Rendering SU(3) intuitive: Symmetries of Lorentz tensors

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Laboratoire des Solides Irradiés,
CEA-DSM-IRAMIS, CNRS UMR 7642, Ecole Polytechnique,
28, Route de Saclay, F-91128-Palaiseau CEDEX, France

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Abstract. Under a Lorentz transformation of the electromagnetic field $(\mathbf{E}, c\mathbf{B})$ the quantity $\mathbf{E}^2 - c^2\mathbf{B}^2$ remains preserved. A Lorentz transformation does not preserve the energy density $\varepsilon_0\,(\mathbf{E}^2 + c^2\mathbf{B}^2)$ of the field. One can thus ask which are the transformations that preserve the quantity $\mathbf{E}^2 + c^2\mathbf{B}^2$. We show that the symmetry group of such transformations is isomorphic to SU(3). We can thus use these transformations of the electromagnetic field as a model to visualize what happens in SU(3) by analogy, just like we can use the rotation group SU(2) as a model to visualize what happens in the isospin group SU(2) by analogy.

PACS. 02.20.-a Group Theory

1 Introduction: Lorentz symmetry

1.1 Lorentz symmetry of the electromagnetic field

This paper really starts in Section 2. The reader should not feel disheartend if he does not master the subject matter recalled in the Introduction. He will still be able to grasp the essence of the paper, which does not really build on the contents of the Introduction. They will only be used from time to time to make some comparisons within a broader context. For the reader who wants to fill some gaps in his knowledge, a very clear introduction to spinors in the representation SU(2) of the three-dimensional rotation group in $\mathbb{R}^3$ is given in the first part of reference [1]. This is also explained in Chapter 3 of reference [2]. In the second part of reference [1] we also show how these ideas can be extended to rotation groups in vector spaces $\mathbb{R}^n$, of arbitrary dimension $n > 3$. It also indicates how we can apply the same ideas to the homogeneous Lorentz group in $\mathbb{R}^4$. This is actually worked out in Chapter 4 of [2]. The goal of these references is to explain the subject matter in such a way that it does not leave the reader frustrated behind with a feeling that he may well check mechanically the algebra of the proofs, but just cannot figure out what might be going on behind the scenes. Their motivation is exactly to confer all the insight needed to dissipate any kind of feeling of impenetrability, which one encounters unfortunately all too often in austere standard presentations of what has been called the *Gruppenpest*. Reading these references will allow the reader to understand every single detail of the reminders given in the present Introduction.

As explained in reference [2], SL(2,$\mathbb{C}$), the group of all complex 2×2 matrices $\mathbf{L}$ with $\det \mathbf{L} = 1$, is a representation of the homogeneous Lorentz group $\mathscr{L}$. Within SL(2,$\mathbb{C}$) a fourvector (v_t, v_x, v_y, v_z) is represented by:

$$\mathbf{V} = v_t \mathbb{1} + \mathbf{v}{\cdot}\boldsymbol{\sigma} = \left[\begin{array}{cc} v_t + v_z & v_x - \imath v_y \\ v_x + \imath v_y & v_t - v_z \end{array} \right] \quad \rightsquigarrow \quad \det \mathbf{V} = v_t^2 - v_x^2 - v_y^2 - v_z^2. \tag{1}$$

Here $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the three Pauli matrices, whereby $\mathbf{v}{\cdot}\boldsymbol{\sigma}$ is not a true scalar product, but just a convenient shorthand for $v_x\sigma_x + v_y\sigma_y + v_z\sigma_z$. In fact, in SU(2) $\subset$ SL(2,$\mathbb{C}$) a vector $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$ is represented by the matrix:

$$\mathbf{X} = [\,\mathbf{r}{\cdot}\boldsymbol{\sigma}\,] = \left[\begin{array}{cc} z & x - \imath y \\ x + \imath y & -z \end{array} \right] \quad \rightsquigarrow \quad -\det \mathbf{X} = x^2 + y^2 + z^2. \tag{2}$$

We can see from this that $\sigma_x, \sigma_y, \sigma_z$ just correspond to $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ in the mapping: $\mathbf{r} \rightarrow \mathbf{X}$. Therefore $\mathbf{r}{\cdot}\boldsymbol{\sigma}$ is a vector, not a scalar as the notation $[\,\mathbf{r}{\cdot}\boldsymbol{\sigma}\,]$ may seem to suggest. A general homogeneous Lorentz transformation is in this representation SL(2,$\mathbb{C}$) represented by:

$$\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{with:} \quad \det \mathbf{L} = ad - bc = 1. \tag{3}$$

It transforms the four-vector $\mathbf{V}$ "quadratically" according to:

$$\mathbf{V} \to \mathbf{L}\mathbf{V}\mathbf{L}^{\dagger}. \tag{4}$$

As $\det \mathbf{L} = 1$, this transormation preserves indeed $\det \mathbf{V} = v_t^2 - v_x^2 - v_y^2 - v_z^2$. Elements $\mathbf{L}_1$ of the Lorentz group themselves transform linearly:

$$\mathbf{L}_1 \to \mathbf{L}\mathbf{L}_1. \tag{5}$$

It is hereby understood that $\mathbf{L}$ works like a group automorphism that transforms a group element into another group element according to Eq. 5. The homogeneous Lorentz group contains not only boosts, but also the full group of rotations SO(3). The reason for this is that the product $\mathbf{L}_2\mathbf{L}_1$ of two non-collinear boosts is not a pure boost but the product $\mathbf{R}\mathbf{L}$ of a rotation and a boost. Hence the product of boosts $\mathbf{L}_2\mathbf{L}_1\mathbf{L}^{-1}$ is a pure rotation. Finally, a tensor $\mathbf{T}$ is transformed according to:

$$\mathbf{T} \to \mathbf{L}\mathbf{T}\mathbf{L}^{-1}, \tag{6}$$

which preserves $\det \mathbf{T}$. We can now play with these matrices and combine the four-gradient and the four-potential:

$$[\, \tfrac{\partial}{\partial ct}\mathbb{1} - \nabla\cdot\boldsymbol{\sigma}\,]\,[\, \tfrac{V}{c}\mathbb{1} - \mathbf{A}\cdot\boldsymbol{\sigma}\,] = \underbrace{[\,\frac{1}{c^2}\frac{\partial V}{\partial t} + \nabla\cdot\mathbf{A}\,]\,\mathbb{1}}_{\text{Lorentz gauge}} \; - \; \underbrace{\frac{1}{c}[\,(\nabla V + \frac{\partial \mathbf{A}}{\partial t})\cdot\boldsymbol{\sigma}\,]}_{\frac{1}{c}\mathbf{E}\cdot\boldsymbol{\sigma}} \; + \; \underbrace{\imath[\,(\nabla\wedge\mathbf{A})\cdot\boldsymbol{\sigma}\,]}_{\imath\mathbf{B}\cdot\boldsymbol{\sigma}} \tag{7}$$

With the Lorentz gauge condition, we obtain thus:

$$[\,\frac{\partial}{\partial ct}\mathbb{1} - \nabla\cdot\boldsymbol{\sigma}\,]\,[\,\frac{V}{c}\mathbb{1} - \mathbf{A}\cdot\boldsymbol{\sigma}\,] = \frac{1}{c}[\,(\mathbf{E} + \imath c\mathbf{B})\cdot\boldsymbol{\sigma}\,]. \tag{8}$$

We recover thus automatically the expressions for the Lorentz gauge condition, and for the electric and magnetic fields in terms of the potentials. The term $\mathbf{E} + \imath c\mathbf{B}$ is the electromagnetic field tensor. The presence of $\imath$ in an expression signals that it is a pseudo-vector or a pseudo-scalar. The vector $\mathbf{E}$ and pseudo-vector $\mathbf{B}$ are the symmetric and anti-symmetric three-component parts of the six-component field tensor. We see thus that symmetry is enough to recover all the definitions. It summarizes in a sense the reason why we need the theory of relativity by showing that Lorentz symmetry is the symmetry that is compatible with the structure of the Maxwell equations. Applying Eq. 2 to the electromagnetic field tensor yields:

$$-\det[\,(\mathbf{E} + \imath c\mathbf{B})\cdot\boldsymbol{\sigma}\,] = \mathbf{E}^2 - c^2\mathbf{B}^2 + 2\imath c\mathbf{E}\cdot\mathbf{B} = |\,\mathbf{E} + \imath c\mathbf{B}\,|_{\mathsf{E}}^2\,. \tag{9}$$

Here $|\cdot|_{\mathsf{E}}$ denotes the extrapolation to $\mathbb{C}^3$ of the Euclidean norm function $|\,\mathbf{r}\,|_{\mathsf{E}}^2 = \sqrt{x^2+y^2+z^2}$ of $\mathbb{R}^3$. This extrapolation is of course no longer a true norm function, because it is no longer true that $\forall (x,y,z) \in \mathbb{C}^3 : x^2+y^2+z^2 \geq 0$. To obtain a proper norm function for $\mathbb{C}^3$, we must therefore rather consider the extrapolation $|\,\mathbf{r}\,|_{\mathsf{H}}^2 = \sqrt{xx^* + yy^* + zz^*}$, because $\forall (x,y,z) \in \mathbb{C}^3 : xx^* + yy^* + zz^* \geq 0$. This is the Hermitian norm. From Eqs. 6 and 9 it can be seen that a Lorentz transformation of an electromagnetic field preserves the Euclidean norm rather than the Hermitian norm of the electromagnetic field. This implies that Lorentz transformations preserve simultaneously both $\mathbf{E}^2 - c^2\mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$. If we assume that $\mathbf{E} \perp \mathbf{B}$, which is the case for an electromagnetic wave, then $-\det[\,(\mathbf{E}+\imath c\mathbf{B})\cdot\boldsymbol{\sigma}\,] = \mathbf{E}^2 - c^2\mathbf{B}^2$. For an electromagnetic field $\mathbf{E} \perp \mathbf{B}$ is thus true in all frames from the moment that it is true in one frame.

1.2 Lorentz symmetry of angular momentum

All four-vectors transform the same way under Lorentz transformations. Also all six-component tensors transform the same way under Lorentz transformations. In fact, what we did in Eq. 7 can also be done for angular momentum:[1]

[1] Unfortunately, the symbol E for the energy can be confused with $|\mathbf{E}|$. It would of course be preposterous to change the notations in this paper with respect to the standard notations for this reason. The context should each time be clear enough to avoid any possible confusion. But to avoid such a confusion we have adopted the notation $\mathcal{E}$ for the energy density.

$$\frac{1}{c}[\,ct\mathbb{1} - \mathbf{r}\cdot\boldsymbol{\sigma}\,]\,[\,E\mathbb{1} + c\mathbf{p}\cdot\boldsymbol{\sigma}\,] = \underbrace{[\,(Et - \mathbf{p}\cdot\mathbf{r})\,\mathbb{1}\,]}_{\text{phase of the wave-function}} - \underbrace{[\,\gamma m_0 c\,(\mathbf{r} - \mathbf{v}t)\cdot\boldsymbol{\sigma}\,]}_{\text{centre of mass}} + \underbrace{-\imath[\,(\mathbf{r}\wedge\mathbf{p})\cdot\boldsymbol{\sigma}\,]}_{\text{angular momentum}} \tag{10}$$

We discover this way that angular momentum is in reality a part of a tensor. Just like we cannot consider a magnetic field as an isolated quantity under Lorentz transformations but part of a tensor, we cannot consider angular momentum as an isolated quantity under Lorentz transformations. We must consider it as part of a tensor. This tensor must transform just like the electromagnetic field according to Eqs. 12 below. Both angular momentum and the magnetic field are axial vectors (as indicated by the prefactor $\imath$). In the second term of Eq. 10 we have rewritten the energy E under the form $E = \gamma m_0 c^2$ to highlight the motion of the centre of mass $\mathbf{r} - \mathbf{v}t$. Such expressions are used in the definition of the spin current. The scalar part $(Et - \mathbf{p}\cdot\mathbf{r})\,\mathbb{1}$ is a Lorentz invariant, but does not disappear here like the Lorentz gauge. The tensor part is again of the form $[\,(\mathbf{C} + \imath\mathbf{D})\cdot\boldsymbol{\sigma}\,]$. Due to the universality of these structures, what we are going to develop does not only apply to the electromagnetic field, but to a Lorentz tensor in general.

2 The symmetry group of the energy density of the electromagnetic field

2.1 The relation with U(3)

The energy density $\mathcal{E}$ of an electromagnetic field is $\mathcal{E} = \varepsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2$. Using $\varepsilon_0\mu_0 = \frac{1}{c^2}$ this can be rewritten as: $\mathcal{E} = \varepsilon_0(\mathbf{E}^2 + c^2\mathbf{B}^2)$. We could now ask for the symmetry group $\mathscr{G}$ of all the transformations which preserve the energy density. This is of course also the group of all the transformations which preserve $(\mathbf{E}^2 + c^2\mathbf{B}^2)$. As transpires from the discussion in Section 1 this is a different symmetry group than the Lorentz group which rather preserves $(\mathbf{E}^2 - c^2\mathbf{B}^2)$. If we note the electromagnetic field as $\mathbf{E} + \imath c\mathbf{B}$ like in SL(2,$\mathbb{C}$), then the quantity $\mathbf{E}^2 + c^2\mathbf{B}^2$ corresponds to the square of Hermitian norm of $\mathbf{E} + \imath c\mathbf{B}$, because $(\mathbf{E} + \imath c\mathbf{B})^*\cdot(\mathbf{E} + \imath c\mathbf{B}) = (\mathbf{E} - \imath c\mathbf{B})\cdot(\mathbf{E} + \imath c\mathbf{B}) = \mathbf{E}^2 + c^2\mathbf{B}^2$.[2] The symmetry group $\mathscr{G}$ we are looking for is thus the group of transformations that preserve the Hermitian norm and its square $(\mathbf{E} + \imath c\mathbf{B})^*\cdot(\mathbf{E} + \imath c\mathbf{B})$.

Let us now introduce a new matrix formalism whereby we write $(\mathbf{E} + \imath c\mathbf{B})$ under the form of a 3×1 matrix:

$$\mathbf{F} = \begin{bmatrix} E_x + \imath B_x \\ E_y + \imath B_y \\ E_z + \imath B_z \end{bmatrix}, \quad \text{such that} \quad \mathcal{E} = \mathbf{F}^\dagger\mathbf{F}. \tag{11}$$

Let $\mathbf{M}$ be a matrix of the three-dimensional unitary group U(3). By definition this implies that $\mathbf{M}^\dagger = \mathbf{M}^{-1}$. If we transform $\mathbf{F}\to\mathbf{MF}$, then $\mathbf{F}^\dagger \to \mathbf{F}^\dagger\mathbf{M}^\dagger = \mathbf{F}^\dagger\mathbf{M}^{-1}$, and thus $\mathbf{F}^\dagger\mathbf{F} \to \mathbf{F}^\dagger\mathbf{M}^{-1}\mathbf{MF} = \mathbf{F}^\dagger\mathbf{F}$. The group U(3) can thus be interpreted as a group of transformations of an electromagnetic field that preserve its energy density.

2.2 Lorentz transformations

The group SU(3) contains the group SO(3) of the rotations of $\mathbb{R}^3$, but as SU(3) is compact, it cannot contain the other transformations of the homogeneous Lorentz group $\mathscr{L}$, which is not a compact group. In fact, boosts preserve the Euclidean norm $\mathbf{F}^\top\mathbf{F}$ and therefore both $\mathbf{E}^2 - c^2\mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$, while the transformations of SU(3) $\subset$ U(3) preserve the Hermitian norm and therefore $\mathbf{E}^2 + c^2\mathbf{B}^2$. The boosts for the electromagnetic field are thus a complex generalization of SO(3) to $\mathbb{C}^3$, which we will note as [SO(3),$\mathbb{C}$]. In fact, this complex generalization is the complete homogeneous Lorentz group $\mathscr{L}$.[3] The Lorentz transformation of the electromagnetic field by a boost with velocity $\mathbf{v}$ is given by:

$$\begin{array}{rclcrcl} \mathbf{E}'_{\parallel} & = & \mathbf{E}_{\parallel} & , & \mathbf{E}'_{\perp} & = & \gamma(\mathbf{E}_\perp + \boldsymbol{\beta}\wedge c\mathbf{B}_\perp), \\ c\mathbf{B}'_{\parallel} & = & c\mathbf{B}_{\parallel} & , & c\mathbf{B}'_{\perp} & = & \gamma(c\mathbf{B}_\perp - \boldsymbol{\beta}\wedge \mathbf{E}_\perp). \end{array} \tag{12}$$

[2] We want to study the symmetry group of all the transformations which preserve the energy density, but the formulation of this symmetry in terms of the quantity $\mathbf{E}^2 + c^2\mathbf{B}^2$ is much more simple and elegant. In order not to burden the formulation we will therefore introduce the *abus de langage* to call also $\mathbf{E}^2 + c^2\mathbf{B}^2$ the energy density of the electromagnetic field.

[3] As rotations belong to SO(3) $\subset$ [SO(3),$\mathbb{C}$], all Lorentz transformations belong to [SO(3),$\mathbb{C}$], i.e. $\mathscr{L}\subset$ [SO(3),$\mathbb{C}$]. We also have [SO(3),$\mathbb{C}$] $\subset\mathscr{L}$, such that $\mathscr{L}$ = [SO(3),$\mathbb{C}$]. In fact, consider a matrix $\mathbf{M}\in$ [SO(3),$\mathbb{C}$]. We have then $\mathbf{M}^\top = \mathbf{M}^{-1}$. Therefore: $(\mathbf{MF})^\top(\mathbf{MF}) = \mathbf{F}^\top\mathbf{M}^\top\mathbf{MF} = \mathbf{F}^\top\mathbf{M}^{-1}\mathbf{MF} = \mathbf{F}^\top\mathbf{F}$. The transformation $\mathbf{M}$ preserves thus $[\,\mathbf{E} + \imath c\mathbf{B}\,]^2$ such that $\mathbf{M}\in\mathscr{L}$.

Here $\boldsymbol{\beta} = \mathbf{v}/c$ and $\gamma = (1-\beta^2)^{-1/2}$ as usual. The indices $\perp$ and $\parallel$ indicate the components that are perpendical and parallel to the boost velocity $\mathbf{v}$ respectively. Working Eq. 12 out for the example of a boost with velocity $\mathbf{v} = v\mathbf{e}_z$ along the z-axis, and combining the components E_j and cB_j into $E_j + \imath cB_j$ yields:

$$\mathbf{L}_z(v) = \begin{bmatrix} \gamma & \imath\beta\gamma & 0 \\ -\imath\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{13}$$

for the transformation matrix of the boost operating on $\mathbf{F}$. From:

$$[\,\mathbf{L}_z(v)\,]^\top = \begin{bmatrix} \gamma & -\imath\beta\gamma & 0 \\ \imath\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\,\mathbf{L}_z(v)\,]^\dagger = \begin{bmatrix} \gamma & \imath\beta\gamma & 0 \\ -\imath\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

it is easily seen that $[\,\mathbf{L}_z(v)\,]^\top = [\,\mathbf{L}_z(v)\,]^{-1} \neq [\,\mathbf{L}_z(v)\,]^\dagger$ such that $\mathbf{L}_z(v) \notin \mathrm{SU}(3)$. The boosts $\mathbf{L}_x(v)$ and $\mathbf{L}_y(v)$ are defined *mutatis mutandis*. The rotations *en bloc* of $\mathbf{E} + \imath c\mathbf{B}$ are of the type:

$$\mathbf{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

which illustrates a rotation over an angle α around the z-axis. The rotations $\mathbf{R}_x(\alpha)$, $\mathbf{R}_y(\alpha)$ have again *mutatis mutandis* similar expressions. These rotations belong thus, as already stated to $\mathrm{SU}(3) \cap \mathrm{SO}(3)$.

2.3 Counting the number of free real parameters

Let us note the matrix $\mathbf{M} \in \mathrm{SU}(3)$ by means of its three lines, which are the 1×3 line matrices $\mathbf{v}_j$:

$$\mathbf{M} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} \quad \Rightarrow \quad \mathbf{M}^\dagger = \begin{bmatrix} \mathbf{v}_1^\dagger & \mathbf{v}_2^\dagger & \mathbf{v}_3^\dagger \end{bmatrix}. \tag{16}$$

This permits simultaneously to note $\mathbf{M}^\dagger$ in terms of its three columns $\mathbf{v}_j^\dagger$, which are the 3×1 column vectors obtained by transposing the line vectors $\mathbf{v}_j^*$. Expressing that $\mathbf{M}\mathbf{M}^\dagger = \mathbb{1}$ shows that vectors $\mathbf{v}_j$ must satisfy the conditions $\mathbf{v}_j{\cdot}\mathbf{v}_k^* = \delta_{jk}$. One can make an analogous reasoning, whereby one starts from $\mathbf{M}$ expressed in its column matrices $\mathbf{w}_k$ and uses then $\mathbf{M}^\dagger\mathbf{M} = \mathbb{1}$. We find then $\mathbf{w}_j^*{\cdot}\mathbf{w}_k = \delta_{jk}$ This means that all line vectors $\mathbf{v}_j$ must be normalized to 1 and mutually orthogonal according to the Hermitian norm. Also all column vectors $\mathbf{w}_j$ must be normalized to 1 and mutually orthogonal according to the Hermitian norm. We can thus construct an arbitrary matrix $\mathbf{M} \in \mathrm{SU}(3)$ by constructing an arbitrary basis for $\mathbb{C}^3$ by the Gram-Schmidt procedure. The basis vectors will then constitute the columns (or the lines) of the SU(3) matrix.

For the first basis vector we can choose an arbitrary vector $\mathbf{a} + \imath\mathbf{b} \in \mathbb{C}^3$, where $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$. We can thus choose all six real parameters $(a_x, a_y, a_z, b_x, b_y, b_z)$ at will. But we must normalize $\mathbf{a} + \imath\mathbf{b}$ to 1, such that we are left with five free real parameters. This defines a first complex basis vector $\boldsymbol{\eta}_1 \in \mathbb{C}^3$. The second basis vector $\boldsymbol{\eta}_2 \in \mathbb{C}^3$ must be normalized to 1 (one real constraint). It must also be orthogonal to the first one, which implies $<\boldsymbol{\eta}_1, \boldsymbol{\eta}_2> = 0$. This corresponds to two real constraints: $\Re(<\boldsymbol{\eta}_1, \boldsymbol{\eta}_2>) = 0$ & $\Im(<\boldsymbol{\eta}_1, \boldsymbol{\eta}_2>) = 0$.[4] To choose $\boldsymbol{\eta}_2$ we are thus left with three free real parameters. The third basis vector $\boldsymbol{\eta}_3 \in \mathbb{C}^3$ must be normalized to 1 and orthogonal to the two previous ones. This leaves us with one free real parameter. But to be elegible for constituting the third column of an SU(3) matrix $\mathbf{M}$, its phase must be adjusted to satisfy also the condition $\det \mathbf{M} = 1$. After choosing the first two basis vectors, the third basis vector is thus already uniquely determined. We can no longer choose it. There are thus no free real parameters left to choose the third basis vector. The group SU(3) has thus in total eight free real parameters.

We can generalize this reasoning to SU(n) to show that SU(n) has: $(2n-1) + (2n-3) + \cdots 3 = n^2 - 1$ free real parameters (see Appendix). We can use *mutatis mutatndis* exactly the same method to count the number of free real parameters that define a matrix $\mathbf{M}$ of the group SO(n), which is defined by $\mathbf{M}^\top\mathbf{M} = \mathbb{1}$ (see Appendix). This number is $n(n-1)/2$. Elements of SO(n) and SU(n) can thus both be visualized by their action on a canonical basis, where the basis vectors are all normalized to one and mutually orthogonal according to the Euclidean and the Hermitian norm

[4] We discover here the reason why we cannot define the scalar product $<\mathbf{z}_1, \mathbf{z}_2>$ of two complex vectors $\mathbf{z}_1 \in \mathbb{C}^n$, $\mathbf{z}_2 \in \mathbb{C}^n$ in the vector space $(\mathbb{C}^n, \mathbb{C})$ as $\frac{1}{2}\sum_{j=1}^{n}(z_{1j}z_{2j}^* + z_{2j}z_{1j}^*)$. The space orthogonal to a vector would be equivalent to $(\mathbb{R}^{2n-1}, \mathbb{R})$. Every choice of a new basis vector in the Gram-Schmidt procedure must have two free real parameters less than the previous one to make sure the procedure yields the correct dimension n for the complex vector space $(\mathbb{C}^n, \mathbb{C})$.

respectively. Such a basis is called in German a *Vielbein*, which could be translated as "multipod" in English. The idea behind this rather expressive terminology is that an orthonormal basis of $\mathbb{R}^n$ has n legs, which are the n basis vectors which constitute it. There is a on-to-one corrspondence between group elements and the vielbeins obtained by them by acting on the canonical basis.

2.4 Magnetic monopoles

One transformation which preserves the energy density and does not belong to the homogeneous Lorentz group $\mathscr{L}$ is easily found. The transformation $h : \mathbf{E} \rightarrow c\mathbf{B}$, $c\mathbf{B} \rightarrow -\mathbf{E}$ corresponds to $\mathbf{E} + \imath c\mathbf{B} \rightarrow \imath(\mathbf{E} + \imath c\mathbf{B})$, such that it is easily seen to preserve $(\mathbf{E} + \imath c\mathbf{B})^\dagger(\mathbf{E} + \imath c\mathbf{B})$. As it transforms $-(\mathbf{E}^2 - c^2\mathbf{B}^2)$ into $(\mathbf{E}^2 - c^2\mathbf{B}^2)$ and $\mathbf{E} \cdot \mathbf{B}$ into $-\mathbf{E} \cdot \mathbf{B}$, it is obviously not a Lorentz transformation, such that it does not belong to $\mathscr{L}$. There is indeed no Lorentz transformation that can transform a pure electric field into a pure magnetic field and *vice versa*.

The symmetry transformation $\mathbf{E} \rightarrow c\mathbf{B}$, $c\mathbf{B} \rightarrow -\mathbf{E}$ is introduced in the discussion of magnetic monopoles, but as we see here it is all but a trivial transformation. It is not a Lorentz transformation. It is a Lorentz-forbidden "rotation" around the direction of Poynting vector.[5] Another, different transformation that would preserve the energy density is: $\mathbf{E} \rightarrow c\mathbf{B}$, $c\mathbf{B} \rightarrow \mathbf{E}$. This is the transformation: $\mathbf{F} \rightarrow \imath\mathbf{F}^*$ and cannot be written in the form of a matrix multiplication in the formalism whereby one expresses the field in terms of $\mathbf{F} \in \mathbb{C}^3$. We will see below that the transformations $\mathbf{I}_j(\pi/2)$ are almost of the form of h. They are just an itty-bitty more subtle.[6]

3 Classification of the elements of the group SL(3,$\mathbb{C}$)

3.1 Preliminary considerations

SL(3,$\mathbb{C}$) is the group of all complex 3×3 matrices $\mathbf{M}$ with $\det \mathbf{M} = 1$. As both the Lorentz group and SU(3) are part of SL(3,$\mathbb{C}$) it is perhaps worth to classify the transformations of SL(3,$\mathbb{C}$). The group SL(3,$\mathbb{C}$) is 16-dimensional, because $\det \mathbf{M} = 1$ imposes two real conditions on a complex 3×3 matrix $\mathbf{M}$, *viz.* $\Re(\det \mathbf{M}) = 1$ and $\Im(\det \mathbf{M}) = 0$. We must thus find 16 basic transformations. Of these, three correspond to boosts (which belong to the complex group $[\,\mathrm{SO}(3),\mathbb{C}\,]\ ) = \mathscr{L}$, three to the plain rotations in $\mathbb{R}^3$, which belong to SO(3) $\cap$ SU(3), and five to transformations of SU(3)$\setminus$ SO(3) that are not plain rotations in $\mathbb{R}^3$. This way we account for 11 parameters. There remain thus five parameters to identify.

3.2 "Infinitesimal generators"

Let G be a Lie group. Let us note a representation matrix $[\,\mathbf{D}(g)\,](\boldsymbol{\lambda})$ of a group element $g \in G$ as $\mathbf{M}(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2 \cdots \lambda_{\dim(G)}) \in \mathbb{R}^{\dim(G)}$ is a set of independent real group parameters.[7] Here $\dim(G)$ is the dimension of the group, which is of course different from the dimension of the representation, which is the rank of the matrix $\mathbf{M}$. E.g. in SU(2) the dimension of the representation in 2, while the dimension of the group is 3. These two dimensions are in general still different from the dimension of the vector space the transformations might be acting on. E.g. the homogeneous Lorentz group is six-dimensional, it acts on four-dimensional space-time, and the dimension of the representation SL(2,$\mathbb{C}$) is two. In the group SO(3), the three different types of dimensions all take the same value three, such that we may think that there is only one concept of dimension. The resulting absence of disambiguation can stirr confusion in one's first contact with the group theory.

[5] Assume e.g. that $\mathbf{E} = E\mathbf{e}_x$ and $c\mathbf{B} = cB\mathbf{e}_y$. A geometrical rotation over $\pi/2$ will turn this into $\mathbf{E} = E\mathbf{e}_y, c\mathbf{B} = -cB\mathbf{e}_x$. This is entirely different from the transformation considered here, which changes the nature of the fields. In fact, h changes a vector $\mathbf{E}$ into an axial vector $c\mathbf{B}$ and vice versa. The transformation h is a so-called involution, which means that h^2 is equal to the identity element.

[6] The subtlety will reside in the way one handles the rule $\det(\mathbf{M}) = 1$ of SU(3). The rule $\det(\mathbf{M}) = 1$ makes that we must be careful in how we write a transformation. If we do not take into acount the planar symmetry of the electro-magnetic field, we could write the transformation $\mathbf{E} + \imath c\mathbf{B} \rightarrow \imath(\mathbf{E} + \imath c\mathbf{B})$ as:

$$\mathbf{H} = \begin{bmatrix} \imath & & \\ & \imath & \\ & & \imath \end{bmatrix}, \quad \text{with:} \quad \mathbf{H}^\dagger = \mathbf{H}^{-1}, \quad \text{but:} \quad \det \mathbf{H} = -\imath \neq 1 \Rightarrow \mathbf{H} \notin \mathrm{SU}(3). \tag{17}$$

As indicated, the definition of SU(3) includes the supplementary condition $\det \mathbf{M} = 1$, such that the transformation h in its transcription according to Eq. 17 does not belong to SU(3). In Footnote 15 we will come back on this issue.

[7] There are different choices possible for such sets. E.g. in SU(2) we can take the three Euler angles, but we can also take the rotation axis and the rotation angle.

In Lie groups G one uses so-called "infinitesimal generators". Textbooks [3,4] explain then the following. We consider a neighbourhood of the identity element $\mathbb{1}$. In this neighbourhood, all parameters λ_j are small. We now consider the group elements, whereby only one parameter λ_j is allowed to vary and to be different from zero. We let λ_j vary between 0 and 1. The matrices $\mathbf{M}(0,0,\cdots,0,\lambda_j,0,\cdots 0,0)$ will then be one-parameter sets of group elements. They will describe a one-dimensional curve on the group manifold. An infinitesimal generator is then:

$$\imath\,\left[\,\frac{\partial}{\partial\lambda_j}\,\mathbf{M}(0,0,\cdots,0,\lambda_j,0,\cdots 0,0)\,\right]_{\lambda_j=0}. \tag{18}$$

The infinitesimal generators belong to the tangent space of the group manifold at the identity element. The idea behind defining the infinitesimal generators and the Lie algebra is thus to construct a basis for the tangent space to the manifold of the Lie group. The elements of the Lie algebra, the tangent vectors, are thus objects that are completely different from the elements of the Lie group which are group elements. As they belong to the tangent space rather than to the group itself, "infinitesimal generators" are not generators. Furthermore, the matrices $\mathbf{M}(0,0,\cdots,0,d\lambda_j,0,\cdots 0,0)-\mathbb{1}$ are infinitesimal, but the quantities defined by Eq. 18 are not. The "infinitesimal generators" are thus also not infinitesimal. Let us drop a further small comment on the prescription based on Eq. 18. With the definition given in Eq. 18 and the examples given in textbooks, the procedure to define the Lie algebra looks clearly outlined. But an example is not a definition. What one does not spell out is that one wants an orthonormal basis for the tangent space and how the various one-parameter sets must be chosen to achieve that goal. That not speciffying this very clearly can lead to difficulties does not transpire from the textbook examples. An example that shows that defining this choice may not always as simple as in these examples is given by the parameterization of a rotation matrix $\mathbf{R}(\alpha_1,\alpha_2,\alpha_3)$ of SU(2) in terms of its three Euler angles α_j (see e.g. Eq. 1.2.29 of reference [4]):

$$\mathbf{R}(\alpha_1,\alpha_2,\alpha_3)=\left[\begin{array}{cc} e^{-\imath(\alpha_1+\alpha_3)/2}\cos(\alpha_2/2) & -\imath e^{-\imath(\alpha_1-\alpha_3)/2}\sin(\alpha_2/2)\\ -\imath e^{\imath(\alpha_1+\alpha_3)/2}\sin(\alpha_2/2) & e^{\imath(\alpha_1+\alpha_3)/2}\cos(\alpha_2/2)\end{array}\right]. \tag{19}$$

The infinitesimal generators of SU(2) are the three Pauli matrices, but blindly applying the prescription of Eq. 18 by calculating the partial derivatives with respect to α_j will not yield the desired result. The three Euler angles are not a good choice for the set of parameters to be used. This shows that it takes some geometrical insight in the structure of the group to make the good choices which will tease out the infinitesimal generators correctly. With a group that has been defined completely abstractly like SU(3), we do not have that geometrical insight. We develop this idea a bit more in the Appendix. The insight we will gain in SU(3) from the present study will enable us to derive the infinitesimal generators by using the procedure of Eq. 18. To illustrate this, we will derive the infinitesimal generators for the group elements we will come across. Even with the additional insight, it can prove tricky enough (see Footnote 13).

With respect to further use we can observe that:

$$\begin{array}{cc} \imath\left[\,\frac{d}{d\alpha}\sin\alpha\,\right]_{\alpha=0}=\imath & \imath\left[\,\frac{d}{d\alpha}\cos\alpha\,\right]_{\alpha=0}=0\\[2ex] \imath\left[\,\frac{d}{d\beta}\gamma\,\right]_{\alpha=0}=0 & \imath\left[\,\frac{d}{d\beta}\beta\gamma\,\right]_{\alpha=0}=\imath \end{array} \tag{20}$$

The infinitesimal generator for the rotation $\mathbf{R}_z$ around the z-axis in the Oxy plane is therefore:

$$\mathbf{R}_z \rightsquigarrow \overset{\circ}{\mathbf{R}}_z=\left[\begin{array}{ccc} 0 & -\imath & 0\\ \imath & 0 & 0\\ 0 & 0 & 0\end{array}\right]\in \mathrm{su}(3), \tag{21}$$

Here su(3) is the notation for the Lie algebra of SU(3). We use the superscript $\circ$ to identify the infinitesimal generators. The infinitesimal generator for the boost $\mathbf{L}_z$ along the z-axis is:

$$\mathbf{L}_z \rightsquigarrow \overset{\circ}{\mathbf{L}}_z=\left[\begin{array}{ccc} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{array}\right]\in\ \mathrm{sl}(3,\mathbb{C})\setminus\ \mathrm{su}(3), \tag{22}$$

3.3 SL(3,$\mathbb{C}$) $\setminus$ (SU(3) $\cup\,\mathscr{L}$) is not an empty set

Let us now show that SL(3,$\mathbb{C}$) contains more than $\mathscr{L}$ and SU(3). Matrices $\mathbf{M}\in\mathscr{L}$ are characterized by $\mathbf{M}\mathbf{M}^\top=\mathbb{1}$, while matrices $\mathbf{M}\in$ SU(3) are characterized by $\mathbf{M}\mathbf{M}^\dagger=\mathbb{1}$. The matrix:

$$\mathbf{Q}_z(v) = \begin{bmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{with:} \quad \det \mathbf{Q}_z(v) = 1, \quad \text{and} \quad \mathbf{Q}_z^{-1} = \mathbf{Q}_z(-v) = \begin{bmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

belongs very obviously to SL(3,ℂ), but not to [SO(3),ℂ] (because $\mathbf{Q}_z^\top \neq \mathbf{Q}_z^{-1}$) nor to SU(3) (because $\mathbf{Q}_z^\dagger \neq \mathbf{Q}_z^{-1}$). The infinitesimal generator of $\mathbf{Q}_z$ is:

$$\mathbf{Q}_z \rightsquigarrow \overset{\circ}{\mathbf{Q}}_z \begin{bmatrix} 0 & \imath & 0 \\ \imath & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{sl}(3,\mathbb{C}). \tag{24}$$

It is $\imath$ times the infinitesimal generator of $\mathbf{S}_z$ (see below) and anti-Hermitian. We can derive three such matrices $\mathbf{Q}_x$, $\mathbf{Q}_y$, $\mathbf{Q}_z$. There remain thus two basic matrices of SL(3,ℂ)\ SU(3) which we have not yet identified. We must also further specify the five matrices of SU(3)\ [SO(3),ℂ] .

3.4 The meaning of SU(3)

The group that preserves the energy density allows actually for a lot of freedom. Imagine an initial field $(\mathbf{E}_0, c\mathbf{B}_0)$. Let us call $\sqrt{\mathbf{E}_0^2 + c^2\mathbf{B}_0^2} = F$.[8] We can then define a new field $(\mathbf{E}, c\mathbf{B})$ by choosing an angle χ_0, and taking $|\mathbf{E}| = F \cos\chi_0$, $|c\mathbf{B}| = F \sin\chi_0$. The orientations of $\mathbf{E}$ and $c\mathbf{B}$ can then further be taken at will. In other words, we we can just choose two vectors $\mathbf{E}$ and $c\mathbf{B}$ with relative orientations and magnitudes at will. The only constraint we must satisfy is that $\mathbf{E}^2 + c^2\mathbf{B}^2$ remains constant.

A matrix $\mathbf{M} \in$ U(3), satisfies the condition $\mathbf{M}\mathbf{M}^\dagger = \mathbb{1}$. To belong to SU(3) it must satisfy also the condition $\det\mathbf{M} = 1$. For preserving the energy density of the electromagnetic field, adding this restriction $\det\mathbf{M} = 1$ is at first sight not in order. The addition of the supplementary condition $\det\mathbf{M} = 1$ feels like a curse. We thought it was a nice idea that the group SU(3) would be the symmetry group of the energy density of a Lorentz tensor. But with this supplementary condition $\det\mathbf{M} = 1$ the whole idea seems to fall apart. It even saddles us with uncomfortable questions like: what is the condition $\det\mathbf{M} = 1$ good for? What purpose does it serve and what does it mean?

Fortunately, this is not the end of the story and we will be able to answer these questions. In fact, we have been unsubtle as $\mathbf{E}$ and $c\mathbf{B}$ define a plane through the origin, unless when $\mathbf{E} \parallel c\mathbf{B}$.[9] When $\mathbf{E} \parallel c\mathbf{B}$ they do not define a plane but just a straight line $\{\mathbf{r} \in \mathbb{R}^3 \parallel \mathbf{r} = \lambda\mathbf{u} \;\&\; \lambda \in \mathbb{R}\}$, whereby $\mathbf{u}$ is a unit vector. We can then choose another arbitrary straight line defined by a unit vector $\mathbf{w}$ that intersects the first straight line at the origin. Together the two straight lines will define a plane spanned by the unit vectors $\mathbf{u}$ and $\mathbf{w}$. There is thus always at least one plane. Taking the x-axis and y-axis of the reference frame within this plane, the electromagnetic field tensor becomes:

$$\mathbf{F} = \begin{bmatrix} E_x + \imath B_x \\ E_y + \imath B_y \\ 0 \end{bmatrix}. \tag{25}$$

The transformations:

$$\mathbf{M}_z(\chi_z) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & e^{\imath\chi_z} \end{bmatrix} \in \mathrm{U}(3), \tag{26}$$

have then absolutely no impact on the value of $\mathbf{E} + \imath c\mathbf{B}$. After performing a number of in-plane transformations on $\mathbf{E} + \imath c\mathbf{B}$ we can therefore apply $\mathbf{M}_z(\chi_z)$ to the result, whereby we choose the value of χ to make sure that $\det\mathbf{M} = 1$ in the global final result. This will not change the value of $\mathbf{E} + \imath c\mathbf{B}$. There is only one such value χ_z (mod 2π). The other values lead to $\det\mathbf{M} \neq 1$ and are useless. We can thus restrict U(3) to a smaller group. We argued above that U(3) $\subset \mathscr{G}$ because every element of U(3) preserves the energy density. Hence we have SU(3) $\subset$ U(3) $\subset \mathscr{G}$. If we can now also show $\mathscr{G} \subset$ SU(3), then we will have proved that $\mathscr{G} =$ SU(3). We are back on track.

In fact, the electromagnetic plane is an object we turn around in a three-dimensional space. There is always a direction of space, *viz.* that of the Poynting vector, along which the component of the field is zero. Consider an electromagnetic field in its canonical form $(\mathbf{E}_0, c\mathbf{B}_0)$, given by Eq. 25. It defines the plane $(\mathbf{E}_0, c\mathbf{B}_0)$. As we will show

[8] Note that $\mathbf{E}_0$ and $c\mathbf{B}_0$ do not need to be orthogonal in $\mathbb{R}^3$ to enable us to calculate the square F^2 of the Hermitian norm of $\mathbf{E} + \imath c\mathbf{B}$ according to the rule $\mathbf{E}_0^2 + c^2\mathbf{B}_0^2$.

[9] When $\mathbf{E} = \mathbf{0}$ or $c\mathbf{B} = \mathbf{0}$, we consider also that $\mathbf{E} \parallel c\mathbf{B}$.

below, within the limits of the constraint that $\mathbf{E}^2+c^2\mathbf{B}^2$ must remain constant, the relative magnitudes and orientations of $\mathbf{E}$ and $c\mathbf{B}$ within this plane can be changed at will by transformations $\mathbf{M}_x(\chi_x)$ and $\mathbf{M}_y(\chi_y)$. We can then perform a further transformation $\mathbf{M}_z(\chi_z)$ that does not affect the result and whose angle $\chi_z = -\chi_x - \chi_y$ is choosen to make $\det[\,\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y)\mathbf{M}_z(\chi_z)\,] = 1$. Furthermore, we can replace the transformations $\mathbf{M}_x(\chi_x)$, $\mathbf{M}_y(\chi_y)$, $\mathbf{M}_z(\chi_z)$, which belong to U(3) but not to SU(3), by transformations $\mathbf{I}_x(\zeta_x)$, $\mathbf{I}_y(\zeta_y)$, $\mathbf{I}_z(\zeta_z)$ which all belong to SU(3), which equally satisfy $\det[\,\mathbf{I}_x(\zeta_x)\mathbf{I}_y(\zeta_y)\mathbf{I}_z(\zeta_z)\,] = 1$ and whereby the combined transformation $\mathbf{I}_x(\zeta_x)\mathbf{I}_y(\zeta_y)\mathbf{I}_z(\zeta_z)$ has exactly the same effect on $\mathbf{E} + \imath c\mathbf{B}$ as $\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y)\mathbf{M}_z(\chi_z)$.

Consider now a second field $\mathbf{E}' + \imath c\mathbf{B}'$ that defines a different plane $(\mathbf{E}', c\mathbf{B}')$. The plane $(\mathbf{E}', c\mathbf{B}')$ can be obtained from the plane $(\mathbf{E}_0, c\mathbf{B}_0)$ by a simple rotation $\mathbf{R}$ around the axis defined by the intersection of the two planes. The rotations $\mathbf{R}$ belong to SU(3) and fulfill the condition $\det \mathbf{R} = 1$. Let us now apply $\mathbf{R}^{-1}$ to $(\mathbf{E}', c\mathbf{B}')$. This rotation will produce a field $(\mathbf{E}, c\mathbf{B})$ in the plane defined by $(\mathbf{E}_0, c\mathbf{B}_0)$. In general $(\mathbf{E}, c\mathbf{B})$ will be different from $(\mathbf{E}_0, c\mathbf{B}_0)$. But as we said, we will show that $(\mathbf{E}, c\mathbf{B})$ can be transformed to $(\mathbf{E}_0, c\mathbf{B}_0)$ by a transformation $\mathbf{I} = \mathbf{I}_x(\zeta_x)\mathbf{I}_y(\zeta_y)\mathbf{I}_z(\zeta_z)$ with $\det(\mathbf{I}) = 1$. The transformation $\mathbf{T} = \mathbf{I}\mathbf{R}^{-1}$, with $\det \mathbf{T} = 1$ transforms thus $(\mathbf{E}', c\mathbf{B}')$ into $(\mathbf{E}_0, c\mathbf{B}_0)$. Consequently, the transformation $\mathbf{T}^{-1}$ with $\det(\mathbf{T}^{-1}) = 1$ transforms a field from its canonical form $(\mathbf{E}_0, c\mathbf{B}_0)$ to $(\mathbf{E}', c\mathbf{B}')$. All physically relevant configurations of an electromagnetic field with a given energy density can thus be obtained from its canonical form by a transformation of SU(3). The group U(3) = SU(3) × U(1), contains an extra degree of freedom, contained in U(1) which is redundant for the physics due to the fact that the electromagnetic field remains confined to a plane. The group U(1) contains thus redundant information: it is a gauge group. What we do in U(3) to fix the gauge is to impose the gauge condition $\det \mathbf{M} = 1, \forall \mathbf{M} \in$ SU(3). The symmetry group for the energy density of the electromagnetic field is thus really SU(3).

Let us now give the proofs we promised. We have already pointed out that the rotations $\mathbf{R}_j(\alpha)$ belong to SU(3). Consider a vector $\mathbf{E} + \imath c\mathbf{B} = (E_x + \imath cB_x)\mathbf{e}_x$. The "phase transformations":

$$\mathbf{M}_x(\chi_x) = \begin{bmatrix} e^{\imath\chi_x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{with:} \quad \mathbf{M}\mathbf{M}^\dagger = \mathbb{1}, \tag{27}$$

preserve $\mathbf{E}^2 + c^2\mathbf{B}^2$. The vectors transformed, $E_x\mathbf{e}_x$, and $cB_x\mathbf{e}_x$ are and remain all parallel to the x-axis. In spite of this obvious parallelism, the formalism permits to consider these two x-components as mutually orthogonal within the complex plane. We could compare using $\mathbb{C}$ instead of $\mathbb{R}$ to having a fan that is folded up and aligned with the x-axis (which corresponds to using $\mathbb{R}$). Both $E_x\mathbf{e}_x$ and $cB_x\mathbf{e}_x$ are aligned with the x-axis and our vision is one-dimensional. The fact that we write E_x and cB_x under the form $E_x + \imath cB_x$ opens the fan such that what we considered as a one-dimensional subspace has become two-dimensional with respect to $\mathbb{R}$. But we can still describe it as one-dimensional with respect to $\mathbb{C}$. The transformation in Eq. 27 corresponds to:

$$\begin{bmatrix} E'_x \\ cB'_x \end{bmatrix} = \begin{bmatrix} \cos\chi_z & -\sin\chi_z \\ \sin\chi_z & \cos\chi_z \end{bmatrix} \begin{bmatrix} E_x \\ cB_x \end{bmatrix}. \tag{28}$$

The transformation does not change the orientation in physical space of these components but it permits changing their relative strengths, thereby taking care of keeping $E_x^2 + c^2B_x^2$ constant. In fact, if we define the angle χ_0 by:

$$\begin{bmatrix} E_x \\ cB_x \end{bmatrix} = \sqrt{E_x^2 + c^2B_x^2} \begin{bmatrix} \cos(\chi_0) \\ \sin(\chi_0) \end{bmatrix}, \tag{29}$$

then the transformation Eq. 27 changes this into:

$$\begin{bmatrix} E'_x \\ cB'_x \end{bmatrix} = \sqrt{E_x^2 + c^2B_x^2} \begin{bmatrix} \cos(\chi_0 + \chi_z) \\ \sin(\chi_0 + \chi_z) \end{bmatrix}. \tag{30}$$

Unfortunately the transformation Eq. 27 does not comply with the rule $\det \mathbf{M} = 1$. However, the combined transformations:

$$\mathbf{I}_z(\zeta_z) = \begin{bmatrix} e^{\imath\zeta_z} & 0 & 0 \\ 0 & e^{-\imath\zeta_z} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{with:} \quad \mathbf{I}_z^\dagger(\zeta_z) = \mathbf{I}_z^{-1}(\zeta_z), \tag{31}$$

do comply with the rule $\det \mathbf{I}_z(\zeta_z) = 1$. The corresponding infinitesimal generator is:

$$\overset{\circ}{\mathbf{I}}_z = \imath \left[\frac{\partial \mathbf{I}_z(\zeta_z)}{\partial \zeta_z} \right]_{\zeta_z=0} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{32}$$

The definitions of the transformations $\mathbf{I}_x(\zeta_x)$ and $\mathbf{I}_y(\zeta_y)$ are obtained by cyclic permuation:

$$\mathbf{I}_x(\zeta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\imath\zeta_x} & 0 \\ 0 & 0 & e^{-\imath\zeta_x} \end{bmatrix}, \quad \mathbf{I}_y(\zeta_y) = \begin{bmatrix} e^{-\imath\zeta_y} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\imath\zeta_y} \end{bmatrix}. \tag{33}$$

The infinitesimal generator corresponding to $[\,\mathbf{I}_z(\zeta_z)\,]^{-1}$ is algebraically identical to σ_z of SU(2), but it must be clear that $\mathbf{I}_z(\zeta_z)$ is by no means a rotation around the z-axis of physical space.[10] It is a rotation around an abstract z-axis in an abstract space as further explained in Footnote 11. The fact that $\mathbf{I}_x(\zeta_x)$, $\mathbf{I}_y(\zeta_y)$, $\mathbf{I}_z(\zeta_z)$ are mutually commuting and $\mathbf{I}_z(\zeta)\mathbf{I}_x(\zeta)\mathbf{I}_y(\zeta) = \mathbb{1}$, implies that we need only two of them to generate all their products. Only two of them are thus independent. This explains also why one needs only two diagonal infinitesimal generators in the Lie algebra su(3). In fact, as $\mathbf{I}_z(\zeta)\mathbf{I}_x(\zeta)\mathbf{I}_y(\zeta) = \mathbb{1}$ the sum of the three infinitesimal generators is zero. By combining $\mathbf{I}_x(\zeta_x)$, $\mathbf{I}_y(\zeta_y)$, $\mathbf{I}_z(\zeta_z)$, we obtain:

$$\mathbf{I}_x(\zeta_x)\mathbf{I}_y(\zeta_y)\mathbf{I}_z(\zeta_z) = \begin{bmatrix} e^{\imath(\zeta_z-\zeta_y)} & 0 & 0 \\ 0 & e^{\imath(\zeta_x-\zeta_z)} & 0 \\ 0 & 0 & e^{\imath(\zeta_y-\zeta_x)} \end{bmatrix} \in \mathrm{SU}(3). \tag{34}$$

It is obvious that $\det[\,\mathbf{I}_x(\zeta_x)\mathbf{I}_y(\zeta_y)\mathbf{I}_z(\zeta_z)\,] = 1$. Let us now consider the matrices $\mathbf{M}_j$. We could take the rotation angles χ_x of $\mathbf{M}_x(\chi_x)$ and χ_y of $\mathbf{M}_y(\chi_y)$ at will, but must the choose the angle χ_z of $\mathbf{M}_z(\chi_z)$ such that $\chi_x + \chi_y + \chi_z = 0$. We haven then:

$$\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y)\mathbf{M}_z(\chi_z) = \begin{bmatrix} e^{\imath\chi_x} & 0 & 0 \\ 0 & e^{\imath\chi_y} & 0 \\ 0 & 0 & e^{\imath\chi_z} \end{bmatrix} \quad \text{with:} \quad \det[\,\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y)\mathbf{M}_z(\chi_z)\,] = 1. \tag{35}$$

But this equivalent to stating that $\exists(\zeta_x, \zeta_y, \zeta_z) \in \mathbb{R}^3 : \chi_x = \zeta_z - \zeta_y, \chi_y = \zeta_x - \zeta_z, \chi_z = \zeta_y - \zeta_x$, In fact, this is equivalent to $\zeta_x = \frac{1}{2}(\chi_x + \chi_y - \chi_z)$ (cycl.). As $\chi_x + \chi_y + \chi_z = 0$, this simplifies actually to $\zeta_x = -\chi_x$ (cycl.). If we make sure that $\chi_x + \chi_y + \chi_z = 0$ then $\det[\,\mathbf{I}_x(\zeta_x)\mathbf{I}_y(\zeta_y)\mathbf{I}_z(\zeta_z)\,] = 1$. This is thus equivalent to $\det[\,\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y)\mathbf{M}_z(\chi_z)\,] = 1$. But $\mathbf{I}_j(\zeta_j)$ belong to SU(3), while $\mathbf{M}_j(\chi_j)$ do not belong to SU(3).

The way the product of transformations $\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y)$ acts in the Oxy plane is illustrated in Fig.1. It illustrates how the operation $\mathbf{M}_x(\pi/4)\mathbf{M}_y(\pi/3)$ of an electromagnetic field $(\mathbf{E}, \mathbf{0})$, with $\mathbf{E} \parallel (\cos(\pi/6), \sin(\pi/6))$ and $c\mathbf{B} = \mathbf{0}$, transforms it into an electromagnetic field $(\mathbf{E}', c\mathbf{B}')$. The way the transformation $\mathbf{I}_z(\zeta_z)$ acts on the Oxy plane is illustrated in Fig.2.

Let us now prove that we can indeed change at will the relative strengths and orientations of $\mathbf{E}$ and $c\mathbf{B}$ within the plane $(\mathbf{E}, c\mathbf{B})$ by using the transformations $\mathbf{M}_j(\chi_j)$. This will complete the prove that $\mathscr{G} \subset \mathrm{SU}(3)$. Consider two arbitray vectors $\mathbf{c}$ and $\mathbf{d}$, and a vector $\mathbf{F} = F\mathbf{e}_x = E_x\mathbf{e}_x$ in a canonical plane, whereby $\mathbf{F}^2 = \mathbf{c}^2 + \mathbf{d}^2$. Here $\mathbf{c} = (c_x, c_y, 0)$ and $\mathbf{d} = (d_x, d_y, 0)$. We calculate $\ell_x = \sqrt{c_x^2 + d_x^2}$, $\ell_y = \sqrt{c_y^2 + d_y^2}$, $\ell_z = 0$. The vector $\boldsymbol{\ell} = (\ell_x, \ell_y, \ell_z)$ has the same length F as $\mathbf{F}$. There exists a rotation $\mathbf{R}(\alpha)$ in $\mathbb{R}^2$ that turns $\mathbf{F}$ into $\boldsymbol{\ell}$.[11] We can now define the angles χ_j as follows: $c_x = \cos\chi_x\ell_x$, $d_x = \sin\chi_x\ell_x$, $c_y = \cos\chi_y\ell_y$, $d_y = \sin\chi_y\ell_y$. The combined transformation $\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y)$ will turn (ℓ_x, ℓ_y) into $\mathbf{c} + \imath\mathbf{d}$. We can now choose χ_z such that $\chi_x + \chi_y + \chi_z = 0$. The in-plane transformation $\mathbf{T}_2 = \mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y)\mathbf{M}_z(\chi_z)\mathbf{R}(\alpha)$ of SU(3) permits thus to transform the vector $\mathbf{F}$ into an arbitrary vector $\mathbf{c} + \imath\mathbf{d}$ of the same length $\mathbf{F}^2 = \mathbf{c}^2 + \mathbf{d}^2$.

We can the same way transform $\mathbf{F}$ into $\mathbf{C} + \imath\mathbf{D}$, whereby $\mathbf{C}^2 + \mathbf{D}^2 = \mathbf{F}^2$. Let us note this transformation as $\mathbf{T}_1 \in \mathrm{SU}(3)$. The transformation $\mathbf{T}_2\mathbf{T}_1^{-1} \in \mathrm{SU}(3)$ permits then to transform $\mathbf{C} + \imath\mathbf{D}$ into $\mathbf{c} + \imath\mathbf{d}$, whereby $\mathbf{C}^2 + \mathbf{D}^2 = \mathbf{c}^2 + \mathbf{d}^2$. This clearly shows that the matrices $\mathbf{M}_j$ and $\mathbf{R}_k$ generate all norm-preserving transformations. We have also made

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[10] The infinitesimal generator of a rotation $\mathbf{R}_z(\alpha)$ around the z-axis in $\mathbb{R}^3$ is in SU(3) not given by σ_z, but by the non-diagonal matrix with imaginary entries given in Eq. 21. Similarly, the infinitesimal generator of a rotation $\mathbf{R}_x(\alpha)$ around the x-axis in $\mathbb{R}^3$ is in SU(3) not given by σ_x, but also a matrix with off-diagonal and imaginary entries.

[11] This is a rotation around the origin O of the plane. It is typical of the mathematics in vector spaces based on $\mathbb{C}$ that we consider only such planes and not a third dimension that would embed this plane in $\mathbb{R}^3$, such that we could call the rotation a rotation around an axis in this space $\mathbb{R}^3$. We encounter this also with the $2^\nu \times 2^\nu$ matrices in the spinor representations of the rotation groups SO(n) as discussed in [1]. We refer the reader in this respect to Eq. (103) in [1], where we refrain from introducing the terminology σ_z for the operator in the function space that is algebraically identical to σ_z. Instead of that we use $\mathbb{1}$, σ_x, $\imath\sigma_y$, and $-\imath\sigma_x\sigma_y$. This is necessary to fit the step from a representation with $2^\nu \times 2^\nu$ matrices to a representation with $2^{\nu+1} \times 2^{\nu+1}$ matrices into a Peano induction scheme in an intelligible way. If we embedded the (E_x, cB_x) plane into $\mathbb{R}^3$ with a basis $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$, $\boldsymbol{\eta}_3$ we would have to represent (E_x, cB_x) as $E_x\boldsymbol{\eta}_1 + cB_x\boldsymbol{\eta}_2$. We could then claim that a rotation in the plane $(E_x, cB_x) \equiv (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ would be a rotation around $\boldsymbol{\ell} \wedge \mathbf{F} \parallel \boldsymbol{\eta}_3$ in this abstract vector space $\mathbb{R}^3$. The space is really abstract because in physical space $E_x\mathbf{e}_x$ and $cB_x\mathbf{e}_x$ are parallel.

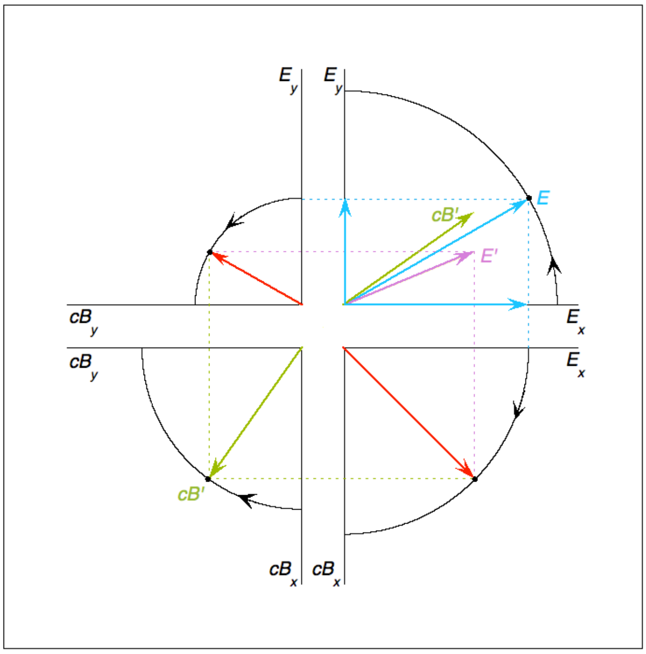

Fig. 1. Illustration of the action of the transformation $\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y)$ in the plane $(\mathbf{E}, c\mathbf{B})$. For simplicity we have illustrated the special case that $c\mathbf{B} = \mathbf{0}$. In the figure we have taken $\chi_x = \pi/4$, $\chi_y = \pi/3$, $\mathbf{E} \parallel (\cos(\pi/6), \sin(\pi/6))$, $c\mathbf{B} = \mathbf{0}$. The transformation results in an electromagnetic field $(\mathbf{E}', c\mathbf{B}')$. The figure consists of four quadrants. The first quadrant shows the initial field $\mathbf{E}$ and the transformed field $(\mathbf{E}', c\mathbf{B}')$. We first decompose $\mathbf{E}$ into its components $E_x\mathbf{e}_x$ and $E_y\mathbf{e}_y$, as shown by the blue vectors along the x- and y-axis. The component E_y is carried over to the second quadrant as shown by the horizontal blue dotted line. In the second quadrant, this (blue) component E_y is rotated in the complex $E_y + \imath cB_y$ plane to $E'_y + \imath cB'_y$ (represented by the red vector) by the operation $\mathbf{M}_y(\pi/3)$. The component E_x is carried over to the fourth quadrant as shown by the vertical blue dotted line. There the component E_x is then rotated in the complex $E_x + \imath cB_x$ plane to $E'_x + \imath cB'_x$ (represented by the red vector) by the operation $\mathbf{M}_x(\pi/4)$. In the third quadrant, we reconstruct $c\mathbf{B}'$ from its components (cB'_x, cB'_y), by recollecting them from the second and the fourth quadrants as indicated by the green dotted lines. We reconstruct $\mathbf{E}'$ in the first quadrant from its components (E'_x, E'_y) by recollecting them from the second and the fourth quadrants, as indicated by the magenta dotted lines. To represent the result for $c\mathbf{B}'$ in the fourth quadrant also within the first quadrant we must refect the quadrant with respect to the direction of the diagonal of the second and the fourth quadrant over π (i.e. the line $y = -x$) such that the axis cB_x aligns with the axis E_x and the axis cB_y with the axis E_y). This is different from mirroring with respect to the origin (which would map cB_x onto E_y and cB_y onto E_x). All the operations shown preserve the square of the Hermitian norm: $|\,\mathbf{E}' + \imath c\mathbf{B}'\,|_{\mathsf{H}}$. In the general case, where $c\mathbf{B} \neq \mathbf{0}$ we must also use the third quadrant as shown in Fig. 2

sure that $\det[\,\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y\mathbf{M}_z(\chi_z)\,] = 1$. Using the transformations $\mathbf{M}_x(\chi_x)$, $\mathbf{M}_y(\chi_y)$, $\mathbf{M}_z(\chi_z)$ is equivalent to using the transformations $\mathbf{I}_x(\zeta_x)$, $\mathbf{I}_y(\zeta_y)$, $\mathbf{I}_z(\zeta_z)$ as we have shown above.[12]

We have thus shown that $\mathscr{G} \subset \mathrm{SU}(3)$, such that $\mathscr{G} = \mathrm{SU}(3)$. SU(3) is thus the symmetry group of the energy density we were looking for. With the operations $\mathbf{I}_j$ and the rotations of $\mathbb{R}^3$ we can this way construct any operation of the group $\mathscr{G} = \mathrm{SU}(3)$. In other words, the group $\mathscr{G}$ of norm-preserving operations is generated by operations of the type $\mathbf{I}_x$, $\mathbf{I}_y$, $\mathbf{I}_z$ and $\mathbf{R}_x$, $\mathbf{R}_y$, $\mathbf{R}_z$. SU(3) contains thus the three operations $\mathbf{I}_x$, $\mathbf{I}_y$, $\mathbf{I}_z$, the three rotations $\mathbf{R}_x$, $\mathbf{R}_y$, $\mathbf{R}_z$ and three remaining operations. As the products of $\mathbf{I}_j$ and $\mathbf{R}_k$ are not rotations and not of the type $\mathbf{I}_x$, the remaining operations must be of the type $\mathbf{I}_j\mathbf{R}_k$.

Let us now try to find the three lacking infinitesimal generators. We know that we should find an infinitesimal generator:

[12] Everything is here a consequence of the fact that the fields are planar. In fact, this implies that in a reference plane whose Oxy plane we have made coincide with the plane of the field, we can choose c_x, c_y, d_x, d_y at will (within the constraint $\mathbf{c}^2 + \mathbf{d}^2 = \mathbf{F}^2$), while $(c_z, d_z) = (0, 0)$. This in turn implies that we can then satisfy the condition $\det(\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y\mathbf{M}_z(\chi_z)) = 1$ by making a proper choice of χ_z. In terms of infinitesimal generators, $\det(\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y\mathbf{M}_z(\chi_z)) = 1$ is expressed by $\mathring{\mathbf{I}}_x + \mathring{\mathbf{I}}_y + \mathring{\mathbf{I}}_z = 0$. In the case of the strong interaction this is translated into an expressive language of three colors, which must forcedly add up to white. In the case of the electromagnetic force, we can also express it by saying that the field (and a photon) have only two linear polarizations.

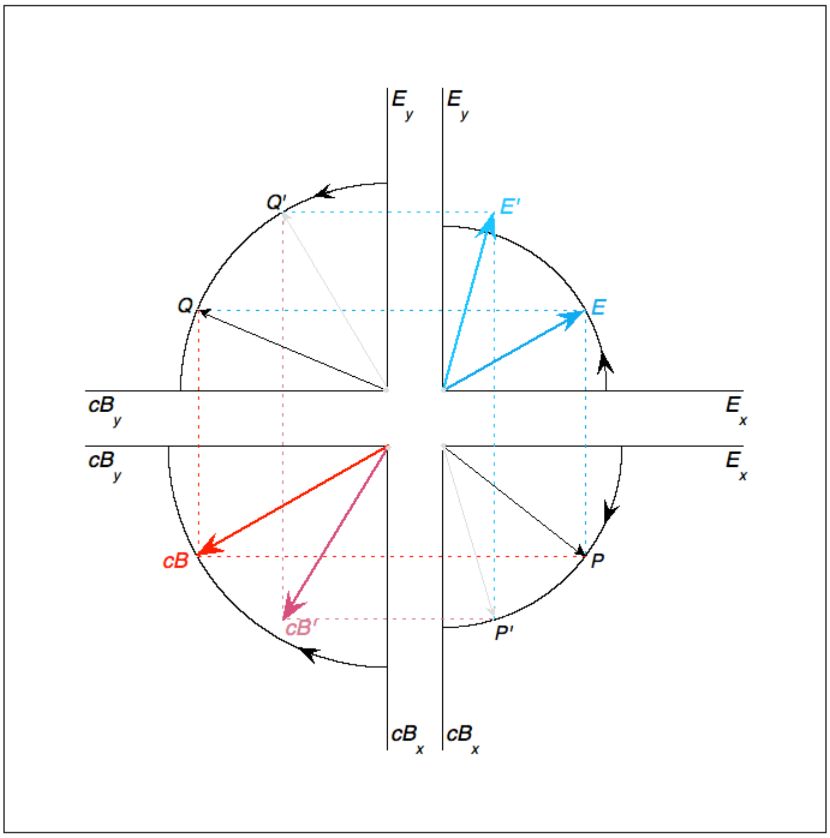

Fig. 2. Illustration of the action of the transformation $\mathbf{I}_z(\zeta_z)$ in the plane $(\mathbf{E}, c\mathbf{B})$. This is equivalent to $\mathbf{M}_x(\chi_x)\mathbf{M}_y(\chi_y)$, with the constraint: $\chi_x = -\chi_y$, whereby we note χ_x as ζ_z. In the figure we have taken $\zeta_z = -\pi/5$, $\mathbf{E} = (6\cos(\pi/6), 6\sin(\pi/6))$, $c\mathbf{B} = (8\cos(\pi/3), 8\sin(\pi/3))$. The transformation results in an electromagnetic field $(\mathbf{E}', c\mathbf{B}')$. The figure consists of four quadrants. The positive orientation for the angles in each quadrant is shown by the arrows on the circles. We have not presented the magnetic field in the first quadrant in order not to burden the figure. The representation of $c\mathbf{B}$ in the first quadrant can be obtained by reflecting the third quadrant with respect to the diagonal $y = -x$. In the first quadrant we decompose $\mathbf{E} = (E_x, E_y)$ into its components E_x and E_y, as shown by the dark blue dotted lines. In the third quadrant we decompose $c\mathbf{B} = (cB_x, cB_y)$ into its components cB_x and cB_y, as shown by the red dotted lines. The values of E_x and cB_x are combined in the fourth quadrant to define the point $P(E_x, cB_x)$. The values of E_y and cB_y are combined in the second quadrant to define the point $Q(E_y, cB_y)$. The transformation $\mathbf{I}_z(\zeta_z)$ turns P over ζ_z to $P'(E'_x, cB'_x)$ and Q over $-\zeta_z$ to $Q'(E'_y, cB'_y)$. Then we reconstruct $\mathbf{E}' = (E'_x, E'_y)$ in the first quadrant and $c\mathbf{B}' = (cB'_x, cB'_y)$ in the third quadrant from the components of P' and Q' as shown by the light blue and magenta dotted lines. Comparison with the positive sense of orientation in each quadrant shows that $\mathbf{E}$ and $c\mathbf{B}$ are rotating in opposite directions. The transformation increases the amplitude of the field $\mathbf{E}$. Concomitantly it decreases the amplitude of the field $c\mathbf{B}$ to keep $\mathbf{E}^2 + c^2\mathbf{B}^2$ constant.

$$\overset{\circ}{\mathbf{S}}_z = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{su}(3). \tag{36}$$

We could thus cheat a bit and work backwards to this result. It is then tempting to propose:

$$\mathbf{S}_z = \begin{bmatrix} \imath\cos\alpha & -\imath\sin\alpha & 0 \\ -\imath\sin\alpha & -\imath\cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{with:} \quad \det\mathbf{S}_z = 1, \quad \text{and} \quad \mathbf{S}_z^{-1} = \begin{bmatrix} -\imath\cos\alpha & \imath\sin\alpha & 0 \\ \imath\sin\alpha & \imath\cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{S}_z^{\dagger}. \tag{37}$$

This can be decomposed as:

$$\mathbf{S}_z = \begin{bmatrix} \imath & 0 & 0 \\ 0 & -\imath & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_z(\pi/2)\mathbf{R}_z(\alpha). \tag{38}$$

Applying the recipe of Eq. 18 yields the tangent vector:

$$\mathbf{S}_z(\alpha) \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{39}$$

But this is wrong because it is not a tangent vector at the identity element. It is a tangent vector at $\mathbf{I}_z(\pi/2)$. In fact:

$$\lim_{\alpha\to 0} \mathbf{S}_z(\alpha) = \begin{bmatrix} \imath & 0 & 0 \\ 0 & -\imath & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \mathbb{1}. \tag{40}$$

A correct solution is the following:

$$\mathbf{S}_z(\zeta_z) = [\,\mathbf{R}_z(\pi/4)\,]\,[\,\mathbf{I}_z(-\zeta_z)\,]\,[\,\mathbf{R}_z(-\pi/4)\,] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-\imath\zeta_z} & 0 & 0 \\ 0 & e^{\imath\zeta_z} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}. \tag{41}$$

This leads indeed to:

$$\lim_{\zeta_z\to 0} \mathbf{S}_z(\zeta_z) = \mathbb{1}, \quad \mathbf{S}_z(\zeta_z) \rightsquigarrow \overset{\circ}{\mathbf{S}}_z = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{su}(3). \tag{42}$$

This existence of $\mathbf{S}_z(\zeta_z)$ and its construction are relying entirely on the fact that $\mathbf{R}_z(\pi/4)$ and $\mathbf{I}_z(-\zeta_z)$ do not commute.[13]

The nine 3×3 matrices $\mathbf{E}_{km}$, $k \in [1,3] \cap \mathbb{N}$, $m \in [1,3] \cap \mathbb{N}$, defined by $(\mathbf{E}_{km})_{ij} = \delta_{ki}\delta_{mj}$ are a basis for the vector space of 3×3 matrices. The infinitesimal generators of S_z and R_z span the same subspace as $\mathbf{E}_{12}$ and $\mathbf{E}_{21}$. Similarly the infinitesimal generators of S_x and R_x span the same subspace as $\mathbf{E}_{23}$ and $\mathbf{E}_{32}$, while the infinitesimal generators

[13] To find this solution we have used rather fishy heuristics, by taking inspiration from the geometry of SU(2). In SU(2), a rotation $\mathbf{R}(\mathbf{n}, \varphi)$ around an axis $\mathbf{n}$ over an angle φ is given by the Rodrigues formula:

$$\mathbf{R}(\mathbf{n}, \varphi) = \cos(\varphi/2)\,\mathbb{1} - \imath\sin(\varphi/2)\,[\,\mathbf{n}{\cdot}\boldsymbol{\sigma}\,] = \begin{bmatrix} \cos(\varphi/2) - \imath n_z \sin(\varphi/2) & -\imath(n_x - \imath n_y)\sin(\varphi/2) \\ -\imath(n_x + \imath n_y)\sin(\varphi/2) & \cos(\varphi/2) + \imath n_z \sin(\varphi/2) \end{bmatrix}. \tag{43}$$

The matrix:

$$\mathbf{R}_y(\pi/2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \tag{44}$$

represents therefore in SU(2) a rotation over $\pi/2$ around the y-axis. When a 2×2 matrix $\mathbf{M}_2$ is embedded inside a 3×3 matrix $\mathbf{M}_1$ according to:

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{M}_2 & \mathbf{0}^\top \\ \mathbf{0} & 1 \end{bmatrix}, \tag{45}$$

then we will note this as $\mathbf{M}_1 \cong \mathbf{M}_2$. Here $\mathbf{0}$ is the 2×1 matrix $[\,0\;0\,]$, This is an equivalence between a subalgebra of SU(3) and the algebra of SU(2) that identifies $\mathbf{M}_1$ with $\mathbf{M}_2$ on the mere basis of a purely formal algebraic identity, which is devoid of any geometrical meaning. In our heuristics to find Eq. 41 we have identified $\mathbf{R}_z(\pi/4) \cong \mathbf{R}_y(\pi/2)$, although this is geometrically spoken non-sense. Despite their algebraic identity $\mathbf{R}_z(\pi/4)$ and $\mathbf{R}_y(\pi/2)$ have completely different geometrical meanings in the two formalisms, But the identification helps us to find solutions in SU(3) by reasoning on SU(2) with whose geometry we are more familiar. In the algebra of SU(2) the rotation $\mathbf{R}_y(\pi/2)$ turns the vector $\mathbf{e}_z$ (represented by σ_z) into the vector $\mathbf{e}_x$ (represented by σ_x). As Eq. 4 shows, this is expressed by $\sigma_x = [\,\mathbf{R}_y(\pi/2)\,]\,\sigma_z\,[\,\mathbf{R}_y(\pi/2)\,]^\dagger$ in the group formalism. When we translate this back to SU(3), this yields Eq. 41. This is the result we needed as it permits us to obtain an infinitesimal generator $\overset{\circ}{\mathbf{S}}_z \cong \sigma_x$ starting from an infinitesimal generator $\overset{\circ}{\mathbf{I}}_z \cong \sigma_z$. Geometrically what happens in SU(3) has nothing to do with what happens in SU(2). What happens in SU(3) is that the non-commutativity $[\,\mathbf{R}_z(\pi/4)\,]\,[\,\mathbf{I}_z(-\zeta_z)\,] \neq [\,\mathbf{I}_z(-\zeta_z)\,]\,[\,\mathbf{R}_z(\pi/4)\,]$ creates a new type of displacement $\mathbf{I}_z(-\zeta_z) \to [\,\mathbf{R}_z(\pi/4)\,]\,[\,\mathbf{I}_z(-\zeta_z)\,]\,[\,\mathbf{R}_z(\pi/4)\,]^{-1}$ within the Oxy plane. We will come back on this in Section 5.

of S_y and R_y span the same subspace as $\mathbf{E}_{13}$ and $\mathbf{E}_{31}$. The three matrices $\overset{\circ}{\mathbf{I}}_j$ span the same subspace as $\mathbf{E}_{11}$, $\mathbf{E}_{22}$, and $\mathbf{E}_{33}$. Therefore the infinitesimal generators we obtained forcedly span the tangent space to the group manifold. But to do so, one must use also imaginary coefficients in the linear combinations. The infinitesimal generators we derived constitute a basis for SU(3) whereby the coefficients in the linear combinations can be kept real. SU(3) has thus five generating operations (three rotations and two of the three transformations $\mathbf{I}_j$) and eight infinitesimal generators. It is the group one obtains by extending SO(3) with the "scrambler" operations $\mathbf{I}_j(\zeta_j)$, which unlock $\mathbf{E}$ and $c\mathbf{B}$ one from another, while keeping them within the constraint that $\mathbf{E}^2 + c^2\mathbf{B}^2$ must remain constant.

3.5 Philosophical motivation

There is a kind of geometrical nomansland between physics and mathematics nobody seems to be interested in. It permits to render algebra more intuitive by relying on its geometrical meaning. Mathematicians consider that intuition endangers the mathematical rigor of their algebraic proofs, such that they prefer not to mention what is going on behind the scenes of the algebra of their proofs. As explained by Dieudonné [5], they experience serious difficulties and feelings of alienation with such presentations themselves. For physicists it is worse. It forces them to use this algebra as a blackbox. In doing so they have become so much used to think that mathematics is only about carrying out algebraic calculations, that they cannot even imagine that there could be a geometrical counterpart to the algebra they use. They teach their students to shut up and calculate. Despite the lack of interest from both sides, figuring out the geometry that corresponds to the algebra is important because it provides additional insight into the meaning of the mathematics. To understand the physics, one must always go after the geometry hidden behind the algebra, as this will open the door to insight in its physical meaning. It is a very useful strategy. The present work has been entirely motivated by this idea and shows that the strategy works. There is also such a geometrical nomansland that gives additional insight into the meaning of the algebra of quantum mechanics [1,2,6].

3.6 The meaning of SL(3,$\mathbb{C}$)

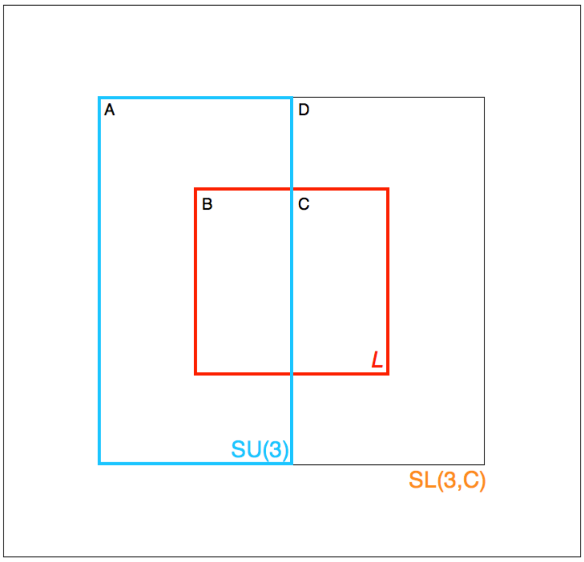

Fig. 3. Venn diagram of the group SL(3,$\mathbb{C}$) and its subgroups. The letters A, B, C and D design sets that are all mutually disjoint. (The sets A and D are thus represented by non-convex polygons). The group SO(3) is the set B. The set $B \cup C$, surrounded by a red line in the figure is the homogeneous Lorentz group $\mathscr{L}$. Its subset $C = \mathscr{L} \setminus$ SO(3) contains thus all the boosts and the products of a boost and a rotation (different from the identity element). The set generated by SO(3) and the operations $\mathbf{I}_j(\zeta_j)$ is SU(3), which corresponds to $A \cup B$. The set generated by $\mathscr{L}$ and the operations $\mathbf{I}_j(\zeta_j)$ is SL(3,$\mathbb{C}$) and corresponds thus to $A \cup B \cup C \cup D$. SU(3) and SL(3,$\mathbb{C}$) are thus extensions of SU(2) and SL(2,$\mathbb{C}$) by the operations $\mathbf{I}_j(\zeta_j)$.

Although this is somewhat beyond the scope of this paper, we can classify further the elements of SL(3,$\mathbb{C}$). The situation is summarized by the Venn diagram in Fig. 3. First of all, we note that $\overset{\circ}{\mathbf{L}}_j = -\imath \overset{\circ}{\mathbf{R}}_j, \forall j \in [1,3] \cap \mathbb{N}$. The additional infinitesimal generators we are looking for could thus be of the types:

$$\overset{\circ}{\mathbf{K}}_z = -\imath \overset{\circ}{\mathbf{I}}_z = \begin{bmatrix} \imath & 0 & 0 \\ 0 & -\imath & 0 \\ 0 & 0 & 0 \end{bmatrix} \cong \imath\sigma_z, \quad \text{and:} \quad \overset{\circ}{\mathbf{C}}_z = -\imath \overset{\circ}{\mathbf{S}}_z = \begin{bmatrix} 0 & -\imath & 0 \\ -\imath & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cong -\imath\sigma_x. \tag{46}$$

The complement of SU(3) in SL(3,$\mathbb{C}$) would then be obtained from SU(3) by mutiplication with $-\imath$. We have thus $\overset{\circ}{\mathbf{L}}_z \cong -\imath\sigma_y$ and $\overset{\circ}{\mathbf{K}}_z \cong \imath\sigma_z$. We can use again the quirky heuristics explained in Footnote 13. In SU(2), we can obtain $\imath\sigma_z$ from $-\imath\sigma_y$ by a rotation $\mathbf{R}_x(-\pi/2)$ over $-\pi/2$ around the x-axis. We therefore consider an operation $\mathbf{U}$ in SU(3) defined by $\mathbf{U} \cong \mathbf{R}_x(-\pi/2)$, to construct:

$$\begin{aligned} \mathbf{K}_z(v) = \mathbf{U}\,[\,\mathbf{L}_z(v)\,]\,\mathbf{U}^\dagger \quad &= \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & \imath & 0 \\ \imath & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & \imath\beta\gamma & 0 \\ -\imath\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\imath & 0 \\ -\imath & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tfrac{1}{\sqrt{2}} \\ &= \begin{bmatrix} \gamma(1+\beta) & 0 & 0 \\ 0 & \gamma(1-\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{47}$$

Checking the algebra is greatly simplified by observing that $\mathbf{L}_z(v) \cong \gamma\mathbb{1} - \beta\gamma\sigma_y$, such that we only need to check the calculations on σ_y. The operation $\mathbf{U}$ belongs to SU(3), because $\det \mathbf{U} = 1$ & $\mathbf{U}^{-1} = \mathbf{U}^\dagger$. Its decomposition into more familiar group elements follows from:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \imath \\ \imath & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{\imath\pi/4} & 0 \\ 0 & e^{-\imath\pi/4} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \cong \mathbf{R}_z(\pi/4)\,\mathbf{I}_z(\pi/4)\,\mathbf{R}_z(-\pi/4). \tag{48}$$

The operator $\mathbf{K}_z(v)$ leads indeed to the desired infinitesimal operator. We have also $\overset{\circ}{\mathbf{L}}_z \cong -\imath\sigma_y$ and $\overset{\circ}{\mathbf{C}}_z \cong -\imath\sigma_x$. We can obtain $-\imath\sigma_x$ from $-\imath\sigma_y$ in SU(2) by a rotation over $-\pi/2$ around the z-axis. We use therefore now a transformation $\mathbf{I}_z(\pi/4) \cong \mathbf{R}_z(-\pi/2)$:

$$\begin{aligned} \mathbf{C}_z(v) = \mathbf{I}_z(\pi/4)\,[\,\mathbf{L}_z(v)\,]\,\mathbf{I}_z(-\pi/4) \quad &= \tfrac{1}{\sqrt{2}} \begin{bmatrix} e^{\imath\pi/4} & 0 & 0 \\ 0 & e^{-\imath\pi/4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\imath\beta\gamma & 0 \\ \imath\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-\imath\pi/4} & 0 & 0 \\ 0 & e^{\imath\pi/4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tfrac{1}{\sqrt{2}} \\ &= \begin{bmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{49}$$

This is the desired result. Obviously, $\mathbf{C}_z(v)$ is just the group element $\mathbf{Q}_z(-v)$ we guessed in Eq. 23. The group SL(3,$\mathbb{C}$) is thus generated by the rotations $\mathbf{R}$, the boosts $\mathbf{L}$, and the transformations $\mathbf{I}$. It is the extension of $\mathscr{L} \equiv$ SL(2,$\mathbb{C}$) by the operations $\mathbf{I}_j(\zeta_j)$.[14] The mixed products of the basic operations are providing additional infinitesimal generators because have different characters. Any group automorphism: $T_f, f \in G : \forall g \in G, T_f(g) = f^{-1} \circ g \circ f$ will produce an equivalent basis of infinitesimal generators, and will thus induce a change of basis in tangent space.

4 Intermezzo - Important caveats

We can consider what we have done as didactically interesting. But we would like to point out that we must remain cautious and refrain from using these results to lay down claims about the physics. In Heisenbergs conception of using SU(2) to define isospin, the rotation group is assumed not to operate on $\mathbb{R}^3$ but on some internal space of the particles. Something similar has been postulated for the group SU(3) used in particle physics. The symmetries are thus very abstract and impenetrable. The statements about the isospin group SU(2) and the elementary-particle group SU(3) actually imply that the parameters of the elementary particles the groups work on are not defined. They are some unidentified parameters in some unidentified space. But working on concepts that are not defined is not recommended practice in dayly life actions, let alone in science. One should try to figure out what these parameters are.

[14] It is an interesting question if SU(n) could be always obtained from SO(n) by such a "scrambler" extension scheme that completely unlocks all vectors of a two-dimensional plane. This is extremely likely, because the dimension of the group is $n^2 - 1$ and $\det \mathbf{M} = 1$, $\forall \mathbf{M} \in$ SU(n). But this has to be proved. One may also wonder what are the groups that would completely unlock higher-dimensional subspaces.

The level of abstraction in SU(3) looked even worse than in the case of isospin, because we even did not have an analogous group in physical space to visualize it. The present findings improve on this situation. The group which describes the symmetry properties of the energy density of the electromagnetic field in physical space $\mathbb{R}^3$ can now be used to visualize the transformation group SU(3) used in particle physics by analogy, just like we could use the rotation group SU(2) in physical space $\mathbb{R}^3$ to visualize the isospin group by analogy. The abstraction of SU(3) discouraged us from asking any questions about the possible physical meaning the group could have in $\mathbb{R}^3$ (or any other space). It was too impenetrable. It looked impossible to figure out what was going on behind the scenes of the algebra. We did not perceive any firm conceptual ground we could stand on to even start reasoning about the problem. The abstraction disempowered us. That situation has now changed.

However, this does not mean that we must engage head over heels into fomenting a rebellion against the current doctrine. We must rather try to blend our results into it, without getting carried away towards jumping to conclusions. We can certainly not claim that our findings would give us a cogent re-interpretation of SU(3) as it occurs in particle physics in terms of electromagnetic fields. This is *a priori* completely unlikely. The considerations apply to any Lorentz tensor $\mathbf{C} + \imath c\mathbf{D}$. We think that we can definitely conclude that the group SU(3) acts on a tensor, because Lorentz symmetry implies that all physical quantities are scalars, four-component vectors, six-component tensors, four-component axial vectors or pseudo-scalars. The whole problem remains thus to identify this tensor.

The fact that the transformation h belongs to SU(3)[15] raises a question about the way we must interpret the group SU(3). We could interpret it just as a mathematical gimmick, in the sense that it does not describe transformations that occur in nature. For the electromagnetic field, it could then be merely a convenient way to relate various configurations of the electromagnetic field which we can put together *manually*. Nature itself would not do it. Perhaps for the electromagnetic field the rôle of the group is limited to such a mathematical gimmick. It is an interesting thought that for the strong interaction, the transformations could really occur in nature without manual human intervention. But for the moment, this is all speculation. We are not sure that SU(3) acts on the field tensor of the strong interaction. It could act on any type of tensor, e.g. the one we defined in Eq. 10 and which contains the angular momentum.

We have seen that Lorentz transformations form $[\,\mathrm{SO}(3),\mathbb{C}\,]$, an extension of SO(3) to $\mathbb{C}^3$ which ceases to be a compact group, while SU(3) is compact. Hence the transformations we are considering here within the framework of SU(3) are spatially confined, in contrast with boosts in free space which can result in motions that cover distances that can be arbirarily large.

In any case, we have now an analogy we can use to visualize the structure of SU(3) within a less abstract context. We can visualize SU(3) by an analogy in terms of transformations of an electromagnetic field, because there is an isomorphism between the physics of a class of transformations of an electromagnetic field and the physics of the strong intractions at work in baryons. As we have intuition for the electromagnetic field, this could help us in dealing with the transformations that occur in particle physics, whose meaning remains otherwise rather abstract and impenetrable.

One application of the analogy has consisted in obtaining the Lie algebra from the infinitesimal generators of SU(3) like we do it for the rotation group or the Lorentz group, by calculating the tangent vectors to one-parameter subsets of the group manifold. In textbooks another approach is used. One lets an abstract basis descend from heaven, letting the reader check by purely algebraic means that it is indeed a correct basis. The difficulties one may encounter in trying to built an approach that provides some more geometrical insight are again a consequence of the abstraction of SU(3). We have given a hint about the difficulties one may encounter in Subsection 3.2. In fact, we can parametrize the SU(3) matrices like one can parameterize an SO(3) matrix in terms of a rotated triad of the three basis vectors, but when we did this for SU(3), we ended up in the puzzling situation that we could not figure out which one-parameter sets we had to choose to obtain the full set of infinitesimal generators. We outline this briefly in the Appendix.

The procedure given in Eq. 18 is ill-defined, in that it does not give a clear prescription of the way one must select the group parameters to obtain an orthonormal basis for the tangent space. This can lead to several difficulties. Some of the parameter sets may contain singularities, in the same way spherical coordinates contain a singularity at the North and South poles of the sphere. Also the curves we define on the manifold with the one-parameter sets are not forcedly everywhere mutually orthogonal. We have seen in the example of the Euler angles that α_1 and α_3 produce tangent vectors that are parallel. In an abstract formulation of the group, without any geometrical clues, its is hard to foresee where the problems will occur and to figure out how we can avoid them. We may have to cover the group with an atlas of several sets of parameters in order to avoid the singularities. We will need then the transformations between the various members of the atlas.

One's first guess would be that SU(3) is a little big brother of the group SU(2). That sounds all right and reasonable as long as we keep it abstract. But when we start to inspect applications and try to give geometrical meaning to the

[15] The transformation matrix $\mathbf{H}$ of h for the field in the canonical form $(\mathbf{E}_0, c\mathbf{B})$ can be written as:

$$\mathbf{H} = \begin{bmatrix} \imath & & \\ & \imath & \\ & & -1 \end{bmatrix}, \quad \text{with:} \quad \mathbf{H}^\dagger = \mathbf{H}^{-1}, \quad \text{but:} \quad \det \mathbf{H} = 1 \Rightarrow \mathbf{H} \in \mathrm{SU}(3). \tag{50}$$

In this transcription we have $h \in \mathrm{SU}(3)$.

groups, things immediately go wrong. In fact, the group SU(2) represents the rotations of $\mathbb{R}^3$. It is thus directly related to SO(3), of which it is a double covering. How we construct SU(2) from scratch based on its geometrical meaning is explained in reference [1]. But when we try to extrapolate this idea to rotation groups of $\mathbb{R}^n$, any analogy one might anticipate with some group SU(k), $k \in \mathbb{N}$ turns out to be chimerical. In fact, the next step along this line of thought would be to consider SO(4). But the spinor representation corresponding to SO(4) is a six-dimensional representation of rank 4 and can thus not be SU(3). Noting $\nu = \lfloor \frac{n}{2} \rfloor$, the spinor representation corresponding to SO(n) is a $n(n-1)/2$-dimensional representation of rank 2^ν, while SU(n) is a n^2-1-dimensional representation of rank n.[16]

Hence considering SU(3) as a generalization of SU(2) is not as straightforward as one might have guessed. SU(3) just does not fit conceptually into the logical chains 2^ν, $\nu \in \mathbb{N}$ for the ranks, or $n(n-1)/2$, $n \in \mathbb{N}$ for the dimensions. We use the group SU(2) to describe angular momentum. But we have seen that this neglects the fact that angular momentum is part of a tensor. In this respect, we could consider perhaps SU(2) as a group we can use to describe a non-relativistic approximation to the description of tensors. In such a non-relativistic approximation, we could describe the angular momentum within the framework of SU(3) by the group SO(3) $\subset$ SU(3). We can then study the corresponding rotation group in an alternative way by SU(2). The electron has a magnetic moment. At relativistic velocities it should thus also have an electric dipole moment, but it could be so tiny that it has never interfered with our experiments.

5 The three subalgebras of the type su(2)

Most textbooks mention that the Lie algebra su(3) contains three subalgebras su(2).[17] We may feel stunned when we hear this and wonder what it may mean. We get the impression that there are some hidden connections. But nothing in textbooks would help us further out in clarifying what these connections are. We are forced to accept that it is just an factual truth that comes out of some abstract algebra. To demystify these issues we turn towards the corresponding geometry. What this actually implies is that we have three basic planes, Oxy (cycl.), which we can label by the index ℓ of the unit vectors $\mathbf{u}_\ell$ along their Poynting vectors. There are three basic planes Ox_jx_k with normals $\mathbf{u}_\ell$ whereby $jk\ell$ can be cyclically permuted. We can then label the planes with the single index ℓ instead of the double index jk. There is a whole subalgebra su(2) for each of the planes ℓ labeled by ℓ. Using the imagery of the fan we introduced after Eq. 27 we can consider the three-dimensional space SU(2) associated with $\mathbf{u}_\ell$ as a three-dimensional fan along $\mathbf{u}_\ell$ we unfold. Of course, three-dimensional fans do not exist for real, perhaps an umbrella comes close to the idea. Anyway we think that the language we use will nevertheless make the point.

We can no longer describe such a fan with the number field $\mathbb{C}$ as for its real bidimensional realization, which is why we must now use the quaternions σ_x, σ_y, σ_z. We have in this paper given all our examples based on the Oxy plane with normal $\mathbf{u}_k = \mathbf{e}_z$. E.g. We have rotations $\mathbf{R}_z(\alpha)$ in the Oxy plane. The infinitesimal generator corresponding to these rotations ressembles σ_y. Next we have the transformations $\mathbf{I}_z(\zeta_z)$ in the Oxy plane which scramble the electric and the magnetic fields. The corresponding infinitesimal generator ressembles σ_z. The third infinitesimal generator only intervenes when we start considering products $\mathbf{I}_z(\zeta_z)$ of and $\mathbf{R}_z(\alpha)$. These products do not commute, and therefore we must consider products $\mathbf{R}_z(\alpha)^{-1}\mathbf{I}_z(\zeta_z)^{-1}\mathbf{R}_z(\alpha)\mathbf{I}_z(\zeta_z) \neq \mathbb{1}$.[18] In the Lie algebra this is expressed in terms of commutators. It is these commutators which are related to the matrix σ_x of the subalgebra su(2). The x-axis that would correspond to such a "σ_x-operator" just does not belong to physical space, just like the "σ_z-operator" does not belong to physical space, while the "σ_y-operator" is a true rotation in physical space $\mathbb{R}^3$, but around the z-axis rather than around the y-axis.

The three Pauli matrices σ_j related to the symmetries in the Oxy plane are all defined within a fan that is folded along the z-axis and that we must unfold in order to be able to see the dazzle of the symmetry. They belong thus really to what physicists call a hidden internal space of symmetries, but this does not mean that this "abstract" internal space

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[16] It is worth comparing the dimensions and the ranks of the representations. For SO(n), with $n \geq 3$, the dimensions of the groups follow the series $3, 6, 10, 15, 21, 28, 35, 45, \ldots$. For SU($n$), with $n \geq 3$, the dimensions follow the series $8, 15, 24, 35, 48, \ldots$. For SO($n$), with $n \geq 3$, the ranks follow the series $3, 4, 4, 8, 8, 16, 16, \ldots$. For SU($n$), with $n \geq 3$, the ranks follow the series $3, 4, 5, 6, \ldots$. The spinor representations of SO(n) can therefore at the very best be subgroups of SU(2^ν).

[17] It is dangerous to state that the Lie group SU(3) contains three subgroups SU(2). It contains three subgroups, but these subgroups are linearly acting on meaningful vectors, not quadratically like SU(2).

[18] We are used to associate non-commutativity with curvature. The Oxy plane is of course not curved. The curvature occurs in the four-dimensional manifold of the quantities (E_x, cB_x, E_y, cB_y) subjected to the constraint that $\mathbf{E}^2 + c^2\mathbf{B}^2$ is a constant. This is a hypersphere and in the symmetry group of this hypersphere $\mathbf{I}_z(\zeta_z)$ and $\mathbf{R}_z(\alpha)$ do not commute, which introduces the curvature. We have actually taken advantage of this non-commutativity to find the construction presented in Eq. 41. The lesson we may take from this is that non-commutativity can be expressed mathematically in terms of curvature, and that curvature can be described mathematically by introducing additional dimensions. But these additional mathematical dimensions do not need to be additional physical dimensions, as we see here with SU(3). Hence, it is a non-issue that the additional seven dimensions of string theory are "not observed".

must remain beyond intuition, because we can identify it with what happens in the Oxy plane. As shown here, with some fan-tasy we can really describe SU(3) with vivid imagery and colors. In fact, all the operations corresponding to one of the subalgebras su(2) are taking place in the Oxy plane. We may note that in each of the subalgebras the indices of the operators σ_x, σ_y, σ_z, are related to the type of the operations, not to the indices we might use to note the orientations of the planes. Hence, when we make cyclic permutations of the planes Oxy, the operators σ_x, σ_y, and σ_z in the corresponding subalgebras are not co-permuted. We may ask why there are no further operators that enter the scene when we start making products of operators of different su(2) subalgebras. The algebraic answer to that question would presumably consist in pointing out laconically that it just factually rolls out of the algebra. The geometrical answer resides in our proof in Subsection 3.4 that $\mathscr{G}$ = SU(3). Whatever we do, we always end up with a plane defined by $\mathbf{E}$ and $c\mathbf{B}$ while the transformations between different planes are already taken care of by SO(3) $\subset$ SU(3).

6 A possible link?

The ideas of the present Section are more speculative and really need validation before they can be adopted. They are about what could be the next step in implementing the strategy outlined in Subsection 3.5. There must be such a next step because the algebra must mean something in the physics of the real world. One should never give in to Heisenberg's morbid doctrine that quantum mechanics would be beyond human understanding. We should never accept that something looks impenetrable.

The identification of the three subgroups we proposed in Section 5 draws us into turbulent waters. Physicists call the 3×1 column matrices of SU(3) spinors. A 3×1 column matrix taken from a 3×3 SU(3) matrix represents only a part of the information about the whole matrix, and thus in general only a part of the information about a general SU(3) group element. Most of the time a single column drawn from a matrix will indeed not contain the complete information about the matrix. E.g. SL(2,$\mathbb{C}$) is a six-parameter group. A single column of an SL(2,$\mathbb{C}$) matrix can thus not possibly contain the full information about a general group element. A notable exception is SU(2) where a single column represents all the information about the group element, because the first column constructed by the Gram-Schmidt procedure outlined in the Appendix, defines already the full *Vielbein*. The 4×4 column matrices of the Dirac theory also contain the complete information about the group element, such that we have here an exception as well. A column of an SU(3) matrix can only contain five parameters while the group is defined by eight parameters. It can thus not possibly contain the complete information. It is for this reason that we prefer to use the term "spinor" only for a number of columns that together represent the complete information [1,2].

The construction of a parameterization of the group by the Gram-Schmidt procedure in the Appendix shows that the columns of an SU(3) matrix are basis vectors. The 3×1 column matrices $\mathbf{F}$ are exactly such column vectors. The group acts linearly on them, just like it acts linearly on group elements and we could call them in an *abus de langage* thus also spinors, even if they do not represent the full information. There is however a very important difference between a true spinor formalism, like SU(2) and a formalism like that of SU(3). In a true spinor formalism like SU(2), vectors and tensors of physical space transform quadratically rather than linearly under the transformations of the group, as we explained in Section 1. This is the reason why one often states that a spinor is the square root of a vector. Now SU(3) acts linearly, not quadratically on the Lorentz tensors $\mathbf{F}$. The interpretation of SU(3) as a symmetry group for Lorentz tensors makes it therefore appear to belong to an entirely different world of concepts than SU(2). The situation within SU(3) looks much more to that within the rotation group SO(3) of $\mathbb{R}^3$, which turns vectors, than that within SU(2), which turns spinors representing group elements, and is thus an automorphism group.

This linearity of the action of the group on tensors raises difficult questions. There seems to be a contradiction in the fact that SU(3) acts linearly on tensors and vectors (which are the real and imaginary components of the tensors), while on the other hand it contains groups SU(2) which are supposed to act quadratically on vectors and tensors. If the fields $\mathbf{E} + \imath c\mathbf{B}$ are transformed linearly, what are then the quantities in the SU(2) subgroups that are transformed quadratically? Are they perhaps purely mathematical, physically meaningless quantities? One would be inclined to opt for this solution. But we enter then on collision course with the way Gell-Mann's scheme is used to interpret the elementary particles as representations of the group SU(3). Gell Mann really uses the SU(2) subgroups as working on physically meaningful quantities.

The way out of this paradox is that a group acts on two different types of vector spaces as explained in Footnote 19. These vector spaces can even have different dimensions as shown by the example of the Lorentz group. We want to eleborate this in some more detail. This will give us a link between the two conceptually very different worlds of the two types of vector spaces and this way resolve the paradox. The crucial point is that SU(2) and the Dirac theory act quadratically on both vector spaces, while SU(3) acts quadratically on one space and linearly on the other one. Let us consider an electromagnetic field $\mathbf{E} = E\mathbf{e}_x$, $c\mathbf{B} = \mathbf{e}_z \wedge \mathbf{E} = E\mathbf{e}_y$. This field is represented by:

$$\mathbf{F} = F \begin{bmatrix} 1 \\ \imath \\ 0 \end{bmatrix} \tag{51}$$

Consider now the operator:

$$\mathbf{P}(\varphi) = \begin{bmatrix} e^{-\imath\varphi} & & \\ & e^{-\imath\varphi} & \\ & & e^{2\imath\varphi} \end{bmatrix} \quad \rightsquigarrow \quad \overset{\circ}{\mathbf{P}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}, \tag{52}$$

whose infinitesimal generator is proportional to Gell-Mann's matrix λ_8. Let us now render this dynamical by putting $\varphi = \omega t$. This is completly analogous to the way we describe orbits of planets in Newtonian dynamics by replacing $\mathbf{r} \in \mathbb{R}^3$ by $\mathbf{r}(t)$. We have shown in reference [2] that putting $\varphi = \omega t$ in Eq. 43 permits to derive the Dirac equation. Let now the operator $\mathbf{P}(\omega t)$ act on $\mathbf{F}$. This will then describe the polarization of a circularly polarized electromagnetic wave. This permits us to identify the varying electromagnetic field of a circularly polarized photon with a group element of SU(3). Other group elements could then describe polarization dynamics that are not available to photons. We have now:

$$\frac{d}{dt}\mathbf{P}(\omega t) = -\imath\omega\, \overset{\circ}{\mathbf{P}}\, \mathbf{P}(\omega t). \tag{53}$$

Let us now consider the action of a static, fixed group element $\mathbf{G}$ on $\mathbf{P}(\omega t)$ We will have then:

$$\frac{d}{dt}\mathbf{G}\mathbf{P}(\omega t) = -\imath\omega\, \mathbf{G}\overset{\circ}{\mathbf{P}}\mathbf{P}(\omega t) = -\imath\omega\, \mathbf{G}\overset{\circ}{\mathbf{P}}\mathbf{G}^{-1}\, \mathbf{G}\mathbf{P}(\omega t). \tag{54}$$

We see thus that while the group elements $\mathbf{P}$ and the fields $\mathbf{F}$ transform linearly according to $\mathbf{P} \rightarrow \mathbf{G}\mathbf{P}$ and $\mathbf{F} \rightarrow \mathbf{G}\mathbf{F}$, the infinitesimal generators $\overset{\circ}{\mathbf{P}}$, which are vectors of tangent space, transform quadratically according to $\overset{\circ}{\mathbf{P}} \rightarrow \mathbf{G}\overset{\circ}{\mathbf{P}}\mathbf{G}^{-1}$, just like vectors in SU(2).[19] With other types of polarization dynamics $\mathbf{O}(t)$ we would have the same scheme: $\mathbf{O}(\varOmega t) \rightarrow \mathbf{G}\mathbf{O}(\varOmega t)$, $\overset{\circ}{\mathbf{O}} \rightarrow \mathbf{G}\overset{\circ}{\mathbf{O}}\mathbf{G}^{-1}$, $\frac{d}{dt}\mathbf{O}(\varOmega t) = \imath\varOmega\overset{\circ}{\mathbf{O}}\,\mathbf{O}(\varOmega t)$. We see thus that SU(3) acts linearly on the "vectors" (tensors $\mathbf{F}$) of physical space while it acts quadratically on the vectors of tangent space. This suggests that we could interpret SU(3) in terms of polarization modes of fields within particles. The Lie algebra is then a generalized angular-momentum algebra.[20]

7 Appendix - Parameter sets for SO(n) and SU(n)

7.1 SO(n)

We have explained in Subsection 2.3 that we can obtain a matrix of SU(n) or of SO(n) by taking the basis vectors of a given basis as its columns. Parameter sets for SO(n) and SU(n) can thus be constructed by constructing a basis for $\mathbb{R}^n$

[19] As explained in reference [1], there are two types of vectors that transform quadratically in SU(2): *viz.* the vectors of $\mathbb{R}^3$ and the vectors of the three-dimensional tangent space to the group manifold (the infinitesimal generators, which are here proportional to the angular momentum operators). While one could confuse them in SU(2) because they look algebraically identical, they can no longer be confused in other groups. E.g. the tangent space to the Lorentz group is six-dimensional, while its physical space is four-dimensional. The Lorentz group acts quadratically on the four-dimensional vectors of space time, but also on the six-dimensional vectors of its tangent space. Hence all groups act like a spinor formalism on tangent space, transforming the tangent vectors "quadratically", while they may not act as a spinor formalism on the vectors of physical space and transform them linearly. Examples exhibiting such a linear behaviour would be SO(3) en SU(3).

[20] Gell-Mann's construction uses ladder operators to construct various representations of the group. Each particle in the multiplet corresponds to a representation. Quantization is counterintuive as we are used to quantities that are continuous in physical space. In the case of angular momentum we get the impression that the directions of physical space are not a continuum. This is all the more puzzling as in the theory this discreteness is derived from the group theory of a continuous group, which makes it look as though the theory contains a contradiction. The answer is that quantization is wired into the mathematics by associating stable states with group representations. Why this is so is not laid down in the mathematics. But once we accept the principle, quantization corresponds to the fact that the representations of finite rank of a group are forcedly discrete and thus "quantized". We can also construct these representations in physical space by making tensor products of representations as explained in [2]. The ranks of these tensor products are forcedly integer numbers and thus quantized. In the construction of the multiplets we encounter exactly the same phenomenology of quantization as with angular momentum. We start from a group SU(3) that is continuous and end up with a discontinous set of members of a multiplet. The discontinuity seems to contradict the continuity we started from. Note that the quantum numbers also exist if we do not multiply them by $\hbar$.

or $\mathbb{C}^n$ by the Gram-Schmidt procedure. The difference between SU(n) and SO(n) resides in the vector spaces they are acting on and on the definition we use for the scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in these vector spaces. For SO(n) we have $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$ and we use the Euclidean scalar product $\mathbf{a} \cdot \mathbf{b}$. For SU($n$) we have $\mathbf{a} \in \mathbb{C}^n$ and $\mathbf{b} \in \mathbb{C}^n$ and we use the Hermitian scalar product $< \mathbf{a}, \mathbf{b} >= \mathbf{a}^*{\cdot}\mathbf{b}$. We will explain here how to construct the corresponding bases. Let us note the canonical basis $\mathbf{e}_j, j \in [1,n] \cap \mathbb{N}$. We are going to define an arbitrarily oriented new basis. The parameters we will use to define this new basis will become the free parameters that define the SO(n) matrix. We will do this iteratively in steps. The first step will be to define a new orthonormal basis $\boldsymbol{\eta}_j, j \in [1,n] \cap \mathbb{N}$ for $\mathbb{R}^n$. But this will only be the first step of the iteration procedure. To highlight the iterative character of the procedure, we will note $\mathbf{e}_j, j \in [1,n] \cap \mathbb{N}$ as $\mathbf{e}_j^{(0)}, j \in [1,n] \cap \mathbb{N}$. The superscript 0 is here an index for the step in the iterative procdure. We will thus similarly note $\boldsymbol{\eta}_j, j \in [1,n] \cap \mathbb{N}$ as $\boldsymbol{\eta}_j^{(1)}, j \in \mathbb{R}^n$. We will also note $\mathbb{R}^n$ as V_n. We observe how the spherical coordinates in $\mathbb{R}^3$ are defined by extending the definition of polar coordinates in $\mathbb{R}^2$ to $\mathbb{R}^3$ by following the scheme:

$$\mathbf{e}_n \to \mathbf{e}_{n+1} = \begin{bmatrix} \cos\phi_n \\ \sin\phi_n\,[\,\mathbf{e}_n\,] \end{bmatrix}. \tag{55}$$

From $|\mathbf{e}_n| = 1$ it follows then also $|\mathbf{e}_{n+1}| = 1$. We can extrapolate this extension scheme to $\mathbb{R}^n$ to obtain hyperspherical coordinates. We use $R = 1$ all the time:

$$\boxed{\begin{array}{lcl} x &=& \cos\phi \\ y &=& \sin\phi \end{array}}\,, \quad \boxed{\begin{array}{lcl} x &=& \sin\theta\cos\phi \\ y &=& \sin\theta\sin\phi \\ z &=& \cos\theta \end{array}}\,, \quad \cdots \quad \boxed{\begin{array}{lcl} x_n &=& \cos\phi_{n-1} \\ x_{n-1} &=& \sin\phi_{n-1}\cos\phi_{n-2} \\ x_{n-2} &=& \sin\phi_{n-1}\sin\phi_{n-2}\cos\phi_{n-3} \\ &\vdots& \\ x_2 &=& \sin\phi_{n-1}\sin\phi_{n-2}\sin\phi_{n-3}\cdots\sin\phi_2\sin\phi_1 \\ x_1 &=& \sin\phi_{n-1}\sin\phi_{n-2}\sin\phi_{n-3}\cdots\sin\phi_2\cos\phi_1 \end{array}}\,. \tag{56}$$

We must then redub (ϕ, θ) as (ϕ_1, ϕ_2) to make their notation suitable for use in a general iterative scheme. We have thus $n-1$ free parameters $\phi_1, \phi_2, \cdots \phi_{n-1}$ to choose a first basis vector:

$$\boldsymbol{\eta}_1^{(1)} = \begin{bmatrix} \cos\phi_{n-1} \\ \sin\phi_{n-1}\cos\phi_{n-2} \\ \sin\phi_{n-1}\sin\phi_{n-2}\cos\phi_{n-3} \\ \vdots \\ \sin\phi_{n-1}\sin\phi_{n-2}\sin\phi_{n-3}\cdots\sin\phi_2\sin\phi_1 \\ \sin\phi_{n-1}\sin\phi_{n-2}\sin\phi_{n-3}\cdots\sin\phi_2\cos\phi_1 \end{bmatrix}. \tag{57}$$

The parameters $\phi_1, \phi_2, \cdots \phi_{n-1}$ are curvilinear coordinates for the hypersphere. They satisfy the constraint $\boldsymbol{\eta}_1^{(1)}{\cdot}\boldsymbol{\eta}_1^{(1)} = 1$ automatically. The $n-1$ vectors $\boldsymbol{\eta}_{j+1}^{(1)} = \frac{\partial}{\partial\phi_j}[\,\boldsymbol{\eta}_1^{(1)}\,], j \in [1, n-1] \cap \mathbb{N}$ satisfy then automatically the conditions $\boldsymbol{\eta}_j^{(1)}{\cdot}\boldsymbol{\eta}_1^{(1)} = 0$. This identity follows from $\frac{\partial}{\partial\phi_j}[\,\boldsymbol{\eta}_1^{(1)}{\cdot}\boldsymbol{\eta}_1^{(1)}\,] = 0$. The vectors $\boldsymbol{\eta}_{j+1}^{(1)} = \frac{\partial}{\partial\phi_j}[\,\boldsymbol{\eta}_1^{(1)}\,], j \in [1, n-1] \cap \mathbb{N}$ do not yet satisfy the condition $\boldsymbol{\eta}_j^{(1)}{\cdot}\boldsymbol{\eta}_j^{(1)} = 1$, but the normalization factor is a trivial common factor that occurs in the non-zero entries. We continue to use the notation $\boldsymbol{\eta}_{j+1}^{(1)}, j \in [1, n-1] \cap \mathbb{N}$ for the vectors obtained after renormalization. These $n-1$ vectors span now an orthonormal basis for the $(n-1)$-dimensional subspace V_{n-1} of $V_n \equiv \mathbb{R}^n$ that is orthogonal to $\boldsymbol{\eta}_1^{(1)}$. We keep from now on $\boldsymbol{\eta}_1^{(1)}$ with its $n-1$ free parameters $\phi_1, \phi_2, \cdots \phi_{n-1}$ fixed. But we will reorient the basis $\boldsymbol{\eta}_1^{(1)}, j = [2, n] \cap \mathbb{N}$ within the subspace V_{n-1}. We can consider thus the $n-1$ parameters $\phi_1, \phi_2, \cdots \phi_{n-1}$ as being used to select an orientation of the subspace V_{n-1} within V_n. The subspace V_{n-1} is a hyperplane and $\phi_1, \phi_2, \cdots \phi_{n-1}$ define the normal to the hyperplane. Within the $(n-1)$-dimensional subspace V_{n-1} itself we can reorient the basis at will, just like we reoriented the n basis vectors $\boldsymbol{\eta}_j^{(1)}, j \in [1,n] \cap \mathbb{N}$ in $\mathbb{R}^n = V_n$ with respect to $\mathbf{e}_j^{(0)}, j \in [1,n] \cap \mathbb{N}$ at will.

We therefore consider them as a new canonical basis $\mathbf{e}_j^{(1)}, j \in [1, n-1] \cap \mathbb{N}$ of $V_{n-1} \equiv \mathbb{R}^{n-1}$. With respect to this basis we introduce now second-level hyperspherical coordinates $\theta_1, \theta_2, \cdots \theta_{n-2}$:

$$\boldsymbol{\eta}_1^{(2)} = \begin{bmatrix} \cos\theta_{n-2} \\ \sin\theta_{n-2}\cos\theta_{n-3} \\ \sin\theta_{n-2}\sin\theta_{n-3}\cos\theta_{n-4} \\ \vdots \\ \sin\theta_{n-2}\sin\theta_{n-3}\sin\theta_{n-4}\cdots\sin\theta_2\sin\theta_1 \\ \sin\theta_{n-2}\sin\theta_{n-3}\sin\theta_{n-4}\cdots\sin\theta_2\cos\theta_1 \end{bmatrix}. \tag{58}$$

They constitute $n-2$ new free parameters to define the *Vielbein*. Again, for the sake of generality, we must change the nomenclature and redub $\phi_1, \phi_2, \cdots \phi_{n-1}$ as $\phi_1^{(1)}, \phi_2^{(1)}, \cdots \phi_{n-1}^{(1)}$ and $\theta_1, \theta_2, \cdots \theta_{n-2}$ as $\phi_1^{(2)}, \phi_2^{(2)}, \cdots \phi_{n-2}^{(2)}$. We could have introduced these notations right from the start, but this would certainly have looked puzzling at that stage of the development. We can now complete the basis for $\mathbb{R}^{n-1}$ by introducing $\boldsymbol{\eta}_{j+1}^{(2)} = \frac{\partial}{\partial\theta_j}[\,\boldsymbol{\eta}_1^{(2)}\,], j \in [1, n-2] \cap \mathbb{N}$. We can now carry on with $\boldsymbol{\eta}_j^{(2)}, j \in [2, n-1] \cap \mathbb{N}$ in a way completely analogous to what we did with $\boldsymbol{\eta}_j^{(1)}, j \in [2, n] \cap \mathbb{N}$. That is, we have used $\phi_1^{(2)}, \phi_2^{(2)}, \cdots \phi_{n-2}^{(2)}$ to define an orientation of a subspace V_{n-2} within V_{n-1}. Within this subspace itself $\boldsymbol{\eta}_j^{(2)}, j \in [2, n-1] \cap \mathbb{N}$ define the new canonical basis $\mathbf{e}_j^{(2)}, j \in [1, n-2] \cap \mathbb{N}$. This way we obtain iterative scheme of spaces V_n, V_{n-1}, $\cdots V_1$, whereby we introduce progressively $n + (n-1) + \cdots + 2 + 1$ free parameters to define the orientations of the subspaces V_{n-j-1} within the subspaces V_{n-j}. We construct this way a basis $\boldsymbol{\eta}_1^{(k)}, k \in [1, n] \cap \mathbb{N}$ for $\mathbb{R}^n$. It is defined by $n(n-1)/2$ free real parameters which define the complete orientation of a general *Vielbein*. The *Vielbein* stands in one-to-one correspondence with the SO(n) matrix that has to be applied to the canonical *Vielbein* to obtain it. It is this matrix which transforms $\mathbf{e}_j^{(0)}, j \in [1, n] \cap \mathbb{N}$ into $\boldsymbol{\eta}_1^{(k)}, k \in [1, n] \cap \mathbb{N}$. The basis vectors $\boldsymbol{\eta}_1^{(k)}, k \in [1, n] \cap \mathbb{N}$ of the *Vielbein* are the columns of the SO(n) matrix.

7.2 SU(n)

We start now from:

$$\boldsymbol{\eta}_1^{(1)} = \begin{bmatrix} e^{\imath\chi_1^{(1)}} & \cos\phi_{n-1}^{(1)} \\ e^{\imath\chi_2^{(1)}} & \sin\phi_{n-1}^{(1)}\cos\phi_{n-2}^{(1)} \\ e^{\imath\chi_3^{(1)}} & \sin\phi_{n-1}^{(1)}\sin\phi_{n-2}^{(1)}\cos\phi_{n-3}^{(1)} \\ \vdots & \\ e^{\imath\chi_{n-1}^{(1)}} & \sin\phi_{n-1}^{(1)}\sin\phi_{n-2}^{(1)}\sin\phi_{n-3}^{(1)}\cdots\sin\phi_2^{(1)}\sin\phi_1^{(1)} \\ e^{\imath\chi_n^{(1)}} & \sin\phi_{n-1}^{(1)}\sin\phi_{n-2}^{(1)}\sin\phi_{n-3}^{(1)}\cdots\sin\phi_2^{(1)}\cos\phi_1^{(1)} \end{bmatrix}. \tag{59}$$

The phase parameters can be choosen at will. They do not affect the normalization of $\boldsymbol{\eta}_1^{(1)}$. We have now $2n-1$ free parameters, *viz.* the $n-1$ parameters from SO(n) and n phases. We must now consider a subspace V_{2n-3} of V_{2n-1}. Here the dimensions $2n-1$ and $2n-3$ are defined with respect to $\mathbb{R}$. We can complete the column vector $\boldsymbol{\eta}_1^{(1)}$ with $n-1$ other column vectors, which will depend on $2n-3$ free real parameters, scattered over $n-1$ column matrices. This can be done by using $e^{\imath\chi_j^{(2)}}\frac{\partial}{\partial\phi_j^{(1)}}[\,\boldsymbol{\eta}_1^{(1)}\,]$. That is, we first consider $n-1$ vectors of $\mathbb{R}^n$ obtained from $\boldsymbol{\eta}_1^{(1)}$ by partial derivation with respect to $\phi_j^{(1)}, j \in [1, n-1] \cap \mathbb{N}$ to constitute $n-1$ columns. We will be able to reorient these column vectors in $\mathbb{R}^{n-1}$ by defining $n-2$ free parameters $\phi_j^{(2)}, j \in [1, n-2] \cap \mathbb{N}$. To these $n-2$ parameters we can add $n-1$ phases $\chi_j^{(2)}, j \in [1, n-1] \cap \mathbb{N}$. We have then $2n-3$ free real parameters. The chain of spaces is now V_{2n-1}, $V_{2n-3}\cdots 1$. However, the last parameter must be used to render the determinant of the matrix equal to 1. The total number of free parameters is thus $(2n+1) + (2n-3) + \cdots + 5 + 3 = n^2 - 1$.

7.3 Application to SU(3) to show the limits of an abstract approach

By using this procedure we can derive the following parameterization for a matrix $\mathbf{M}$ of SU(3):

$$\mathbf{M} = \left[\begin{array}{ll} e^{\imath\chi_1^{(1)}}\cos\phi_2^{(1)} & e^{\imath\chi_1^{(1)}}[-e^{\imath\chi_1^{(2)}}\sin\phi_2^{(1)}\cos\phi_1^{(2)}] \\ e^{\imath\chi_2^{(1)}}\sin\phi_2^{(1)}\cos\phi_1^{(1)} & e^{\imath\chi_2^{(1)}}[\;e^{\imath\chi_1^{(2)}}\cos\phi_2^{(1)}\cos\phi_1^{(1)}\cos\phi_1^{(2)} - e^{\imath\chi_2^{(2)}}\sin\phi_1^{(1)}\sin\phi_1^{(2)}] \\ e^{\imath\chi_3^{(1)}}\sin\phi_2^{(1)}\sin\phi_1^{(1)} & e^{\imath\chi_3^{(1)}}[\;e^{\imath\chi_1^{(2)}}\cos\phi_2^{(1)}\sin\phi_1^{(1)}\cos\phi_1^{(2)} + e^{\imath\chi_2^{(2)}}\cos\phi_1^{(1)}\sin\phi_1^{(2)}] \end{array}\right. \cdots \tag{60}$$

$$\left.\begin{array}{l} e^{-\imath(\chi_2^{(1)}+\chi_3^{(1)})}[\;e^{-\imath\chi_2^{(2)}}\sin\phi_2^{(1)}\sin\phi_1^{(2)}] \\ e^{-\imath(\chi_1^{(1)}+\chi_3^{(1)})}[-e^{-\imath\chi_2^{(2)}}\cos\phi_2^{(1)}\cos\phi_1^{(1)}\sin\phi_1^{(2)} - e^{-\imath\chi_1^{(2)}}\sin\phi_1^{(1)}\cos\phi_1^{(2)}] \\ e^{-\imath(\chi_1^{(1)}+\chi_2^{(1)})}[-e^{-\imath\chi_2^{(2)}}\cos\phi_2^{(1)}\sin\phi_1^{(1)}\sin\phi_1^{(2)} + e^{-\imath\chi_1^{(2)}}\cos\phi_1^{(1)}\cos\phi_1^{(2)}] \end{array}\right],$$

This requires no insight. All it takes is following the algebraic procedure outlined above, without caring about the geometry. It is completely abstract and we do not need to know what it means. But as the reader may check, within such an abstract approach deriving the infinitesimal generators by bluntly taking the partial derivatives with respect to all parameters will not yield a full set of infinitesimal generators. It is not obvious what kind of one-parameter sets one must choose. What he needs to understand is that what he wants is an orthonormal basis for tangent space to the group manifold. This basis must be defined in the space of linear operators L($\mathbb{C}^3$,$\mathbb{C}^3$) from $\mathbb{C}^3$ to $\mathbb{C}^3$. The orthogonality must thus also be defined, not in $\mathbb{C}^3$, but in the space L($\mathbb{C}^3$,$\mathbb{C}^3$). The scalar product of two matrices **A** and **B** in L($\mathbb{C}^n$,$\mathbb{C}^n$) is defined by:

$$< \mathbf{A}, \mathbf{B} > = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk}^* b_{jk} = \mathrm{Tr}\,(\mathbf{A}^\dagger \mathbf{B}). \tag{61}$$

Here Tr stands for the trace. The first identity is just based on the fact that the matrices $\mathbf{E}_{jk}$, with $j \in [1,n] \cap \mathbb{N}, k \in [1,n] \cap \mathbb{N}$, which are defined by $(\mathbf{E}_{jk})_{\ell,m} = \delta_{j\ell}\delta_{km}$, are a basis for L($\mathbb{C}^n$,$\mathbb{C}^n$). The second identity follows by straightforward calculation from the first one. Once we have an orthonormal basis for the tangent space, any other orthonormal basis of the tangent space obtained by an appropriate change of basis from the first one will be as good a choice as the initial one. It is for this reason that it is quite practical to shortcut the parametrization procedure and to derive an abstract expression for the infinitesimal generators right ahead from a few algebraic rules as is done in textbooks. However one may wonder then how all this would work according to the prescription of Eq. 18 and why all at once one does no longer use this prescription anymore. We also have no longer any clue as to what the defining parameters are. We do no longer know the full details of the relationship between the Lie algebra and the Lie group. Very specifically, we do not know which parameter has been varied within the Lie group according to the method based on Eq. 18 in order to obtain a given infinitesimal generator that has been obtained by the abstract methods. We also do not know how the abstract basis is oriented with respect to the intuitive basis that we could derive by using the prescription from Eq. 18. This is all unnecessarily puzzling.

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