Beyond Amplitudes' Positivity and the Fate of Massive Gravity

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We constrain effective field theories by going beyond the familiar positivity bounds that follow from unitarity, analyticity, and crossing symmetry of the scattering amplitudes. As interesting examples, we discuss the implications of the bounds for the Galileon and ghost-free massive gravity. The latter is ruled out by our theoretical bounds when combined with the experimental constraints on the graviton mass and from fifth-force experiments, given the impossibility to consistently implement the Vainshtein mechanism. We also show that the Galileon theory must contain symmetry-breaking terms that are at most one-loop suppressed compared to the symmetry-preserving ones.

I. INTRODUCTION AND SUMMARY

The idea that physics at low energy can be described in terms of light degrees of freedom alone is one of the most satisfactory organising principle in physics, which goes under the name of Effective Field Theory (EFT). A quantum field theory (QFT) can be viewed as the trajectory in the renormalization group (RG) flow from one EFT to another one, each being well described by an approximate fixed point where the local operators are classified mainly by their scaling dimension. The effect of ultraviolet (UV) dynamics is systematically accounted for in the resulting infrared (IR) EFT by integrating out the heavy degrees of freedom which generate an effective Lagrangian made of infinitely many local operators. Yet, EFTs are predictive even when the UV dynamics is unknown, because in practice only a finite number of operators contributes, at a given accuracy, to observable quantities. The higher the operator dimension, the smaller the effect at low energy.

In fact, extra information about the UV is always available: the underlying Lorentz invariant microscopic QFT is unitary, causal and local. These principles are stirred in the fundamental properties of the S-matrix such as unitarity, analyticity, crossing symmetry, and polynomial boundedness. These imply a UV-IR connection in the form of dispersion relations that link the (forward) amplitudes in the deep IR with the discontinuity across the branch cuts integrated all the way to infinite energy [1, 2]. Unitarity ensures the positivity of such discontinuities, and in turn the positivity of (certain) Wilson coefficients associated to the operators in the EFT Lagrangian. This UV-IR connection can be used to show that "wrong-sign" Wilson coefficients in the IR can not be generated by a Lorentz invariant, unitary, casual, and local UV completion, as it was emphasised e.g. in [3]. These positivity bounds have found several applications, including the proof of the a-theorem [4, 5], the study of chiral perturbation theory [6], WW-scattering and theories of composite Higgs [7–11], as well as bounds on quantum gravity [12], massive gravity [13–15], Galileons [15–18], inflation [19, 20], the weak gravity conjecture [21, 22] and conformal field theory [23–25]. The approach has been recently extended to particles of arbitrary spin [15], with applications to massive gravity and the EFT of a Goldstino [26–28], and the formulation of a general no-go theorem on the leading energy-scaling behavior of the amplitudes in the IR [15]. Ref.'s [16, 29, 30] extended this technique beyond the forward limit, providing an infinite series of positivity constraints for amplitudes of arbitrary spin.

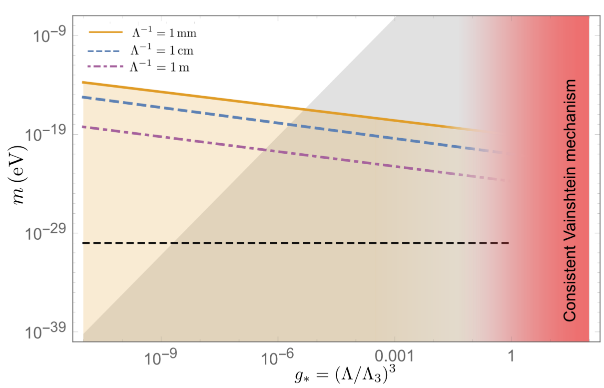

FIG. 1. Exclusion region for massive gravity. The gray region is theoretically excluded by our lower bound Eq. (38) with accuracy $\delta = 1\%$. The colored lines represent the physical cutoff away from the red region, where a Vainshtein mechanism could be consistently implemented. Fifth-force experiments probe the mm scale (excluding the orange region); the horizontal dashed line represents the experimental upper bound on the graviton mass.

In this paper we show that bounds stronger than standard positivity constraints can be derived by taking into account the irreducible IR cross-sections under the dispersive integral, which are calculable within the EFT. In models where the forward amplitude is suppressed (e.g. Galileons), or the high-energy scattering is governed by strong, but soft, dynamics (massive gravity, dilaton, WZW-like theories [38]), as well as models with suppressed $2 \to 2$ (but enhanced $2 \to 3$), our bounds are dramatically stronger. These bounds can be used to place rigorous upper limits on the cutoff scale for certain EFT's or constrain the relevant couplings, in a way that is somewhat reminiscent of the revived S-matrix bootstrap approach in four dimensions [31]. The procedure we use was originally suggested in [17], and later employed to estimate order-of-magnitude bounds [15, 16]; here we extend these arguments to sharp inequalities and bring this technique beyond amplitudes' positivity.

We discuss explicitly two relevant applications of the bounds: the EFT for a weakly broken Galileon [32, 33], and the ghost-free massive gravity theory [34, 35], known also as dRGT massive gravity, or Λ_3-theory (Λ_3 is the mass scale that remains in the decoupling limit for the scalar Galileon mode). Despite the encouraging recent results on the positivity conditions that ghost-free massive gravity must satisfy [13], our constraints provide a much stronger, and yet theoretically robust, *lower bound* on the graviton mass m. Indeed, our dispersion relations imply that the forward elastic amplitudes, that are *suppressed* by m at fixed Λ_3, must nevertheless be larger than a factor times the *unsuppressed* elastic or inelastic cross-sections. Resolving this tension requires a non-trivial lower bound for the graviton mass. Under the customarily accepted assumption that Λ_3 is the cutoff of the theory in Minkowski background, i.e. away from all massive sources, this lower bound reads $m \gtrsim 100$ keV, which is grossly excluded observationally. Even relaxing this assumption and lowering the cutoff even further (i.e. taking hierarchically separated values for the actual cutoff Λ and the scale Λ_3 evaluated in Minkowski), we show that the dRGT massive gravity theory does not survive the combination of our bound with the experimental constraints on the graviton mass, unless a (presently unknown) mechanism other than the Vainshtein screening is at play. We anticipate these results in Fig. 1.

In the following, we begin by deriving the new bounds in full generality, and then apply them to the Galileon theory, showing that Galileon-symmetry-breaking terms can not be arbitrarily small. This naturally leads us to ghost-free massive gravity, where we find the most dramatic implications of the bounds; other applications are discussed in the conclusion.

II. DISPERSION RELATIONS

Let us consider the center-of-mass 2-to-2 scattering amplitude $\mathcal{M}^{z_1z_2z_3z_4}(s,t)$ where the various polarization functions are labeled z_i. The Mandelstam variables[1] are defined by $s = -(k_1+k_2)^2$, $t = -(k_1+k_3)^2$, $u = -(k_1+k_4)^2$ and satisfy $s+t+u = 4m^2$, where m is the mass of the scattered particles (all of the same species for easy of presentation). Our arguments will require finite $m \neq 0$. Yet, they hold even for some massless theories (scalars, spin-1/2 fermions, and softly broken $U(1)$ gauge theories), which have a smooth limit $m \to 0$ at least for the highest helicities, so that the bound can be derived with an arbitrarily small but finite mass, before taking the zero limit. We call,

$$\mathcal{M}^{z_1z_2}(s) \equiv \mathcal{M}^{z_1z_2z_1z_2}(s, t=0)\,, \tag{1}$$

the forward elastic amplitude at $t = 0$, and integrate $\mathcal{M}^{z_1z_1}(s)/(s-\mu^2)^3$ along a closed contour Γ in the complex s-plane, enclosing all the physical IR-poles s_i associated to the stable light degrees of freedom exchanged in the scattering (or its crossed-symmetric process), together with the point $s = \mu^2$ lying on the real axis between $s = 0$ and $s = 4m^2$,

$$\Sigma_{\rm IR}^{z_1z_2} \equiv \frac{1}{2\pi i}\oint_\Gamma ds \frac{\mathcal{M}^{z_1z_2}(s)}{(s-\mu^2)^3}\,, \tag{2}$$

see Fig. 2. The $\Sigma_{\rm IR}^{z_1z_2}$ is nothing but the sum of the IR residues,

$$\Sigma_{\rm IR}^{z_1z_2} = \sum \operatorname*{Res}_{s=s_i,\mu^2}\left[\frac{\mathcal{M}^{z_1z_2}(s)}{(s-\mu^2)^3}\right]\,, \tag{3}$$

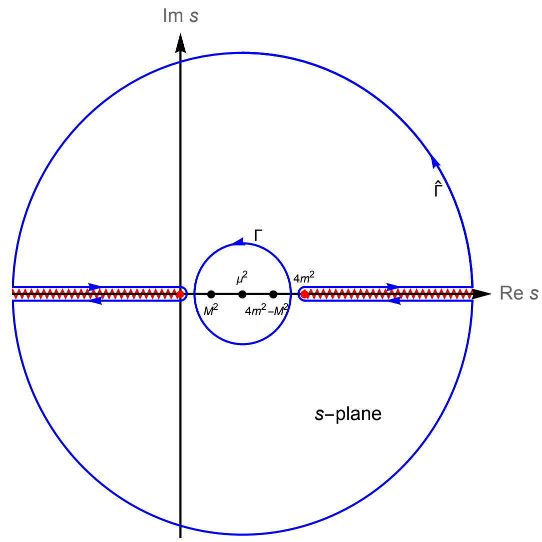

FIG. 2. Integration contours in the complex s-plane at fixed $t = 0$, with poles at $s_1 = M^2$ and $s_2 = 4m^2 - M$. The point $s = \mu^2$ is on the real axis between the branch-cuts shown in red.

[1] We use the mostly-plus Minkowski metric $(-,+,+,+)$, work with the relativistic normalization of one-particle states $\langle p, z|p'z'\rangle = (2\pi)\delta^3(p-p')2E(p)\delta_{zz'}$, and define the $\mathcal{M}$ operator from the S-matrix operator, $S = 1 + (2\pi)^4\delta^4(\sum k_i)i\mathcal{M}$.

and it is therefore calculable within the EFT. Using the Cauchy integral theorem we deform the contour integral into $\widehat{\Gamma}$ that runs just around the s-channel and u-channel branch-cuts, and goes along the big circle eventually sent to infinity.

The polynomial in the denominator of Eq. (2) has the lowest order that ensures the convergence of the dispersive integral in the UV, a consequence of the Froissart-Martin asymptotic bound $|\mathcal{M}(s\to\infty)| < \text{const}\cdot s\log^2 s$, which is always satisfied in any local massive QFT [36, 37]. Thus $\lim_{s\to\infty}|\mathcal{M}(s)|/s^2 \to 0$, and we can drop this contribution and write $\Sigma_{IR}^{z_1z_2}$ as an integral of the discontinuity $\text{Disc}\mathcal{M}^{z_1z_2}(s) \equiv \mathcal{M}^{z_1z_2}(s+i\epsilon) - \mathcal{M}^{z_1z_2}(s-i\epsilon)$ along the branch-cuts,

$$\Sigma_{\text{IR}}^{z_1z_2} = \frac{1}{2\pi i}\left(\int_{4m^2}^{\infty} ds + \int_{-\infty}^{0} ds\right)\frac{\text{Disc}\mathcal{M}^{z_1z_2}(s)}{(s-\mu^2)^3}\,. \quad (4)$$

The integral along the u-channel branch-cut runs over non-physical values of $s=(-\infty,0)$, but can be expressed in terms of another physical amplitude, involving anti-particles (identified by a bar over the spin label, i.e. $\bar{z}$), and related to the former by crossing. Indeed, crossing particle 1 and 3 in the forward elastic limit $t=0$, implies [15],

$$\begin{aligned}&\mathcal{M}^{z_1z_2}(s) = \mathcal{M}^{-\bar{z}_1z_2}(u=-s+4m^2) \quad \text{(helicity basis)}\\ &\mathcal{M}^{z_1z_2}(s) = \mathcal{M}^{\bar{z}_1z_2}(u=-s+4m^2) \quad \text{(linear basis)}\end{aligned}$$

We will work in the helicity basis notation and recall that for $-\bar{z}\to\bar{z}$ we recover the results for linear polarizations. Moreover, for particles that are their own antiparticles, $\bar{z}=z$.

Finally, amplitudes are real functions of complex variables, i.e. $\mathcal{M}(s)^* = \mathcal{M}(s^*)$, so that the discontinuities are proportional to the imaginary part, and one obtains the dispersion relation between IR and UV:

$$\Sigma_{\text{IR}}^{z_1z_2} = \int_{4m^2}^{\infty}\frac{ds}{\pi}\left(\frac{\text{Im}\mathcal{M}^{z_1z_2}(s)}{(s-\mu^2)^3} + \frac{\text{Im}\mathcal{M}^{-\bar{z}_1z_2}(s)}{(s-4m^2+\mu^2)^3}\right)\,. \quad (5)$$

III. POSITIVITY AND BEYOND

Unitarity of the S-matrix implies the optical theorem,

$$\text{Im}\mathcal{M}^{z_1z_2}(s) = s\sqrt{1-4m^2/s}\cdot\sigma_{\text{tot}}^{z_1z_2}(s) > 0\,, \quad (6)$$

where $\sigma_{\text{tot}}^{z_1z_2}(s)$ is the total cross-section $\sigma_{\text{tot}}^{z_1z_2} = \sum_X \sigma^{z_1z_2\to X}$. So, the imaginary parts in the integrand Eq. (5) are strictly positive for any theory where particles 1 and 2 are interacting, as long as $0<\mu^2<4m^2$. Thus one obtains the rigorous positivity bound,

$$\Sigma_{\text{IR}}^{z_1z_2} > 0\,. \quad (7)$$

Since $\Sigma_{\text{IR}}^{z_1z_2}$ is calculable in the IR in terms of the Wilson coefficients, Eq. (7) provides a non-trivial constraint on the EFT.

As a simple example consider the theory of a pseudo-Goldstone boson π, from an approximate global $U(1)$ symmetry which is broken spontaneously in the IR. The effective Lagrangian reads $\mathcal{L}_{\text{EFT}} = -\frac{1}{2}(\partial\pi)^2 + \frac{\lambda}{\Lambda^4}[(\partial\pi)^2 + \ldots]^2 - \epsilon^2\pi^2\left(\Lambda^2 + c(\partial\pi)^2 + \ldots\right)$, where Λ is the cutoff and $\lambda\sim O(1)$ (or even larger should the underlying dynamics be strongly coupled). The parameters that break the approximate Goldstone shift symmetry $\pi\to\pi+const$ are instead suppressed, naturally, by $\epsilon\ll 1$. In any case, from an EFT point of view, both signs of λ are consistent with the symmetry; however $\Sigma_{\text{IR}} = \lambda/2\Lambda^4$, so that only $\lambda>0$ is consistent with the positivity bound (7). Unitary, local, causal and Lorentz invariant UV completions can generate only positive λ in the IR [3]. Important for our arguments below, is that this statement is irrespective of the soft deformations $\sim\epsilon$: the limit $\epsilon\to 0$ is smooth.

Like in the previous example, Σ_{IR} is often calculable within the tree-level EFT where the only discontinuity in the amplitude $\mathcal{M}^{\text{EFT}}$ are simple poles. In such a case we can use again Cauchy theorem on the tree-level EFT amplitude so that Σ_{IR} is more promptly calculated as minus the residue at infinity [13],

$$\Sigma_{\text{IR}}^{z_1z_2} = -\operatorname*{Res}_{s=\infty}\left[\frac{\mathcal{M}^{\text{EFT}}(s)}{(s-\mu^2)^3}\right] \quad (8)$$

up to small tiny corrections. In addition, for amplitudes that scale as $\mathcal{M}^{\text{EFT}}(s)\sim s^2$ for large s and $t=0$ (as in e.g. the Galileon and ghost-free massive gravity), we have $\Sigma_{\text{IR}}^{z_1z_2} = 1/2(\partial^2\mathcal{M}^{\text{EFT}}/\partial s^2)|_{m^2\ll s}$. In this case, the left-hand side of the dispersion relation (5) is μ^2-independent and one can thus drop μ^2 from the right-hand side too.

So far we invoked very general principles of QFT and derived positivity constraints on EFT's. We can in fact extract more than positivities by noticing that the total cross-section on the right-hand side of the dispersion relation contains an irreducible contribution from the IR physics, which is calculable within the EFT by construction. The other contributions, e.g. those from the UV, are uncalculable with the EFT but are nevertheless always strictly positive, by unitarity. Moreover, each final state X in the total cross-section contributes positively too. Therefore, an exact inequality follows from truncating the right-hand side of (5) at some energy $E^2\ll\Lambda^2$ below the cutoff Λ of the EFT,

$$\Sigma_{\rm IR}^{z_1 z_2} > \sum_X \int_{4m^2}^{E^2} \frac{ds}{\pi}\sqrt{1-4\frac{m^2}{s}}\left[\frac{s\sigma^{z_1 z_2\to X}(s)}{(s-\mu^2)^3}+\frac{s\sigma^{-\bar z_1 z_2\to X}(s)}{(s-4m^2+\mu^2)^3}\right]_{\rm IR} . \tag{9}$$

Both sides are now calculable, hence the subscript IR. The $\Sigma_{\rm IR}^{z_1z_2}$ must not only be positive but *strictly bigger* than something which is itself positive and calculable within the EFT. Moreover we can retain any subset X of final states, independently on whether they are elastic or inelastic: the more channels and information are retained in the IR the more refined the resulting bound will be.

The information provided by our bound (9), is maximized in theories where the elastic forward amplitude $\mathcal{M}^{z_1z_2}$, which appears in the lefthand side, is parametrically suppressed compared to the non-forward or inelastic ones (that is $\mathcal{M}^{z_1z_2z_1z_2}(s,t\neq 0)$, $\mathcal{M}^{z_1z_2z_3z_4}(s,t)$, or more generally $\mathcal{M}^{z_1z_2\to X}$), that appear in the righthand side. This tension results in constraints on the couplings and masses of the EFT, that include and go beyond the positivity of $\Sigma_{\rm IR}$. Galileons, for examples, have a suppressed forward amplitude: the would-be leading *stu*-term actually vanishes at $t=0$ and the amplitude is thus sensitive to the small Galileon-symmetry breaking terms. On the other hand, neither the Galileon elastic cross-section nor the right-hand side of (9) are suppressed. Massive gravity, dilaton, WZW-like theories [38], as well as other models where $2\to 2$ is suppressed while $2\to 3$ is not, are other simple examples of theories that get non-trivial constraints from our bound Eq. (9). Even in situations without parametric suppression, our bound carries important information: it links elastic and inelastic cross sections that might depend on different Wilson coefficients in the EFT.

Amplitudes in an EFT means finite, yet systematically improvable, accuracy δ in the calculation. The main source of error for small masses is the truncation of the tower of higher dimensional operators. For example, working to the leading order (LO) in powers of $(E/\Lambda)^2$, $(m/E)^2$ (and hence $(\mu/E)^2$), the bound (9) takes a simpler form

$$\Sigma_{\rm IR,LO}^{z_1z_2} > \sum_X \int^{E^2}\frac{ds}{\pi s^2}\left[\sigma^{z_1z_2\to X}(s)+\sigma^{z_1-\bar z_2\to X}(s)\right]_{\rm IR,LO} \times\left[1+o\left(\frac{m}{E}\right)^2+o\left(\frac{E}{\Lambda}\right)^2\right] \tag{10}$$

where the error from the truncation

$$o\left(\frac{E}{\Lambda}\right)^2=\left(c_{\rm UV}+o(1)\frac{g_*^2}{16\pi^2}\ln\frac{E}{\Lambda}\right)\left(\frac{E}{\Lambda}\right)^2+\dots \tag{11}$$

is controlled by the (collective) coupling g_* of the IR-theory that renormalizes the higher dimensional operators that come with (unknown) Wilson coefficients $c_{\rm UV}\sim o(1)$.[2] The IR-running effects from Λ to E are an irreducible (yet improvable) source of error, whereas the UV contribution is model dependent.

Choosing E at or slightly below the cutoff scale Λ gives just an order of magnitude estimate for the bound [15, 16]. A rigorous bound can instead be defined even for large couplings $g_*\sim 4\pi$ and $c_{\rm UV}\sim 1$, just by choosing a sufficiently small $(E/\Lambda)^2$. Percent accuracy can be achieved already with $E/\Lambda\sim 1/10$. Of course, nothing except more demanding calculations prevents us to reduce the error by working to all order in the mass, or including next-to-next-to…next-to-LO corrections so that the truncation of the EFT (or the running couplings) affects the result only by an even smaller relative error, loops − factors × $o(E/\Lambda)^n$ [3].

IV. GALILEON

Let us consider the amplitude

$$\mathcal{M}(s,t)=g_*^2\left[-3\frac{stu}{\Lambda^6}+\epsilon^2\frac{s^2+t^2+u^2}{2\Lambda^4}+\dots\right] \tag{12}$$

for a single scalar π whose hard-scattering limit is $o(s^3)$, whereas the forward scattering is $o(s^2)$ and suppressed by $\epsilon^2\ll 1$. The cutoff Λ corresponds to a physical threshold for new states propagating on-shell, i.e. the location of the first non-analyticity in the complex s-plane which is not accounted by loops of π. We have factored out the overall coupling constant g_*^2 to make clear the distinction between the physical cutoff Λ and other scales not associated to physical masses, such as decay constants, see Appendix A.

Eq. (12) gives $\Sigma_{\rm IR}=g_*^2\epsilon^2/\Lambda^4$ and $\sigma^{\pi\pi\to\pi\pi}=3g_*^4s^5/(320\pi\Lambda^{12})+\dots$ [4] so that the bound (10) is

$$\epsilon^2 > \frac{3}{40}\left(\frac{g_*^2}{16\pi^2}\right)\left(\frac{E}{\Lambda}\right)^8 , \tag{13}$$

[2] In our perspective, $c_{\rm UV}\gg 1$ would just signal the misidentification of what the actual LO hard-scattering amplitude is, and would require inclusion of the operators with large $c_{\rm UV}$ within the LO amplitude.

[3] In addition, the LO may possibly receive corrections from the logarithmic running of LO couplings. In the examples where our bounds are interesting, symmetry are often at play and the LO operators do not actually get renormalized, except from small explicit breaking effects.

[4] Curiously, there is a mild violation of the naive-dimensional analysis estimate $\epsilon^2_{\rm NDA} > \left(9g_*^2/16\pi^2\right)(E/\Lambda)^8$ [15] due to a 10% cancellation in the phase-space integral $1/2\int_{-1}^1 d\cos\theta|stu|^2$ which returns $s^6(1/3+1/5-1/2)=s^6/30$ rather than $o(1)s^6$.

up to the relative error (11). The lesson to be learnt here is that s^2-terms in the amplitude can not be too much suppressed compared to the the s^3-terms.

Choosing e.g. a 30%-accuracy on the overall factor $(E/\Lambda)^8 \sim 10^{-2}$ (loosely speaking corresponding to a "3-sigma" claim), one gets that $\epsilon^2 > 10^{-3}(1 \pm 30\%)$ for a fully strongly coupled theory $g_* = 4\pi$. A claim valid at "1-sigma" corresponds to setting $E \sim \Lambda$, that is accepting $o(1)$ corrections: $\epsilon^2 \gtrsim g_*^2/(16\pi^2)$.

The weakly broken Galileon Lagrangian [32, 33]

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\pi)^2\big[1 + \frac{c_3}{\Lambda^3}\Box\pi + \frac{c_4}{\Lambda^6}\left((\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2\right) + c_5\,(\ldots)\,\big] + \frac{\lambda}{4\Lambda^4}[(\partial\pi)^2]^2 - \frac{m^2}{2}\pi^2 \quad (14)$$

has suppressed Galileon symmetry-breaking terms $\lambda \ll c_3^2, c_4$ and $m^2 \ll \Lambda^2$. It reproduces the scattering amplitude (12) with the identification

$$c_3^2 - 2c_4 = 4g_*^2\,, \qquad \frac{\lambda}{\Lambda^4} + \frac{c_3^2 m^2}{2\Lambda^6} = \frac{g_*^2\epsilon^2}{\Lambda^4} = \Sigma_{\rm IR}\,. \quad (15)$$

In the massless limit, or more generally for $c_3^2 m^2/\Lambda^2 \ll \lambda$ (which is fully natural given that λ preserves a shift symmetry while m^2 does not), the bound (13) tells that λ is not only positive, but (parametrically) at most one-loop factor away from $(c_3^2 - 2g_4)/4$

$$\lambda > \frac{3}{640}\frac{\left(c_3^2 - 2c_4\right)^2}{16\pi^2}\left(\frac{E}{\Lambda}\right)^8 . \quad (16)$$

For a massive Galileon with negligible λ one gets that $c_3 > 0$ and that the mass is bounded below

$$m^2 > \Lambda^2\left(\frac{3}{320}\right)\frac{\left(c_3 - 2c_4/c_3\right)^2}{16\pi^2}\left(\frac{E}{\Lambda}\right)^8 \quad (17)$$

where $(E/\Lambda)^8 \sim 10^{-2}$ for a 30% accuracy on the overall factor.

V. MASSIVE GRAVITY

The previous bounds on Galileons are unfortunately not directly applicable to cosmological models of modified gravity, which contain other IR degrees of freedom affecting $\Sigma_{\rm IR}$ significantly, such as e.g. a massless graviton like in Horndeski theories [39]. In that case both size of the inequality would be ill-defined at the Coulomb singularity $t = 0$ because of the massless spin-2 state exchanged in the t-channel. Alternative ideas or extra assumptions are needed for a massless graviton, see e.g. [12, 40, 47, 48].

In a massive gravity theory the situation is instead more favourable as a finite graviton mass has a double role: it regulates the IR singularity and tips the s^2-term (vanishing in the forward and decoupling limit) to either positive or negative values, depending on the parameters of the theory that get therefore constrained by the positivity of $\Sigma_{\rm IR} > 0$ [13]. Notice that one can not directly interpret the results obtained above for the scalar Galileon mode as the longitudinal component of the massive graviton as the IR dynamics is different: for example, the helicity-2 mode in t-channel gives a contribution to the amplitude that is as large as the contribution from the Galileon scalar modes. Ghost-free massive gravity has to be studied in its completeness [34, 35] (for reviews see [41, 42]),

$$S = \int d^4x\sqrt{-g}\left[\frac{m_{\rm Pl}^2}{2}R - \frac{m_{\rm Pl}^2 m^2}{8}V(g,h)\right] \quad (18)$$

where $m_{\rm Pl} = (8\pi G)^{-1/2}$ is the reduced Planck mass, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is an effective metric written in term of the Minkowski metric $\eta_{\mu\nu}$ (with mostly + signature) and a spin-2 graviton field $h_{\mu\nu}$ in the unitary gauge, R is the Ricci scalar for $g_{\mu\nu}$, and $V(g,h) = V_2 + V_3 + V_4$ is the soft graviton potential

$$V_2(g,h) = b_1\langle h^2\rangle + b_2\langle h\rangle^2 \quad (19)$$
$$V_3(g,h) = c_1\langle h^3\rangle + c_2\langle h^2\rangle\langle h\rangle + c_3\langle h\rangle^3 \quad (20)$$
$$V_4(g,h) = d_1\langle h^4\rangle + d_2\langle h^3\rangle\langle h\rangle + d_3\langle h^2\rangle^2 + d_4\langle h^2\rangle\langle h\rangle^2 + d_5\langle h\rangle^4 \quad (21)$$

with $\langle h\rangle \equiv h_{\mu\nu}g^{\mu\nu}$, $\langle h^2\rangle \equiv g^{\mu\nu}h_{\nu\rho}g^{\rho\sigma}h_{\sigma\mu}, \ldots$ The coefficients depend on just two parameters, c_3 and d_5, after imposing the ghost-free conditions

$$b_1 = 1\,,\ b_2 = -1 \quad (22)$$
$$c_1 = 2c_3 + \frac{1}{2}\,,\ c_2 = -3c_3 - \frac{1}{2}\,, \quad (23)$$
$$d_1 = -6d_5 + \frac{3}{2}c_3 + \frac{5}{16}\,,\ d_2 = 8d_5 - \frac{3}{2}c_3 - \frac{1}{4}\,, \quad (24)$$
$$d_3 = 3d_5 - \frac{3}{4}c_3 - \frac{1}{16}\,,\ d_4 = -6d_5 + \frac{3}{4}c_3\,. \quad (25)$$

Since the graviton is its own antiparticle, it is convenient to work with linear polarizations that make the crossed amplitudes, and in turn the bound, neater [12, 13, 15]. For example, the LO bound with linear polarizations reads

$$\Sigma_{\rm IR\,,LO}^{z_1z_2} > \sum_X \frac{2}{\pi}\int^{E^2}\frac{ds}{s^2}\left[\sigma^{z_1z_2\to X}(s)\right]_{\rm IR\,,LO}\,. \quad (26)$$

Adopting the basis of polarizations reported in the appendix B, we have two tensor polarizations (T, T') that do not grow with energy, two vector polarizations (V, V') that grow linearly with energy, and one scalar polarization (S) that grows quadratically with the energy.

We calculate the amplitudes for different initial and final state configurations and find that $\Sigma_{\rm IR}^{z_1z_2} \sim m^2/\Lambda_3^6$ is suppressed by the small graviton mass, where $\Lambda_3 \equiv (m^2 m_{\rm Pl})^{1/3}$. On the other hand, we find that the cross-sections are not generically suppressed by m: hence, a small mass is at odd with our bound Eq. (26). Resolving this tension results into non-trivial constraints on the theory, beyond the positivity bounds derived in [13]. In particular, the amplitudes for SS, VV, VV', VS elastic scatterings, have the following suppressed residues,

$$\Sigma_{\rm IR}^{SS} = \frac{2m^2}{9\Lambda_3^6}\left(7-6c_3(1+3c_3)+48d_5\right) > 0 \qquad (27)$$

$$\Sigma_{\rm IR}^{VV} = \frac{m^2}{16\Lambda_3^6}\left(5+72c_3-240c_3^2\right) > 0 \qquad (28)$$

$$\Sigma_{\rm IR}^{VV'} = \frac{m^2}{16\Lambda_3^6}\left(23-72c_3+144c_3^2+192d_5\right) > 0 \qquad (29)$$

$$\Sigma_{\rm IR}^{VS} = \frac{m^2}{48\Lambda_3^6}\left(91-312c_3+432c_3^2+384d_5\right) > 0\,. \qquad (30)$$

In contrast the hard-scattering limits of the amplitudes, that enters the RHS of Eq. (26), is unsuppressed. For $s, t \gg m^2$,

$$\begin{aligned}
\mathcal{M}^{SSSS} &= \frac{st(s+t)}{6\Lambda_3^6}\left(1-4c_3(1-9c_3)+64d_5\right)\\
\mathcal{M}^{VVVV} &= \frac{9st(s+t)}{32\Lambda_3^6}(1-4c_3)^2\\
\mathcal{M}^{VV'VV'} &= \frac{3t^3}{32\Lambda_3^6}(1-4c_3)^2\\
\mathcal{M}^{VSVS} &= \frac{3t}{4\Lambda_3^6}\Big(c_3(1-2c_3)(s+st-t^2)\\
&\qquad -\frac{5s^2+5st-9t^2}{72}\Big)
\end{aligned} \qquad (31)$$

It is convenient to recall also the bound

$$\frac{m^2}{36\Lambda_3^6}\left(35+60c_3-468c_3^2-192d_5\right) > 0 \qquad (32)$$

which follows from the positivity of the residue of maximally-mixed ST polarizations, i.e. (with a slighly abuse of notation) $\Sigma_{\rm IR}^{TT}+\Sigma_{\rm IR}^{SS}+2\Sigma_{\rm IR}^{TSTS}+4\Sigma_{\rm IR}^{TTSS} > 0$, where the expressions for these $\Sigma_{\rm IR}$, that we have explicitly reproduced, are given in [13].

At this point we chose the arbitrary energy scale E of Eq. (26) between the cutoff, $E \ll \Lambda$, so that the EFT calculation for the right-hand side of (26) is trustworthy, and the mass $E \gg m$, so that the EFT hard-scattering amplitudes Eq. (31) are dominating the cross-sections. We define,

$$\delta \equiv \left(\frac{E}{\Lambda}\right)^2 \qquad (33)$$

that controls the accuracy of the EFT calculation, and obtain,

$$F_i(c_3,d_5) > \left(\frac{4\pi m_{\rm Pl}}{m}\right)\left(\frac{g_*}{4\pi}\right)^4 \delta^6\,, \qquad (34)$$

where we have defined

$$g_* \equiv \left(\frac{\Lambda}{\Lambda_3}\right)^3\,. \qquad (35)$$

The functions $F_i(c_3,d_5)$ are

$$\begin{aligned}
F_{SS} &= \left[960\,\frac{7-6c_3(1+3c_3)+48d5}{\left(1-4c_3(1-9c_3)+64d_5\right)^2}\right]^{3/2},\\
F_{VV} &= \left[\left(\frac{2560}{27}\right)\frac{5+72c_3-240c_3^2}{(1-4c_3)^4}\right]^{3/2},\\
F_{VV'} &= \left[\left(\frac{896}{9}\right)\frac{23-72c_3+144c_3^2+192d_5}{(1-4c_3)^4}\right]^{3/2},\\
F_{VS} &= \left[\frac{80640\left(91-312c_3+432c_3^2+384d_5\right)}{1975-29808c_3(1-2c_3)(1-4c_3+8c_3^2)}\right]^{3/2}.
\end{aligned} \qquad (36)$$

The four inequalities following from Eq. (34), are the main result of this section: they imply *lower bounds* on the graviton mass, which can not be arbitrarily small compared to $4\pi m_{\rm Pl}$ (which, incidentally, is the largest cutoff for quantum gravity) for a fixed g_*. These bounds represent the very much improved, sharper, and more conservative version of the rough estimate done in [15]. As we discuss below, g_* cannot be taken arbitrarily small either.

Implications

The bounds Eq. (34) can be read in two ways: as constraints on the plane (c_3, d_5) of the graviton potential parameters, for a given graviton mass m and a given ratio $(\Lambda/\Lambda_3)^3 = g_*$, or as an absolute constraint on g_* for a given m, independently of c_3 and d_5, by finding the maximum of $F_i(c_3, d_5)$.

We begin with a discussion of the bounds on the parameters c_3 and d_5 inside F_i. The experimental *upper limit* on the graviton mass is extremely stringent, $m \lesssim 10^{-32} - 10^{-30}$ eV, depending on the type of experiment and theory assumptions behind it (see [45] for a critical discussion). Taking this as input, we show in Fig. 3 the constraints on c_3 and d_5, for a given g_*; the colored regions being allowed by our constraints. The yellow region are nothing but the standard positivity constraints (27, 28, 29, 30, 32). For $g_* \gtrsim 10^{-9}$ (corresponding to situations where Λ and Λ_3 are less than a factor $\sim$ 1000 away from each other, see Eq. (35), our bounds do not admit any solution in the (c_3, d_5)-plane. Notice that as g_* gets bigger the constraints from F_{VV} or $F_{VV'}$ alone single out essentially a narrow band around the line $c_3 = 1/4$, in agreement with the causality arguments of Ref.s [56, 57]. Similarly, the constraint from F_{SS} alone, converge quickly on the line $1 - 4c_3(1-9c_3) + 64d_5 = 0$. The intersection point $(c_3, d_5) = (1/4, -9/256)$ (red point in Fig. 3) is finally removed by F_{VS}.

In substance, the intersection region in the left panel of Fig. 3 is empty, while a small island (delimited by a solid black line) survives in the right plot with smaller g_*. To find the maximum value of g_* that allows for a

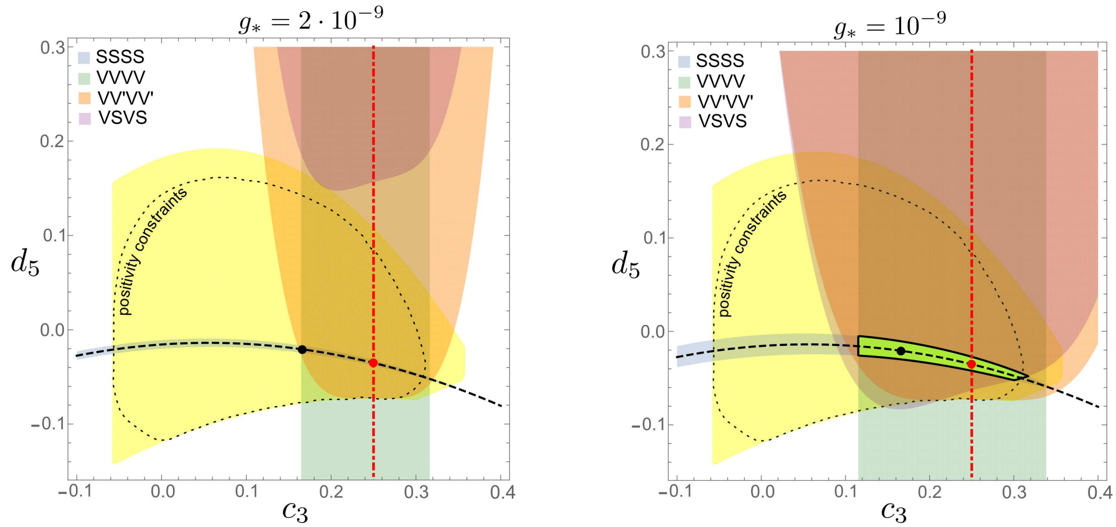

FIG. 3. Exclusion plot in the (c_3, d_5) plane for ghost-free massive gravity, for fixed accuracy $\delta = 1\%$, mass $m = 10^{-30}$ eV, and coupling $g_* = 1(2) \cdot 10^{-9}$ in the right (left) panel. Yellow region: allowed by standard positivity constraints, Eqs.(27, 28, 29, 30, 32), whose optimized versionfrom Ref. [13] is dotted. Other colored regions are allowed by our new bounds. Dash-dotted red line (dashed black line): regions of vanishing F_{VV} (F_{SS}), where the respective bounds vanish: indeed in the red dot $(c_3, d_5) = (1/4, -9/256)$ the vector and scalar mode decouple from the tensors, but not from each other, and in the black dot the scalar mode decouples from the tensor mode and itself.

solution, we write Eq. (34) as

$$m > 1.2 \cdot 10^{12}\mathrm{eV} \left(\frac{g_*}{1}\right)^4 \left(\frac{\delta}{1\%}\right)^6 \frac{1}{F_i(c_3, d_5)}\,, \tag{37}$$

and note that F_{VS} is continuous on the compact region allowed by the positivity constraints (27, 28, 29, 30, 32). The F_{VS} has thus a maximum value $F_{VS}(\hat{c}_3, \hat{d}_5) = 1.95 \cdot 10^7$ at $(\hat{c}_3, \hat{d}_5) \simeq (0.19, 0.15)$ inside that region (in fact, at the boundary), which implies the lower bound

$$m > 10^{-30}\,\mathrm{eV} \left(\frac{g_*}{2 \cdot 10^{-9}}\right)^4 \left(\frac{\delta}{1\%}\right)^6 , \tag{38}$$

independently of any values of c_3 and d_5. We recall that the direct experimental constraints on the graviton mass are $m \gtrsim 10^{-30}$eV. This situation is summarized in Fig. 1 and implies that $g_* \gtrsim 10^{-9}$ is excluded. Even slightly stronger bounds can be obtained by working with the non-elastic channels. In fact, as we now discuss, g_* can not be taken to such small values anyway.

The crucial question now is: what is the physical meaning of g_*, the relation between the cutoff Λ and the scale Λ_3? Can the UV completion be arbitrarily weakly coupled $g_* \lesssim 10^{-9}$[15]? To our knowledge, most literature of massive gravity has so far identified the cutoff Λ with the scale Λ_3, so that one expects a sizable $g_* \approx 0.1 - 4\pi$. These values are now grossly excluded by our bounds.

What about hierarchical values for Λ and Λ_3 corresponding to tiny values for g_*? From a theoretical point of view Λ and Λ_3 scale differently with $\hbar \neq 1$, so that their ratio actually changes when units are changed, in such a way that indeed g_* scales like a coupling constant (see Appendix A). This is fully analogous to the difference between a vacuum expectation value v (VEV) and the masses of new particles $\sim$ *coupling* $\times\, v$, e.g. the W-boson mass $m_W \sim gv$. The crucial point is that the cutoff, which is a physical scale Λ differs from Λ_3, which does not have the right dimension to represent a cutoff. Since $\Lambda_3^{-1} = 15\,\mathrm{km}\,\left(m/10^{-30}\mathrm{eV}\right)^{-2/3}$, a very small coupling g_* translates into a very low cutoff (large in distance units)

$$\Lambda \simeq (15000\,\mathrm{km})^{-1} \left(\frac{g_*}{10^{-9}}\right)^{1/3} \left(\frac{m}{10^{-30}\mathrm{eV}}\right)^{2/3} . \tag{39}$$

This is grossly *inconsistent* with the precise tests of general relativity, which go down to the mm scale or even below, see e.g. [50, 51] and references in [45]. The tension between our bounds, direct limits on the graviton mass, and from fifth force experiments is not resolvable (see Fig. 1), and this excludes massive gravity.

One might naively think that some screening effect, e.g. the Vainshtein mechanism [52, 53], could resolve this tension: after all the cutoff in Eq. (39) holds in Minkowski space and non-necessarily in regions near massive bodies, such as the earth, where non-linearities are important. However, the Vainshtein mechanism relies, crucially, on the assumption that the tower of effective operators is such that only building blocks of the type $\partial\partial\pi/\Lambda_3^3$ are unsuppressed (we work here for simplicity with the Stueckelberg mode π in the decoupling limit), whereas terms with more than two derivatives per field, like $(\partial/\Lambda)^n\partial\partial\pi/\Lambda_3^3$, are small. This assumption is consistent with NDA for $\Lambda \gg \Lambda_3$ i.e. for $g_* \gg 1$, but this is exactly the region ruled out by our bounds, as discussed above.

On the other hand, assuming that the Vainshtein mechanism works along with $\Lambda \ll \Lambda_3$, translates into extra highly non-generic assumptions about the UV-completion, see [46]. More importantly, the very same assumption that allows one to trust the Vainshtein mechanism, that is trusting the prediction at distances much smaller than Λ_3^{-1}, and a fortiori smaller than Λ^{-1}, would at the same time allows one to choose E in (26) to much larger values than Λ as well. In practice, the assumption that would justify the call for a Vainshtein mechanism would allow us to set, effectively, $g_* \sim 1$ again.

In summary, either $g_* \ll 1$ or Vainshtein can be taken, but not both. This is after all obvious: Vainshtein requires effectively a larger calculability cutoff, $\Lambda \to \Lambda_3$, so our bounds become more effective too.

All in all, our theoretical bounds (37, 38) rule out ghost-free massive gravity for good, unless some new clever mechanism beyond Vainshtein's would allow to lower even further the cutoff. In turn, ghost-free massive gravity is no longer a candidate for explaining the cosmic acceleration since that requires $m \sim H_0 \sim 10^{-33}$ eV.

VI. OUTLOOK

Positivity bounds are statements that arise from first principles such unitarity, analyticity, and crossing symmetry of the Lorentz invariant S-matrix. They have proven to be very useful because they set non-perturbative theoretical constraints even in strongly coupled theories, and give information that goes well beyond the mere use of symmetries. In this paper we went beyond positivity bounds and derived rigorous inequalities for amplitudes that are calculable in the IR via an EFT approach. The dispersive integral in the IR is not only positive but calculable, with an error from truncating the EFT towers of higher-dimensional operators that can be tamed thanks to separation of scales, which is what makes the EFT useful in the first place.

Our results are simple and general, and they can be applied straightforwardly to several EFTs. The phenomenological applications to interesting theories such as the weakly-broken Galileon and the ghost-free massive gravity that we explored in this paper are extremely rewarding. Our results, taken at face value, rule out dRGT massive gravity by combining our bound with the experimental lower bound on the graviton mass, and fifth force experiments, see Fig. 1. In the region where the Vainshtein mechanism could be at play, our bounds in fact require $m \gtrsim 100$ keV. Needless to say, our bounds neither apply to Lorentz-violating models of massive gravity (e.g. [59]), nor to theories with a massless graviton.

There are several directions where our bounds can find fruitful applications. The most immediate ideas involve theories with Goldstone particles, e.g., the EFT for the Golstino from SUSY breaking, the R-axion from R-symmety breaking, and the dilaton from scale-symmetry breaking. In these theories there exist universal couplings that are set by the various decay constants, but include also other non-universal parameters whose sizes and signs are often not accessible with the standard positivity bounds. Our results would allow instead to relate the non-universal parameters to the decay constants and extract thus non-trivial information on these EFTs, which are also phenomenologically interesting, see e.g. [26–28, 54, 55]. Another direction would be theories that have suppressed 2-to-2 amplitudes but unsuppressed 2-to-3 amplitudes, as those discussed e.g. in [38].

One important open question, that for the time being remains elusive, is whether it is possible (at least under extra assumptions) to extend our results to theories with massless particles and with spin $J \geq 2$. If that would be the case, the bounds would provide new insights on the long-distance universal properties of the UV-completion of quantum gravity. The bounds would also apply to IR modifications of General Relativity such as Horndeski-like theories, where the graviton remains massless.

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Appendix A: g_*-counting via $\hbar$-counting

In this appendix we recall how dimensional analysis is useful to extract the scaling with respect to coupling constants.

Rescaling the units from $\hbar = 1$ to $\hbar \neq 1$ while keeping $c = 1$ reintroduces a conversion factor between energy (or momentum) units $\mathcal{E}$ and length (or time) units ℓ, i.e. $\ell = \hbar/\mathcal{E}$. With canonically normalized kinetic terms, we have the following scaling with $\hbar$: $[A] = \mathcal{E}[\hbar]^{-1/2}$, $[\partial] = \mathcal{E}[\hbar]^{-1}$, $[m] = \mathcal{E}$, and $[g_*] = [\hbar]^{-1/2}$, where g_* is (a collective name for) coupling constant(s) and m a physical mass. Notice that Higgs quartic coupling λ is really a coupling squared $[\lambda] = [g_*^2]$. Quantum corrections scale indeed like powers of the dimensionless quantity $g_*^2\hbar/(16\pi^2)$ or $\lambda\hbar/(16\pi^2)$, so that they are important for $g_*^2 \sim 16\pi^2/\hbar \sim \lambda$ when there is no large dimensionless number (such as e.g. the number of species). Extending this dimensional analysis to fermions is immediate to see that Yukawa couplings scale like $\hbar^{-1/2}$ too.

More importantly, the relation between VEVs, couplings, physical masses and the associated Compton lengths is

$$[\lambda^{-1}] = \left[\frac{m}{\hbar}\right] = [g_*\langle A\rangle]\,. \tag{A1}$$

A coupling times a VEV is nothing but an inverse physical length which can be converted to a physical mass by plugging in the conversion factor, aka $\hbar$. In other words, the appearance of the coupling in (A1) tells us that parametrically *VEV's are to masses (or Compton lengths) like apples are to oranges.* [5] The immediate consequence of this trivial excercise is that the reduced Planck mass $m_{\rm Pl}$ has units of a VEV, $[m_{\rm Pl}] = [A]$, and not of a physical mass scale, in full analogy with the axion decay constant $[f_a] = [A]$. The UV-completion of general relativity should enter at some physical energy $g_* m_{\rm Pl}\hbar$ which is parametrically different than $m_{\rm Pl}$, even after setting $\hbar = 1$, because of the coupling g_*.

But what is left behind is the correct counting of g_*-insertions. This reasoning with $\hbar \neq 1$ is useful to keep track of the appropriate g_* insertions; the structure of a generic Lagrangian that automatically reproduces it is,

$$\mathcal{L} = \frac{\Lambda^4}{g_*^2}\widehat{\mathcal{L}}\left(\frac{\partial}{\Lambda}, \frac{g_* A}{\Lambda}, \frac{g_*\psi}{\Lambda^{3/2}}\right) \tag{A2}$$

where Λ is a physical mass scale and $\widehat{\mathcal{L}}$ is a polynomial with dimensionless coefficient, and we have finally set back $\hbar = 1$.The Lagrangian (A2) accounts for the intuitive fact that any field insertion in a given non-trivial process requires including a coupling constant as well. A class of simple theories with only one coupling and one scale [43] are those where all coefficients of $\widehat{\mathcal{L}}$ are of the same order (except for symmetries that can naturally suppress a subset of the parameters). They represent a generalization of the naive counting of 4π-factors, routinely used in strongly coupled EFT's in particle physics (see e.g. [44]), which goes under the name of naive dimensional analysis (NDA).

With the g_*-counting at hand, we immediately recognize that the scale $\Lambda_3^3 = (m^2 m_{\rm Pl})$ traditionally used in massive gravity is not parametrically a physical threshold: it misses a coupling constant. This is made manifest by the fact that the graviton mass is a physical mass scale but $m_{\rm Pl}$ is only a VEV. Alternatively, in the decoupling limit the coefficient of the cubic Galileon must carry a coupling g_*, that is $[c_3] = [g_*]$ to match the general scaling (A2). The actual correct parametric scaling is thus $\Lambda^3 = g_*\Lambda_3^3$. A weakly coupled theory corresponds to suppressed Λ relative to Λ_3, like a weakly coupled UV completion of general relativity corresponds to states entering much earlier than $4\pi m_{\rm Pl}$.

[5] We thank Riccardo Rattazzi who inspired this adage, with his interventions at the J. Hopkins workshop in Budapest in 2017.

Appendix B: Polarizations

We adopt the following basis of linear polarizations

$$\left(\epsilon^T(\mathbf{k}_1)\right)^{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu\nu}\,, \tag{B1}$$

$$\left(\epsilon^{T'}(\mathbf{k}_1)\right)^{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu\nu}\,, \tag{B2}$$

$$\left(\epsilon^V(\mathbf{k}_1)\right)^{\mu\nu} = \frac{1}{\sqrt{2}m}\begin{pmatrix} 0 & k_1^z & 0 & 0 \\ k_1^z & 0 & 0 & E \\ 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 \end{pmatrix}^{\mu\nu}\,, \tag{B3}$$

$$\left(\epsilon^{V'}(\mathbf{k}_1)\right)^{\mu\nu} = \frac{1}{\sqrt{2}m}\begin{pmatrix} 0 & 0 & k_1^z & 0 \\ 0 & 0 & 0 & 0 \\ k_1^z & 0 & 0 & E \\ 0 & 0 & E & 0 \end{pmatrix}^{\mu\nu}\,, \tag{B4}$$

$$\left(\epsilon^S(\mathbf{k}_1)\right)^{\mu\nu} = \sqrt{\frac{2}{3}}\begin{pmatrix} \frac{k_1^{z\,2}}{m^2} & 0 & 0 & \frac{k_1^z E}{m^2} \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ \frac{k_1^z E}{m^2} & 0 & 0 & \frac{E^2}{m^2} \end{pmatrix}^{\mu\nu} \tag{B5}$$

which are associated to the particle $k_1^\mu = (E, \mathbf{k}_1) = (E_1, 0, 0, k_1^z)$ which lies along the z-axis and has $E^2 = \mathbf{k}_1^2 + m^2$. They are real, symmetric, traceless, orthogonal, transverse to k_1, and normalized to $\epsilon^*_{\mu\nu}\epsilon^{\nu\mu} = 1$ [6]. The polarizations associated to the other momenta k_i^μ in the 2-to-2 scattering in the center of mass frame are obtained by a Lorentz transformation of those in (B1), e.g.

$$\left(\epsilon^V(\mathbf{k}_3)\right)^{\mu\nu} = R^\mu{}_{\mu'}R^\nu{}_{\nu'}\left(\epsilon^V(\mathbf{k}_1)\right)^{\mu'\nu'} \tag{B6}$$

with $R^\mu{}_{\mu'}$ the rotation along the y-axis by $\cos\theta = 1 + 2t/(s-4m^2)$ such that $k_3 = Rk_1$. While this definition is valid and legitimate, it corresponds effectively to consider k_1 as the canonical reference vector, rather than $(m,0,0,0)^T$, upon which constructing the massive one-particle states via boosting. Alternatively, it means that the standard Lorentz transformation that sends $(m,0,0,0)^T$ to k is a boost along the z-axis followed by rotation that sends $\hat{z}$ to $\hat{k}$ (like done e.g. for massless particle in [58]), rather than the sequence rotation-boost-rotation usually adopted for massive states [58].

[6] We are taking the same matrix entries of [13], except that that we have removed the i factor from the vector polarizations and taken all upper Lorentz indexes. We checked that our choice satisfy the completeness relation. The i factor is never important in elastic amplitudes, but it should actually be included whenever considering mixed-helicity states that include vector components, as done in [13]

The advantage of our convention is that it removes the little group matrix that would otherwise act on the polarization indexes $z = T, T', V, V', S$ when performing the rotations that send k_1 to k_i. (The Wigner rotation must be adapted accordingly too). For massless particles the differences between the two conventions is essentially immaterial as the little group acts just like phases.

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