# Integrability of W( sl d )-symmetric Toda conformal field theories I : Quantum geometry

Raphaël Belliard, Bertrand Eynard

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# Integrability of $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric Toda conformal field theories I : Quantum geometry

Raphaël Belliard[1], Bertrand Eynard[2,3],

[1] Deutsches Elektronen-Synchrotron Theorie Group,
Notkestrasse 85, 22607 Hamburg, Germany
[2] Institut de Physique Théorique, Université Paris Saclay,
CEA, CNRS, F-91191 Gif-sur-Yvette, France
[3] Centre de Recherches Mathématiques, Université de Montréal,
2920, Chemin de la tour, 5357 Montréal, Canada.

**Abstract**

In this article which is the first of a series of two, we consider $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric conformal field theory in topological regimes for a generic value of the background charge, where $\mathcal{W}(\widehat{\mathfrak{sl}_d})$ is the W-algebra associated to the affine Lie algebra $\widehat{\mathfrak{sl}}_d$. In such regimes, the theory admits a free field representation. We show that the generalized Ward identities assumed to be satisfied by chiral conformal blocks with current insertions can be solved perturbatively in topological regimes. This resolution uses a generalization of the topological recursion to non-commutative, or quantum, spectral curves. In turn, special geometry arguments yields a conjecture for the perturbative reconstruction of a particular chiral block.

# Contents

**1 Lightning review of conformal field theory** **2**

**2 W-algebras and associated conformal field theories** **9**

2.1 From Virasoro to W-algebras . . . . . . . . . . 9
2.2 Operator product expansions . . . . . . . . . . 10
2.3 W-algebra generators . . . . . . . . . . 12
2.4 Ward identities . . . . . . . . . . 14
2.5 Classical limit $Q \longrightarrow 0$ and quantization . . . . . . . . . . 15

**3 Quantum geometry** **16**
3.1 Topological regime and quantum spectral curve . . . . . . . . . . . . . . . 16
3.2 Fermionic description and notion of sheets . . . . . . . . . . . . . . . . . 18
3.3 2-points function . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20

**4 Topological recursion** **21**
4.1 Ward identities in the $\varepsilon \longrightarrow 0$ expansion . . . . . . . . . . . . . . . . . 21
4.2 Bethe roots and kernel . . . . . . . . . . . . . . . . . . . . . . . . . . 22
4.3 Recursion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 23
4.4 Special geometry and free energies . . . . . . . . . . . . . . . . . . . . . 25

**5 Conclusion** **28**

**6 Appendix** **28**
6.1 Loop insertion operator . . . . . . . . . . . . . . . . . . . . . . . . . . 28
6.2 Differential Hirota identities . . . . . . . . . . . . . . . . . . . . . . . 29
6.3 $K_\mu$ has no monodromy around $S_\mu$ . . . . . . . . . . . . . . . . . . . . . 30

# 1 Lightning review of conformal field theory

Conformal field theories in two dimensions have appeared in the physics litterature as powerful tools to study numerous systems, from critical (possibly quantum) statistical models in two dimensions (in some thermodynamic limit) to the worldsheet conformal symmetry of string theories [2]. They have the particularity to exhibit infinite dimensional conformal algebras of symmetries, namely extensions of the Virasoro algebra (that are not necessarily Lie algebras as we shall see). It was in turn argued in some cases [1] that there exists an underlying structure of quantum integrable system with commuting transfer matrices and such.

Mathematically speaking, all these constructions assume the existence of a certain set of functions called $M$-point correlation functions, for some integer $M \in \mathbb{N}^*$, defined on $M$ copies of a given connected Riemann surface $\Sigma$, and denoted formally as

$$\left\langle \prod_{j=1}^{M} \Phi_{\alpha_j,\overline{\alpha}_j}(p_j) \right\rangle \tag{1-1}$$

for distinct generic points $p_1, \ldots, p_M \in \Sigma$ (the correlation function is not analytic in those points), called the punctures, to which are associated labels $\alpha_1, \overline{\alpha}_1, \ldots, \alpha_M, \overline{\alpha}_M$, called the charges, in the linear dual $\mathfrak{h}^*$ of a Cartan subalgebra $\mathfrak{h}$ of the considered reductive complex Lie algebra $\mathfrak{g}$. They are moreover assumed to be smooth on the generic locus of $\Sigma^M$ and to satisfy a set of axioms, written here for our purpose.

- *Axiom 1 : Holomorphic factorization.* For any $M \in \mathbb{N}^*$, much like a Hodge decomposition, or a separation of variables, there exists a sequence of objects called conformal blocks $\{\mathcal{F}_\gamma\}_{\gamma \in B_\mathcal{G}}$ such that

$$\left\langle \prod_{j=1}^{M} \Phi_{\alpha_j,\overline{\alpha}_j}(p_j) \right\rangle = \sum_{\gamma,\gamma' \in B_\mathcal{G}} C^{\gamma,\gamma'} \mathcal{F}_\gamma(\mathbf{z})\, \mathcal{F}_{\gamma'}(\overline{\mathbf{z}}) \tag{1-2}$$

  where we introduced the vector notation $\mathbf{z} = (z_1, \dots, z_M)$. $B_\mathcal{G}$ is the set of labels parametrizing this basis of conformal blocks and it contains in particular the data of $\mathcal{G}$, a channel, namely a certain choice of unicellular trivalent graph on the considered Riemann surface satisfying $\partial\mathcal{G} = \{z_1, \dots, z_M\}$ and $\pi_1(\Sigma - \mathcal{G}, o) = 0$ with respect to a chosen reference point $o \in \Sigma$. The label $\gamma$ (resp. $\gamma'$) contains furthermore the data of the so-called charges $(\alpha_1, \dots, \alpha_M)$ (resp. $(\overline{\alpha}_1, \dots, \overline{\alpha}_M)$). This axiom allows to reduce the problem to its holomorphic (often called chiral) and anti-holomorphic (anti-chiral) parts.

In Physics, one wishes the correlation functions that are reconstructed in this way to be modular invariant and this constrains admissible root geometries for the Lie algebras. It is known that in the case of $\mathfrak{g} = \mathfrak{sl}_2$, modular invariant partition function are classified by simply laced (ADE) Dynkin diagrams but such a statement does not exist for higher rank Lie algebras.

In this chapter we will be interested solely in studying chiral, or holomorphic, conformal blocks denoted $\mathcal{F}_\gamma(\mathbf{z}) = \left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j) \right\rangle$, where we introduced the vertex operators $V_{\alpha_i}$ merely as a notation (although the vertex operator formalism [13] is the right way to give a precise meaning to the bracket $\langle \cdot \rangle$ and operator product expansions to come). To do so (and as is customary, in quantum mechanics, to mimic the interaction of an observer with the system) we introduce a probe, a so-called chiral spin-one current $\mathbf{J}(\widetilde{x})$ valued in the dual $\mathfrak{g}^*$ of the Lie algebra and defined for points $\widetilde{x} \in \widetilde{\Sigma}$ in the universal cover $\widetilde{\Sigma} \longrightarrow \Sigma$. It can be seen as being multivalued on $\Sigma$ and generically having irregular singularities at the $z_j$'s. From the perspective of Toda quantum field theory, this relates to the fact that we are only focusing on a topological heavy limit for the considered charges.

In this quantum theory, the vertex operators are interpreted as the matter content with which the current interacts. This interaction is such that to configurations of points on the Riemann surface, where the operators and currents are inserted, are associated correlations, describing the entanglement of the particles.

The following axioms are analytic and algebraic requirements these correlations should satisfy as functions of these configurations of points.

Let us fix once and for all the Lie algebra we consider to be $\mathfrak{g} \underset{def}{=} \mathfrak{sl}_d$ and choose a set of simple roots $\mathfrak{R}_0 \underset{def}{=} \{\mathfrak{e}_1, \dots, \mathfrak{e}_{d-1}\}$. We will denote by $\mathfrak{R}_+$ (resp. $\mathfrak{R}_-$) the corresponding set of positive (resp. negative) roots. Introducing the *minimal* invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}^* \times \mathfrak{g}^*$ (giving length 2 to simple roots), let us consider the algebra generated by a central element $K$ together with the harmonics $(\mathbf{J}^{(n)})_{n\in\mathbb{Z}}$, or modes, obtained by decomposing the chiral current around any generic point $\widetilde{x}_0 \in \widetilde{\Sigma}$ (with local coordinate $t = x - x_0$) as

$$\mathbf{J}(\widetilde{x}) \underset{def}{=} \sum_{n\in\mathbb{Z}} \frac{\mathbf{J}^{(n)}(\widetilde{x}_0)}{(x - x_0)^{n+1}} \tag{1-3}$$

**Remark 1.1** *We will drop the explicit writing of the dependence of the modes in the generic point $x_0$ when no confusion is possible. Modes can only be compared when evaluated at the same point.*

These generators satisfy the commutation relations

$$[\mathbf{J}^{(n)}, \mathbf{J}^{(m)}] = [\mathbf{J}, \mathbf{J}]^{(n+m)} + (\mathbf{J}^{(n)}, \mathbf{J}^{(m)})\delta_{n+m,0}K \tag{1-4}$$

where the $\mathbf{J}$ symbols in $[\mathbf{J}, \mathbf{J}]$ are generically evaluated at different Lie algebra elements and therefore have non-trivial Lie bracket. The so-called affine Kac-Moody algebra at level $\kappa \in \mathbb{C}$ denoted $\widehat{\mathfrak{g}}_\kappa$ is then defined as the Lie algebra $\widehat{\mathfrak{g}}_\kappa \underset{def}{=} \widehat{\mathfrak{g}}\big/(\kappa - K)$, where $\widehat{\mathfrak{g}}$, called the generic affine Kac-Moody algebra associated to $\mathfrak{g}$, is defined as the central extension of vector spaces

$$0 \longrightarrow \mathbb{C}K \longrightarrow \widehat{\mathfrak{g}}_\kappa \longrightarrow \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\partial \longrightarrow 0 \tag{1-5}$$

where $\mathcal{L}(\mathfrak{g})$ is the *loop algebra* of $\mathfrak{g}$ denoted $\mathcal{L}(\mathfrak{g}) \underset{def}{=} \mathfrak{g}((t))$ (endowed with the natural Lie algebra structure coming from $\mathfrak{g}$) and the extra generator $\partial$ is defined to satisfy

$$[\partial, M] = \frac{\mathrm{d}}{\mathrm{dt}}M, \quad \text{(and thus } [\partial, K] = 0) \tag{1-6}$$

for any $M \in \mathcal{L}(\mathfrak{g}) = \mathfrak{g}((t))$. $K$ is a central element assumed to act trivially as multiplication by $\kappa$ (using the fact that the short exact sequence splits, this is equivalent to focusing on diagonalizable $\widehat{\mathfrak{g}}$-module with finite weight spaces [28] and restricting ourselves to the highest weight representations they define). The levels $\kappa$ are in general not constrained but they are for example in the case where the conformal field theory can be extended to a certain three dimension topological field theory named Chern-Simons theory on a given 3-manifold $M$ whose boundary $\partial M = \Sigma$ is the Riemann surface. Then the levels are often required to make two copies of Chern-Simons on $M$ equivalent if they yield the same conformal field theory on $\partial M$. They are then parametrized by maps $M_\Sigma M \longrightarrow G$ considered up to homotopy, where $M_\Sigma M$ denotes gluing of the copies of $M$ along their identical boundary $\Sigma$ with matching of orientations. In the case where $G = SU(2)$, $\Sigma$ is the Riemann sphere and $M \subset \mathbb{R}^3$ is the unit ball, since gluing in this case yields a 3-sphere, the levels of the Kac-Moody algebras of interest are then parametrized by the third homotopy group given by $\pi_3(SU(2)) \simeq \mathbb{Z}$. Constraints can also arise from the representation theory of the Kac-Moody algebra, indeed, when the levels under consideration are positive integers, $\widehat{\mathfrak{g}}$ admits unitary highest weight representations whose highest weights are dominant integral (quantization condition).

**Definition 1.1** *Insertions of currents*
*By insertions of currents into chiral correlation functions we mean that we consider an infinite yet countable set of additionnal* $(\mathfrak{g}^*)^{\otimes n}$*-valued functions of interest denoted*

$$\left\langle\left\langle \mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\right\rangle\right\rangle \underset{def}{=} \frac{\left\langle \mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\prod_{j=1}^M V_{\alpha_j}(z_j)\right\rangle}{\left\langle \prod_{j=1}^M V_{\alpha_j}(z_j)\right\rangle} \tag{1-7}$$

This will allow us to define our algebra of symmetries $\mathcal{W}(\widehat{\mathfrak{sl}_d})$ by requiring some closure condition under Lie bracket. It takes the form of some prescriptions for the singular behaviour the obtained functions are assumed to exhibit near the divisor where two of the points at which currents are inserted have base point projections that come together, or when one of them goes to one of the punctures $z_1, \ldots, z_M$ (where the *vertex operators* are located).

- *Axiom 2 : Operator product expansion.*

  Keeping the notation $(\cdot,\cdot)$ for the form on $\mathfrak{g}\times\mathfrak{g}$ dual to minimal invariant bilinear form on $\mathfrak{g}^* \times \mathfrak{g}^*$,

$$\mathbf{J}(\widetilde{x}\cdot E)\mathbf{J}(\widetilde{y}\cdot F) \underset{x\sim y}{=} -\kappa\frac{(E,F)}{(x-y)^2} + \frac{\mathbf{J}(\widetilde{y}\cdot[E,F])}{x-y} + \,:\mathbf{J}(\widetilde{x}\cdot E)\mathbf{J}(\widetilde{y}\cdot F)\underset{x=y}{:} + \mathcal{O}(x-y) \tag{1-8}$$

$$\mathbf{J}(\widetilde{x}\cdot E)\,V_{\alpha_j}(z_j) \underset{x\sim z_j}{=} b\,\frac{\alpha_j(E)}{x-z_j}V_{\alpha_j}(z_j) + :\mathbf{J}(\widetilde{x}\cdot E)\,V_{\alpha_j}(z_j)\underset{x=z_j}{:} + \mathcal{O}(x-z_j) \tag{1-9}$$

for $\widetilde{x},\widetilde{y}\in\widetilde{\Sigma}$, with some Lie algebra elements $E,F\in\mathfrak{g}$ and some puncture index $j\in\{1,\dots,M\}$. We also introduced the parameter $b\in\mathbb{C}^*$ and the linear notation $\mathbf{J}(\widetilde{x}\cdot E)\underset{def}{=}\mathbf{J}(\widetilde{x})(E)$ to relate with our notations in the study of Fuchsian differential systems for the evaluation [4], [6]. We denote the normal ordering operation as $:A(\widetilde{x})B(\widetilde{y})\underset{x=y}{:}$ defined as the next to singular term when the basepoints of $\widetilde{x},\widetilde{y}\in\widetilde{\Sigma}$ come together (but are not necessarily such that $\widetilde{x}=\widetilde{y}$). The last asymptotic equality defines $V_{\alpha_j}$ as a primary field, that is an eigenvector of the zero mode $\mathbf{J}^{(0)}$ that is anihilated by all positive modes $\mathbf{J}^{(n)}$, $n>0$, and moreover, the presence of a simple pole means that we consider only regular singularities. Irregular singularities in the $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})$ case of Liouville conformal field theory were studied in [27].

**Definition 1.2** *Background charge*
*Define the background charge as the number* $Q\underset{def}{=}b+b^{-1}$

These asymptotic relations are to be understood as holding when inserted into correlation functions, that is to say that they are meromorphic conditions on the functions we denoted $\langle\langle\mathbf{J}\cdot\dots\cdot\mathbf{J}\rangle\rangle$. They are strong requirements as $\mathbf{J}$ contains for example both data of the Lie bracket and the minimal invariant bilinear form on $\mathfrak{g}$.

Recall that the Virasoro algebra $Vir$ is the infinite dimensional Lie algebra that generates the conformal transformations of the complex plane. It is defined as the central extension of vector spaces

$$0\longrightarrow\mathbb{C}c\longrightarrow Vir\longrightarrow\mathrm{Der}_{\mathbb{C}}\longrightarrow 0 \tag{1-10}$$

where we introduced the Lie algebra $\mathrm{Der}_{\mathbb{C}}$ of holomorphic derivations of the field of Laurent series on the complex plane as well as the central element $c$ ($c$ stands for

Casimir) called the central charge. $Vir$ is generated by $c$ together with the elements $(L_n)_{n\in\mathbb{Z}}$ satisfying the famous commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \quad (n,m)\in\mathbb{Z}^2 \tag{1-11}$$

where for any $n \in \mathbb{Z}$, $L_n$ generates the one-parameter family of local conformal transformations $(z \longmapsto t\, z^{n+1})_{t\in\mathbb{C}}$ e.g. $L_0$ is the dilation operator. It goes to the Witt algebra in the zero central charge limit $c \longrightarrow 0$. The generators of the Virasoro algebra can be gathered into a meromorphic stress-energy tensor (anticipating on the next axiom by denoting the variable by $x$ and not one of its preimages $\widetilde{x}$ by the universal covering map) around a base point $x_0 \in \Sigma$ by

$$T(x) \underset{x\sim x_0}{=} \sum_{n\in\mathbb{Z}} \frac{L_n(x_0)}{(x-x_0)^{n+2}}, \tag{1-12}$$

such that the Virasoro canonical commutation relations are translated in the following *operator product expansion*

$$T(x')T(x) \underset{x'\sim x}{=} \frac{c/2}{(x'-x)^4} + \frac{2\,T(x)}{(x'-x)^2} + \frac{\partial T(x)}{x'-x} + \mathcal{O}(1) \tag{1-13}$$

and similarly, the operator product expansion of the stress-energy tensor with the chiral current is defined to be

$$T(x)\mathbf{J}(\widetilde{y}) \underset{x\sim y}{=} \frac{\mathbf{Q}}{(x-y)^3} + \frac{\mathbf{J}(\widetilde{y})}{(x-y)^2} + \frac{\partial\mathbf{J}(\widetilde{y})}{x-y} + \mathcal{O}(1) \tag{1-14}$$

where $\mathbf{Q} \underset{def}{=} Q\rho$, $\rho \underset{def}{=} \frac{1}{2}\sum_{\mathfrak{r}\in\mathfrak{R}_+}\mathfrak{r}$ being the Weyl vector, to be again understood as identities holding when inserted into correlation functions. The coefficient 1 in front of the second order pole in the last expression tells us that the current $\mathbf{J}$ has spin (conformal weight) 1.

**Remark 1.2** *One might be afraid that such a decomposition for the stress-energy tensor would create singularities of infinite order in some operator product expansion appearing in the theory but a requirement of the vertex operator algebra formalism is that any admissible field $V_\alpha$ should be anihilated by all high enough modes of $T$, see [13] for details. In particular, define an admissible ground state as a vector $|0\rangle \in \mathcal{A}$ in the considered representation satisfying the so-called Virasoro constraints*

$$\forall n \geq -1, \qquad L_n|0\rangle = 0 \tag{1-15}$$

*In particular, if we were to assume that $L_n^\dagger = L_{-n}$, then the Virasoro constraints would yield that the expected value of the stress-energy tensor vanishes*

$$\langle 0|T(x)|0\rangle = 0 \tag{1-16}$$

*namely that we have conformal symmetry in this ground state at the quantum level. We will not however be assuming the existence of such a ground state in our study.*

The next axiom is at the heart of the method we adopt to study conformal field theories. As was mentioned in the introduction, a path integral formulation of the problem with a Lagrangian allows for the derivation of Schwinger-Dyson equations. Their counterparts in this non-perturbative definition of conformal field theories are the following conformal Ward identities.

- *Axiom 3 : Generalized conformal Ward identities.*

  For any generic $\widetilde{x}_1, \cdots, \widetilde{x}_n \in \widetilde{\Sigma}$, $\left\langle\left\langle T(x)\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\right\rangle\right\rangle$ is a holomorphic function of the variable $x \in \Sigma - \{z_1, \ldots, z_M, x_1, \ldots, x_n\}$.

This is (again !) an axiom prescribing some analytic conditions for the functions of interest. We will be applying similar ideas for the generating series of generators of $W_n \underset{def}{=} \mathcal{W}(\widehat{\mathfrak{sl}_d})$ for which the conformal Ward identities together with the operator products expansions yield the so-called loop equations.

The two last axioms deal with how one can reconstruct the full theory from its chiral and anti-chiral parts. We will not be needing them in the context of this work but we still state them for completeness.

- *Axiom 4 : Single-valuedness.*

  The $M$-point correlation functions have no monodromy around cycles in the moduli space of configurations of $M$ distinct points on the Riemann surface $\Sigma$.

- *Axiom 5 : Fusion and crossing symmetries.*

  The decomposition of the real correlation functions in terms of the conformal blocks requires in particular a choice of channel, a unicellular trivalent graph, on the base Riemann surface $\Sigma$ and different choices of such channels should lead to the same correlation function after reconstruction. This is often referred to as the *associativity* of the operator product expansions.

There is no general proof that all these axioms are actually compatible. We will therefore proceed by necessary condition, assuming these axioms to be compatible and satisfied, to define the algebra $\mathcal{W}(\widehat{\mathfrak{sl}_d})$ and the so-called insertions of W-generators in these chiral correlation functions with currents. In turn, this will yield $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric, or Toda, conformal Ward identities. In the second part we will define the associated quantum geometry through the quantum spectral curve. This will turn out to be the initial data needed to run the topological recursion of [23] in this context and we will show that it constructs perturbatively solutions to the $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric conformal Ward identities. In the sequel to this paper we shall exhibit an explicit realization of this framework using $\beta$-deformed two-matrix models.

# 2 W-algebras and associated conformal field theories

## 2.1 From Virasoro to W-algebras

A conformal field theory [33] is a quantum field theory defined on a Riemann surface $\Sigma$ and endowed with an action of the product $\mathcal{A} \times \mathcal{A}'$ of two extensions $Vir \subset A$, $Vir \subset \mathcal{A}'$ of the Virasoro algebra (they need not be the same).

Let us stress at this point that these two extensions $\mathcal{A}$ and $\mathcal{A}'$ act respectively upon the holomorphic and the anti-holomorphic dependence of the observables defined on the Riemann surface. We will here only be interested in the chiral theory, that is in the action of $\mathcal{A}$ and in the meromorphic properties of the soon to be defined chiral correlation functions.

We will be interested particularly in the extension $Vir \subset \mathcal{A} \underset{def}{=} \mathcal{W}(\widehat{\mathfrak{sl}_d})$ defined from the affine Lie algebra $\widehat{\mathfrak{sl}_d}$, using a higher rank generalization of the Sugawara construction [34], namely the quantum Miura transform, and defining generating functions whose modes generate $\mathcal{W}(\widehat{\mathfrak{sl}_d})$. Let us mention that our method is not directly generalizable to general reductive complex Lie algebra since it relies on the explicit expression of the W-generators that the quantum Miura transform yields and that such a definition does not work in the general case where one has to quantize the Poisson algebra underlying the Drinfeld-Sokolov hierarchy associated to the Lie algebra under consideration (equivalent to the quantum Drinfeld-Sokolov reduction) [15],[25].

The idea behind W-algebras is that they allow for a better encoding of some representations of $Vir$. Indeed, there are spaces representing both the W-algebra and the Virasoro algebra that decompose as an infinite direct sums of irreducible representations of $Vir$ but as a finite direct sums of irreducible representations of $\mathcal{W}(\widehat{\mathfrak{sl}_d})$. In particular, the operator product expansions they satisfy should be expressible in terms of these generators only. We will see two different situations in which this is possible but we will not get any further in studying the representation theory of W-algebras and refer the reader the [15], [16].

The definition of the W-algebra generators involves non-commutative geometry and in order for the algebra to close, we will restrict the current $\mathbf{J}$ to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_d$. The choice of level becomes irrelevant as it can be reabsorbed in the definition of $\mathbf{J}$ and $\mathcal{W}(\widehat{\mathfrak{sl}_d})$ then appears as a subalgebra $\mathcal{W}(\widehat{\mathfrak{sl}_d}) \subset \overline{\mathcal{U}}\left(\widehat{\mathfrak{h}}\right)$ of a completion of the universal envelopping algebra of the affine Kac-Moody algebra defined from the commutative Lie algebra $\mathfrak{h}$.

The background charge plays the role of a quantization parameter and noticing that the W-algebra for generic values of $Q$ reduces to a Casimir algebra in the limit $Q \longrightarrow 0$ will allow for the interpretation of $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric conformal field theory as a quantization of the $\mathfrak{sl}_d$ Fuchsian system [4],[6],[7].

Before defining these algebras, let us review a few generalities on W-algebras.

## 2.2 Operator product expansions

We will throughout this text consider the Lie algebra $\mathfrak{sl}_d$ in its fundamental representation $\mathfrak{sl}_d \subset \mathfrak{gl}_d$. Similarly to the case of the Virasoro algebra, introducing the rank $d-1 = \mathfrak{r}k\,\mathfrak{sl}_d$, the soon to be defined generators of $\mathcal{W}(\widehat{\mathfrak{sl}_d})$, denoted $\{\mathbf{W}_n^{(d_p)}\}_{1\leq p\leq d-1}^{n\in\mathbb{Z}}$, fit for a given $p \in \{1,\ldots,d-1\}$, into a generating function defined around a base point $x_0 \in \Sigma$ by

$$\mathbf{W}^{(d_p)}(\widetilde{x}) = \sum_{n\in\mathbb{Z}} \frac{\mathbf{W}_n^{(d_p)}}{(x-x_0)^{n+d_p}} \quad \text{for}\, p \in \{1,\ldots,d-1\} \tag{2-1}$$

where the $d_p$'s are integer indices defined as follows : since $\mathfrak{h}$ is a commutative Lie algebra, we have an isomorphism $\mathcal{U}(\mathfrak{h}^*) \simeq \mathbb{C}[\mathfrak{h}]$ and moreover, by a theorem of Chevalley, the subspace of this last ring which is invariant under the action of the Weyl group is actually a polynomial ring $\mathbb{C}[\mathfrak{h}]^{\mathfrak{w}} \simeq \mathbb{C}[\sigma_1,\ldots,\sigma_{d-1}]$ where for any index $p \in \{1,\ldots,d-1\}$, we then define $d_p \in \mathbb{N}^*$ as the degree of the invariant polynomial $\sigma_p$. Since we consider $\mathfrak{g} = \mathfrak{sl}_d(\mathbb{C})$, we have $\sigma_p = p+1$ for any $p \in \{1,\ldots,d-1\}$.

We will assume the algebra $\mathcal{W}(\widehat{\mathfrak{sl}_d})$ to be an extension of the Virasoro algebra and will define its generators, denoted $\mathbf{W}^{(k)}$, for any $k \in \{2,\ldots,d\}$.

Following the introduction of [16], the corresponding operator product expansions can be presented schematically as

$$\begin{aligned}\mathbf{W}^{(k)}(\widetilde{x})\mathbf{W}^{(l)}(\widetilde{y}) \underset{x\sim y}{=} & \frac{g^{k,l}}{(x-y)^{k+l}} \\ & + \sum_{s=2}^{d-1} f^{k,l}_{(1),s}\frac{\mathbf{W}^{(s)}(\widetilde{y}) + g^{k,l}_s \partial\mathbf{W}^{(s)}(\widetilde{y}) + \dots}{(x-y)^{k+l-s}} \\ & + \sum_{s,t=2}^{d-1} f^{k,l}_{(2),s,t}\frac{:\mathbf{W}^{(s)}(\widetilde{y})\mathbf{W}^{(t)}(\widetilde{y}): + \dots}{(x-y)^{p+q-s-t}} + \dots\end{aligned} \tag{2-2}$$

We will identify $\mathbf{W}^{(2)} \propto T$ as being proportional to the stress energy tensor generating the Virasoro algebra and we assume moreover the rest of these generating functions, $\mathbf{W}^{(k)}$ for $k \neq 2$, to be primary fields with respect to the Virasoro algebra. In turn they are required to satisfy the operator product expansions

$$T(x)\mathbf{W}^{(k)}(\widetilde{y}) \underset{x\sim y}{=} k\frac{\mathbf{W}^{(k)}(\widetilde{y})}{(x-y)^2} + \frac{\partial\mathbf{W}^{(k)}(\widetilde{y})}{x-y} + \mathcal{O}(1) \tag{2-3}$$

which imply in particular the commutation relations

$$[L_n, \mathbf{W}^{(k)}_m] = [(k-1)n - m]\mathbf{W}^{(k)}_{n+m} \tag{2-4}$$

Let us mention that with $d = 3$, the $\mathfrak{g} = \mathfrak{sl}_3$ case was investigated in [35] and the algebra defined by the corresponding operator product expansions is

$$[L_n, \mathbf{W}^{(3)}_m] = (2n-m)\mathbf{W}^{(3)}_{n+m} \tag{2-5}$$

where we identified the modes of $\mathbf{W}^{(2)}$ with some Virasoro generators and

$$\begin{aligned}[\mathbf{W}^{(3)}_m, \mathbf{W}^{(3)}_m] = & (n-m)[\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)]L_{n+m} \\ & + \frac{c}{3\cdot 5!}n(n^2-1)(n^2-4)\delta_{n+m,0} + \frac{16}{22+c}(n-m)\Lambda_{n+m}\end{aligned} \tag{2-6}$$

where we introduced the symbols

$$\Lambda_n \underset{def}{=} \sum_{k\in\mathbb{Z}} :L_k L_{n-k}: + \frac{1}{5}\nu_n L_n \tag{2-7}$$

$$\text{with} \quad \nu_{2l} \underset{def}{=} (1+l)(1-l) \tag{2-8}$$

$$\text{and} \quad \nu_{2l+1} \underset{def}{=} (2+l)(1-l) \tag{2-9}$$

## 2.3 W-algebra generators

Let us consider the generic situation $Q = b + b^{-1} \neq 0$. As mentioned earlier, for generic values of $Q$, the W-algebra with non-abelian currents does not close. It does however when we restric ourselves to abelian currents defined by the following.

**Definition 2.1** *W-algebra generators*
*Consider the weights $h_i = \omega_1 - e_1 \cdots - e_{i-1}$, $i = 1, \ldots, d$ of the first fundamental representation of $\mathfrak{sl}_d$. The generating functions of generators of the algebra $\mathcal{W}(\widehat{\mathfrak{sl}_d})$ are expressed through the quantum Miura transform*

$$\widehat{\mathcal{E}} = \sum_{k=0}^{d} (-1)^k \mathbf{W}^{(k)} \widehat{y}^{d-k} \underset{def}{=} : (\widehat{y} - \mathbf{J}_1) \cdots (\widehat{y} - \mathbf{J}_d) : \tag{2-10}$$

*where $\widehat{y} \underset{def}{=} Q\partial$ and for any subscript $i = 1, \ldots, d$ we defined $\mathbf{J}_i \underset{def}{=} (h_i, \mathbf{J})$. The non-commutative prescription for evaluating these products at coinciding points has been used. These generating functions could in principle be multivalued on $\Sigma$.*

This definition is to be understood as the identification of the coefficients of the polynomial expression in $\widehat{y}$ obtained by commuting all the derivative symbols to the right.

**Example 2.1**

$$\mathbf{W}^{(1)}(\widetilde{x}) = \sum_{i=1}^{d} \mathbf{J}_i(\widetilde{x}) = 0 \tag{2-11}$$

$$\begin{aligned} \mathbf{W}^{(2)}(\widetilde{x}) &= \sum_{1 \leq i < j \leq d} : \mathbf{J}_i \mathbf{J}_j(\widetilde{x}) : - Q \sum_{i=2}^{d} (i-1) \partial \mathbf{J}_i(\widetilde{x}) & \text{(2-12)} \\ &= -\frac{1}{2} : (\mathbf{J}, \mathbf{J})(\widetilde{x}) : + (\mathbf{Q}, \partial \mathbf{J}(\widetilde{x})) & \text{(2-13)} \end{aligned}$$

*where we used $\sum_{i=1}^{d} h_i = 0$ and the expression*

$$\rho = \frac{1}{2} \sum_{i=1}^{d} (n - 2i + 1) h_i = - \sum_{i=2}^{d} (i-1) h_i \tag{2-14}$$

*of the Weyl vector.*

**Lemma 2.1** *For any $k \in \{1, \dots d\}$, the $k^{th}$ generator $\mathbf{W}^{(k)}$ of $\mathcal{W}(\widehat{\mathfrak{sl}_d})$ is equal to*

$$\sum_{p=1}^{k}(-1)^{k-p}Q^{k-p}\sum_{\substack{1\leq i_1<\cdots<i_p\leq d\\ k\leq i_p}}\ \sum_{\substack{\forall l\in[\![1,p]\!]\\ 0\leq q_l\leq i_l-i_{l-1}-1\\ p+\sum_{l=1}^{p}q_l=k}}\ \prod_{l=1}^{p}\binom{i_l-i_{l-1}-1}{q_l} : \partial^{q_1}\left(\mathbf{J}_{i_1}\cdots\partial^{q_p}\mathbf{J}_{i_p}\right): \tag{2-15}$$

**proof:**

The proof consists in using non-commutative algebra in the ring of differential operators $\mathcal{D}_\Sigma$ overs the base curve to commute all $Q\partial$ symbols to the right before identifying the coefficients of the differential operators. To do so we first identifiy holomorphic functions $f \in \mathcal{O}_\Sigma$ with the degree 0 differential operators $f\cdot \in \mathcal{D}_\Sigma$ of multiplication by $f$ on the left. For any function $f \in \mathcal{O}_\Sigma$ we then have the commutation relation $[Q\partial, f] = Q(\partial f)$ and it recursively yields the non-commutative version of Leibniz formula

$$(Q\partial)^p f = Q^p\sum_{q=0}^{p}\binom{p}{q}(\partial^q f)\partial^{p-q} \tag{2-16}$$

where the equality takes place in $\mathcal{D}_\Sigma$. It is then a straightforward computation to derive the wanted result. $\square$

**Example 2.2** *Straightforward computation for example yields, for $d = 2, 3$,*

- $$\begin{aligned}\widehat{\mathcal{E}}_{d=2} &= (Q\partial)^2-[\mathbf{J}_1+\mathbf{J}_2](Q\partial)+:\mathbf{J}_1\mathbf{J}_2:-Q\left(\partial\mathbf{J}_2\right) && (2\text{-}17)\\ &= (Q\partial)^2-:\mathbf{J}_1^2:+\,Q(\partial\mathbf{J}_1) && (2\text{-}18)\end{aligned}$$

- $$\begin{aligned}\widehat{\mathcal{E}}_{d=3} &= (Q\partial)^3-[\mathbf{J}_1+\mathbf{J}_2+\mathbf{J}_3](Q\partial)^2\\ &+ [:\mathbf{J}_1\mathbf{J}_2+\mathbf{J}_2\mathbf{J}_3+\mathbf{J}_1\mathbf{J}_3:-Q\partial\mathbf{J}_2-2Q\partial\mathbf{J}_3](Q\partial)\\ &+ Q^2(\partial^2\mathbf{J}_3)+Q:[\mathbf{J}_1+\mathbf{J}_2]\left(\partial\mathbf{J}_3\right):+\,Q:\left(\partial\mathbf{J}_2\right)\mathbf{J}_3: && (2\text{-}19)\end{aligned}$$

*and for any $d \in \mathbb{N}^*$,*

$$\begin{aligned}\mathbf{W}^{(3)} &= \sum_{1\leq i<j<k\leq d} :\mathbf{J}_i\mathbf{J}_j\mathbf{J}_k:\\ &- Q\sum_{\substack{1\leq i<j\leq d\\ 3\leq j}}\left[(j-i-1):\mathbf{J}_i\partial\mathbf{J}_j:+(i-1)\partial\left(:\mathbf{J}_i\mathbf{J}_j:\right)\right]\\ &+ Q^2\sum_{i=3}^{d}\binom{i-1}{2}\partial^2\mathbf{J}_i && (2\text{-}20)\end{aligned}$$

For $k \in \{2, \ldots, d\}$, $\mathbf{W}^{(k)}$ therefore involves at most terms of degree $k$ as differential polynomials in $d$ copies of a chosen so-called "chiral $\mathfrak{h}^*$-valued spin-one field" $\mathbf{J}(\widetilde{x})$ as described before. We require as stated in *Axiom* 2 that it satisfies

$$\mathbf{J}(\widetilde{x}\cdot E)\mathbf{J}(\widetilde{y}\cdot F) \underset{x\sim y}{=} -\frac{(E,F)}{(x-y)^2} + :\mathbf{J}(\widetilde{x}\cdot E)\mathbf{J}(\widetilde{y}\cdot F) \underset{x=y}{:} + \mathcal{O}(x-y) \tag{2-21}$$

for Cartan elements $E, F \in \mathfrak{h}$, where $(\cdot,\cdot)$ still denotes the corresponding minimal invariant bilinear form and we redefined the current by a factor of $\kappa^{-1/2}$.

## 2.4 Ward identities

We now generalize *Axiom* 3 to the algebra $\mathcal{W}(\widehat{\mathfrak{sl}_d})$ (not necessarily a Lie algebra) defined by the operator coefficients of the expansions of the generators $\mathbf{W}^{(k)}$, $k \in \{1, \ldots d\}$, around a base point $x_0 \in \Sigma$. We then get that the chiral spin-one current $\mathbf{J}$ should be chosen such that it satisfies

**Definition 2.2** *The generalized Ward identities of this $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric conformal field theory are the axiom defined, for any $k \in \{2, \ldots, d\}$ and any generic ponits $\widetilde{x}_1, \cdots, \widetilde{x}_n \in \widetilde{\Sigma}$ in the universal covering, by requiring that $\left\langle\left\langle \mathbf{W}^{(k)}(\widetilde{x})\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\right\rangle\right\rangle$ is a holomorphic function of the variable $x \in \Sigma - \{z_1, \ldots, z_M, x_1, \ldots, x_n\}$.*

This definition yields that for an admissible chiral current $\mathbf{J}$, the insertion $\langle\langle \mathbf{W}^{(k)}(\widetilde{x})\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\rangle\rangle$ of any of the fields $\mathbf{W}^{(k)}$, $k \in \{2, \ldots, d\}$, should be uniquely valued as a holomorphic function of $x \in \Sigma - \{z_1, \ldots, z_M, x_1, \ldots, x_n\}$. We can therefore drop the upperscript in $\widetilde{x}$ and simply write $\mathbf{W}^{(k)}(x)$ when evaluating the insertion of such a generator.

Replacing the previously computed expression for $\mathbf{W}^{(k)}$ in terms of the current $\mathbf{J}$ in $\langle\langle \mathbf{W}^{(k)}(x)\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\rangle\rangle$ yields that

**Proposition 2.1** *Ward identities as loop equations*

$$\begin{aligned}\left\langle\left\langle \mathbf{W}^{(k)}(x)\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\right\rangle\right\rangle & \\ = \sum_{p=1}^{k}(-1)^{k-p}Q^{k-p} & \sum_{\substack{1\leq i_1<\cdots<i_p\leq d\\ k\leq i_p}} \sum_{\substack{\forall l\in[\![1,p]\!]\\ 0\leq q_l\leq i_l-i_{l-1}-1\\ p+\sum_{l=1}^{p} q_l=k}} \left(\prod_{l=1}^{p}\binom{i_l-i_{l-1}-1}{q_l}\right) \\ & \times \left\langle\left\langle :\partial^{q_1}\left(\mathbf{J}_{i_1}\ldots\partial^{q_p}\mathbf{J}_{i_p}\right)(\widetilde{x}):\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\right\rangle\right\rangle\end{aligned} \tag{2-22}$$

*is a holomorphic function of $x \in \Sigma - \{z_1, \ldots, z_M, x_1, \ldots, x_n\}$.*

For a generic value of $Q$ and specializing to the cases $k = 1, 2$, the expressions for $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$ yield that

**Corollary 2.1**

$$\sum_{i=1}^{d} \left\langle\left\langle \mathbf{J}_i(\widetilde{x})\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\right\rangle\right\rangle = 0 \qquad \textit{and} \tag{2-23}$$

$$\sum_{1\leq i<j\leq d} \left\langle\left\langle :\mathbf{J}_i\mathbf{J}_j(\widetilde{x}):\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\right\rangle\right\rangle - Q\sum_{i=2}^{d}(i-1)\partial_x\left\langle\left\langle \mathbf{J}_i(\widetilde{x})\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\right\rangle\right\rangle \tag{2-24}$$

*is a holomorphic function of* $x \in \Sigma - \{z_1, \ldots, z_M, x_1, \ldots, x_n\}$.

## 2.5 Classical limit $Q \longrightarrow 0$ and quantization

The definition of the generators of $\mathcal{W}(\widehat{\mathfrak{sl}_d})$ involved identifying the coefficients of two differential operators.

**Definition 2.3** *The Casimir algebra* $\mathcal{W}_0(\widehat{\mathfrak{sl}_d})$ *is the associative algebra generated by the modes* $\{\mathbf{W}_{0;n}^{(k)}\}_{\substack{n\in\mathbb{Z}\\ 1\leq k\leq d}}$ *of the generators defined through the following identities :*

$$\sum_{k=0}^{d}(-1)^k\mathbf{W}_0^{(k)}y^{r-k} \quad = \quad :(y-\mathbf{J}_1)\cdots(y-\mathbf{J}_d): \tag{2-25}$$

$$\forall p \in \{1,\ldots,d\}, \quad \mathbf{W}_0^{(k)}(x) \underset{def}{=} \sum_{n\in\mathbb{Z}} \frac{\mathbf{W}_{0;n}^{(k)}(x_0)}{(x-x_0)^{n+k}} \tag{2-26}$$

We get the following classical limit

**Theorem 2.1** *Quantization of Fuchsian differential systems [B.-Eynard]*

$$\sum_{k=0}^{d}(-1)^k\mathbf{W}_0^{(k)}y^{r-k} = \mathrm{Symb}\left(\sum_{k=0}^{d}(-1)^k\mathbf{W}^{(k)}\widehat{y}^{\,r-k}\right) \tag{2-27}$$

*where the symbol of a differential operator* $P(x,\widehat{y}\,) \in \mathbb{C}(x)[\,\widehat{y}\,]$ *is defined as*

$$\mathrm{Symb}\left(P(x,\widehat{y}\,)\right) \underset{def}{=} \lim_{Q\to 0}\left(e^{-xy/Q}P(x,\widehat{y}\,)\cdot e^{xy/Q}\right) \tag{2-28}$$

*and therefore the* $\mathcal{W}(\widehat{\mathfrak{sl}_d})$*-symmetric conformal field theory quantizes the Fuchsian differential system corresponding to this classical limit* $\mathcal{W}_0(\widehat{\mathfrak{sl}_d})$.

More explicitely, putting $Q = 0$ in the Ward identities, the only remaining term of the expression of last proposition is equal to

$$\left\langle\left\langle \mathbf{W}_0^{(k)}(x)\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\right\rangle\right\rangle = \sum_{1\leq i_1<\cdots<i_k\leq r} \left\langle\left\langle :(\mathbf{J}_{i_1}\cdots\mathbf{J}_{i_k})(\widetilde{x}):\mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\right\rangle\right\rangle \tag{2-29}$$

that is the sum over all possible ways to insert $k$ of the $d$ copies of the chiral current $\mathbf{J}$. This is exactly what one would obtain by writing the Ward identities for a Casimir algebra-symmetric conformal field theory and we can read the loop equations on the right hand side [6].

# 3 Quantum geometry

We shall now define the quantum geometry associated to the $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric conformal field theory we are considering. It first consists in a definition of the quantum spectral curve and quantum complex structure, encoded in the 2-points function, when we assume the existence of a topological regime. We will then introduce the relevent topological recursion and show one of the main results of the chapter, namely that it solves the Ward identities.

We will for simplicity restrict ourselves to the case where the Riemann surface is the Riemann sphere $\Sigma = \mathbb{CP}^1$, although most of the reasoning is local and could be generalized to an arbitrary Riemann surface.

## 3.1 Topological regime and quantum spectral curve

This is the main assumption of this study. Let us suppose that all the functions appearing in our construction are now formal series in an expansion parameter $\varepsilon \longrightarrow 0$.

This corresponds to a so-called *heavy limit* where we rescale all the charges simultaneously $\alpha_j \longmapsto \frac{1}{\varepsilon}\alpha_j$. This is equivalent to rescaling the chiral current $\mathbf{J} \longmapsto \frac{1}{\varepsilon}\mathbf{J}$ and ultimately this could be reabsorbed in a redifinition of the background charge $Q \longmapsto \widetilde{Q} \underset{def}{=} \varepsilon Q$. We however consider the limit $\varepsilon \longrightarrow 0$ keeping $Q$ fixed.

Let us now assume that the chiral correlation functions with current insertions admit $\varepsilon \longrightarrow 0$ asymptotic expansions of the form

$$\left\langle\left\langle \mathbf{J}_{i_1}(\widetilde{x}_1)\cdots\mathbf{J}_{i_n}(\widetilde{x}_n)\right\rangle\right\rangle \underset{def}{=} \sum_{g=0}^{\infty} \varepsilon^{2g-2+n}\left(W_{g,n}(\overset{i_1}{x_1},\ldots,\overset{i_n}{x_n}) - \delta_{n,2}\delta_{g,0}\frac{(h_{i_1},h_{i_2})}{(x_1-x_2)^2}\right)$$

$$(3\text{-}1)$$

for all $n \in \mathbb{N}^*$ for which $W_{g,n}$, with $g \in \mathbb{N}$, is a multi-valued meromorphic function on $n$ copies of the universal covering of the punctured Riemann sphere. Such expansions define a topological regime.

**Remark 3.1** *Note that in this article, we make the choice of not writing explicitely the universal covering dependence of the functions appearing as coefficients of topological expansions to lighten notations. The reader should nevertheless keep in mind that these coefficients are defined on the quantum covering whose points are locally described as pairs $(\widetilde{x}, i)$, denoted $\overset{i}{x}$ when appearing as arguments.*

As a consequence, the differential operators obtained by inserting $\widehat{\mathcal{E}}(x)$ into a chiral correlation function with current insertions $\langle\langle \mathbf{J}(\widetilde{x}_1)\cdots\mathbf{J}(\widetilde{x}_n)\rangle\rangle$ also admit asymptotic expansions of a similar form

$$\left\langle\left\langle \widehat{\mathcal{E}}(x)\,\mathbf{J}_{i_1}(\widetilde{x}_1)\cdots\mathbf{J}_{i_n}(\widetilde{x}_n)\right\rangle\right\rangle \underset{def}{=} \sum_{g=0}^{\infty} \varepsilon^{2g-1+n}\mathcal{E}_n^{(g)}(x;\overset{i_1}{x}_1,\ldots,\overset{i_n}{x}_n) \tag{3-2}$$

$$\underset{def}{=} \sum_{k=0}^{d}\sum_{g=0}^{\infty}(-1)^{d-k}\varepsilon^{2g-1+n} \times P_{n,d-k}^{(g)}(x;\overset{i_1}{x}_1,\ldots,\overset{i_n}{x}_n)\widehat{y}^{k} \tag{3-3}$$

**Definition 3.1** *The differential operator $\mathcal{E} \underset{def}{=} \mathcal{E}_0^{(0)}$ is called the quantum spectral curve.*

The Ward identities extended to these formal $\varepsilon$-expansions imply that the operator $\mathcal{E}_n^{(g)}(x;\overset{i_1}{x}_1,\ldots,\overset{i_n}{x}_n)$ is, for all $g, n \in \mathbb{N}$, $n \neq 0$, a meromorphic function of the variable $x \in \mathbb{C}$ with possible singularities at $x = \infty$, $x = x_i$ for some $i \in \{1,\ldots,d\}$ or $x = z_j$ for some $j \in \{1,\ldots,M\}$ and nowhere else.

This definition can be interpreted as exhibiting the quantization of a classical spectral curve, perturbatively this time. Indeed , define a function $E$ of the variables $x, y \in \mathbb{C}$ by the generic assignment

$$E(x,y) \underset{def}{=} \mathrm{Symb}\left(\mathcal{E}_0^{(0)}(x)\right) \tag{3-4}$$

The Riemann surface defined by the equation $E(x,y) = 0$, the character variety of the quantum spectral curve, embeds in $\mathbb{C}^2$ and defines a $d:1$ cover of the complex plane by a meromorphic projection $x : \mathcal{S} \longrightarrow \mathbb{C}$ called the classical spectral curve of the $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric conformal field theory.

Recall that to any classical integrable systems presented in Lax form can be associated its corresponding spectral curve, a meromorphic covering of complex curves, and that this is the starting point to run the topological recursion of [14],[23] in order for example to compute recursively the expansion coefficients of generating functions of derivatives of the $\tau$-function. We wish to upgrade these techniques to the non-commutative, or quantum, case using this operator formalism arising from conformal field theory.

## 3.2 Fermionic description and notion of sheets

A dual point of view to describe the quantum spectral curve is through the solutions of the linear differential equation it defines. Let us therefore consider a set of $d$ independent (multivalued) functions $\psi_j$, for $j \in \{1, \dots, d\}$, satisfying

$$\psi_j \cdot \mathcal{E} = 0 \tag{3-5}$$

where the differential operator acts from the right.

**Theorem 3.1** *The quantum spectral curve decomposes as*

$$\mathcal{E} = (\widehat{y} - Y_1(\widetilde{x})) \cdots (\widehat{y} - Y_d(\widetilde{x})) \tag{3-6}$$

*where for any* $\mu \in \{1, \dots, d\}$,

$$Y_\mu \underset{def}{=} Q\partial \left( \ln \frac{D_{\mu-1}}{D_\mu} \right) = \frac{Q\partial D_{\mu-1}}{D_{\mu-1}} - \frac{Q\partial D_\mu}{D_\mu} \tag{3-7}$$

*with*

$$D_\mu \underset{def}{=} \operatorname*{Det}_{0 \leq i,j \leq \mu-1} \left( (-Q\partial)^i \psi_{j+1} \right) \tag{3-8}$$

*and the convention that* $D_0 \underset{def}{=} 1$.

**Remark 3.2** *Let us mention that such a factorization is in general not unique and that each of these multivalued factors have monodromies cancelling in such a way that the resulting product is a well-defined differential operator on* $\Sigma - \{z_1, \dots, z_M\}$. *In particular, it does not depend on a choice of preimage* $\widetilde{x}$ *of* $x \in \Sigma - \{z_1, \dots, z_M\}$ *in the universal covering.*

**proof:**

If we define recursively the $Y_\mu$'s such that for any $\nu \leq \mu$,

$$\psi_\nu \cdot (\widehat{y} - Y_1(\widetilde{x})) \cdots (\widehat{y} - Y_\nu(\widetilde{x})) = 0 \tag{3-9}$$

then the wanted result is a straightforward corollary of the following

**Lemma 3.1** *For any $\mu \in \{1, \ldots, d\}$,*

$$(\widehat{y} - Y_1(\widetilde{x})) \cdots (\widehat{y} - Y_\mu(\widetilde{x})) = \operatorname*{Det}_{\mu+1} \begin{pmatrix} 1 & \psi_1(\widetilde{x}) & \ldots & \psi_\mu(\widetilde{x}) \\ \widehat{y} & (-Q\partial)\psi_1(\widetilde{x}) & \ldots & (-Q\partial)\psi_\mu(\widetilde{x}) \\ \widehat{y}^2 & (-Q\partial)^2\psi_1(\widetilde{x}) & \ldots & (-Q\partial)^2\psi_\mu(\widetilde{x}) \\ \vdots & & & \vdots \\ \widehat{y}^\mu & (-Q\partial)^\mu\psi_1(\widetilde{x}) & \ldots & (-Q\partial)^\mu\psi_\mu(\widetilde{x}) \end{pmatrix} \frac{1}{D_\mu(\widetilde{x})} \tag{3-10}$$

Indeed, since then the differential operator of the right hand side and the quantum spectral curve have same degree, leading coefficient and space of solutions, we conclude that hey are equal. □

**Definition 3.2** *Master loop equation*
*Define the auxilary operator*

$$\mathcal{U} \underset{def}{=} (\widehat{y} - Y_2(\widetilde{x})) \cdots (\widehat{y} - Y_d(\widetilde{x})) \tag{3-11}$$

*such that the following identity holds*

$$(\widehat{y} - Y_1(\widetilde{x}))\,\mathcal{U} = \mathcal{E} \tag{3-12}$$

*and is called the master loop equation.*

**Remark 3.3** *This terminology comes from the connection existing between the quantum geometry of the $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric conformal field theory we are studying and that of the $\beta(\mathfrak{sl}_d)$-deformed two-matrix model, a generalization of that of [11],[18]. These similarities hide a full correspondence between the theories whose description should appear soon in the sequel [3] of the present paper.*

**Remark 3.4** *The geometrical interpretation of this factorization is that there exists a function $Y$ that is actually multivalued on the punctured sphere in such a way that $Y(\overset{i}{x}) = Y_i(\widetilde{x})$ is its value at a generic point $x \in \Sigma$ taken in the sheet $i \in \{1, \ldots, d\}$. The sheet labelled by $i = 1$ is often called the physical sheet. The quantum sheets therefore label solutions of the quantum spectral curve and in this sense, the function $Y$ is uniquely valued on the quantum spectral curve.*

### 3.3 2-points function

Recall that the 2-points function has the asymptotic expansion

$$\langle\langle \mathbf{J}_{i_1}(\widetilde{x}_1)\mathbf{J}_{i_2}(\widetilde{x}_2)\rangle\rangle = \sum_{g=0}^{\infty} \varepsilon^{2g} \left( W_{g,2}(\overset{i_1}{x_1}, \overset{i_2}{x_n}) - \delta_{g,0} \frac{(h_{i_1}, h_{i_2})}{(x_1 - x_2)^2} \right) \tag{3-13}$$

In the classical formalism of [14],[23], the initial data needed to run the topological recursion included a symmetric bi-differential $\omega_{0,2}$ on two copies of this curve having a double pole with no residue, bi-residue equal to 1 on the diagonal divisor and no other singularities. In the special instance where the spectral curve is a complex curve embedded in $\mathbb{C}^2$, a natural candidate then was the Bergman kernel, or second-kind fundamental form, associated to a choice of Torelli marking, that is to a symplectic basis of real codimension 1 homology cycles. The wanted singularities plus the requirement of vanishing periods on a given half of the symplectic basis fixes the Bergman kernel uniquely.

The quantum setup under here under study is a direct generalization of that of [17] where the construction can be interpreted as solving the Ward identities of a $\mathcal{W}(\widehat{\mathfrak{sl}_2})$-symmetric conformal field theory. The existence of an hyperelliptic involution then simplified the discussion and a structure of quantum Riemann surface with cuts, cycles, holomorphic differential forms and their mutual pairing was defined. In particular, the periods of the quantum Bergman kernel were vanishing on a corresponding half of a symplectic basis of quantum cycles.

This geometric construction is expected to extend to our setup and will be further investigated in subsequent work. By anticipation

**Definition 3.3** *Quantum Bergman kernel*
*We interpret the leading order of the 2-points function of the theory as the quantum Bergman kernel, or second-kind fundamental form, and denote it by*

$$B \underset{def}{=} W_{0,2} \tag{3-14}$$

*Accordingly, we define the third-kind differential form G (up to a function of the variable $\overset{i}{x}$ that will play no role in what follows) by the formula*

$$\partial_z G(\overset{i}{x}, \overset{j}{z}) = B(\overset{i}{x}, \overset{j}{z}) \tag{3-15}$$

# 4 Topological recursion

## 4.1 Ward identities in the $\varepsilon \longrightarrow 0$ expansion

To solve the Ward identities recursively, we must rewrite them order by order in the topological regime. An exceptionnal feature of the structure of these equations is that generically only the two lowest order Ward identities, that we will call *linear and quadratic loop equation*, are needed to reconstruct the chiral correlation functions with current insertions perturbatively. This illustrates the *over-determination* of integrable systems. Classically, the non-generic cases correspond to those where the spectral curves exhibits non-simple ramification points and one then needs the more general formalism of [12].

Recall that using the multi-sheet notation, we denoted the insertion of the $k^{th}$ $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-algebra generator $\mathbf{W}^{(k)}$ at a generic point $x \in \Sigma$ into a chiral correlation function with $n \in \mathbb{N}^*$ current insertions at generic points $x_1, \dots, x_n \in \Sigma$ by

$$\left\langle\left\langle \mathbf{W}^{(k)}(x)\mathbf{J}_{i_1}(\widetilde{x}_1), \dots, \mathbf{J}_{i_n}(\widetilde{x}_n)\right\rangle\right\rangle \underset{def}{=} \sum_{g=0}^{\infty}(-1)^k \varepsilon^{2g-1+n} P_{n,k}^{(g)}(x; \overset{i_1}{x}_1, \dots, \overset{i_n}{x}_n) \tag{4-1}$$

Replacing this expression in the two first conformal Ward identities yields

**Theorem 4.1** *Linear and quadratic loop equations*
*The axioms of the $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric conformal field theory require the linear and quadratic loop equations in the topological limit. Namely for any choice of integers $n, g \in \mathbb{N}$, $n \neq 0$, and any generic choice of points and sheet indices $X = \{\overset{i_1}{x}_1, \dots, \overset{i_n}{x}_n\}$,*

$$P_{n;0}^{(g)}(x, J) = \sum_{i=1}^{d} W_{n+1}^{(g)}(\overset{i}{x}, X) = 0 \tag{4-2}$$

*and*

$$P_{n;1}^{(g)}(x, X) = \sum_{1 \le i < j \le d} \left( W_{n+2}^{(g-1)}(\overset{i}{x}, \overset{j}{x}, X) + \sum_{\substack{I \sqcup I' = X \\ h+h'=g}} W_{1+\#I}^{(h)}(\overset{i}{x}, I) W_{1+\#I'}^{(h')}(\overset{j}{x}, I') \right) - Q \sum_{i=2}^{d} (i-1)\partial_x W_{n+1}^{(g)}(\overset{j}{x}, X) \tag{4-3}$$

*is a holomorphic functions of $x \in \Sigma - \{z_1, \dots, z_M, x_1, \dots, x_n\}$.*

**proof:**

The proof is a done by induction on $2g-2+n$ and it is a straightforward computation. $\square$

## 4.2 Bethe roots and kernel

Generically, a zero of $D_\mu$ for a given $\mu \in \{1,\dots,d\}$ is both a pole of $Y_\mu$ and $Y_{\mu+1}$, with residue $\pm 1$ and is not a zero of any other $D_\nu$ and therefore not a pole of any other $Y_\nu$. This statement is the quantum analog to that saying that generically, there are only two sheets meeting at a branch point of an algebraic curve.

**Definition 4.1** *Bethe roots*
*Let us denote by* $S_\mu \underset{def}{=} \{s \in \mathbb{C} \,|\, D_\mu(s) = 0\}$ *the set of all roots of* $D_\mu$ *for a given* $\mu \in \{1,\dots,d\}$ *and by* $S \underset{def}{=} \bigcup_{\mu=1}^d S_\mu$*, and call them the Bethe roots. We will moreover generically denote* $\mu_s$*, for any root* $s \in S$*, the sheet index such that there exists exactly two functions* $Y_{\mu_s}$ *and* $Y_{\mu_s+1}$*, of which* $s$ *is a pole.*

**Definition 4.2** *Recursion kernel*
*Let us define the recursion kernel* $K_\mu(\overset{i_0}{x}_0, x)$ *as the solution of the following differential equation which is analytic at the zero* $x = s$ *of* $D_\mu$ *(*$\mu_s = \mu$*):*

$$\left(Y_{\mu+1}(\widetilde{x}) - Y_\mu(\widetilde{x}) + Q\partial_x\right)\, K_\mu(\overset{i_0}{x}_0, \widetilde{x}) = \frac{1}{2}\left(G(\overset{i_0}{x}_0, \overset{\mu+1}{x}) - G(\overset{i_0}{x}_0, \overset{\mu}{x})\right) \tag{4-4}$$

*Equivalently, replacing the expressions for the* $Y_\mu$*'s yield*

$$\left(-2\frac{Q\partial D_\mu}{D_\mu} + \frac{Q\partial D_{\mu+1}}{D_{\mu+1}} + \frac{Q\partial D_{\mu-1}}{D_{\mu-1}} + Q\partial_x\right)\, K_\mu(\overset{i_0}{x}_0, \widetilde{x}) = \frac{1}{2}\left(G(\overset{i_0}{x}_0, \overset{\mu+1}{x}) - G(\overset{i_0}{x}_0, \overset{\mu}{x})\right) \tag{4-5}$$

**Remark 4.1** *Such a solution that is analytic at* $s$ *does not necessarily exists and this is a consequence of the existence of a loop insertion operator in the conformal field theory and of the differential Hirota identities satisfied by the* $D_\mu$*'s. We postpone the proof of this claim to the appendix but drop the upperscript in* $\widetilde{x}$ *to simply write* $x$*.*

**Remark 4.2** *The function* $K_\mu$ *thus defined is not unique as one can add* $f(\overset{i_0}{x}_0)\frac{D_\mu(x)^2}{D_{\mu-1}(x)D_{\mu+1}(x)}$ *for any function* $f$*. As we shall see, our main theorem is independent of such a choice.*

## 4.3 Recursion

**Theorem 4.2** *Reconstruction by topological recursion*
*We have the topological recursion*

$$\begin{aligned} W^{(g)}_{n+1}(\overset{i_0}{x}, X) &= \sum_\mu \frac{1}{2\pi i} \oint_{S_\mu} K_\mu(\overset{i_0}{x_0}, x) \Big( W^{(g-1)}_{n+2}(\overset{\mu+1}{x}, \overset{\mu}{x}, X) \\ &\quad + \sum'_{\substack{I \sqcup I' = X \\ h+h'=g}} W^{(h)}_{1+\#I}(\overset{\mu+1}{x}, I)\, W^{(h')}_{1+\#I'}(\overset{\mu}{x}, I') \Big) \end{aligned} \qquad (4\text{-}6)$$

*where $\oint_{S_\mu}$ means integrating along a contour that surrounds all Bethe roots in $S_\mu$ but not any of the other Bethe roots $S - S_\mu$. When there is a finite number of roots or when the sum $\sum_{s\in S}$ can be defined, we may write*

$$\begin{aligned} W^{(g)}_{n+1}(\overset{i_0}{x}, X) &= \sum_{s\in S} \operatorname*{Res}_{x\to s} K_{\mu_s}(\overset{i_0}{x_0}, x) \Big( W^{(g-1)}_{n+2}(\overset{\mu_s+1}{x}, \overset{\mu_s}{x}, X) \\ &\quad + \sum'_{\substack{I \sqcup I' = X \\ h+h'=g}} W^{(h)}_{1+\#I}(\overset{\mu_s+1}{x}, I)\, W^{(h')}_{1+\#I'}(\overset{\mu_s}{x}, I') \Big) \end{aligned} \qquad (4\text{-}7)$$

*thus taking the form of the topological recursion for classical spectral curves defined as finite coverings and whose branchpoints are analogous to the Bethe roots.*

**proof:**

Let us compute the expression

$$\sum_{s\in S} \operatorname*{Res}_{x\to s} K_{\mu_s}(\overset{i_0}{x_0}, x) \Big( W^{(g-1)}_{n+2}(\overset{\mu_s+1}{x}, \overset{\mu_s}{x}, X) + \sum'_{\substack{I \sqcup I' = X \\ h+h'=g}} W^{(h)}_{1+\#I}(\overset{\mu_s+1}{x}, I)\, W^{(h')}_{1+\#I'}(\overset{\mu_s}{x}, I') \Big) \quad (4\text{-}8)$$

where $\sum'_{h,h',I,I'}$ means that we exclude the cases $(h=0, I=\emptyset)$ and $(h'=0, I'=\emptyset)$ from the sum. Let us define the same quantity as the one between parentheses but without the prime symbol $'$ :

$$\mathcal{Q}_{\mu,\nu} \underset{def}{=} W^{(g-1)}_{n+2}(\overset{\mu}{x}, \overset{\nu}{x}, X) + \sum_{\substack{I \sqcup I' = X \\ h+h'=g}} W^{(h)}_{1+\#I}(\overset{\mu}{x}, I)\, W^{(h')}_{1+\#I'}(\overset{\nu}{x}, I') \qquad (4\text{-}9)$$

for sheet indices $\mu, \nu \in \{1, \dots, d\}$. We thus have to compute:

$$\operatorname*{Res}_{x\to s} K_{\mu_s}(\overset{i_0}{x_0}, x) \Big( \mathcal{Q}_{\mu_s,\mu_s+1} - W^{(0)}_1(\overset{\mu_s+1}{x}) W^{(g)}_{n+1}(\overset{\mu_s}{x}, X) - W^{(0)}_1(\overset{\mu_s}{x}) W^{(g)}_{n+1}(\overset{\mu_s+1}{x}, X) \Big) \quad (4\text{-}10)$$

Let us rewrite

$$
\begin{aligned}
2\mathcal{Q}_{\mu_s,\mu_s+1} &= \mathcal{Q}_{\mu_s+1,\mu_s} + \mathcal{Q}_{\mu_s,\mu_s+1} \\
&= \sum_{i\neq j} \mathcal{Q}_{i,j} - \sum_{j\neq \mu_s,\mu_s+1} (\mathcal{Q}_{\mu_s,j} + \mathcal{Q}_{\mu_s+1,j}) - \sum_{i\neq \mu_s,\mu_s+1} (\mathcal{Q}_{i,\mu_s} + \mathcal{Q}_{i,\mu_s+1}) \\
&\quad - \sum_{i\neq j,\, i\neq \mu_s,\mu_s+1,\, j\neq \mu_s,\mu_s+1} \mathcal{Q}_{i,j} \\
&= 2\sum_{i<j} \mathcal{Q}_{i,j} - \sum_{i\neq j,\, (i,j)\neq(\mu_s,\mu_s+1),\, (i,j)\neq(\mu_s+1,\mu_s)} \mathcal{Q}_{i,j} \\
&= 2P^{(g)}_{n;1}(x,X) + 2Q\sum_j j\,\partial_x\, W^{(g)}_{n+1}(\overset{j}{x},X) - \sum_{\substack{i\neq j \\ (i,j)\neq(\mu_s,\mu_s+1),\, (i,j)\neq(\mu_s+1,\mu_s)}} \mathcal{Q}_{i,j} \\
&= +\, 2Q\mu_s\,\partial_x\, W^{(g)}_{n+1}(\overset{\mu_s}{x},X) + 2Q(\mu_s+1)\,\partial_x\, W^{(g)}_{n+1}(\overset{\mu_s+1}{x},X) + \text{reg. at } s
\end{aligned}
\tag{4-11}
$$

where we used the $i \longleftrightarrow j$ symmetry of the symbol $\mathcal{Q}_{i,j}$ and the linear loop equation after introducing $P^{(g)}_{n;1}$. Multiplying by the recursion kernel and taking the residue at the Bethe root $s \in S$ then implies

$$
\begin{aligned}
& \operatorname*{Res}_{x\to s} K_{\mu_s}(\overset{i_0}{x}_0,x)\Big(\mathcal{Q}_{\mu_s,\mu_s+1} - W^{(0)}_1(\overset{\mu_s+1}{x})W^{(g)}_{n+1}(\overset{\mu_s}{x},X) - W^{(0)}_1(\overset{\mu_s}{x})W^{(g)}_{n+1}(\overset{\mu_s+1}{x},X)\Big) \\
= \; & \operatorname*{Res}_{x\to s} K_{\mu_s}(\overset{i_0}{x}_0,x)\Big(-W^{(0)}_1(\overset{\mu_s+1}{x})W^{(g)}_{n+1}(\overset{\mu_s}{x},X) - W^{(0)}_1(\overset{\mu_s}{x})W^{(g)}_{n+1}(\overset{\mu_s+1}{x},X) \\
& + Q\mu_s\,\partial_x\, W^{(g)}_{n+1}(\overset{\mu_s}{x},X) + Q(\mu_s+1)\,\partial_x\, W^{(g)}_{n+1}(\overset{\mu_s+1}{x},X)\Big)
\end{aligned}
\tag{4-12}
$$

Since $W^{(g)}_{n+1}(\overset{\mu_s}{x},X) + W^{(g)}_{n+1}(\overset{\mu_s+1}{x},X)$ is analytic at $s$, we may rewrite:

$$
\begin{aligned}
& \operatorname*{Res}_{x\to s} K_{\mu_s}(\overset{i_0}{x}_0,x)\Big(\mathcal{Q}_{\mu_s,\mu_s+1} - W^{(0)}_1(\overset{\mu_s+1}{x})W^{(g)}_{n+1}(\overset{\mu_s}{x},X) - W^{(0)}_1(\overset{\mu_s}{x})W^{(g)}_{n+1}(\overset{\mu_s+1}{x},X)\Big) \\
= \; & \frac{1}{2}\operatorname*{Res}_{x\to s} K_{\mu_s}(\overset{i_0}{x}_0,x) \\
& \qquad \times \Big(W^{(0)}_1(\overset{\mu_s}{x}) - W^{(0)}_1(\overset{\mu_s+1}{x}) + Q\,\partial_x\Big)\Big(W^{(g)}_{n+1}(\overset{\mu_s}{x},X) - W^{(g)}_{n+1}(\overset{\mu_s+1}{x},X)\Big)
\end{aligned}
\tag{4-13}
$$

Integrating by parts then yields

$$
\begin{aligned}
& \operatorname*{Res}_{x\to s} K_{\mu_s}(\overset{i_0}{x}_0,x)\Big(\mathcal{Q}_{\mu_s,\mu_s+1} - W^{(0)}_1(\overset{\mu_s+1}{x})W^{(g)}_{n+1}(\overset{\mu_s}{x},X) - W^{(0)}_1(\overset{\mu_s}{x})W^{(g)}_{n+1}(\overset{\mu_s+1}{x},X)\Big) \\
= \; & -\operatorname*{Res}_{x\to s}\Big(W^{(g)}_{n+1}(\overset{\mu_s}{x},X) - W^{(g)}_{n+1}(\overset{\mu_s+1}{x},X)\Big)\Big(W^{(0)}_1(\overset{\mu_s+1}{x}) - W^{(0)}_1(\overset{\mu_s}{x}) + Q\,\partial_x\Big) K_{\mu_s}(\overset{i_0}{x}_0,x)
\end{aligned}
\tag{4-14}
$$

and using the defining differential equations of each $K_{\mu_s}$ we get

$$\begin{aligned}
& \operatorname*{Res}_{x\to s} K_{\mu_s}(\overset{i_0}{x}_0, x)\left(\mathcal{Q}_{\mu_s,\mu_s+1} - W_1^{(0)}(\overset{\mu_s+1}{x})W_{n+1}^{(g)}(\overset{\mu_s}{x},X) - W_1^{(0)}(\overset{\mu_s}{x})W_{n+1}^{(g)}(\overset{\mu_s+1}{x},X)\right) \\
= \; & -\frac{1}{2}\operatorname*{Res}_{x\to s}\left(W_{n+1}^{(g)}(\overset{\mu_s}{x},X) - W_{n+1}^{(g)}(\overset{\mu_s+1}{x},X)\right)\left(G(\overset{i_0}{x}_0,\overset{\mu_s+1}{x}) - G(\overset{i_0}{x}_0,\overset{\mu_s}{x})\right). \qquad \text{(4-15)}
\end{aligned}$$

This implies that we have

$$\begin{aligned}
& 2\sum_{\mu}\oint_{S_\mu} K_{\mu}(\overset{i_0}{x}_0, x)\left(\mathcal{Q}_{\mu,\mu+1} - W_1^{(0)}(\overset{\mu+1}{x})W_{n+1}^{(g)}(\overset{\mu}{x},X) - W_1^{(0)}(\overset{\mu}{x})W_{n+1}^{(g)}(\overset{\mu+1}{x},X)\right) \\
= \; & \sum_{\mu}\oint_{S_\mu} W_{n+1}^{(g)}(\overset{\mu}{x},X)G(\overset{i_0}{x}_0,\overset{\mu}{x}) + \sum_{\mu}\oint_{S_\mu} W_{n+1}^{(g)}(\overset{\mu+1}{x},X)G(\overset{i_0}{x}_0,\overset{\mu+1}{x}) \qquad \text{(4-16)} \\
& -\sum_{\mu}\oint_{S_\mu} W_{n+1}^{(g)}(\overset{\mu}{x},X)G(\overset{i_0}{x}_0,\overset{\mu+1}{x}) - \sum_{\mu}\oint_{S_\mu} W_{n+1}^{(g)}(\overset{\mu+1}{x},X)G(\overset{i_0}{x}_0,\overset{\mu}{x}). \qquad \text{(4-17)}
\end{aligned}$$

Now we notice that $W_{n+1}^{(g)}(\overset{\mu}{x},X)$ has poles at the points of $S_\mu$ and $S_{\mu-1}$, and $G(\overset{i_0}{x}_0,\overset{\mu}{x})$ has poles at $x=x_0$ (simple pole with residue $-\delta_{i_0,\mu}$), and at the points of $S_\mu$ and $S_{\mu-1}$. Moving the integration contours in the second line yields

$$\begin{aligned}
& 2\sum_{\mu}\oint_{S_\mu} K_{\mu}(\overset{i_0}{x}_0, x)\left(\mathcal{Q}_{\mu,\mu+1} - W_1^{(0)}(\overset{\mu+1}{x})W_{n+1}^{(g)}(\overset{\mu}{x},X) - W_1^{(0)}(\overset{\mu}{x})W_{n+1}^{(g)}(\overset{\mu+1}{x},X)\right) \\
= \; & 2\pi i\sum_{\mu}\delta_{\mu,i_0}W_{n+1}^{(g)}(\overset{\mu}{x}_0,X) + \delta_{\mu+1,i_0}W_{n+1}^{(g)}(\overset{\mu+1}{x}_0,X) \\
& -\sum_{\mu}\oint_{S_{\mu-1}} W_{n+1}^{(g)}(\overset{\mu}{x},X)G(\overset{i_0}{x}_0,\overset{\mu}{x}) - \sum_{\mu}\oint_{S_{\mu+1}} W_{n+1}^{(g)}(\overset{\mu+1}{x},X)G(\overset{i_0}{x}_0,\overset{\mu+1}{x}) \\
& -\sum_{\mu}\oint_{S_\mu} W_{n+1}^{(g)}(\overset{\mu}{x},X)G(\overset{i_0}{x}_0,\overset{\mu+1}{x}) - \sum_{\mu}\oint_{S_\mu} W_{n+1}^{(g)}(\overset{\mu+1}{x},X)G(\overset{i_0}{x}_0,\overset{\mu}{x}) \qquad \text{(4-18)} \\
= \; & 4\pi i\, W_{n+1}^{(g)}(\overset{i}{x}_0,X) \\
& -\sum_{\mu}\oint_{S_\mu}\left(W_{n+1}^{(g)}(\overset{\mu}{x},X) + W_{n+1}^{(g)}(\overset{\mu+1}{x},X)\right)\left(G(\overset{i_0}{x}_0,\overset{\mu}{x}) + G(\overset{i_0}{x}_0,\overset{\mu+1}{x})\right). \qquad \text{(4-19)}
\end{aligned}$$

The last term vanishes because it has no pole at $S_\mu$ from the linear loop equation thus proving the theorem. $\square$

## 4.4 Special geometry and free energies

This topological recursion procedure applied to a quantum spectral curve is expected to allow the perturbative reconstruction of the chiral correlation function

$$\left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j)\right\rangle \qquad \text{(4-20)}$$

viewed as a function on the moduli space of all such W-symmetric conformal field theories on the Riemann sphere

$$\mathcal{M}_{W(\widehat{\mathfrak{sl}_d})} \underset{def}{=} \left\{ \left( \{(z_j, \alpha_j) \in \mathbb{CP}^1 \times \mathfrak{h}^*\}_{1 \leq j \leq M}, \mathbf{t} \right) \right\} \tag{4-21}$$

where the *times* $\mathbf{t} \underset{def}{=} \{t_0, t_1, t_2, \dots\}$ generically denote the moduli of the potential $V$ such that $\partial V = W_{0,1} + Y$.

We would then like to define the $\tau$-function $\mathfrak{T}_{\mathbf{z},\alpha,\mathbf{t}}$ associated to the W-symmetric conformal field theory as the function on the moduli space $\mathcal{M}_{W(\widehat{\mathfrak{sl}_d})}$ by

$$\ln \mathfrak{T}_{\mathbf{z},\alpha,\mathbf{t}} \underset{def}{=} \sum_{g=0}^{\infty} \varepsilon^{2g-2} \mathcal{F}_g \tag{4-22}$$

where the genus $g$ free energy $\mathcal{F}_g$ is such that for any deformation $\delta \in T^*\mathcal{M}_{W(\widehat{\mathfrak{sl}_d})}$, we have the special geometry relations

$$\delta \mathcal{F}_g = \int_{\delta^*} W_{g,1} \tag{4-23}$$

where we introduced a *generalized cycle* $\delta^*$ dual to the deformation and defined such that

$$\delta W_{g,n} = \int_{\delta^*} W_{g,n+1} \tag{4-24}$$

for any $g, n \in \mathbb{N}$, $n \neq 0$. This needs a systematic definition and study of deformations and associated cycles that should generalize the one introduced in [17] and [19]. In particular, the additional choices that have to be made to define form-cycle type dualities should correspond to parameters of bases of conformal blocks and the natural conjecture

**Conjecture 4.1** *$\tau$-function as conformal block*

$$\mathfrak{T}_{\mathbf{z},\alpha,\mathbf{t}} = \left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j) \right\rangle \tag{4-25}$$

would then go one step further in performing the conformal bootstrap of W-symmetric conformal field theories. Indeed, the remaining problem would then be to recollect these conformal blocks into single-valued smooth correlation functions.

To motivate this conjecture, let us examine formally how the chiral conformal block $\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j)\rangle$ should transform under variations of the moduli. Recall that the vertex operators of Toda conformal field theory are related to the defining chiral current by

$$V_\alpha(z) \underset{def}{=} : \exp\left(b\left(\alpha, \varphi(z)\right)\right): \tag{4-26}$$

where $\varphi$ is the Toda field locally defined to satisfy $\partial\varphi = \mathbf{J}$. This expression implies

$$\begin{aligned}
\frac{\partial}{\partial z_i} V_{\alpha_j}(z_j) &= \delta_{i,j} : b\left(\alpha_j, \mathbf{J}(z_j)\right) V_{\alpha_j}(z_j): & \text{(4-27)}\\
\frac{\partial}{\partial \alpha_i} V_{\alpha_j}(z_j) &= \delta_{i,j} : b\,\varphi(z_j) V_{\alpha_j}(z_j): & \text{(4-28)}
\end{aligned}$$

which immediately yields

$$\begin{aligned}
\frac{\partial}{\partial z_i} \ln \mathfrak{T}_{\mathbf{z},\alpha,\mathbf{t}} &= b\, \frac{\left\langle :(\alpha_i, \mathbf{J}(z_i)) V_{\alpha_i}(z_i): \prod_{\substack{j=1\\ j\neq i}}^{M} V_{\alpha_j}(z_j)\right\rangle}{\left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j)\right\rangle} & \text{(4-29)}\\
&\underset{def}{=} b\,\mathrm{ev}_{(z_i\cdot\alpha_i)} W_1 & \text{(4-30)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \alpha_i} \ln \mathfrak{T}_{\mathbf{z},\alpha,\mathbf{t}} &= b\, \frac{\left\langle :\varphi(z_i) V_{\alpha_i}(z_i): \prod_{\substack{j=1\\ j\neq i}}^{M} V_{\alpha_j}(z_j)\right\rangle}{\left\langle \prod_{j=1}^{M} V_{\alpha_j}(z_j)\right\rangle} & \text{(4-31)}\\
&\underset{def}{=} b \int_{reg}^{z_i} W_1(\overset{*}{x}) dx & \text{(4-32)}
\end{aligned}$$

where the linear evaluation operator and the regularized integral are defined in the only natural way by taking the normal ordered product of the operators located at colliding insertion points. $W_1(\overset{*}{x})$ is defined as the element of $\mathfrak{h}^*$ satisfying the relation $\left(h_a, W_1(\overset{*}{x})\right) = W_1(\overset{a}{x})$ for any sheet index $a \in \{1, \ldots, d\}$.

Moreover, the wave function reconstructed from topological recursion applied to the quantum spectral curve of the $W(\widehat{\mathfrak{sl}_d})$-symmetric conformal field theory would then be defined as

$$\Psi(D;\varepsilon) \underset{def}{=} \exp\left(\sum_{n=0}^{\infty}\sum_{g=0}^{\infty} \frac{\varepsilon^{2g-2+n}}{n!} \int_D \cdots \int_D W_{g,n}\, \mathrm{d}\,x_1 \cdots \mathrm{d}\,x_n\right) \tag{4-33}$$

and is expected to be related to correlation functions of the theory with insertions of degenerate fields, in the sense that it should satisfy a KZ type equation.

# 5 Conclusion

We have proved that the Ward identities satisfied by chiral correlation functions of $\mathcal{W}(\widehat{\mathfrak{sl}_d})$-symmetric conformal field theories can be solved perturbatively using a generalization of the topological recursion of [14],[23] that applies to quantum curves. This can also be viewed as a quantization of the setup of [4],[5],[6] and [7] and this corresponds to having $Q \neq 0$ in the conformal field theory [21]. In turn, this yielded a conjecture on how to compute the topological (heavy-limit) expansion of the chiral $M$-point functions of the theory. The sequel to this article will provide a matrix model realization of this theory.

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# 6 Appendix

## 6.1 Loop insertion operator

As is customary in conformal field theory, one can define a linear operator acting on the space of chiral correlation functions with current insertions consisting in the insertion of an additional current at a given point on the surface and in a given quantum sheet.

**Definition 6.1** *Loop insertion operator*
*Denoting* $\Sigma' \underset{def}{=} \Sigma - \{z_1, \dots, z_M\}$ *and by* $\widetilde{\Sigma}'$ *its universal covering, for any* $n \in \mathbb{N}$, *any* $\widetilde{x}_1, \dots, \widetilde{x}_n \in \widetilde{\Sigma}'$ *and any sheet indices* $i_1, \dots, i_n \in \{1, \dots, d\}$, *define the action of the loop insertion operator at the point* $\widetilde{x} \in \widetilde{\Sigma}'$ *in sheet* $i \in \{1, \dots, d\}$, *denoted* $\delta^i_{\widetilde{x}}$, *by*

$$\delta^i_{\widetilde{x}} \cdot \left\langle \left\langle \mathbf{J}_{i_1}(\widetilde{x}_1) \cdots \mathbf{J}_{i_n}(\widetilde{x}_n) \right\rangle \right\rangle \underset{def}{=} \left\langle \left\langle \mathbf{J}_i(\widetilde{x}) \mathbf{J}_{i_1}(\widetilde{x}_1) \cdots \mathbf{J}_{i_n}(\widetilde{x}_n) \right\rangle \right\rangle \tag{6-1}$$

**Remark 6.1** *This definition of a linear loop insertion operator has no reason in general to commute with the infinite sums appearing in the topological expansions of the chiral correlation functions with current insertions. We will however assume this fact to hold such that it satisfies*

$$\delta^i_{\widetilde{x}} \cdot \omega_{g,n}(\overset{i_1}{x}_1, \dots, \overset{i_n}{x}_n) = \omega_{g,n+1}(\overset{i}{x}, \overset{i_1}{x}_1, \dots, \overset{i_n}{x}_n) \tag{6-2}$$

*for any generic values of the arguments. In particular,*

$$\delta^{i_1}_{\tilde{x}_1} \cdot Y(\overset{i_2}{x_2}) = B(\overset{i_1}{x_1}, \overset{i_2}{x_2}) \tag{6-3}$$

## 6.2 Differential Hirota identities

The factors $Y_\mu$ appearing in the factorization of the quantum spectral curve are expressed as logarithmic derivatives $Y_\mu = Q\partial \ln \frac{D_{\mu-1}}{D_\mu}$ where $D_\mu$ is the principal minor

$$D_\mu = \begin{pmatrix} \psi_1 & \cdots & \psi_\mu \\ (-Q\partial)\psi_1 & \cdots & (-Q\partial)\psi_\mu \\ (-Q\partial)^2\psi_1 & \cdots & (-Q\partial)^2\psi_\mu \\ \vdots & & \vdots \\ (-Q\partial)^\mu\psi_1 & \cdots & (-Q\partial)^\mu\psi_\mu \end{pmatrix}, \qquad \mu = 1, \dots, d \tag{6-4}$$

**Proposition 6.1** *Differential Hirota identities*
*For any quantum sheet index $\mu \in \{1, \dots, d\}$,*

$$\frac{D_{\mu-1}D_{\mu+1}}{D_\mu^2} \quad = \quad \frac{\mathrm{d}}{\mathrm{d}\,x}\left(\frac{\widehat{D}_\mu}{D_\mu}\right) \tag{6-5}$$

*where $\widehat{D}_\mu$ is the determinant used to compute $D_\mu$ but with $\psi_\mu$ replaced by $\psi_{\mu+1}$ in the last column.*

**proof:**

This is a straightforward computation. Indeed, the identity of the proposition is equivalent to

$$D_{\mu-1}D_{\mu+1} \quad = \quad \widehat{D}'_\mu D_\mu - \widehat{D}_\mu D'_\mu \tag{6-6}$$

which is exactly a well known result from determinant computation in the form

$$\begin{aligned} \begin{pmatrix} & & * & * \\ & & \vdots & \vdots \\ * & \cdots & * & * \\ * & \cdots & * & * \end{pmatrix} \times \begin{pmatrix} & \cdots & \\ \vdots & & \vdots \\ & \cdots & \end{pmatrix} \quad = \quad & \begin{pmatrix} & & & * \\ & & & \vdots \\ & & & * \\ * & \cdots & * & * \end{pmatrix} \times \begin{pmatrix} & & * & \\ & & \vdots & \\ * & \cdots & * & * \\ & & * & \end{pmatrix} \\ & - \begin{pmatrix} & & * & \\ & & \vdots & \\ & & * & \\ * & \cdots & * & * \end{pmatrix} \times \begin{pmatrix} & & * \\ & & \vdots \\ * & \cdots & * \\ & & * \end{pmatrix} \end{aligned} \tag{6-7}$$

where this is to be understood as an algebraic relation between determinants of matrices all obtained from the same $\mu+1\times\mu+1$ (whose determinant is $D_{\mu+1}$, it is the one represented by the second term of the product before the equal sign) by removing pairs of columns and rows. On the picture, the stars $*$ represent columns and/or rows that are removed before taking the determinant. $\square$

**Remark 6.2** *This is the differential version of the QQ-system of difference relations as appearing in the study of quantum integrable systems.*

## 6.3 $K_\mu$ has no monodromy around $S_\mu$

Recall that we defined the kernel $K_\mu$ associated to the set $S_\mu = \{s\in\Sigma' | D_\mu(s)=0\}$ as satisfying the differential equation

$$\left(-2\frac{Q\partial D_\mu}{D_\mu}+\frac{Q\partial D_{\mu+1}}{D_{\mu+1}}+\frac{Q\partial D_{\mu-1}}{D_{\mu-1}}+Q\partial_x\right)K_\mu(\overset{i_0}{x_0},\widetilde{x}) = G(\overset{i_0}{x_0},\overset{\mu+1}{x}) - G(\overset{i_0}{x_0},\overset{\mu}{x}) \tag{6-8}$$

As was noted before, this defines $K_\mu$ up to terms of the form $f(\overset{i_0}{x_0})\frac{D_\mu(x)^2}{D_{\mu-1}(x)D_{\mu+1}(x)}$ that are irrelevent when computing the $W_{g,n}$'s. The generic solution for $K_\mu$ is therefore of the form

$$\begin{aligned} K_\mu(\overset{i_0}{x_0},\widetilde{x}) &= \frac{D_\mu(\widetilde{x})^2}{D_{\mu-1}(\widetilde{x})D_{\mu+1}(\widetilde{x})}\int^{\widetilde{x}}\frac{D_{\mu-1}(\widetilde{x}')D_{\mu+1}(\widetilde{x}')}{D_\mu(\widetilde{x}')^2}\left(G(\overset{i_0}{x_0},\overset{\mu+1}{x'}) - G(\overset{i_0}{x_0},\overset{\mu}{x'})\right) \\ &+ f(\overset{i_0}{x_0})\,\frac{D_\mu(x)^2}{D_{\mu-1}(x)D_{\mu+1}(x)} \end{aligned} \tag{6-9}$$

The choice of function $f$ can be reabsorbed into a $\overset{i_0}{x_0}$-dependent choice of starting point for the integral. The holomorphic condition we wish to impose on $K_\mu$ ensures it can be integrated globally along a path surrounding the Bethe roots lying in $S_\mu$ and none of the others (the ones in $S-S_\mu$). This is equivalent to requiring that the residue of the integrand appearing in the last equality vanishes at any Bethe root $s\in S_\mu$.

**Theorem 6.1** *Bethe equations*
*For any generic point $\overset{i_0}{x_0}$ in the quantum covering and any Bethe root $s\in S_\mu$,*

$$\mathbf{BE}_{\mu,s}(\overset{i_0}{x_0}) \underset{def}{=} \operatorname*{Res}_{x=s}\left(\frac{D_{\mu-1}(\widetilde{x})D_{\mu+1}(\widetilde{x})}{D_\mu(\widetilde{x})^2}\left(G(\overset{i_0}{x_0},\overset{\mu+1}{x}) - G(\overset{i_0}{x_0},\overset{\mu}{x})\right)\right) = 0 \tag{6-10}$$

**proof:**

Recall that the loop insertion operator is such that $\delta_{\widetilde{x}_1}^{i_1} \cdot Y(\overset{i_2}{x}_2) = B(\overset{i_1}{x}_1, \overset{i_2}{x}_2)$. Taking a primitive on both sides of this equality yields $\delta_{\widetilde{x}_1}^{i_1} \cdot \ln \frac{D_{\mu-1}(\widetilde{x}_2)}{D_\mu(\widetilde{x}_2)} = G(\overset{i_1}{x}_1, \overset{i_2}{x}_2)$ and in turn

$$G(\overset{i_0}{x}_0, \overset{\mu+1}{x}) - G(\overset{i_0}{x}_0, \overset{\mu}{x}) = \delta_{\widetilde{x}_0}^{i_0} \ln \frac{D_{\mu-1}(\widetilde{x}) D_{\mu+1}(\widetilde{x})}{D_\mu(\widetilde{x})^2} \tag{6-11}$$

Replacing this into the expression appearing in the statement of the theorem implies

$$\begin{aligned} \mathbf{BE}_{\mu,s}(\overset{i_0}{x}_0) &= \operatorname*{Res}_{x=s} \left( \frac{D_{\mu-1}(\widetilde{x}) D_{\mu+1}(\widetilde{x})}{D_\mu(\widetilde{x})^2} \delta_{\widetilde{x}_0}^{i_0} \ln \frac{D_{\mu-1}(\widetilde{x}) D_{\mu+1}(\widetilde{x})}{D_\mu(\widetilde{x})^2} \right) && \text{(6-12)} \\ &= \delta_{\widetilde{x}_0}^{i_0} \left( \operatorname*{Res}_{x=s} \frac{D_{\mu-1}(\widetilde{x}) D_{\mu+1}(\widetilde{x})}{D_\mu(\widetilde{x})^2} \right) && \text{(6-13)} \end{aligned}$$

which is indeed equal to zero by the differential Hirota identities since it allows to realize the term whose residue has to be taken as a total derivative. □

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