435	15.14
NSGAII Low			1.4250	0.7551	3.0300	4.58
NSGAII High	SBX	Uniform	1.1632	0.5621	2.4632	17.50
NSGAII Medium			1.1525	0.5119	2.5599	10.50

Table B.1. (*Continued*)

EE	Cross. Op.	Mut. Op.	Average	Median	Variance	Av. imp. (%)
NSGAII Low			1.5589	0.8574	3.3644	−4.39
NSGAII High	BLX	Polynomial	1.3678	0.8189	2.3991	2.98
NSGAII Medium			1.3571	0.7556	2.5721	−5.38
NSGAII Low			1.5393	1.0070	2.3887	−3.07
NSGAII High	BLX	Uniform	1.5059	1.0064	2.2449	−6.81
NSGAII Medium			1.4299	0.8688	2.4835	−11.04
NSGAII Low			1.6251	1.1078	2.4379	−8.82
SPEA2 High	Uniform	Polynomial	1.1595	0.4096	3.9791	
SPEA2 Medium			1.1438	0.4172	3.8321	
SPEA2 Low			1.2503	0.4826	3.9048	
SPEA2 High	Uniform	Uniform	1.1526	0.4320	3.8641	0.59
SPEA2 Medium			1.1576	0.4416	3.8475	−1.21
SPEA2 Low			1.3602	0.6366	3.6763	−8.79
SPEA2 High	SBX	Polynomial	0.8493	0.3523	2.2745	26.76
SPEA2 Medium			0.9831	0.4126	2.4459	14.05
SPEA2 Low			1.3162	0.6190	3.0601	−5.27
SPEA2 High	SBX	Uniform	0.8915	0.4177	2.0401	23.11
SPEA2 Medium			1.0871	0.5124	2.1727	4.96
SPEA2 Low			1.5497	0.8858	2.9585	−23.94
SPEA2 High	BLX	Polynomial	1.3171	0.6766	2.7992	−13.59
SPEA2 Medium			1.3748	0.7181	2.8934	−20.20
SPEA2 Low			1.4919	0.9483	2.4961	−19.32
SPEA2 High	BLX	Uniform	1.3846	0.7827	2.7387	−19.41
SPEA2 Medium			1.4241	0.8060	2.8795	−24.51
SPEA2 Low			1.5499	1.0293	2.4349	−23.96
SMPSO High	N/A	Polynomial	0.8211	0.5162	1.1562	
SMPSO Medium			1.0785	0.7184	1.1759	
SMPSO Low			1.5171	1.1020	1.9934	
SMPSO High	N/A	Uniform	1.0178	0.6961	1.2953	−23.96
SMPSO Medium			1.1753	0.8275	1.2073	−4.04
SMPSO Low			1.5785	1.2221	1.9816	−8.97

Table B.2. Sensitivity of stability indicator to evolutionary operators.

Crossover: SBX $\eta_c = 20$, BLX $\alpha = 0.5$, Uniform Prob. = 0.5
Mutation: Uniform Perturb. = 1, Polynomial $\eta_m = 20$

ST	Cross. Op.	Mut. Op.	Average	Median	Variance	Av. imp. (%)
NSGAII High	Uniform	Polynomial	2.5756	2.0301	4.0390	
NSGAII Medium			2.3863	1.9149	3.0975	
NSGAII Low			3.2434	2.8207	4.2727	
NSGAII High	Uniform	Uniform	3.9037	3.2838	7.9404	−51.56
NSGAII Medium			3.6572	3.0151	6.1054	−53.26
NSGAII Low			5.0242	4.2081	9.4289	−54.90
NSGAII High	SBX	Polynomial	2.1713	1.7496	2.6931	15.70
NSGAII Medium			2.2796	1.9733	2.4405	4.47

Table B.2. (*Continued*)

ST	Cross. Op.	Mut. Op.	Average	Median	Variance	Av. imp. (%)
NSGAII Low			3.6430	3.0543	5.7428	−12.32
NSGAII High	SBX	Uniform	2.3214	1.8288	3.2417	9.87
NSGAII Medium			2.4464	2.0672	2.8112	−2.52
NSGAII Low			4.0692	3.3929	6.7888	−25.46
NSGAII High	BLX	Polynomial	3.3514	2.8410	5.6230	−30.12
NSGAII Medium			3.4755	2.8704	5.6367	−45.64
NSGAII Low			4.7928	4.1389	8.7753	−47.77
NSGAII High	BLX	Uniform	3.8479	3.2192	7.9766	−49.40
NSGAII Medium			3.6560	3.0118	6.1538	−53.21
NSGAII Low			4.9783	4.1836	9.2820	−53.49
SPEA2 High	Uniform	Polynomial	1.7629	1.6965	0.9544	
SPEA2 Medium			1.7243	1.5875	0.9450	
SPEA2 Low			2.0391	1.7974	1.5289	
SPEA2 High	Uniform	Uniform	1.7761	1.7071	0.8998	−0.75
SPEA2 Medium			1.7817	1.6331	0.9120	−3.33
SPEA2 Low			2.6003	2.3076	2.4584	−27.52
SPEA2 High	SBX	Polynomial	1.5555	1.4582	0.7858	11.77
SPEA2 Medium			2.1069	1.7175	2.5325	−22.19
SPEA2 Low			3.2902	2.5046	6.5230	−61.35
SPEA2 High	SBX	Uniform	1.7092	1.5509	0.9235	3.05
SPEA2 Medium			2.6156	1.9594	4.1602	−51.69
SPEA2 Low			4.3845	3.3307	9.7292	−115.02
SPEA2 High	BLX	Polynomial	3.0978	2.6761	4.5804	−75.72
SPEA2 Medium			3.5888	3.0108	6.1195	−108.13
SPEA2 Low			4.6482	4.0915	8.4107	−127.96
SPEA2 High	BLX	Uniform	3.1861	2.6886	5.3176	−80.73
SPEA2 Medium			3.5544	2.9632	6.1035	−106.14
SPEA2 Low			4.7665	4.0284	9.0661	−133.75
SMPSO High	N/A	Polynomial	2.2425	2.0529	1.6246	
SMPSO Medium			3.7007	3.1776	5.7124	
SMPSO Low			5.1728	4.4513	10.1250	
SMPSO High	N/A	Uniform	2.9281	2.6096	3.3223	−30.57
SMPSO Medium			4.1833	3.5362	7.2630	−13.04
SMPSO Low			5.4670	4.8265	10.7605	−5.69

Table B.3. Sensitivity of extreme risk indicator to evolutionary operators.

Crossover: SBX $\eta_c = 20$, BLX $\alpha = 0.5$, Uniform Prob. = 0.5
Mutation: Uniform Perturb. = 1, Polynomial $\eta_m = 20$

ER	Cross. Op.	Mut. Op.	Average	Median	Variance	Av. imp. (%)
NSGAII High	Uniform	Polynomial	2.0830	1.2043	5.1970	
NSGAII Medium			1.8719	1.1469	3.7793	
NSGAII Low			2.2642	1.5608	3.9371	
NSGAII High	Uniform	Uniform	2.3169	1.7856	3.2135	−11.23
NSGAII Medium			2.1479	1.5919	2.8992	−14.74
NSGAII Low			2.4696	2.1120	2.9153	−9.07

Table B.3. (*Continued*)

ER	Cross. Op.	Mut. Op.	Average	Median	Variance	Av. imp. (%)
NSGAII High	SBX	Polynomial	1.6968	1.0344	2.9890	18.54
NSGAII Medium			1.7059	1.0417	2.6472	8.87
NSGAII Low			2.2077	1.6325	3.3042	2.49
NSGAII High	SBX	Uniform	1.7922	1.0784	3.1922	13.96
NSGAII Medium			1.7727	1.1261	2.7121	5.30
NSGAII Low			2.3677	1.8052	3.5897	−4.57
NSGAII High	BLX	Polynomial	2.0500	1.4572	3.0578	1.58
NSGAII Medium			2.0273	1.4075	2.9418	−8.30
NSGAII Low			2.3400	1.9451	2.8815	−3.35
NSGAII High	BLX	Uniform	2.3029	1.7524	3.2588	−10.56
NSGAII Medium			2.1429	1.5886	2.8309	−14.48
NSGAII Low			2.4656	2.1127	2.8610	−8.89
SPEA2 High	Uniform	Polynomial	1.6308	0.8174	4.1723	
SPEA2 Medium			1.6519	0.8571	3.8335	
SPEA2 Low			1.8729	1.0022	4.3176	
SPEA2 High	Uniform	Uniform	1.6353	0.8311	4.0881	−0.28
SPEA2 Medium			1.6869	0.9193	3.7906	−2.12
SPEA2 Low			2.0581	1.2684	4.1035	−9.89
SPEA2 High	SBX	Polynomial	1.3136	0.7291	2.4065	19.45
SPEA2 Medium			1.5371	0.9002	2.6449	6.95
SPEA2 Low			2.0103	1.3141	3.4594	−7.34
SPEA2 High	SBX	Uniform	1.3980	0.8398	2.2632	14.27
SPEA2 Medium			1.7086	1.1332	2.4663	−3.43
SPEA2 Low			2.3539	1.8263	3.3893	−25.68
SPEA2 High	BLX	Polynomial	1.9473	1.2652	3.3444	−19.41
SPEA2 Medium			2.0443	1.3856	3.2797	−23.75
SPEA2 Low			2.2768	1.8320	3.0319	−21.57
SPEA2 High	BLX	Uniform	2.0947	1.3967	3.6256	−28.44
SPEA2 Medium			2.1156	1.5002	3.2975	−28.07
SPEA2 Low			2.3710	1.9655	2.9180	−26.59
SMPSO High	N/A	Polynomial	1.3114	0.9860	1.1583	
SMPSO Medium			1.7062	1.3493	1.4805	
SMPSO Low			2.2328	1.9063	2.3973	
SMPSO High	N/A	Uniform	1.6031	1.2745	1.4233	−22.24
SMPSO Medium			1.8295	1.5042	1.5045	−7.22
SMPSO Low			2.3209	2.0886	2.3663	−3.95

Table B.4. Sensitivity of unrelaized returns to evolutionary operators.

Crossover: SBX $\eta_c = 20$, BLX $\alpha = 0.5$, Uniform Prob. = 0.5
Mutation: Uniform Perturb. = 1, Polynomial $\eta_m = 20$

UR	Cross. Op.	Mut. Op.	Average	Median	Variance	Av. imp. (%)
NSGAII High	Uniform	Polynomial	2.5901	2.0154	6.4126	
NSGAII Medium			2.4786	1.8850	5.4828	
NSGAII Low			2.7654	2.1186	5.8123	
NSGAII High	Uniform	Uniform	2.7843	2.1664	5.1318	−7.50
NSGAII Medium			2.7714	2.1087	5.3960	−11.81
NSGAII Low			3.1060	2.4013	6.2799	−12.32
NSGAII High	SBX	Polynomial	2.3015	1.8486	3.7305	11.14
NSGAII Medium			2.3730	1.8612	4.2165	4.26
NSGAII Low			2.7954	2.1399	5.8025	−1.08
NSGAII High	SBX	Uniform	2.3871	1.8727	4.1781	7.84
NSGAII Medium			2.4427	1.8925	4.5125	1.45
NSGAII Low			2.9210	2.2511	6.2307	−5.63
NSGAII High	BLX	Polynomial	2.5922	2.0100	4.3465	−0.08
NSGAII Medium			2.6769	2.0854	5.0529	−8.00
NSGAII Low			3.0245	2.3310	6.0111	−9.37
NSGAII High	BLX	Uniform	2.7965	2.1580	5.1894	−7.97
NSGAII Medium			2.7715	2.1149	5.2273	−11.82
NSGAII Low			3.0988	2.3914	6.3012	−12.06
SPEA2 High	Uniform	Polynomial	2.2843	1.7146	5.3432	
SPEA2 Medium			2.2655	1.6740	5.1199	
SPEA2 Low			2.4537	1.8015	5.5949	
SPEA2 High	Uniform	Uniform	2.2812	1.7026	4.9974	0.14
SPEA2 Medium			2.2822	1.6808	4.9082	−0.74
SPEA2 Low			2.6052	1.9488	5.7174	−6.18
SPEA2 High	SBX	Polynomial	1.9386	1.6319	2.2516	15.14
SPEA2 Medium			2.1857	1.7556	3.2872	3.52
SPEA2 Low			2.6194	2.0236	5.1601	−6.75
SPEA2 High	SBX	Uniform	1.9979	1.6440	2.5751	12.54
SPEA2 Medium			2.3241	1.7801	3.8877	−2.59
SPEA2 Low			2.8702	2.1505	6.0742	−16.98
SPEA2 High	BLX	Polynomial	2.5211	1.9340	4.3368	−10.37
SPEA2 Medium			2.6821	2.0709	4.9709	−18.39
SPEA2 Low			2.9411	2.2753	5.5599	−19.86
SPEA2 High	BLX	Uniform	2.6168	1.9877	5.0101	−14.56
SPEA2 Medium			2.7256	2.0608	5.4047	−20.31
SPEA2 Low			2.9931	2.3030	5.7821	−21.98
SMPSO High	N/A	Polynomial	1.8212	1.4248	2.1816	
SMPSO Medium			2.2934	1.8028	3.4946	
SMPSO Low			2.8357	2.2378	5.2096	
SMPSO High	N/A	Uniform	2.1179	1.6185	2.8641	−16.29
SMPSO Medium			2.4117	1.8846	3.6406	−5.16
SMPSO Low			2.9282	2.3369	5.3500	−3.26

Table B.5. Sensitivity of HV to evolutionary operators.

Crossover: SBX $\eta_c = 20$, BLX $\alpha = 0.5$, Uniform Prob.= 0.5
Mutation: Uniform Perturb.= 1, Polynomial $\eta_m = 20$

HV	Cross. Op.	Mut. Op.	Average	Median	Variance	Av. imp. (%)
NSGAII High	Uniform	Polynomial	0.2924	0.2771	0.0255	
NSGAII Medium			0.3763	0.3861	0.0326	
NSGAII Low			0.5223	0.5677	0.0192	
NSGAII High	Uniform	Uniform	0.3436	0.3593	0.0176	17.50
NSGAII Medium			0.4671	0.4975	0.0168	24.14
NSGAII Low			0.5155	0.5391	0.0142	−1.30
NSGAII High	SBX	Polynomial	0.3132	0.3151	0.0192	7.10
NSGAII Medium			0.4042	0.4224	0.0239	7.42
NSGAII Low			0.5325	0.5659	0.0188	1.95
NSGAII High	SBX	Uniform	0.3341	0.3344	0.0186	14.26
NSGAII Medium			0.4295	0.4634	0.0217	14.13
NSGAII Low			0.5562	0.5763	0.0140	6.50
NSGAII High	BLX	Polynomial	0.3485	0.3669	0.0212	19.19
NSGAII Medium			0.4457	0.4772	0.0186	18.44
NSGAII Low			0.4796	0.4990	0.0165	−8.17
NSGAII High	BLX	Uniform	0.3417	0.3558	0.0174	16.87
NSGAII Medium			0.5134	0.5382	0.0146	−1.70
NSGAII Low			0.4650	0.4967	0.0168	23.58
SPEA2 High	Uniform	Polynomial	0.2404	0.1984	0.0307	
SPEA2 Medium			0.2578	0.2074	0.0341	
SPEA2 Low			0.4025	0.4276	0.0260	
SPEA2 High	Uniform	Uniform	0.2486	0.2121	0.0310	3.40
SPEA2 Medium			0.2865	0.2438	0.0338	11.12
SPEA2 Low			0.4831	0.5344	0.0220	20.03
SPEA2 High	SBX	Polynomial	0.2556	0.2315	0.0206	6.33
SPEA2 Medium			0.3289	0.3127	0.0225	27.58
SPEA2 Low			0.4452	0.4821	0.0195	10.61
SPEA2 High	SBX	Uniform	0.2829	0.2552	0.0223	17.68
SPEA2 Medium			0.3706	0.3770	0.0218	43.74
SPEA2 Low			0.4746	0.5086	0.0182	17.92
SPEA2 High	BLX	Polynomial	0.3472	0.3715	0.0265	44.42
SPEA2 Medium			0.4064	0.4343	0.0214	57.66
SPEA2 Low			0.4447	0.4632	0.0174	10.49
SPEA2 High	BLX	Uniform	0.3207	0.3349	0.0222	33.39
SPEA2 Medium			0.4258	0.4545	0.0210	65.18
SPEA2 Low			0.4807	0.5035	0.0161	19.44
SMPSO High	N/A	Polynomial	0.3668	0.3849	0.0154	
SMPSO Medium			0.3989	0.4217	0.0134	
SMPSO Low			0.3981	0.4262	0.0146	
SMPSO High	N/A	Uniform	0.3690	0.3861	0.0154	0.59
SMPSO Medium			0.3997	0.4226	0.0133	0.21
SMPSO Low			0.3980	0.4257	0.0143	−0.03

Table B.6. Sensitivity of spread to evolutionary operators.

Crossover: SBX $\eta_c = 20$, BLX $\alpha = 0.5$, Uniform Prob. = 0.5
Mutation: Uniform Perturb. = 1, Polynomial $\eta_m = 20$

Spread	Cross. Op.	Mut. Op.	Average	Median	Variance	Av. imp. (%)
NSGAII High	Uniform	Polynomial	1.2004	1.0596	0.0601	
NSGAII Medium			1.0859	1.0204	0.0343	
NSGAII Low			0.9776	0.9520	0.0354	
NSGAII High	Uniform	Uniform	1.1686	1.1577	0.0422	2.65
NSGAII Medium			0.9886	0.9832	0.0332	8.96
NSGAII Low			0.8507	0.8357	0.0157	12.98
NSGAII High	SBX	Polynomial	1.1925	1.0803	0.0543	0.66
NSGAII Medium			1.0640	1.0123	0.0262	2.02
NSGAII Low			0.9853	0.9564	0.0247	−0.79
NSGAII High	SBX	Uniform	1.1967	1.1004	0.0510	0.31
NSGAII Medium			1.0873	1.0266	0.0311	−0.13
NSGAII Low			0.9666	0.9256	0.0268	1.13
NSGAII High	BLX	Polynomial	1.2282	1.2017	0.0569	−2.32
NSGAII Medium			1.0658	1.0301	0.0319	1.86
NSGAII Low			0.9459	0.9321	0.0150	3.24
NSGAII High	BLX	Uniform	1.1666	1.1511	0.0421	2.81
NSGAII Medium			0.9867	0.9812	0.0334	9.14
NSGAII Low			0.8521	0.8392	0.0155	12.83
SPEA2 High	Uniform	Polynomial	1.0866	1.0209	0.0257	
SPEA2 Medium			1.0388	1.0123	0.0101	
SPEA2 Low			1.0611	1.0507	0.0195	
SPEA2 High	Uniform	Uniform	1.0863	1.0188	0.0269	0.32
SPEA2 Medium			1.0445	1.0134	0.0121	−0.33
SPEA2 Low			1.0473	1.0280	0.0306	1.22
SPEA2 High	SBX	Polynomial	1.0381	1.0024	0.0097	4.74
SPEA2 Medium			1.0020	0.9978	0.0060	3.75
SPEA2 Low			0.9879	0.9858	0.0142	6.83
SPEA2 High	SBX	Uniform	1.0556	1.0026	0.0174	3.13
SPEA2 Medium			1.0106	0.9983	0.0154	2.93
SPEA2 Low			0.9570	0.9557	0.0175	9.74
SPEA2 High	BLX	Polynomial	1.2061	1.1400	0.0630	−10.68
SPEA2 Medium			1.0759	1.0274	0.0315	−3.35
SPEA2 Low			0.9859	0.9741	0.0162	7.02
SPEA2 High	BLX	Uniform	1.1762	1.1143	0.0572	−7.93
SPEA2 Medium			1.0290	0.9997	0.0369	1.16
SPEA2 Low			0.9073	0.9016	0.0192	14.43
SMPSO High	N/A	Polynomial	1.0336	1.0000	0.0386	
SMPSO Medium			0.9481	0.9525	0.0139	
SMPSO Low			0.8888	0.8945	0.0087	
SMPSO High	N/A	Uniform	1.0332	1.0000	0.0380	0.03
SMPSO Medium			0.9482	0.9527	0.0139	−0.01
SMPSO Low			0.8889	0.8953	0.0091	−0.02

References

1. H. Markowitz, Portfolio selection, *The Journal of Finance* **7**(1) (1952) 77–91.
2. C. Zopuonidis and M. Doumpos, Multicriteria decision systems for financial problems, *TOP*, **21**(2) (2013) 241–261.
3. T. Chang, N. Meade, J. Beasley and Y. Sharaiha, Heuristics for cardinality constrained portfolio optimization, *Computers & Operations Research*, **27**(13) (2000) 1271–1302.
4. K. Anagnostopoulos and G. Mamanis, A portfolio optimization model with three objectives and discrete variables, *Computers & Operations Research* **37**(7) (2010) 1285–1297.
5. C. C. Lin and Y. T. Liu, Genetic algorithms for portfolio selection problems with minimum transaction lots, *European Journal of Operational Research* **185**(1) (2008) 393–404.
6. S. C. Chiam, K. C. Tan and A. A. Mamum, Evolutionary multi-objective portfolio optimization in practical context. *International Journal of Automation and Computing* **5**(1) (2008) 67–80.
7. Y. Peng, G. Kou, Y. Shi and Z. Chen, A descriptive framework for the field of data mining and knowledge discovery, *International Journal of Information Technology & Decision Making.* **7**(4) (2008) 639–684.
8. Y. Huang and G. Kou, A kernel entropy manifold learning approach for financial data analysis, *Decision Support Systems* **64** (2014) 31–42.
9. G. Kou, Y. Peng and G. Wang, Evaluation of clustering algorithms for financial risk analysis using MCDM methods, *Information Sciences* **275** (2014) 1–12.
10. F. J. Fabozzi, P. N. Kolm, D. A. Pachamanova and S. M. Focardi, Robust Portfolio optimization, *The Journal of Portfolio Management* **33**(3) (2007) 40–48.
11. S. García, D. Quintana, I. M. Galván and P. Isasi, Time-stamped resampling for robust evolutionary portfolio optimization, *Expert Systems with Applications* **39**(12), 10 (2012) 11722–11730.
12. K. Deb, A. Pratap, S. Agarwal and T. Meyarivan, A fast and elitist multiobjective genetic algorithm: NSGA-II, *IEEE Transactions on Evolutionary Computation* **6**(2) (2002) 182–197.
13. E. Zitzler, M. Laumanns and L. Thiele, SPEA2: Improving the Strength Pareto Evolutionary Algorithm., Technical Report 103, Computer Engineering and Networks Laboratory (TIK), Swiss Federal Institute of Technology (ETH), Zurich, Switzerland (2001).
14. P. Skolpadungket, K. Dahal and N. Harnpornchai, Portfolio optimization using multi-objective genetic algorithms, in *Proc. 2007 IEEE Congress on Evolutionary Computation*, Singapore (2007), pp. 516–523.
15. S. Kukkonen and J. Lampinen, GDE3: The third evolution step of generalized differential evolution, *in IEEE Congress on Evolutionary Computation (CEC'2005)*, Edinburgh, Scotland (2005), pp. 443–450.
16. A. Nebro, J. J. Durillo, J. M. García-Nieto, C. A. Coello Coello, F. Luna and E. Alba, SMPSO: A new PSO-based metaheuristic for multi-objective optimization, *in: 2009 IEEE Sympo. Computational Intelligence in Multicriteria Decision-Making (MCDM 2009)*, Nashville, TN, USA (IEEE Press, 2009), pp. 66–73.
17. Q. Zhang and H. Li, MOEA/D: A multi-objective evolutionary algorithm based on decomposition, *IEEE Transactions on Evolutionary Computation* **11**(6) (2007) 712–731.
18. K. Metaxiotis and K. Liagkouras, Multiobjective evolutionary algorithms for portfolio management: A comprehensive literature review, *Expert Systems with Applications* **39**(14) (2012) 11685–11698.
19. K. Liagkouras and K. Metaxiotis, A new probe guided mutation operator and its application for solving the cardinality constrained portfolio optimization problem, *Expert Systems with Applications* **41**(14) (2014) 6274–6290.

20. K. Lwin, R. Qu and G. Kendall, A learning-guided multi-objective evolutionary algorithm for constrained portfolio optimization, *Applied Soft Computing Journal* **24** (2014) 757–772.
21. S. Babaei, M. M. Sepheri and E. Babaei, Multi-objective portfolio optimization considering the dependence structure of asset returns, *European Journal of Operational Research* **244**(2) (2015) 525–539.
22. W. Yue, Y. Wang and C. Dai, An evolutionary algorithm for multiobjective fuzzy portfolio selection models with transaction cost and liquidity, *Mathematical Problems in Engineering*, Articel. No. 569415 (2015).
23. S. K. Mishra, G. Panda, S. Meher and R. Majhi, Constraint Robust Portfolio selection by multiobjective evolutionary genetic algorithm, *International Journal of Electronics Signals and Systems* **1**(3) (2012) 57–62.
24. R. Michaud, *Efficient Asset Management* (Harvard Business School Press, Boston, MA, 1998).
25. T. M. Idzorek, Developing robust asset allocations, *Working Paper of Ibbotson Associates* (2006).
26. G. Hassan and C. Clack, Multiobjective robustness for portfolio optimization in volatile environments, in *Proc. Genetic and Evolutionary Computation Conference (GECCO'2008)* (ACM Press, Atlanta, USA, 2008), pp. 1507–1514.
27. S. García, D. Quintana, I. M. Galván and P. Isasi, Portfolio optimization using SPEA2 with resampling, in *Proc. 12th Int. Conf. Intelligent Data Engineering and Automated Learning (IDEAL'11)*, eds H. Yin, W. Wang and V. Rayward-Smith (Springer-Verlag, Berlin, Heidelberg, 2011), pp. 127–134.
28. H. G. Beyer and B. Sendhoff, Robust optimization — a comprehensive survey, *Computer Methods in Applied Mechanics and Engineering* **196**(33-34) (2007) 3190–3218.
29. K. Deb and H. Gupta, Introducing robustness in multi-objective optimization, *Evolutionary Computation* **14**(4) (2006) 463–494.
30. A. Gaspar-Cunha and J. A. Covas, Robustness in multi-objective optimization using evolutionary algorithms, *Computational Optimization and Applications* **39**(1) (2008) 75–96.
31. H. Markowitz, *Portfolio Selection: Efficient Diversification of Investments* (John Wiley & Son, NY, 1959).
32. T. J. Chang, S. C. Yang and K. J. Chang, Portfolio optimization problems in different risk measures using genetic algorithm, *Expert Systems with Applications* **36**(7) (2009) 10529–10537.
33. D. C. Montgomery, *Design and Analysis of Experiment* (Wiley, New York, 2001).
34. P. C. Mahalanobis, On the generalized distance in statistics, *Proceedings of the National Institute of Sciences* **2** (1936) 49–55.
35. C. Blum and A. Roli, Metaheuristics in combinatorial optimization: Overview and conceptual comparison, *ACM Computing Surveys* **35**(3) (2003) 268–308.
36. M. Clerc and J. Kennedy, The particle swarm — explosion, stability, and convergence in a multidimensional complex space, *IEEE Transactions on Evolutionary Computation* **6**(1) (2002) 58–73.
37. E. Zitzler and L. Thiele, An evolutionary algorithm for multiobjective optimization: The Strength Pareto approach, *Technical Report* 43, ss, Gloriastrasse 35, CH-8092 Zurich, Switzerland (1998).
38. J. Knowles, A summary-attainment-surface plotting method for visualizing the performance of stochastic multiobjective optimizers, in *5th Int. Conf. Intelligent Systems Design and Applications (ISDA'05)*, Wroclaw, Poland (2005), pp. 552–557.