Spinors for everyone

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Abstract. It is hard to find intuition for spinors in the literature. We provide this intuition by explaining all the underlying ideas in a way that can be understood by everybody who knows the definition of a group, complex numbers and matrix algebra. We first work out these ideas for the representation SU(2) of the three-dimensional rotation group in $\mathbb{R}^3$. In a second stage we generalize the approach to rotation groups in vector spaces $\mathbb{R}^n$ of arbitrary dimension $n > 3$, endowed with an Euclidean metric. The reader can obtain this way an intuitive understanding of what a spinor is. We discuss the meaning of making linear combinations of spinors.

PACS. 02.20.-a Group Theory

1 Introduction

1.1 Motivation

Many people have difficulties in apprehending the concept of spinors. In search for enlightenment, the reader will discover that it is very hard to find a clear definition of what a spinor is in the literature. E.g. Cartan [1] states in his monograph: *"A spinor is a kind of isotropic vector"*. Using a terminology "a kind of" can hardly be considered as a valid part of a clear definition. And a literature search reveals that this is an ever recurring theme. In all the various presentations I was able to consult, one just develops the algebra and states at the end of it that certain quantities introduced in the process are spinors. This is completely at variance with the usual practice, where the definition of a concept *precedes* the theorem about that concept. This way of introducing spinors leaves us without any clue as to what is going on behind the scenes, e.g. in the form of a conceptual mental image of what a spinor is supposed to be. What we are hitting here are actually manifestations of a state of affairs described by Michael Atiyah (winner of the Fields medal in 1966), when he declared:

□ *"No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the "square root" of geometry and, just as understanding the square root of -1 took centuries, the same might be true of spinors"* [2].

□ *" . . . the geometrical significance of spinors is still very mysterious. Unlike differential forms, which are related to areas and volumes, spinors have no such simple explanation. They appear out of some slick algebra, but the geometrical meaning is obscure . . . "* [3].

What is going on here? In algebraic geometry, geometry and algebra go hand in hand. We have a geometry, an algebra and a dictionary in the form of a one-to-one correspondence that translates the algebra into the geometry and vice versa. As may transpire from what Atiyah says, the problem with the spinor concept is thus that in the approaches which are presented in textbooks the algebra and the geometry have not been developed in parallel. It is all "algebra first". We have only developed the algebra and neglected the geometry and the dictionary. The approach has even been so asymmetrical that we are no longer able to guess the geometry from the algebra.

It is perhaps worth asking here a provocative question. Spinors occur in the representation SU(2)[1] of the three-dimensional rotation group in $\mathbb{R}^3$. As it uses spinors, which seem particularly difficult to understand, SU(2) appears

[1] The notation SU(2) implies that it is the special unitary group of dimension 2. This means that its complex 2×2 matrices $\mathbf{M}$ satisfy the conditions $\det \mathbf{M} = 1$ and $\mathbf{M}^\dagger = \mathbf{M}^{-1}$. By expressing these conditions for an arbitrary 2×2 matrix it is easy to show that such a matrix has the structure given in Eq. 3 below. In Section 2 we will derive a representation for the group of rotations in $\mathbb{R}^3$ that will turn out to be SU(2).

like a mystery representation of the three-dimensional rotation group. Now here is the question: How on Earth can it be that there is something mysterious about the three-dimensional rotation group? Is it not mere Euclidean geometry? This seems to suggest that there might be something simple that we have missed and has escaped our attention.

We will see that this indeed true. In this article we will restore the balance between the algebra and the geometry by providing the reader also with the geometry and the dictionary. This way, he will be able to clearly understand the concept of spinors in SU(2). The reader will see that the strategy followed to solve the riddle what the square root of a vector could mean is somewhat analogous to the one that solves the puzzle what the square root of -1 could mean (see Footnote 14 in Subsection 2.4). We will define the spinor concept in its own right and show afterwards that one can define an isomorphism that allows to interpret a spinor as "squaring to a vector". However, we will see that the terminology "square root" used by Atiyah is only a loose metaphor, and in the generalization of the approach to groups of rotations in $\mathbb{R}^n$, with $n > 3$ (see Sections 3 and 4) the metaphor will become even more loose. The present paper is an improvement of our presentation of spinors given in Chapter 3 of reference [4] and an extension of it to the group of rotations in $\mathbb{R}^n$. There is of course some overlap with reference [4] but not everything is systematically reproduced here.

1.2 Purpose and outline

Let us try to build the theory of spinors starting from the geometry. This way the underlying ideas will become clear in the form of "visual" geometrical clues. This will suffice for what the reader will need to know about spinors in the rotation and the Lorentz group for applications in quantum mechanics. When he will have understood this, he should in principle be able to design or complete the proofs himself based on these ideas. We will therefore not strive for a formal perfection of our presentation. Our presentation may in this respect be considered as clumsy or deficient from the viewpoint of mathematical rigor, but mathematically rigorous presentations have their own inconvenience, viz. that they may render it very difficult to perceive what is going on behind the scenes. Other people than professional mathematicians may want to use and understand certain mathematical results and they might find it very hard to come to grips with the austere presentation. As the example of spinors in the light of Atiyah's statement shows, it is even possible to write a perfect mathematical account of something without having properly understood it. Our aim is not to give a perfect formal account of the mathematical theory. Our aim is rather to confer to the reader all the insight he needs to work out the mathematical theory himself. We want to render the ideas so clear and utterly obvious that the reader will become fluent enough to flawlessly derive all further developments without any substantial difficulty. The self-learning that will intervene in carrying out this exercise will certainly help him much better to become acquainted with the subject matter than reading and checking the algebra of an exhaustive and formally perfect account of it in a book. All one has to know is complex numbers, matrix algebra, vector spaces, the Hermitian norm and the definition of a group.[2]

To develop the ideas we will start from a simple specific case and then see how we can generalize it. We will this way discover and take the ideas and the difficulties one by one, while in a general abstract approach many of the underlying ideas may become hidden. The simplest case in point is the representation SU(2) for the rotations in $\mathbb{R}^3$. We understand very well the formalism for SO(3).[3] We rotate vectors, coded as 3×1 matrices by multiplying them to the left by 3×3 rotation matrices. It is natural to try to carry over this idea to SU(2) and to attempt to make sense of SU(2) by analogy with what happens in SO(3). But as we will see such heuristics are an impasse. The spinors, which are the 2×1 matrices on which the 2×2 SU(2) rotation matrices are operating do not correspond to images of vectors of $\mathbb{R}^3$ or $\mathbb{C}^3$.

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[2] Let us introduce here some notation. We will note by $\mathrm{F}(A, B)$ the set of all mappings from the set A to the set B. We will note by $\mathrm{L}(V, W)$ the set of all linear mappings from the vector space V to the vector space W. They correspond thus to $k \times m$ matrices if $\dim V = k$ and $\dim W = m$. One notes $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$ often as $\mathrm{M}_n(\mathbb{R})$ in the literature, while one notes $\mathrm{L}(\mathbb{C}^n, \mathbb{C}^n)$ often as $\mathrm{M}_n(\mathbb{C})$.

[3] The n-dimensional rotation group in $\mathbb{R}^n$, the matrix group $\mathrm{SO}(n)$ that represents it in $\mathbb{R}^n$ and the matrix group we will construct in Sections 3-4 are three different mathematical objects. For convenience we will note the n-dimensional rotation group in $\mathbb{R}^n$ by its most intuitive representation $\mathrm{SO}(n)$. This way we will speak about the spinors of $\mathrm{SO}(n)$ although in reality they are not concepts that occur in the representation $\mathrm{SO}(n)$, but in the representation that acts on a subset $\mathscr{S} \subset \mathbb{C}^{2^\nu}$, and will be constructed in Sections 3-4. Here $\nu = \lfloor \frac{n}{2} \rfloor$. We will use this notation throughout the paper. The quantity ν enters naturally into the discussions as will become clear when we go along with developing the presentation.

1.3 Preliminary caveat: Spinors do not build a vector space

As we will see in Section 2, spinors in SU(2) do not build a vector space but a curved manifold. This is almost never clearly spelled out.[4] This remark will remain true in the generalization of the ideas to the n-dimensional rotation group SO(n) in $\mathbb{R}^n$, to be discussed in Sections 3 and 4.

It is important to point out that it is even a transgression within the context of pure group theory to make linear combinations of rotation matrices in SO(3). A linear combination of rotation matrices will in general no longer be a rotation matrix. Within $\mathrm{L}(\mathbb{C}^3, \mathbb{C}^3)$ or $\mathrm{L}(\mathbb{R}^3, \mathbb{R}^3)$ we can nevertheless try to find a meaning for such linear combinations, because the matrices are operating on elements of a vector space $\mathbb{R}^3$ or $\mathbb{C}^3$, yielding again elements of the same vector space $\mathbb{R}^3$ or $\mathbb{C}^3$. The matrix group SO(3) is embedded within the matrix group $\mathrm{L}(\mathbb{R}^3, \mathbb{R}^3)$. The linear combinations of the matrices can then be defined in $\mathrm{L}(\mathbb{R}^3, \mathbb{R}^3)$ by falling back on the meaning of linear combinations of vectors in the image space. But in SU(2) this will not be possible as the spinors are not building a vector space (see Remark 1 in Subsection 2.1 below).

The caveat we are introducing here is actually much more general. In group representation theory one introduces purely formal expressions $\sum_j c_j \mathbf{D}(g_j)$ which build the so-called group ring. Here $\mathbf{D}(g_j)$ are the representation matrices of the group elements $g_j \in G$ of the group $(G, \circ)$ with operation $\circ$ and c_j are elements of a number field $\mathbb{K}$, which can e.g. be $\mathbb{R}$ or $\mathbb{C}$. This is purely formal as in the definition of a representation $\mathbf{D}$ we define the operation $\mathbf{D}(g_j)\mathbf{D}(g_k) = \mathbf{D}(g_j \circ g_k)$, but we do not define the operation $\sum_j c_j \mathbf{D}(g_j)$ as corresponding to $\mathbf{D}(\sum_j c_j g_j)$, for the very simple reason that $\sum_j c_j g_j$ is in general not defined. A good text book should insist thus on the fact that introducing $\sum_j c_j \mathbf{D}(g_j)$ is purely formal [6]. To illustrate this, we could ask the question what the meaning of the sum of two permutations p and q:

$$\begin{pmatrix} 1 & 2 & \cdots & j & \cdots & n \\ p_1 & p_2 & \cdots & p_j & \cdots & p_n \end{pmatrix} + \begin{pmatrix} 1 & 2 & \cdots & j & \cdots & n \\ q_1 & q_2 & \cdots & q_j & \cdots & q_n \end{pmatrix}, \tag{1}$$

in the permutation group S_n is supposed to be. To illustrate this further, imagine the group $(G, \circ)$ of moves of a Rubik's cube. It is obvious in this example that $g_j \circ g_k$ is defined while $g_j + g_k$ is not. Giving meaning to $g_j + g_k$ requires introducing new definitions. This can e.g. be done by introducing sets of Rubik's cubes. E.g. we can define $g_j + g_k$ as the set $\{g_j, g_k\}$. This way, we can give a meaning to expressions of the type $\sum_j c_j g_j$, with $c_j \in \mathbb{N}$. Giving meaning to $\sum_j c_j g_j$, with $c_j \in \mathbb{C}$ can perhaps be achieved by interpreting the sets further in terms of geometrical configurations. We will dwell further on this issue of making linear combinations of spinors in Subsection 2.3.

1.4 Ideals

A concept that is very instrumental in reminding us of the no-go zone of linear combinations of spinors is the concept of an ideal. The spinors $\boldsymbol{\phi}$ of SO(n) build a set $\mathscr{I}$ such that for all rotation matrices $\mathbf{R}$ (which work on them by left multiplication), $\mathbf{R}\boldsymbol{\phi}$ also belongs to the set: $\forall \boldsymbol{\phi} \in \mathscr{I}, \forall \mathbf{R} \in G : \mathbf{R}\boldsymbol{\phi} \in \mathscr{I}$. One summarizes this by stating that $\mathscr{I}$ is a left ideal. Here G can stand for SU(2) or SO(n). This does not imply that the set of spinors would be a vector space, such that: $\neg(\forall \boldsymbol{\phi}_1 \in \mathscr{I}, \forall \boldsymbol{\phi}_2 \in \mathscr{I}, \forall c_1 \in \mathbb{C}, \forall c_2 \in \mathbb{C} : c_1\boldsymbol{\phi}_1 + c_2\boldsymbol{\phi}_2 \in \mathscr{I})$. The group SO(3) contains two trivial ideals, which are topologically disconnected, *viz.* the proper rotations and the reversals (which include the reflections), because it is impossible to change a left-handed frame into a right-handed frame by a proper rotation.

2 Construction of SU(2)

2.1 The geometrical meaning of spinors

The idea behind the meaning of a 2×1 spinor of SU(2) is that we will no longer rotate vectors, but that we will "rotate" rotations. To explain what we mean by this, we start from the following diagram for a group G:

[4] A consequence of this is that physicists believe that the linearity of the Dirac equation (and the Schrödinger equation) implies the superposition principle in quantum mechanics, which is wrong because the spinors are not building a vector space. In this respect Cartan stated that physicists are using spinors like vectors. This confusion plays a major rôle in one of the meanest paradoxes of quantum mechanics, *viz.* the double-slit experiment [5].

$$
\begin{array}{|c|cccccc|}
\hline
\circ & g_1 & g_2 & g_3 & \cdots & g_j & \cdots \\
\hline
g_1 & g_1 \circ g_1 & g_1 \circ g_2 & g_1 \circ g_3 & \cdots & g_1 \circ g_j & \cdots \\
g_2 & g_2 \circ g_1 & g_2 \circ g_2 & g_2 \circ g_3 & \cdots & g_2 \circ g_j & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \\
\hline
g_k & g_k \circ g_1 & g_k \circ g_2 & g_k \circ g_3 & \cdots & g_k \circ g_j & \cdots \\
\hline
\vdots & \vdots & \vdots & \vdots & & \vdots & \\
\hline
\end{array}
\; \leftarrow T_{g_k} \tag{2}
$$

This diagram tries to illustrate a table for group multiplication. Admittedly, we will not be able to write down such a table for an infinite group, but we will only use it to render more vivid the ideas. Such a table tells us everything about the group we need to know: One can check on such a table that the group axioms are satisfied, and one can do all the necessary calculations. For the rotation group, we do not need to know how the rotations work on vectors. We might need to know how they work on vectors to construct the table, but once this task has been achieved, we can forget about the vectors. The infinite table in Eq. 2 defines the whole group structure. When we look at one line of the table - the one flagged by the arrow - we see that we can conceive a group element g_k in a hand-waving way as a "function" $g_k : G \rightarrow G$ that works on the other group elements g_j according to: $g_k : g_j \rightarrow g_k(g_j) = g_k{\circ}g_j$. We can thus identify g_k with a function. More rigorously, we can say that we represent the group element g_k by a group automorphism $T_{g_k} : g_j \rightarrow T_{g_k}(g_j) = g_k{\circ}g_j$. A rotation operates in this representation not on a vector, but on other rotations. We "turn rotations" instead of vectors. This is a construction which always works: The automorphisms of a group G are themselves a group that is isomorphic to G, such that they can be used to represent G.

It can be easily seen that this idea about the meaning of a spinor is true. As we will show below in Eq. 8, the general form of a rotation matrix $\mathbf{R}$ in SU(2) is:

$$\mathbf{R} = \left[\begin{array}{cc} \xi_0 & -\xi_1^* \\ \xi_1 & \xi_0^* \end{array} \right], \tag{3}$$

A 2×1 spinor $\boldsymbol{\phi}$ can then be shown to be just a stenographic notation for a 2×2 SU(2) rotation matrix $\mathbf{R}$ by taking its first column $\hat{\mathrm{c}}_1(\mathbf{R})$:

$$\mathbf{R} = \left[\begin{array}{cc} \xi_0 & -\xi_1^* \\ \xi_1 & \xi_0^* \end{array} \right] \rightarrow \boldsymbol{\phi} = \hat{\mathrm{c}}_1(\mathbf{R}) = \left[\begin{array}{c} \xi_0 \\ \xi_1 \end{array} \right], \tag{4}$$

This is based on the fact that the first column of $\mathbf{R}$ contains already the whole information about $\mathbf{R}$ and that $\mathbf{R}_1\hat{\mathrm{c}}_1(\mathbf{R}) = \hat{\mathrm{c}}_1(\mathbf{R}_1\mathbf{R})$. Instead of $\mathbf{R}' = \mathbf{R}_1\mathbf{R}$, we can write then $\boldsymbol{\phi}' = \mathbf{R}_1\boldsymbol{\phi}$ without any loss of information. We can alternatively also use the second column $\hat{\mathrm{c}}_2(\mathbf{R})$ as a shorthand and as a (so-called conjugated) spinor. This way, a spinor has a well-defined geometrical meaning. As already stated, it is just a group element. This is all that spinors in SU(2) are about. Spinors code group elements. Within SU(2), 2×2 rotation matrices operate on 2×1 spinor matrices. These spinor matrices represent themselves the rotations that are "rotated".

Stating that a spinor in SU(2) is a rotation is actually an *abus de langage*. A spinor is, just like a 3×3 SO(3) rotation matrix, an unambiguous representation of a rotation within the group theory. But due to the isomorphism we can merge the concepts and call the matrix or the spinor a rotation, in complete analogy with what we proposed in Footnote 3. For didactical reasons we can consider a spinor as conceptually equivalent to a system of "generalized coordinates" for a rotation.

We should not be surprised by the removal of the vectors from the formalism in favour of the group elements themselves as described above. Group theory is all about this kind of abstraction. We try to obtain general results from just a few abstract axioms for the group elements, without bothering about their intuitive meaning in a more specific context of a practical realization. And as far as representations are concerned, we do not have to get back to a specific context. We always have a representation at hand in the form of group automorphisms. This is a well-known fact, but in its general abstract formulation this fact looks indeed very abstract. Here we can see what this abstract representation in terms of automorphisms intuitively means in the context of the specific example of the rotation group. The idea is then no longer abstract: We can identify the 2×2 matrices $\mathbf{R}$ of SU(2) with the group automorphisms T_{g_k}, and the 2×1 rotation matrices $\boldsymbol{\phi}_j$ with the group elements g_j, such that $g_j \rightarrow g_k \circ g_j = T_{g_k}(g_j)$ is algebraically represented by: $\boldsymbol{\phi}_j \rightarrow \mathbf{R}\boldsymbol{\phi}_j$.

□ Remark 1. From this it must be already obvious that spinors in SU(2) do not build a vector space as we stressed in the Introduction, more particularly in its Subsection 1.3. The three-dimensional rotation group is not a vector space but a curved manifold (because the group is non-Abelian). We cannot try to find a meaning for a linear combination $\sum_j c_j\mathbf{R}_j$ of SU(2) matrices $\mathbf{R}_j$, in analogy to what we can do with 3×3 matrices in SO(3), where we can fall back on the fact that 3×1 matrices of the image space correspond to elements of a vector space $\mathbb{R}^3$ or $\mathbb{C}^3$. The reason for this is that the spinors $\boldsymbol{\phi}_j$ do not build a vector space, such that we cannot define $\sum_k c_k\mathbf{R}_k$ by falling back on some definition for $\sum_j c_j \boldsymbol{\phi}_j$ in the image space. And the very reason why we cannot define $\sum_j c_j \boldsymbol{\phi}_j = \sum_j c_j \hat{\mathrm{c}}_1(\mathbf{R}_j) = \hat{\mathrm{c}}_1(\sum_j c_j \mathbf{R}_j)$, is that we cannot define $\sum_j c_j \mathbf{R}_j$. In trying to define linear combinations of SU(2) matrices or spinors we hit thus a vicious circle from which we cannot escape. Furthermore, the relation between spinors and vectors of $\mathbb{R}^3$ is not linear as may have already transpired from Atiyah's statement cited above and as we will explain below (see Subsection 2.5). This frustrates all attempts to find a meaning for a linear combination of spinors in SU(2) based on the meaning of the linear combination with the same coefficients in SO(3). Trying to make sense of linear combinations of spinors is therefore an impasse.[5] We will illustrate this in Subsection 2.3.

□ Remark 2. We will extrapolate the idea that the representation theory "rotates rotations rather than vectors" to SO(n), such that we will then obtain a good geometrical intuition for the group theory. If we can extrapolate also the idea that spinors are group elements to SO(n), we will then obtain also a very good intuition for spinors. We can then e.g. also understand why spinors constitute an ideal $\mathscr{I}$. The ideal is then just the group and the group is closed with respect to the composition of rotations.

□ Remark 3. However, we will not be able to realize this dream completely. The idea that spinors are just rotations will give us a very nice intuition for them in SU(2). But as we will discover, the interpretation in SU(2) of a single column matrix as a shorthand for the whole information needed to define a group element unambiguously is not general, as it relies on a coincidence in $\mathbb{R}^3$. As we will see in Sections 3-4, for rotations in $\mathbb{R}^n$ (with $n > 3$), a single column will not suffice to represent all the information contents of the rotation matrix. It is therefore that we always will prefer to define a spinor in SO(n) as the full rotation matrix (or perhaps as a set of column matrices that completely define a rotation), rather than as a single column matrix.[6] It is in this sense that it will remain then true that a spinor is a set of coordinates for a rotation. In the general approach the main idea will thus be to consider the formalism just as a formalism of rotation matrices and the column matrices as auxiliary sub-quantities which encase only a subset of the complete information.

But for the moment we want to explore the idea of a single-column spinor that contains the complete information about a rotation in SU(2), where the intuitively attractive idea that a column spinor represents a group element is viable. It remains to explain in which form the information about the rotation is encoded within this column matrix. This is done in several steps.

2.2 Generating the group from reflections

The first step is deciding that we will generate the whole group of rotations and reversals from reflections, based on the idea that a rotation is the product of two reflections as explained in Fig. 1. We therefore need to cast a reflection into the form of a 2×2-matrix. The coordinates of the unit vector $\mathbf{a} = (a_x, a_y, a_z)$, which is the normal to the reflection plane that defines the reflection A, should be present as parameters within the reflection matrix $\mathbf{A}$ but we do not know how. Therefore we decompose heuristically the matrix $\mathbf{A}$ that codes the reflection A defined by $\mathbf{a}$ linearly as $a_x\tau_x + a_y\tau_y + a_z\tau_z$, where τ_x, τ_y, τ_z are unknown matrices, as summarized in the following diagram:

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[5] In quantum mechanics, probabilities are a part of a probability charge-current four-vector, while probability amplitudes ψ correspond to spinors. As the probabilities are quadratic expressions in terms of the probability amplitudes, the relation between (four-)vectors and spinors is thus indeed not linear. It is the fact that it is so difficult to give meaning to a linear combination of spinors which makes it so difficult to make sense of the superposition $\psi_1 + \psi_2$ in the double-slit experiment. What adds to the confusion is that physicists are even not aware of the fact that $\psi_1 + \psi_2$ is *a priori* a meaningless quantity. A physicist may be puzzled by what we are stating here because the wave functions which occur in the Schrödinger equation are scalars rather than spinors, but we can consider them as a spinors whereby we drop the matrix part and only keep the phase because the rotation axis remains always parallel to $\mathbf{e}_z$, such that the matrix part we are dropping is a constant (see the Rodrigues formula Eq. 8 below). This is actually how the Schrödinger equation relates to the Pauli equation.

[6] This is not at all unusual. Consider the jump diffusion of a particle on the vertices $(\pm1, \pm1, \pm1)$ of a cube. The rate equations for the probabilities $p(\mathbf{r}_j)$ to find the particle at the vertex $\mathbf{r}_j$ are then $\frac{dp(\mathbf{r}_j)}{dt} = \frac{1}{\tau}\left[\,-3p(\mathbf{r}_j) + \sum_{\mathbf{r}_k \in S_j} p(\mathbf{r}_k)\,\right]$. Here τ is the so-called relaxation time and S_j is the set of the three first neighbours of $\mathbf{r}_j$. Note the set of indices k of $\mathbf{r}_k \in S_j$ as J_j. The coupled rate equations can then be written in matrix form as $\frac{d}{dt}\mathbf{P} = \frac{1}{\tau}\mathbf{MP}$, where $\mathbf{P}$ is the column matrix $[\,p(\mathbf{r}_1), p(\mathbf{r}_2), \cdots p(\mathbf{r}_8)\,]^\top$, and $M_{jk} = -3\delta_{jk} + \sum_{k \in J_j} \delta_{jk}$. To solve the coupled equations one must diagonalize the matrix $\mathbf{M}$. It is easy to see that $[\,\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_8\,]^\top$ is a generalized eigenvector of $\mathbf{M}$ with eigenvalue $\lambda = -2$. We call it a generalized eigenvector because its entries are vectors rather than scalars. But we can represent this generalized eigenvector under the form of three columns, where each column corresponds to one of the coordinates x_k, y_k or z_k of $\mathbf{r}_k$ of the generalized eigenvector. These three columns are three different true scalar eigenvectors corresponding to the same eigenvalue $\lambda = -2$. The eigenvalue is thus degenerate and in its three-dimensional eigenvector space we can choose a basis at will. The possible choices correspond to different orientations of the basis. The individual columns will then hardly have a meaning, but the meaning of the vectors defined by the combined three columns does not depend on the choice of the reference frame and is geometrically completely clear. Another example is given on pp. 42-45 of reference [4]. The analogy is not perfect because in the rotation group individual columns never represent components of vectors.

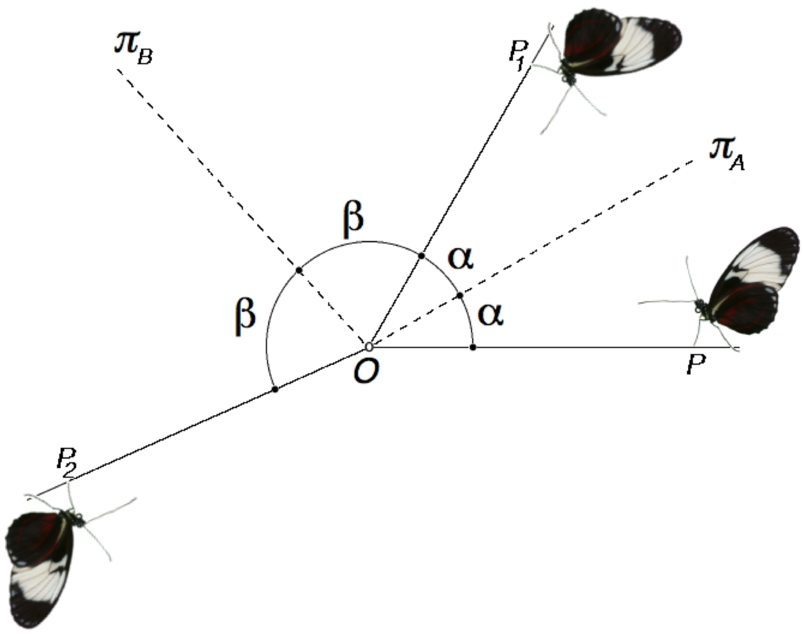

Fig. 1. A rotation R in $\mathbb{R}^3$ as the product of two reflections A and B defined by their reflection planes π_A and π_B. The planes π_A and π_B in $\mathbb{R}^3$ intersect along a straight line ℓ defined by $\ell = \{\, \mathbf{r} \in \mathbb{R}^3 \parallel (\exists \lambda \in \mathbb{R})(\mathbf{r} = \lambda \mathbf{n}) \,\}$. The plane of the figure is taken perpendicular to the line ℓ and intersects ℓ in the point O. We use the names π_A and π_B of the planes to label their intersections with the plane of the figure. The position vector OP of the point P to be reflected, is at an angle α with respect to π_A. We call $A(P) = P_1$ and $B(P_1) = P_2$. The position vector OP_1 is at angle β with respect to π_B. The angle between π_A and π_B is then $\alpha + \beta$. As can be seen from their operations on the Heliconius butterfly, reflections have negative parity, but the product of two reflections conserves the parity. The product of the two reflections is therefore a rotation $R = B \circ A$, with axis ℓ and rotation angle $2(\alpha + \beta)$. Only the relative angle $\alpha + \beta$ between π_A and π_B appears in the final result, not its decomposition into α and β. Hence, the final result will not be changed when we turn the two planes together as a whole around ℓ keeping $\alpha + \beta$ fixed (After [4]).

$$
\begin{tikzcd}
\text{unit vector } \mathbf{a} = (a_x, a_y, a_z) \in \mathbb{R}^3 \arrow[r, "\mathbf{a}\text{ defines a}", "\text{reflection } A"'] \arrow[d, "\text{definition}"] & 2 \times 2 \text{ complex reflection matrix } \mathbf{A} \arrow[d, "\text{Dirac's heuristics}"] \\
\mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z & \mathbf{A} = \underbrace{a_x \tau_x + a_y \tau_y + a_z \tau_z}_{\text{noted as } \mathbf{a}\cdot\boldsymbol{\tau}} \arrow[l, "\text{analogy of}"', "\text{decompositions}"]
\end{tikzcd}
\tag{5}
$$

If we know the matrix τ_x, this will tell us where and with which coefficients a_x pops up in $\mathbf{A}$. The same applies *mutatis mutandis* for τ_y and τ_z. The matrices τ_x, τ_y, τ_z, we use to code this way reflection matrices within $\mathbb{R}^3$, can be found by expressing isomorphically through $\mathbf{A}\mathbf{A} = \mathbf{a}^2 \mathbb{1} = \mathbb{1}$ what defines a reflection, viz. that the reflection operator A is idempotent. We find out that this can be done provided the three matrices simultaneously satisfy the six conditions $\tau_\mu \tau_\nu + \tau_\nu \tau_\mu = 2\delta_{\mu\nu}\mathbb{1}$, i.e. provided we take e.g. the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ for τ_x, τ_y, τ_z. Physicists among the readers will recognize that this construction is completely analogous to the one that introduces the gamma matrices in the

Dirac equation.[7] We will discuss in Subsection 2.6 that the solution $(\tau_x, \tau_y, \tau_z) = (\sigma_x, \sigma_y, \sigma_z)$ is not unique and that there are many other possible choices. But we follow here the tradition to adopt the choice of the Pauli matrices. The reflection matrix $\mathbf{A}$ is thus given by:

$$A \to \mathbf{A} = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z = \left[\begin{array}{cc} a_z & a_x - \imath a_y \\ a_x + \imath a_y & -a_z \end{array} \right] \hat{=} \mathbf{a}\cdot\boldsymbol{\sigma}. \tag{7}$$

The symbol $\hat{=}$ serves here to flag that the notation $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ is a purely conventional shorthand for $a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$. It does not express a true scalar product involving $\mathbf{a}$, but just exploits the mimicry with the expression for a scalar product to introduce the shorthand.[8] By expressing a rotation as the product of two reflections, one can then derive the well-known Rodrigues formula:

$$\mathbf{R}(\mathbf{n}, \varphi) = \mathbf{BA} = \left[\begin{array}{cc} b_z & b_x - \imath b_y \\ b_x + \imath b_y & -b_z \end{array} \right] \left[\begin{array}{cc} a_z & a_x - \imath a_y \\ a_x + \imath a_y & -a_z \end{array} \right] = \cos(\varphi/2)\,\mathbb{1} - \imath\sin(\varphi/2)\,[\,\mathbf{n}\cdot\boldsymbol{\sigma}\,], \tag{8}$$

for a rotation by an angle φ around an axis defined by the unit vector $\mathbf{n}$. To derive this result it suffices to consider two reflections A (with matrix $[\mathbf{a}\cdot\boldsymbol{\sigma}]$) and B (with matrix $[\mathbf{b}\cdot\boldsymbol{\sigma}]$) whose planes contain $\mathbf{n}$, and which have an angle $\varphi/2$ between them, and to use the algebraic identity $[\mathbf{b}\cdot\boldsymbol{\sigma}]\,[\mathbf{a}\cdot\boldsymbol{\sigma}] = (\mathbf{b}\cdot\mathbf{a})\,\mathbb{1} + \imath(\mathbf{b}\wedge\mathbf{a})\cdot\boldsymbol{\sigma}$. There is an infinite set of such pairs of planes, and which precise pair one chooses from this set does not matter.

We can now appreciate also why SU(2) is a double covering of SO(3). Consider the matrix product:

$$\mathbf{BA} = \left[\begin{array}{cc} b_z & b_x - \imath b_y \\ b_x + \imath b_y & -b_z \end{array} \right] \left[\begin{array}{cc} a_z & a_x - \imath a_y \\ a_x + \imath a_y & -a_z \end{array} \right], \tag{9}$$

in the derivation of the Rodrigues equation in Eq. 8. Imagine that we keep A fixed and increase the angle $\chi = \varphi/2$ between the reflection planes π_A and π_B of A and B from $\chi = 0$ onwards. Of course φ is the angle of the rotation $R = BA$. This means that the reflection plane π_B with normal vector $\mathbf{b}$ that defines B is rotating. In the matrix product that occurs in Eq. 9, the numbers in the matrix $\mathbf{A}$ would remain fixed, while the numbers in the blue matrix $\mathbf{B}$ would be continuously changing like the digits that display hundredths of seconds on a wrist watch. When the starting value of the angle $\chi = \varphi/2$ between the reflection planes π_B and π_A is zero, the reflection planes are parallel, $\pi_B \parallel \pi_A$, and the starting value of $\mathbf{b}$ is $\mathbf{b} = \mathbf{a}$. When $\chi = \varphi/2$ reaches the value π, the rotating reflection plane π_B will have come back to its original position parallel to the fixed reflection plane π_A, and the resulting rotation BA will correspond to a rotation over an angle $\varphi = 2\chi = 2\pi$. As far as group elements are concerned, we have thus made a full turn both of the reflection B and the rotation BA when π_B will have made a turn of $\chi = \pi$ in $\mathbb{R}^3$. This is because we need to rotate a plane in $\mathbb{R}^3$ only over $\chi = \pi$ to bring it back to its original position. The consequence of this is that we can define any plane π_U (or a reflection U) always equivalently by two normal unit vectors $\mathbf{u}$ and $-\mathbf{u}$. These full turns of B and $R = BA$ within the group must be parameterized with a "group angle" $\varphi_G = 2\pi$ if we want to express the periodicity within the rotation group in terms of trigonometric functions. However, for the normal vector $\mathbf{b}$ which we have used to define B and which belongs to $\mathbb{R}^3$ this is different. For $\chi = \varphi/2 = 0$, its starting value is $\mathbf{b} = \mathbf{a}$. For $\chi = \varphi/2 = \pi$, its value has become $\mathbf{b} = -\mathbf{a}$, such that we obtain $\mathbf{R} = -\mathbb{1}$ in Eq. 9. There is nothing wrong with that because both the normal vectors $\mathbf{b} = \mathbf{a}$ and $\mathbf{b} = -\mathbf{a}$ define the same plane $\pi_B \parallel \pi_A$. Each group element g is thus represented by two matrices $\mathbf{G}$ and $-\mathbf{G}$. As the group elements B and $R = BA$ have recovered their initial values we have $\varphi_G = 2\pi$. In general, we have $\varphi_G = 2\chi = \varphi$. Only after a rotation over a "group angle" $\varphi_G = 4\pi$, which corresponds to a rotation of π_B over an angle $\chi = \varphi/2 = 2\pi$ will we obtain the values $BA = \mathbb{1}$ and $\mathbf{b} = \mathbf{a}$. It is often presented as a mystery of quantum mechanics that one must turn the wave function over $\varphi_G = \varphi = 4\pi$ before we obtain the starting configuration $(\chi = \varphi/2 = 2\pi)$ again. We can see from a proper understanding of the group theory that this is quite trivial. Most textbooks mystify this subject matter by invoking topological arguments. This link is clear and explained in [4], where we compare a full turn on the group

[7] That the reflection matrix is linear in a_x, a_y, a_z within SU(2) is special and not general. It is typical of the spinor-based representations we present in this paper. Dirac's representation is based on an analogous construction for the Lorentz group. A counter-example is the expression for a reflection matrix $\mathbf{A}$ in SO(3), which is quadratic in the parameters a_x, a_y, and a_z:

$$\mathbf{A} = \mathbb{1} - 2 \left[\begin{array}{c} a_x \\ a_y \\ a_z \end{array} \right] \otimes \left[\begin{array}{ccc} a_x & a_y & a_z \end{array} \right]. \tag{6}$$

Writing $\mathbf{A}$ this way permits to verify immediately algebraically that it corresponds to $A(\mathbf{v}) = \mathbf{v} - (\mathbf{a}\cdot\mathbf{v})\,\mathbf{a}$. Writing $\mathbb{1}$ as $(a_x^2 + a_y^2 + a_z^2)\,\mathbb{1}$ shows that the expression is purely quadratic. This is due to the fact that vectors in SO(3) are rank-2 tensor products of the spinors of SU(2) as we will discuss in this paper.

[8] We may note that we have defined the reflection matrices without defining a "vector space" on which they would be working. They are defined *en bloc*. This complies with the idea of expressed in Remark 3 of Subsection 2.1.

with a full turn on a Moebius ring. However, in the physical illustration of this topological argument by Feynman [8], Dirac [9] or Misner *et al.* [10] the connection between the topological argument and its physical illustration is hard to see. It is e.g. very difficult to follow how disentangling the threads in the work of Misner *et al.* would make the point.

2.3 Fleshing out the caveat: A superposition principle for spinors?

2.3.1 An SU(2)-specific approach

We can now illustrate how the procedure of summing spinors is geometrically obscure. Consider a rotation R_1 over an angle φ around the axis defined by the unit vector $\mathbf{n}$, and a rotation R_2 over an angle χ around the axis defined by the unit vector $\mathbf{s}$. Using Eq. 8, we have then:

$$\begin{aligned} \boldsymbol{\phi}_1 &= \hat{c}_1(\mathbf{R}_1) = \begin{bmatrix} \cos(\varphi/2) - \imath n_z \sin(\varphi/2) \\ -\imath n_x \sin(\varphi/2) + n_y \sin(\varphi/2) \end{bmatrix}, \\ \boldsymbol{\phi}_2 &= \hat{c}_1(\mathbf{R}_2) = \begin{bmatrix} \cos(\chi/2) - \imath s_z \sin(\chi/2) \\ -\imath s_x \sin(\chi/2) + s_y \sin(\chi/2) \end{bmatrix}. \end{aligned} \tag{10}$$

Summing $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ as though they were vectors is algebraically perfectly feasible. We obtain:

$$\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 = \begin{bmatrix} \cos(\varphi/2) + \cos(\chi/2) - \imath(n_z \sin(\varphi/2) + s_z \sin(\chi/2)) \\ -\imath((n_x \sin(\varphi/2) + s_x \sin(\chi/2)) + (n_y \sin(\varphi/2) + s_y \sin(\chi/2)) \end{bmatrix}. \tag{11}$$

But what does the result mean geometrically? The quantity $\boldsymbol{\phi} = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2$ cannot represent a rotation because $\boldsymbol{\phi}^\dagger \boldsymbol{\phi} \neq 1$. It is therefore not a true spinor. It corresponds obviously to $\hat{c}_1(\mathbf{R}_1 + \mathbf{R}_2)$, and as explained in Remark 1 of Subsection 2.2 we cannot interpret $\mathbf{R}_1 + \mathbf{R}_2$ like we can interpret a sum of rotation matrices in SO(3), because the spinors do not build a vector space. To interpret $\mathbf{R}_1 + \mathbf{R}_2$ we would need an interpretation of sums of spinors, and to interpret sums of spinors we would need an interpretation of sums of rotation matrices. When we try to transpose the ideas from SO(3) to SU(2) we end thus up running in circles.

But suppose now that we try to normalize the result in Eq. 11 to 1 as physicists do routinely. The result will then remain a linear combination of spinors but it is now a special one, whereby the coefficients used in the linear combination preserve the normalization. One must then find a rationale to explain what the geometrical idea behind such a procedure could be. Mind in this respect that we have no idea about the geometrical meaning of $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2$ in the first place. How do we justify defining a procedure on a quantity that is undefined? The procedure remains thus geometrically impenetrable, and we have rendered the situation worse. We have now also concealed the fact that there are conceptual problems with making linear combinations of spinors, because the final quantity obtained is now (almost always) algebraically identical to a true spinor. Let us prove this. To normalize $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2$ according to the Hermitian norm we calculate:

$$(\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2)^\dagger(\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2) = 2\,[\,1 + \cos(\varphi/2)\cos(\chi/2) + (\mathbf{n}\cdot\mathbf{s})\,\sin(\varphi/2)\sin(\chi/2)\,]. \tag{12}$$

Here:

$$\cos(\Omega/2) = \cos(\varphi/2)\cos(\chi/2) + (\mathbf{n}\cdot\mathbf{s})\,\sin(\varphi/2)\sin(\chi/2), \tag{13}$$

allows for a geometrical interpretation: Ω is the rotation angle of the product rotation $R_2 \circ R_1$ as shown e.g. in Appendix C of the monograph of Jones [11]. We are already running into trouble here, because it is certainly conceivable that $1 + \cos(\Omega/2) = 0$. The result $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2$ is then zero, such that it cannot be normalized to 1. This happens e.g. when we define $\mathbf{R}_2(\mathbf{s}, \chi)$ by: $\mathbf{s} = \mathbf{n}$ and $\chi = \varphi + 2\pi$. This is actually the only way this can happen, because $\boldsymbol{\phi}_1 = -\boldsymbol{\phi}_2$ implies $\mathbf{R}_1 = -\mathbf{R}_2$, such that $\mathbf{s} = \mathbf{n}$ and $\chi = \varphi + 2\pi$. This example is the absolute proof for the fact that the sum of two spinors is not a spinor. Let us now continue carrying out the algebra keeping this in mind and check whether there could be other problems. Writing the sum $\mathbf{R}_1 + \mathbf{R}_2$ in the form of the Rodrigues equation Eq. 8 makes it clear that the vector:

$$\mathbf{v} = \mathbf{n}\sin(\varphi/2) + \mathbf{s}\sin(\chi/2), \tag{14}$$

plays a prominent rôle in the algebra. Let us now assume that $1 + \cos(\Omega/2) \neq 0$ and calculate the result of normalizing the purely formal algebraic sum $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2$ to 1. This yields:

$$\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 \rightsquigarrow \frac{1}{\sqrt{2(1 + \cos(\Omega/2))}} \begin{bmatrix} \cos(\varphi/2) + \cos(\chi/2) - \imath v_z \\ -\imath(v_x + \imath v_y) \end{bmatrix}. \tag{15}$$

Let us now try to identify the right-hand side with a spinor $\boldsymbol{\psi}$ representing a rotation $R(\mathbf{u},\alpha)$ over an angle α around an axis defined by the unit vector $\mathbf{u}$:

$$\boldsymbol{\psi} = \left[\begin{array}{c} \cos(\alpha/2) - \imath \sin(\alpha/2) u_z \\ -\imath \sin(\alpha/2)(u_x + \imath u_y) \end{array} \right]. \tag{16}$$

Very obviously, the rotation angle α must then be given by:

$$\alpha = 2 \arccos \left[\frac{\cos(\varphi/2) + \cos(\chi/2)}{\sqrt{2(1+\cos(\Omega/2))}} \right] = 2 \arccos \left[\frac{\cos(\varphi/2) + \cos(\chi/2)}{2\cos(\Omega/4)} \right]. \tag{17}$$

But we must check whether this is a meaningful expression. The rotation angle α will only be defined if $|\cos(\varphi/2) + \cos(\chi/2)| \leq \sqrt{2(1+\cos(\Omega/2))}$. To check this, we square both sides and rewrite 2 as $\cos^2(\varphi/2)+\sin^2(\varphi/2)+\cos^2(\chi/2)+\sin^2(\chi/2)$. We obtain then the inequality:

$$\begin{aligned} &\cos^2(\varphi/2) + \cos^2(\chi/2) + 2\cos(\varphi/2)\cos(\chi/2) \leq \\ &\sin^2(\varphi/2) + \cos^2(\varphi/2) + \sin^2(\chi/2) + \cos^2(\chi/2) + 2\cos(\varphi/2)\cos(\chi/2) + 2\,(\mathbf{n}\cdot\mathbf{s})\,\sin(\varphi/2)\sin(\chi/2), \end{aligned} \tag{18}$$

where we have used the definition of $\cos(\Omega/2)$. Simplification leads to:

$$0 \leq \sin^2(\varphi/2) + \sin^2(\chi/2) + 2\,(\mathbf{n}\cdot\mathbf{s})\,\sin(\varphi/2)\sin(\chi/2) = \mathbf{v}^2, \tag{19}$$

such that the inequality is indeed satisfied.[9] It must be noted now that $|\mathbf{v}|$ can be larger than 1 (but not larger than 2). It is therefore *a priori* not obvious that we can identify:

$$\frac{\mathbf{v}}{\sqrt{2(1+\cos(\Omega/2))}} = \sin(\alpha/2)\,\mathbf{u}, \tag{20}$$

where $\mathbf{u} \parallel \mathbf{v}$ is a unit vector. But the calculations that occur in the simplification from Eq. 18 to Eq. 19 show $\mathbf{v}^2 = 2(1+\cos(\Omega/2)) - |\cos(\varphi/2) + \cos(\chi/2)|^2$, such that we have indeed $|\mathbf{v}| \leq \sqrt{2(1+\cos(\Omega/2))}$. We can thus calculate the unit vector $\mathbf{u} \parallel \mathbf{v}$ from:

$$\mathbf{u} = \frac{\mathbf{n}\sin(\varphi/2) + \mathbf{s}\sin(\chi/2)}{\sqrt{\sin^2(\varphi/2) + \sin^2(\chi/2) + 2\,(\mathbf{n}\cdot\mathbf{s})\,\sin(\varphi/2)\sin(\chi/2)}}. \tag{21}$$

While the normalized sum of two spinors can this way be interpreted in terms of a well-defined rotation $R(\mathbf{u},\alpha)$, it is not obvious what this is kind of operation $(R_1, R_2) \rightarrow R(\mathbf{u},\alpha)$ is then supposed to mean geometrically. The meaning of the unit vector $\mathbf{u}$ is at least algebraically clear as the sum of two wedge products. But the definition of the rotation angle α looks impenetrable.

A superposition principle for spinors, i.e. summing and making linear combinations of them with a wave picture in mind, like physicists routinely do, is thus an all but self-evident procedure. Within the initial set of underlying ideas this procedure is *a priori* geometrically meaningless despite its misleading apparent algebraic simplicity. We think that interpreting a sum of spinors as presented in this paragraph could be actually a conceptual impasse. The use of the superposition principle in physics requires therefore a supplementary geometrical justification. That this caveat is not futile at all can be appreciated from the fact that it is the very introduction of the superposition principle which transforms the spinor formalism, which is in essence purely geometrical and classical, into a much less obvious Hilbert space formalism of quantum mechanics. One of the mysterious creatures we introduce this way is Schrödinger's cat. This need for a justification of the superposition principle is further directly related to the conceptual difficulties encountered under the form of the so-called particle-wave duality in quantum mechanics. In interference we become directly confronted with the fact that the sum of two spinors can be zero when $1 + \cos(\Omega/2) = 0$ as outlined above.

[9] We have seen that when $1 + \cos(\Omega/2) = 0$ occurs, we also have $\cos(\varphi/2) = -\cos(\chi/2)$. This suggests that we could define a non-zero normalized value for $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2$ by a limit procedure, whereby $(\mathbf{s},\chi) \rightarrow (\mathbf{n}, \varphi + 2\pi)$. However, for this definition to be viable, the result must be independent of the path of approach followed by $(\mathbf{s},\chi)$. In any case, if we solve the normalization problem and obtain a "renormalized spinor" whose norm is no longer zero, then we will no longer be able to reproduce destructive interference in the double-slit experiment.

2.3.2 General group-theoretical approach

We may note that the *ad hoc* attempt to interpret the meaning of an element of the group ring presented in paragraph 2.3.1 is specific to SU(2). It does not solve the problem of the meaning of an element of the group ring for the permutations in Eq. 1 or for moves of the Rubik's cube. And even within SU(2) it fails as it is meaningless. We will refrain from interpreting the group theory in terms of a vector space $\mathbb{C}^2$ and proceed as always on the basis of purely group-theoretical considerations. Formal sums of group elements and group rings occur all the time in the group theory (see e.g. [6]). In this context, one encounters e.g. *formal* identities:

$$g \circ (\, h_1 + h_2 \cdots h_n \,) = (\, h_1 + h_2 \cdots h_n \,) \circ g. \tag{22}$$

Here $g, h_1, h_2, \cdots h_n$ are all group elements. In fact, all this expresses is an identity for sets:

$$g \circ \{h_1, h_2, \cdots h_n\} = \{h_1, h_2, \cdots h_n\} \circ g. \tag{23}$$

From a purely group-theoretical viewpoint, we can thus interpret sums of group elements in terms of sets. Here the coefficients in the linear combinations are all equal to one. We can extend this idea further and allow for integer values. We could e.g. imagine that we have a collection of 3000 Rubik's cubes and that 2000 of the cubes have the configuration of group element g_1 and 1000 the configuration of group element g_2. We could then note the collection as $2000g_1 + 1000g$. But let us now normalize. In the formal use of sums, all group elements are considered to be mutually orthogonal. We obtain then $c_1 g_1 + c_2 g_2$, whereby $|c_1|^2 = 2/3$ corresponds to the frequency of g_1 in the collection and $|c_2|^2 = 1/3$ the frequency of g_1 in the collection. This is exactly the rule used by physicists to interpret mixed states in quantum mechanics, such that physics seems to thrive on the alternative approach based on sets we have introduced here. We think that this is a much more simple way of interpreting linear combinations of spinors than the one presented in paragraph 2.3, which we qualified as a possible conceptual impasse. This interpretation remains faithful to the general scheme that we consider only the group structure and do not interpret the group elements in terms of their action on vectors.

We have then a simple interpretation for the wave function of Schrödingers cat. The wave function does not describe a cat that is half dead and half alive, but an ensemble of cats whereby half of the cats are dead and half of them alive. In this approach, the algebra cannot be taken literally. We must refrain from insisting on carrying out the algebra to the very end, by treating spinors like vectors, as in the example where we obtained $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 = \mathbf{0}$. The mixed state must be considered as a juxtaposition, rather than as a real algebraic sum. It would be preferable to write $\{c_1\psi_1, c_2\psi_2, \cdots, c_p\psi_p\}$ rather than $c_1\psi_1 + c_2\psi_2 + \cdots + c_p\psi_p$, because carrying out the algebra by brute force anyway is just wrong mathematics and the notation $\{c_1\psi_1, c_2\psi_2, \cdots, c_p\psi_p\}$ just sticks to the real meaning. However, this solution of refraining from brute-force algebra raises a severe issue in the case of interference, e.g. in the double-slit experiment, where $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 = \mathbf{0}$ leads indeed to a zero physical intensity, suggesting that the procedure should be taken seriously anyway. Interpreting $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 = \mathbf{0}$ in terms of sets is here certainly not meaningful. It would imply that the union of two non-empty sets would be empty. This apparent contradiction can be solved as follows. When we have a differential equation for spinors $\boldsymbol{\phi}$, we could first try to find all solutions by making the calculations as for vectors. As the spinors are embedded in the vector space, spinor solutions will be a subset of the pool of vector solutions. One can check afterwards which of these solutions allow for an alternative interpretation in terms of spinors or sets of spinors. An example of this procedure of finding an alternative interpretation is given in reference [5], where the calculation $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 = \mathbf{0}$, which would be valid for vectors, is not valid for spinors, as has been shown above in paragraph 2.3.1. One can then argue that the algebra used to obtain the solution $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 = \mathbf{0}$ is logically flawed for spinors but valid for finding the pool of solutions of the differential equation for vectors. It provides us then with a full set of possible solutions of the differential equation for spinors, which is admittedly obtained by cheating, and therefore must be filtered because not all solutions are forcedly valid. The criterion to be used in the filtering is that one must be able to justify the result by an exact and meaningful reinterpretation in terms of spinors.

The approach based on sets does not yet define all possible linear combinations of group elements, because it limits the values of the coefficients c_j to the interval $[0, 1]$. Quantum mechanics considers linear combinations whereby the coefficients $c_j \in \mathbb{C}$ are limited to a unit hypersphere $|c_j| \leq 1$. A further development that allows for such coefficients could be to consider e.g. the collection of the Rubik's cubes as a geometrical configuration that can be turned in physical space. We have not worked this out but it could make sense for the permutation group S_n as it is the group of rotations and reversals which leave a regular simplex in $\mathbb{R}^{n-1}$ invariant [12], such that the permutation group is a subgroup of the group generated by the reflections of $\mathbb{R}^{n-1}$ and which contains $\mathrm{SO}(n-1)$.

2.4 A parallel formalism for vectors

By construction, the representation SU(2) contains for the moment, (as we explained) *deliberately*, only group elements. Of course, it would be convenient if we were also able to calculate the action of the group elements on vectors. This is our next step. We can figure out how to do this based on the fact that we have already used a unit vector $\mathbf{a}$ to define a reflection A and its corresponding reflection matrix $\mathbf{A}$. Inversely, the reflection A also defines $\mathbf{a}$ up to a sign, such that there exists a one-to-one correspondence between reflections A and the two-member sets of unit vectors $\{\mathbf{a}, -\mathbf{a}\}$ (and the corresponding two-member sets of reflection matrices $\{\mathbf{A}, -\mathbf{A}\}$). This one-to-one correspondence between two-member sets of vectors and reflections will actually impose the formalism for vectors upon us. We can consider that a reflection A and its parameter set $\{\mathbf{a}, -\mathbf{a}\}$ are conceptually the same thing. When a reflection travels around the group, the two-member set of vectors $\{\mathbf{a}, -\mathbf{a}\}$ will travel together with it. Let us explain what we mean by traveling here. In SO(3), a vector $\mathbf{v} \in \mathbb{R}^3$ has a 3×1 representation matrix $\mathbf{V}$. It is transformed by a group element g with 3×3 representation matrix $\mathbf{G}$ into another vector $\mathbf{v}' = g(\mathbf{v}) \in \mathbb{R}^3$: we just calculate the 3×1 representation matrix $\mathbf{V}'$ of $g(\mathbf{v})$ as $\mathbf{V}' = \mathbf{G}\mathbf{V}$. The vector $\mathbf{v}$ travels this way under a group action to another vector $\mathbf{v}'$. The point we want to make is that in SU(2), things are not as simple. Under the action of a group element g with matrix representation $\mathbf{G}$ a reflection A will not travel to another reflection A'.

Let G be the group generated by the reflections. The subgroup of pure rotations $G_+ \subset G$ is the subset obtained from an even number of reflections. The subset $G_- \subset G$ obtained from an odd number of reflections is not a subgroup. It contains the reflections and the reversals. Reflections are of course geometrical objects of a different type than reversals and pure rotations. This transpires also from the fact that a reflection is defined by a unit vector $\mathbf{a} \in \mathscr{S}_2$, where $\mathscr{S}_2$ is the unit sphere in $\mathbb{R}^3$. It is thus defined by two independent real parameters while rotations and reversals are defined by three independent real parameters. Group elements $g_1 \in G$ and $g_2 \in G$ are of the same geometrical type if they are related by a similarity transformation: $\exists g \in G : g_2 = g \circ g_1 \circ g^{-1}$. They have then the same group character.

In general, a new group element gA obtained by operating with an arbitrary group element $g \in G$ on the reflection A will no longer be a reflection that can be associated with a unit vector like it was the case for A, because in general gA can be of a different geometrical type than A.[10] In other words the reflections do not travel according to the rule $A \to gA$. In order to transform a reflection A *always* into another reflection, we must use a similarity transformation: $A \to gAg^{-1}$. Hence, if B and A are reflections, defined by the unit vectors $\mathbf{b}$ and $\mathbf{a}$ then there exists a group element $g \in G$, such that $B = gAg^{-1}$ and $\mathbf{b} = g(\mathbf{a})$. Hence, if A is a reflection operating on $\mathbf{r} \in \mathbb{R}^3$ then the similar reflection B that operates on $g(\mathbf{r}) \in \mathbb{R}^3$ will be represented by $g \circ A \circ g^{-1}$. The reflection plane π_B and normal $\mathbf{b}$ of this reflection B will have the same angles with respect to $g(\mathbf{r})$ as π_A and $\mathbf{a}$ with respect to $\mathbf{r}$. We can move thus this way the reflection A in $\mathbf{r}$ around to group elements B in $g(\mathbf{r})$, and of course the parameter set $\{\mathbf{a}, -\mathbf{a}\}$ will travel with it from $\mathbf{r}$ to $g(\mathbf{r})$ to a parameter set $\{\mathbf{b}, -\mathbf{b}\} = \{g(\mathbf{a}), -g(\mathbf{a})\}$. The ambiguity between $\{\mathbf{a}, -\mathbf{a}\}$ and $\{\mathbf{b}, -\mathbf{b}\}$ is also carried along. For the representation matrices of reflections we have thus:

$$\{[\,\mathbf{b}\cdot\boldsymbol{\sigma}\,], -[\,\mathbf{b}\cdot\boldsymbol{\sigma}\,]\} \equiv \mathbf{B} = \mathbf{G}\mathbf{A}\mathbf{G}^{-1} \equiv \mathbf{G}\,\{\,[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,], -[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]\,\}\,\mathbf{G}^{-1} \text{ if } g \in G, \tag{24}$$

whereby we allow for the ambiguity in the sign of $\mathbf{b}$, because Eq. 24 is not a transformation law for vectors, but for reflections and their associated two-member sets of vectors. Of course the idea would be that $g(\mathbf{a}) = \mathbf{b}, \forall g \in G_+$ and $g(\mathbf{a}) = -\mathbf{b}, \forall g \in G_-$, but the combined presence of $\mathbf{G}$ and $\mathbf{G}^{-1}$ does not permit to reproduce the change of sign in the formalism, because it has been for designed for group elements, not for vectors. This is very clear for $A(\mathbf{a}) = -\mathbf{a}$, while in the formalism $\mathbf{A}\,[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]\,\mathbf{A}^{-1} = [\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$, which is the correct calculation for $A = A \circ A \circ A^{-1}$. On the other hand, a vector $\mathbf{b}$ that is perpendicular to $\mathbf{a}$ is characterized by $[\,\mathbf{b}\cdot\boldsymbol{\sigma}\,][\,\mathbf{a}\cdot\boldsymbol{\sigma}\,] = -[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,][\,\mathbf{b}\cdot\boldsymbol{\sigma}\,]$.[11] Therefore $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,][\,\mathbf{b}\cdot\boldsymbol{\sigma}\,][\,\mathbf{a}\cdot\boldsymbol{\sigma}\,] = -[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,][\,\mathbf{a}\cdot\boldsymbol{\sigma}\,][\,\mathbf{b}\cdot\boldsymbol{\sigma}\,] = -[\,\mathbf{b}\cdot\boldsymbol{\sigma}\,]$, while the vector $\mathbf{b} \in \pi_A$ belongs to the reflection plane and should not change sign under the reflection A. We see thus that in all cases, we get the sign of the reflected vector wrong. We can thus lift the ambiguity and treat the vectors correctly by introducing the sign by brute force:

$$\begin{aligned} [\,\mathbf{b}\cdot\boldsymbol{\sigma}\,] &= +\mathbf{G}\,[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]\,\mathbf{G}^{-1}, \text{ if } g \in G_+, \\ [\,\mathbf{b}\cdot\boldsymbol{\sigma}\,] &= -\mathbf{G}\,[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]\,\mathbf{G}^{-1}, \text{ if } g \in G_-. \end{aligned} \tag{25}$$

In doing so, we quit the formalism for group elements and enter a new formalism for vectors. The transition is enacted by conceiving and elaborating the idea that we can use the matrix $\mathbf{A} = [\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ also as the representation of the unit vector $\mathbf{a}$, since the matrix $\mathbf{A}$ contains the components of the vector $\mathbf{a}$ and the reflection A defines $\mathbf{a}$. To get rid of the ambiguity about the signs of the vectors that exist within the definition of the reflection matrices, it suffices to use $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ as a representation for a unit vector $\mathbf{a}$, and to introduce the rule that $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ is transformed according to:

[10] Group elements that transform a reflection A into an other reflection B, are the identity element $AA = \mathbb{1}$ and rotations R that can be written as $R = BA$. For this to be possible the rotation axis of R must belong to the reflection plane π_A of A.

[11] To see this, consider the rotation R that transforms $\mathbf{e}_x$ into $\mathbf{a}$ and $\mathbf{e}_y$ into $\mathbf{b}$. For the reflections σ_x and σ_y we have $\sigma_x\sigma_y = -\sigma_x\sigma_y$. The similarity transformation based on $\mathbf{R}$ will transform σ_x into the reflection A with matrix representation $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ and σ_y into the reflection B with matrix representation $[\,\mathbf{b}\cdot\boldsymbol{\sigma}\,]$. Applying the similarity transformation to $\sigma_x\sigma_y = -\sigma_x\sigma_y$ proves then the identity.

$$[\,\mathbf{a}\boldsymbol{\cdot}\boldsymbol{\sigma}\,] \to [\,R(\mathbf{a})\boldsymbol{\cdot}\boldsymbol{\sigma}\,] = -\mathbf{R}\,[\,\mathbf{a}\boldsymbol{\cdot}\boldsymbol{\sigma}\,]\,\mathbf{R}^{-1} \text{ under reflections } R \in G_{-}, \tag{26}$$

This will be further justified below. The transformation under other elements $g \in G$ is then obtained by using the decomposition of g into reflections. We have this way developed a parallel formalism for the matrices $\mathbf{A}$, wherein $\mathbf{A}$ takes now a different meaning, *viz.* that of a representation of a unit vector $\mathbf{a}$ and obeys a different kind of transformation algebra, that is no longer linear but quadratic in the transformation matrices. This idea can be generalized to a vector $\mathbf{v}$ of arbitrary length v, which is then represented by $\mathbf{V} = v_x\sigma_x + v_y\sigma_y + v_z\sigma_z$. In fact, the scalar v is a group invariant, because the rotation group is defined as the group that leaves v invariant. We have then $\mathbf{V}^2 = -(\det\mathbf{V})\mathbb{1} = v^2\mathbb{1}$. This idea that within SU(2) a vector $\mathbf{v} \in \mathbb{R}^3$ is represented by a matrix $\mathbf{v}\boldsymbol{\cdot}\boldsymbol{\sigma}$ according to the isomorphism:

$$\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z \longleftrightarrow v_x\sigma_x + v_y\sigma_y + v_z\sigma_z = \begin{bmatrix} v_z & v_x - \imath v_y \\ v_x + \imath v_y & -v_z \end{bmatrix} \hat{=}\, \mathbf{v}\boldsymbol{\cdot}\boldsymbol{\sigma}. \tag{27}$$

was introduced by Cartan [1]. It is a definition that makes it possible to do calculations on vectors. In reading Cartan one could get the impression that we have the leisure to introduce this definition at will. In reality, it is not a matter of mere definition. While introducing the idea as a definition would not lead to errors in the formalism, it would nevertheless be a false presentation of the state of affairs, because it is no longer at our discretion to define things at will. As we can see from the reasoning above, the definition is entirely forced upon us by the one-to-one correspondence between unit vectors and reflections.[12] Using $(\mathbf{v}_1 + \mathbf{v}_2)^2 - \mathbf{v}_1^2 - \mathbf{v}_2^2 = 2\,\mathbf{v}_1\boldsymbol{\cdot}\mathbf{v}_2$, one can derive from the rule $\mathbf{V}^2 = v^2\mathbb{1}$ that $\mathbf{V}_1\mathbf{V}_2 + \mathbf{V}_2\mathbf{V}_1 = 2\,(\mathbf{v}_1\boldsymbol{\cdot}\mathbf{v}_2)\,\mathbb{1}$, which can be seen as an alternative definition of the parallel formalism for vectors. As anticipated above, we can use this result to check the correctness of the rule of Eq. 26 geometrically. It suffices in this respect to observe that the reflection A, defined by the unit vector $\mathbf{a}$, transforms $\mathbf{v}$ into $A(\mathbf{v}) = \mathbf{v} - 2(\mathbf{v}\cdot\mathbf{a})\,\mathbf{a}$. Expressed in the matrices this yields: $\mathbf{V} \to -\mathbf{AVA}$.

We see that the transformation law for vectors $\mathbf{v}$ is quadratic in $\mathbf{A}$ in contrast with the transformation law for group elements g, which is linear: $\mathbf{G} \to \mathbf{AG}$. *Vectors transform thus quadratically as rank-2 tensor products of spinors, whereas spinors transform linearly.*[13] This gives us a full understanding of the relationship between vectors and spinors. It is much easier to understand this relationship in the terms used here, vectors are quadratic expressions in terms of spinors, than in the equivalent terms used by Atiyah, spinors are square roots of vectors.[14] The fact that this solution for the riddle what the meaning of a spinor is, has escaped attention is due to the fact that spinors are in general introduced based on the construction proposed in Eq. 28 below. This construction emphasizes the fact that a spinor is a kind of square root of a vector at the detriment of the notion developed here, that a vector is a rank-2 expression in terms of spinors. But these relations between spinors and vectors are a property that constitutes only a secondary notion, which is not really instrumental in clarifying the concept of a spinor. The essential and clarifying notion is that a spinor corresponds to a rotation.

The reader will notice that the definition $\mathbf{V} = \mathbf{v}\boldsymbol{\cdot}\boldsymbol{\sigma}$ with $\mathbf{V}^2 = v^2\mathbb{1}$ is analogous to Dirac's way of introducing the gamma matrices to write the energy-momentum four-vector as $E\gamma_t + c\mathbf{p}\boldsymbol{\cdot}\boldsymbol{\gamma}$ and postulating $(E\gamma_t + c\mathbf{p}\boldsymbol{\cdot}\boldsymbol{\gamma})^2 = (E^2 - c^2p^2)\mathbb{1}$. In other words, it is the metric that defines the whole formalism, because we are considering groups of metric-conserving transformations (as the definition of a geometry in the philosophy of Felix Klein's Erlangen program). For more

[12] Both in the representation matrices $\mathbf{A} = [\,\mathbf{a}\boldsymbol{\cdot}\boldsymbol{\sigma}\,]$ for reflections A and $\mathbf{V} = [\,\mathbf{v}\boldsymbol{\cdot}\boldsymbol{\sigma}\,]$ for vectors $\mathbf{v}$, the quantities σ_x, σ_y, σ_z are the three Pauli matrices. In the representation ($\mathbf{e}_j \leftrightarrow \sigma_j = [\,\mathbf{e}_j\boldsymbol{\cdot}\boldsymbol{\sigma}\,]$) defined by Eq. 27, the Pauli matrices $\sigma_x, \sigma_y, \sigma_y$ are just the images, *i.e.* the coding of the three basis vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. As clearly indicated in the diagram of Eq. 5, $\boldsymbol{\sigma}$ is a shorthand for the triple $(\sigma_x, \sigma_y, \sigma_z)$. The use of the symbol $\hat{=}$ serves to draw the attention to the fact that the notation $[\,\mathbf{v}\boldsymbol{\cdot}\boldsymbol{\sigma}\,]$ is a purely conventional shorthand for $v_x\sigma_x + v_y\sigma_y + v_z\sigma_z$ which codes the vector $\mathbf{v}$ within the formalism. It is thus analogous to writing $v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z$ pedantically as: $(v_x, v_y, v_z)\boldsymbol{\cdot}(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. The danger of using the convenient shorthand $[\,\mathbf{v}\boldsymbol{\cdot}\boldsymbol{\sigma}\,]$ is that it conjures up the image of a scalar product, while there is no scalar product whatsoever. The fact that $[\,\mathbf{v}\boldsymbol{\cdot}\boldsymbol{\sigma}\,]$ represents the vector $\mathbf{v}$, and that the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ just represent the basis vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$, was clearly stated by Cartan, but physicists have nevertheless over-interpreted the vector $-\frac{\hbar q}{2m_0c}\,[\,\mathbf{B}\boldsymbol{\cdot}\boldsymbol{\sigma}\,]$ as a scalar product $\mathbf{B}\boldsymbol{\cdot}\boldsymbol{\mu}$ in the theory of the anomalous g-factor for the electron. Here $\boldsymbol{\mu}$ would be the magnetic dipole of the electron and $-\mathbf{B}\boldsymbol{\cdot}\boldsymbol{\mu}$ its potential energy with the magnetic field $\mathbf{B}$. In reality $\mathbf{B}\boldsymbol{\cdot}\boldsymbol{\sigma}$ just expresses the magnetic-field pseudo-vector $\mathbf{B}$. The quantity $\frac{\hbar}{2}\boldsymbol{\sigma}$ can never represent the spin, because it is already defined in Euclidean geometry. This reveals that physicists do not only use spinors like vectors: They also use vectors like scalars. To reassure the reader, we may state that the algebra involved with the calculation of the g-factor is correct, and that it is only its interpretation which is clumsy. We have tidied up this problem and proposed a better interpretation elsewhere [13].

[13] Even if reflections $A \in \mathrm{L}(\mathbb{R}^3, \mathbb{R}^3)$ and unit vectors $\mathbf{a} \in \mathbb{R}^3$ are both represented by the same 2×2 matrix $[\,\mathbf{a}\boldsymbol{\cdot}\boldsymbol{\sigma}\,]$, they are obviously completely different quantities, belonging to completely different spaces $\mathrm{L}(\mathbb{R}^3, \mathbb{R}^3)$ and $\mathbb{R}^3$.

[14] It is analogous to the solution proposed by Gauss, Wessel and Argant to solve the problem of the meaning of $\imath = \sqrt{-1}$. As described on p. 118 of reference [14], one defines first $\mathbb{C}$ as $\mathbb{R}^2$, with two operations $+$ and $\times$ defined by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $(x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$. One then shows that $(\mathbb{R}, +, \times)$ is isomorphic to $(\mathscr{R}, +, \times)$ where $\mathscr{R} = \{\,(x, y) \in \mathbb{C} \parallel y = 0\,\} \subset \mathbb{C}$. This permits to identify $\mathbb{R} \equiv \mathscr{R}$ and justifies introducing the notations $x \equiv (x, 0) \in \mathbb{R}$, $\imath \equiv (0, 1)$ and $(x, y) \equiv x + \imath y$. One can prove then that $\imath^2 \equiv (0, 1)^2 = (-1, 0) \equiv -1$.

information about the calculus on the rotation and reversal matrices, we refer the reader to reference [4]. Let us just mention that as a reflection A works on a vector $\mathbf{v}$ according to $\mathbf{V} \rightarrow -\mathbf{AVA} = -\mathbf{AVA}^{-1}$, a rotation $R = BA$ will work on it according to $\mathbf{V} \rightarrow \mathbf{BAVAB} = \mathbf{RVR}^{-1} = \mathbf{RVR}^{\dagger}$. The identity $\mathbf{R}^{-1} = \mathbf{R}^{\dagger}$ explains, in an alternative way, why the representation we end up with is SU(2).

In summary, there are two parallel formalisms in SU(2), one for the vectors and one for the group elements. In both formalisms a matrix $\mathbf{V} = \mathbf{v}{\cdot}\boldsymbol{\sigma}$ can occur but with different meanings. In a formalism for group elements, $\mathbf{v}$ fulfills the rôle of the unit vector $\mathbf{a}$ that defines the reflection A, such that we must have $|\mathbf{v}| = 1$, and then the reflection matrix $\mathbf{V} = \mathbf{A}$ transforms according to: $\mathbf{A} \rightarrow \mathbf{GA}$ under a group element g with matrix representation $\mathbf{G}$. The new group element represented by $\mathbf{GA}$ will then in general no longer be a reflection that can be associated with a unit vector like it was the case for $\mathbf{A}$. In a formalism of vectors, $|\mathbf{v}|$ can be different from 1 and the matrix $\mathbf{V}$ (that represents now a vector) transforms according to: $\mathbf{V} \rightarrow \mathbf{GVG}^{-1} = \mathbf{GVG}^{\dagger}$. Here $\mathbf{GVG}^{\dagger}$ can be associated again with a vector.

We cannot emphasize enough that the vector formalism is a *parallel* formalism that is different from the one for reflections because the reflections defined by $\mathbf{a}$ and $-\mathbf{a}$ are equivalent, while the vectors $\mathbf{a}$ and $-\mathbf{a}$ are not. We have here two concepts that are algebraically identical but not geometrically and this is the source of a lot of confusion. The folklore that one must rotate a wave function by 4π to obtain the same wave function again is part of that confusion. The reflection operator $[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,]$ is a thing that is entirely different from the unit vector $[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,]$, even if their expressions are algebraically identical. By rotating a reflection plane over π we obtain the same reflection, while it takes rotating 2π to obtain the same vector $\mathbf{a}$.

2.5 The quadratic relation between vectors and spinors

2.5.1 Isotropic vectors

We will illustrate the quadratic relationship between spinors and vectors further in what we can consider as the final step in the construction of the formalism. We can picture a rotation R by a rotated triad of three basis vectors $\mathbf{e}'_x = R(\mathbf{e}_x)$, $\mathbf{e}'_y = R(\mathbf{e}_y)$, and $\mathbf{e}'_z = R(\mathbf{e}_z)$. This is a 1-1-correspondence. Triads visualize rotations and *vice versa*. This is a second important idea, that we will be able to carry over to the general case of SO(n): We will code group elements by identifying them with a rotated basis of $\mathbb{R}^n$, a so-called *Vielbein*.[15]

In SU(2) we can code this triad within an isotropic vector $\mathbf{e}'_x + \imath\mathbf{e}'_y = (x, y, z) \in \mathbb{C}^3$. This is also a 1-1-correspondence. From $(x, y, z) \in \mathbb{C}^3$ we can get $\mathbf{e}'_x$ and $\mathbf{e}'_y$ back by taking real and imaginary parts, while $\mathbf{e}'_z = \mathbf{e}'_x \wedge \mathbf{e}'_y$. We can represent thus a rotation by an isotropic vector.

□ Remark 1. Presented this way, this idea may look like a stroke of genius. But in reality, it is just the consequence of embedding $\mathbb{R}^{2n}$ within $\mathbb{C}^{2n}$. We can thus embed $\mathbb{R}^4$ within $\mathbb{C}^4$. Instead of the basis of the mutually orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ of $\mathbb{R}^4$ as a basis for $\mathbb{C}^4$ one can use a coordinate transformation and use the alternative orthogonal basis $\boldsymbol{\eta}_1 = \mathbf{e}_1 + \imath\mathbf{e}_2$, $\boldsymbol{\eta}_1^* = \mathbf{e}_1 - \imath\mathbf{e}_2$ and $\boldsymbol{\eta}_2 = \mathbf{e}_3 + \imath\mathbf{e}_4$, $\boldsymbol{\eta}_2^* = \mathbf{e}_3 - \imath\mathbf{e}_4$ for $\mathbb{C}^4$ (see paragraph 4.6.1). This basis can also be normalized using the Hermitian norm. The subspace spanned by $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ suffices to define the complete *Vielbein* of $\mathbb{R}^4$ and is isomorphic to $\mathbb{C}^2$. The space $\mathbb{R}^3$ is a subspace of $\mathbb{R}^4$, and once we have defined it, it becomes this way possible to treat also $\mathbb{R}^3$ in terms of $\mathbb{C}^2$. This is the reason why will end up with a formalism SU(2). The use of isotropic vectors is thus just a consequence of introducing $\boldsymbol{\eta}_1 = \mathbf{e}_1 + \imath\mathbf{e}_2$, but the idea becomes somewhat concealed by the fact that we work with $\mathbb{R}^3$ instead of $\mathbb{R}^4$, such that we do not have $\boldsymbol{\eta}_2 = \mathbf{e}_3 + \imath\mathbf{e}_4$ to tip us off.

We can now code the isotropic vector (x, y, z) (which codes the triad and thus also the rotation) as the 2×2 matrix $\mathbf{M} = x\sigma_x + y\sigma_y + z\sigma_z$. As for an isotropic vector $x^2 + y^2 + z^2 = 0$, we have $\det(\mathbf{M}) = 0$.[16] This implies that the columns of the matrix $\mathbf{M}$ are proportional. Also the lines of $\mathbf{M}$ are proportional. These proportionalities imply that we can write:

$$\mathbf{M} = \begin{bmatrix} z & x - \imath y \\ x + \imath y & -z \end{bmatrix} = \sqrt{2} \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \otimes [\,-\xi_1, \xi_0\,]\sqrt{2} = 2\,[\,\boldsymbol{\chi} \otimes \dot{\boldsymbol{\psi}}^{\dagger}\,], \tag{28}$$

which introduces the column "spinor" $\boldsymbol{\chi}$ and the conjugated row "spinor" $\dot{\boldsymbol{\psi}}^{\dagger}$. We are putting here the words spinor between quotes, because for the moment it is not yet obvious that they correspond to the same concept as the one we introduced above. We will address this issue very soon. The notation $\dot{\boldsymbol{\psi}}^{\dagger}$ just serves to distinguish row spinors $\dot{\boldsymbol{\psi}}^{\dagger}$

[15] This is a German word meaning "many legs", and the idea is that each basis vector is a leg.

[16] It is often stated in this respect that an isotropic vector has zero length and that it is orthogonal to itself. This is however based on the wrong notion that the extrapolation to $\mathbb{C}^3$ of the Euclidean norm, $|\cdot|_E$ defined by: $\forall (x, y, z) \in \mathbb{R}^3$, $|(x, y, z)|_E = \sqrt{x^2 + y^2 + z^2}$, would still be a correct norm function for $(x, y, z) \in \mathbb{C}^3$. The correct norm to be used for $(x, y, z) \in \mathbb{C}^3$ is the Hermitian norm $|\cdot|_H$ defined by: $\forall (x, y, z) \in \mathbb{C}^3 : |(x, y, z)|_H = \sqrt{xx^* + yy^* + zz^*}$.

from column spinors $\boldsymbol{\chi}$. We will explain below the reason for this rather complicated looking notation $\dot{\boldsymbol{\psi}}^{\dagger}$. The square roots $\sqrt{2}$ are introduced for normalization purposes.[17]

□ Remark 2. When we will try to generalize the formalism to SO(n), we will no longer be able to factorize the matrix of an isotropic vector as done here. For a matrix $\mathbf{M}$ of rank $\rho > 2$ we can no longer conclude from $\det \mathbf{M} = 0$ that there exist $\rho \times 1$ matrices $\boldsymbol{\chi}$ and $1 \times \rho$ matrices $\dot{\boldsymbol{\psi}}^{\dagger}$ such that $\mathbf{M} = \boldsymbol{\chi} \otimes \dot{\boldsymbol{\psi}}^{\dagger}$, because this would imply that all columns of $\mathbf{M}$ are proportional and that all rows of $\mathbf{M}$ are proportional, while it suffices that only two columns and two rows of $\mathbf{M}$ are proportional (See paragraph 4.6.2).

But for the moment we can see how for the specific case of SU(2), the gimmick $\mathbf{M} = \boldsymbol{\chi} \otimes \dot{\boldsymbol{\psi}}^{\dagger}$ permits us to "halve" the formalism. In fact, the isotropic vector that codes the rotation transforms under rotations *quadratically* according to $\mathbf{M} \to \mathbf{RMR}^{-1} = \mathbf{RMR}^{\dagger} = \mathbf{R}\,[\,\boldsymbol{\chi} \otimes \dot{\boldsymbol{\psi}}^{\dagger}\,]\,\mathbf{R}^{\dagger}$, with multiplications on both sides. We could obtain the same result by stipulating that we must transform $\boldsymbol{\chi} \to \mathbf{R}\boldsymbol{\chi}$ and $\dot{\boldsymbol{\psi}}^{\dagger} \to \dot{\boldsymbol{\psi}}^{\dagger}\mathbf{R}^{\dagger}$. Now a spinor $\boldsymbol{\phi}$ that contains the same information as a rotation matrix transforms *linearly* according to $\boldsymbol{\phi} \to \mathbf{R}\boldsymbol{\phi}$, with only left multiplications. On the other hand, an isotropic vector contains the same information as a rotation matrix, because it codes the triad.

Let us now show that the "spinor" formalism for the isotropic vector is algebraically identical to the spinor formalism for the rotations, such that $\boldsymbol{\chi}$ is indeed algebraically a spinor.[18] The reference triad is coded by the isotropic vector $(x, y, z) = \mathbf{e}_x + \imath\mathbf{e}_y = (1, \imath, 0)$, leading to:

$$\mathbf{M} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes [\,0, 1\,]\sqrt{2} \quad \Rightarrow \quad \boldsymbol{\chi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \dot{\boldsymbol{\psi}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{29}$$

This reference triad corresponds to the identity matrix. The corresponding spinor $\boldsymbol{\phi} = \hat{c}_1(\mathbb{1})$, is indeed equal to $\boldsymbol{\chi}$, such that we have checked that the formalism based on multiplying $\boldsymbol{\chi}$ to the left according to $\boldsymbol{\chi} \to \mathbf{R}\boldsymbol{\chi}$ is just identical to the formalism based on multiplying $\boldsymbol{\phi}$ according to $\boldsymbol{\phi} \to \mathbf{R}\boldsymbol{\phi}$.

To summarize, it is not possible in SU(2) to build a linear representation based on vectors because vectors are of rank two in terms of spinor quantities, but is possible to build a linear representation based on spinors by "halving" the formalism. We could also proceed by only right multiplications on $\dot{\boldsymbol{\psi}}^{\dagger}$ according to $\dot{\boldsymbol{\psi}}^{\dagger} \to \dot{\boldsymbol{\psi}}^{\dagger}\mathbf{R}^{\dagger}$, but that would be completely equivalent. The spinor $\dot{\boldsymbol{\psi}}$ transforms like $\boldsymbol{\chi}$, by left multiplication by $\mathbf{R}$, and gives rise to the second column of the matrix in Eq. 3. It contains the same information as $\boldsymbol{\chi}$. Using $\dot{\boldsymbol{\psi}}$ instead of $\dot{\boldsymbol{\psi}}^{\dagger}$ allows us then also to limit ourselves to calculations that contain only left multiplications. In other words, in the notation $\dot{\boldsymbol{\psi}}^{\dagger}$, the symbol † is supposed to flag that it is transformed by right multiplication by $\mathbf{R}^{\dagger}$, while the dot is used to distinguish quantities $\dot{\boldsymbol{\psi}}$ from quantities $\boldsymbol{\chi}$, showing that the quantities $\dot{\boldsymbol{\psi}}$ originally have entered the formalism under the form of row spinors $\dot{\boldsymbol{\psi}}^{\dagger}$. Whereas the formalism $\mathbf{M} \to \mathbf{RMR}^{-1}$ was not linear in the parameters of the rotation matrix $\mathbf{R}$, halving the formalism to $\boldsymbol{\phi} \to \mathbf{R}\boldsymbol{\phi}$ has rendered it linear.

Because a rotation depends only on three independent real parameters, we can normalize these spinors to 1 such that $\xi_0\xi_0^* + \xi_1\xi_1^* = 1$. In fact, the normalization is a consequence of the fact that the matrix in Eq. 3 belongs to

[17] There is some possibility of confusion with the terminology here. From the purely algebraic point of view of matrix algebra, we could call these spinor quantities column "vectors" and row "vectors", but from the geometrical point of view, spinors are not vectors, because they code rotations, and rotations do not build a vector space.

[18] In reality this requires a discussion, because one can consider that it is based on exploiting a misleading algebraic identity between the "spinors" and true spinors which represent rotations, which further extends the confusion between the vector spaces $\mathbb{R}^3$ and $\mathrm{L}(\mathbb{R}^3, \mathbb{R}^3)$ beyond the one created by the algebraic identity of the expressions $[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,]$ for the unit vectors $\mathbf{a}$ and the reflections A with a reflection normal $\mathbf{a}$. We have extended the formalism from $\mathbf{a} \in \mathbb{R}^3$, first to $\mathbf{v} \in \mathbb{R}^3$, and now to $\mathbf{v} \in \mathbb{C}^3$. One might start speculating about extending this formalism for rotations to the larger group generated by the reflections, but we have to abandon this idea because it would introduce ambiguity in the algebra. In fact, the reflection matrix σ_z and the identity element of the rotation group $\mathbb{1}$ become then represented by the same spinor. Furthermore, as the rotations are not building a vector space, we cannot consider these shorthands as basis vectors of some function space, because we are not able to identify the geometrical meaning of the function space wherein the shorthands would be embedded. It would be a jump into the unknown, and therefore *a priori* into the undefined. Similarly, when we have defined that $[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,]$ represents a vector, we have not defined that its column vectors would have a geometrical meaning. In other words, both for $[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,] \in \mathbb{R}^3$ and for $[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,] \in \mathrm{L}(\mathbb{R}^3, \mathbb{R}^3)$ we must operate wild extrapolations of the geometrical definition domain of the algebra before we can reach a situation wherein we could attempt to make a purely algebraic identification between a "spinor" and a true spinor. We can consider all these transgressions and extrapolations as meaningless, scatterbrained algebra and argue that it is this undefined bogus that leads then to the puzzling notion that a spinor would be a square root of a vector. Rather than trying to solve this tantalizing puzzle at any price, we could consider that it is *a priori* completely meaningless. However, we can solve this problem by considering the algebraic identity as an isomorphism under multiplication with 2×2 rotation matrices. We can further appreciate that the "spinor" is not an exact square root of a vector, and the description "square root" will become even more wanting in SO(n) when $n \geq 4$.

SU(2). The spinor contains thus exactly three independent parameters that characterize a rotation (e.g. the three Euler angles, or a rotation axis defined by a unit vector $\mathbf{n}$ and a rotation angle χ). From these spinors and using the identity $\xi_0\xi_0^* + \xi_1\xi_1^* = 1$ we can calculate backwards to (x, y, z). The result is:

$$x = \xi_0^2 - \xi_1^2, \quad y = \imath(\xi_0^2 + \xi_1^2), \quad z = -2\xi_0\xi_1. \tag{30}$$

From this we can recover the basis vectors $\mathbf{e}_x'(x_1, y_1, z_1)$, $\mathbf{e}_y'(x_2, y_2, z_3)$:

$$\begin{array}{ll} x_1 = \frac{1}{2}(\xi_0^2 - \xi_1^2 + \xi_0^{*2} - \xi_1^{*2}), & y_1 = \frac{\imath}{2}(\xi_0^2 + \xi_1^2 - \xi_0^{*2} - \xi_1^{*2}), \\ z_1 = -(\xi_0\xi_1 + \xi_0^*\xi_1^*), & \\ & \\ x_2 = \frac{\imath}{2}(-\xi_0^2 + \xi_1^2 + \xi_0^{*2} - \xi_1^{*2}), & y_2 = \frac{1}{2}(\xi_0^2 + \xi_1^2 + \xi_0^{*2} + \xi_1^{*2}), \\ z_2 = (\xi_0\xi_1 - \xi_0^*\xi_1^*). & \end{array} \tag{31}$$

and from this finally $\mathbf{e}_z' = (x_3, y_3, z_3) = \mathbf{e}_x' \wedge \mathbf{e}_y'$:

$$x_3 = \xi_0\xi_1^* + \xi_0^*\xi_1, \quad y_3 = \imath(\xi_0\xi_1^* - \xi_0^*\xi_1), \quad z_3 = \xi_0\xi_0^* - \xi_1\xi_1^*. \tag{32}$$

We can also calculate ξ_0 and ξ_1 from x, y and z, and this leads to the expressions introduced by Cartan:

$$\left[\begin{array}{c} \xi_0 \\ \xi_1 \end{array} \right] = \left[\begin{array}{c} \pm\sqrt{\frac{x - \imath y}{2}} \\ \\ \pm\sqrt{\frac{-x - \imath y}{2}} \end{array} \right]. \tag{33}$$

This shows how the reference triad of basis vectors is expressed within a spinor. Similar expressions can be derived to show e.g. how the three Euler angles are expressed within a spinor. The Rodrigues formula shows how the rotation axis $\mathbf{n}$ and the rotation angle φ are expressed within the spinor.

In many textbooks, spinors are introduced on the basis of this algebra for the isotropic vector. It is this approach that leads to the idea that a spinor is the square root of a vector.[19] But we can appreciate from our approach that the true meaning of a spinor is not that it is "a kind of isotropic vector" as stated by Cartan, but just a rotation. In generalizing this idea, we can define a spinor to be just a group element. The isotropic vector is merely a secondary tool to express this idea through quite ingenious "slick algebra". The basic idea that a spinor is a rotation is much simpler and developing it requires much less ingenuity.

☐ Remark 3. In reference [4] we discuss also the way SU(2) is introduced in textbooks based on a stereographic projection. We show that this method is in reality conceptually flawed because it only considers the basis vector $\mathbf{e}_z'$, which cannot represent the complete information about a rotation. A rotation of $\mathbf{e}_z$ to $\mathbf{e}_z'$ does not define a unique rotation, as one can afterwards still rotate the basis triad freely around $\mathbf{e}_z'$ over a rotation angle χ.

☐ Remark 4. Many a physicist will be used to the concept of infinitesimal generators used to define the Lie algebra. This can e.g. be used to rewrite the Rodrigues formula Eq. 8 under the form:

$$\mathbf{R}(\mathbf{n}, \varphi) = e^{-\imath[\,\mathbf{n}\cdot\boldsymbol{\sigma}\,]\varphi/2}. \tag{34}$$

In this context, the infinitesimal generators pick up algebraic expressions that are algebraically identical to those for the reflection matrices. We must point out that this algebraic identity is a mere coincidence. The definitions of the Pauli matrices in terms of reflection matrices and in terms of infinitesimal generators are conceptually completely different. One should indeed already feel rather puzzled by the fact that due to the algebraic identity a reflection operator appears to be related to an infinitesimal rotation. The solution of this riddle becomes obvious by considering rotations or Lorentz transformations in $\mathbb{R}^4$. We have then four reflection operators, while there are six infinitesimal generators, such that the two concepts are now clearly seen not to be equivalent. The four reflection operators have four-dimensional vector symmetry and are true generators for the rotation group. The infinitesimal generators have six-dimensional tensor symmetry. They are a vector basis for the six-dimensional tangent space to the Lie group. This explains also why the infinitesimal generators for SU(3) cannot be found by following the strategy outlined in Subsection 2.2.

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[19] The presence of the square roots in Eq. 33 can inspire here also the idea that a spinor is the "square root" of a vector. The Rodrigues equation Eq. 8 can also be expressed as $\mathbf{R}(\mathbf{n}, \varphi) = \frac{1}{2}\, e^{-\imath\varphi/2}\,(\mathbb{1} + [\,\mathbf{n}\cdot\boldsymbol{\sigma}\,]) + \frac{1}{2}\, e^{+\imath\varphi/2}\,(\mathbb{1} - [\,\mathbf{n}\cdot\boldsymbol{\sigma}\,])$. Within this algebraic form the presence of $\varphi/2$ in the exponentials leads also to the idea of a "square root".

2.5.2 Real unit vectors

Eq. 28 is the reason why one says that a spinor is a square root of a vector. We can see that this is only very approximately true as the two spinors $\boldsymbol{\chi}$ and $\dot{\boldsymbol{\psi}}$ are not equal. There is a relation between spinors and vectors that illustrates in a much more direct and less artificial way how vectors are "squares" of spinors. Consider a rotation R with matrix $\mathbf{R}$ that turns the reference triad. The vector $\mathbf{e}'_z = R(\mathbf{e}_z)$ of the rotated reference triad in Eq. 32 can be expressed as:

$$[\,\mathbf{e}'_z{\cdot}\boldsymbol{\sigma}\,] = 2\,\boldsymbol{\chi}\otimes\boldsymbol{\chi}^\dagger - \mathbb{1}. \tag{35}$$

In fact,

$$[\,\mathbf{e}_z{\cdot}\boldsymbol{\sigma}\,] + \mathbb{1} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix}\sqrt{2}. \tag{36}$$

Under the rotation $\mathbf{R}$ this transforms to:

$$[\,\mathbf{e}'_z{\cdot}\boldsymbol{\sigma}\,] + \mathbb{1} = \mathbf{R}\,(\,[\,\mathbf{e}_z{\cdot}\boldsymbol{\sigma}\,] + \mathbb{1}\,)\mathbf{R}^{-1} = \sqrt{2}\begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix} \otimes \begin{bmatrix} \xi_0^* & \xi_1^* \end{bmatrix}\sqrt{2}. \tag{37}$$

Where we have used $\mathbf{R}^{-1} = \mathbf{R}^\dagger$ and $\mathbf{R}\,\mathbb{1}\mathbf{R}^{-1} = \mathbb{1}$ to obtain the desired result. With respect to this identity, introducing the isotropic vectors to argue that vectors are rank-2 quantities in terms of spinors is thus rather a step away from a truly illuminating conceptual understanding of the quadratic relationship. It makes everything more difficult and less clear. We can illustrate this relation between a vector and its spinor in SU(2). We represent the vector by its spherical coordinates as follows:

$$[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,] = \begin{bmatrix} \cos\theta & \sin\theta e^{-\imath\phi} \\ \sin\theta e^{\imath\phi} & -\cos\theta \end{bmatrix}. \tag{38}$$

The rotation required to rotate $\mathbf{e}_z$ to $\mathbf{a}$ along a great circle has axis $\mathbf{n} = (\cos(\phi+\pi/2), \sin(\phi+\pi/2), 0)$ and angle θ. The angle of rotation is counterclockwise when we look at it from the point $(\cos(\phi+\pi/2), \sin(\phi+\pi/2), 0)$. The rotation is thus expressed by:

$$\mathbf{R} = \begin{bmatrix} \cos(\theta/2) & -\imath\sin(\theta/2)e^{-\imath(\phi+\pi/2)} \\ -\imath\sin(\theta/2)e^{\imath(\phi+\pi/2)} & \cos(\theta/2) \end{bmatrix}. \tag{39}$$

One can then check that $[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,] = \mathbf{R}\,[\,\mathbf{e}_z{\cdot}\boldsymbol{\sigma}\,]\,\mathbf{R}^\dagger$, and that:

$$[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,] = 2\,\boldsymbol{\chi}\otimes\boldsymbol{\chi}^\dagger - \mathbb{1}, \quad \text{with:} \quad \boldsymbol{\chi} = \begin{bmatrix} \cos(\theta/2) \\ -\imath\sin(\theta/2)e^{\imath(\phi+\pi/2)} \end{bmatrix}. \tag{40}$$

The spinor $\boldsymbol{\chi}$ that we can associate with $\mathbf{a}$ is thus the rotation required to turn $\mathbf{e}_z$ to $\mathbf{a}$. We can also write $[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,]$ as:

$$[\,\mathbf{a}{\cdot}\boldsymbol{\sigma}\,] = \boldsymbol{\chi}\otimes\boldsymbol{\chi}^\dagger - \dot{\boldsymbol{\psi}}\otimes\dot{\boldsymbol{\psi}}^\dagger \tag{41}$$

This is based on:

$$[\,\mathbf{e}_z{\cdot}\boldsymbol{\sigma}\,] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{42}$$

The various column spinors we obtain are the columns of the rotation matrix. The line spinors are their Hermitian conjugates. The conjugated spinors can be obtained by considering:

$$[\,\mathbf{e}_z{\cdot}\boldsymbol{\sigma}\,] - \mathbb{1} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} = -\sqrt{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}\sqrt{2}. \tag{43}$$

Under the rotation $\mathbf{R}$ this transforms to:

$$[\,\mathbf{e}'_z{\cdot}\boldsymbol{\sigma}\,] - \mathbb{1} = \mathbf{R}\,(\,[\,\mathbf{e}_z{\cdot}\boldsymbol{\sigma}\,] - \mathbb{1}\,)\mathbf{R}^{-1} = -\sqrt{2}\begin{bmatrix} -\xi_0^* \\ \xi_1^* \end{bmatrix} \otimes \begin{bmatrix} -\xi_0 & \xi_1 \end{bmatrix}\sqrt{2}, \tag{44}$$

such that:

$$[\,\mathbf{e}'_z{\cdot}\boldsymbol{\sigma}\,] = \mathbb{1} - 2\,\dot{\boldsymbol{\psi}}\otimes\dot{\boldsymbol{\psi}}^\dagger. \tag{45}$$

The conjugated spinor is thus the alternative spinor obtained by taking the second column of the rotation matrix. We may note that representation matrices of all the basis vectors are linked by a similarity transformation to $[\,\mathbf{e}_z{\cdot}\boldsymbol{\sigma}\,]$ such that they all have eigenvalues 1 and -1.

2.6 Alternative choices of sets of matrices that satisfy the six conditions $\tau_\mu\tau_\nu + \tau_\nu\tau_\mu = 2\delta_{\mu\nu}\mathbb{1}$

2.6.1 Construction based on SU(2)

We have chosen the Pauli matrices as the set of matrices that satisfy the condition $\tau_\mu\tau_\nu + \tau_\nu\tau_\mu = 2\delta_{\mu\nu}\mathbb{1}$. This choice may be the standard one found in textbooks, but it is by far not unique. We can use the fact that σ_x, σ_y, σ_z represent the vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ in order to propose alternative choices τ_x, τ_y, τ_z for a set of matrices that satisfy the conditions $\tau_\mu\tau_\nu + \tau_\nu\tau_\mu = 2\delta_{\mu\nu}\mathbb{1}$. In fact, in the parallel formalism for vector matrices, the condition $\sigma_\mu\sigma_\nu + \sigma_\nu\sigma_\mu = 2\delta_{\mu\nu}\mathbb{1}$ expresses $\mathbf{e}_\mu\cdot\mathbf{e}_\nu = \delta_{\mu\nu}$, which means that $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ is an orthonormal basis. Under a rotation R, the vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ will transform into $\mathbf{e}'_x$, $\mathbf{e}'_y$, $\mathbf{e}'_z$, which will be as good an orthonormal basis for $\mathbb{R}^3$ as $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$. Therefore $\tau_x = [\,\mathbf{e}'_x{\cdot}\boldsymbol{\sigma}\,]$, $\tau_y = [\,\mathbf{e}'_y{\cdot}\boldsymbol{\sigma}\,]$, $\tau_z = [\,\mathbf{e}'_z{\cdot}\boldsymbol{\sigma}\,]$, will also satisfy the condition $\tau_\mu\tau_\nu + \tau_\nu\tau_\mu = 2\delta_{\mu\nu}\mathbb{1}$. This can also be shown algebraically. The representations of the vectors $\mathbf{e}'_x$, $\mathbf{e}'_y$, $\mathbf{e}'_z$ will be:

$$\tau_x = [\,\mathbf{e}'_x{\cdot}\boldsymbol{\sigma}\,] = \mathbf{R}\sigma_x\mathbf{R}^{-1}, \quad \tau_y = [\,\mathbf{e}'_y{\cdot}\boldsymbol{\sigma}\,] = \mathbf{R}\sigma_y\mathbf{R}^{-1}, \quad \tau_z = [\,\mathbf{e}'_z{\cdot}\boldsymbol{\sigma}\,] = \mathbf{R}\sigma_z\mathbf{R}^{-1}, \tag{46}$$

such that:

$$\tau_\mu\tau_\nu + \tau_\nu\tau_\mu = \mathbf{R}\sigma_\mu\mathbf{R}^{-1}\mathbf{R}\sigma_\nu\mathbf{R}^{-1} + \mathbf{R}\sigma_\nu\mathbf{R}^{-1}\mathbf{R}\sigma_\mu\mathbf{R}^{-1} = \mathbf{R}\,[\,\sigma_\mu\sigma_\nu + \sigma_\nu\sigma_\mu\,]\,\mathbf{R}^{-1} = \mathbf{R}\,[\,2\delta_{\mu\nu}\mathbb{1}\,]\,\mathbf{R}^{-1} = 2\delta_{\mu\nu}\mathbb{1}. \tag{47}$$

This is an algebraic proof for the fact that $\tau_x = [\,\mathbf{e}'_x{\cdot}\boldsymbol{\sigma}\,]$, $\tau_y = [\,\mathbf{e}'_y{\cdot}\boldsymbol{\sigma}\,]$, $\tau_z = [\,\mathbf{e}'_z{\cdot}\boldsymbol{\sigma}\,]$ is indeed an alternative set of matrices satisfying $\tau_\mu\tau_\nu + \tau_\nu\tau_\mu = 2\delta_{\mu\nu}\mathbb{1}$. Based on Eq. 3, we can represent a general rotation matrix $\mathbf{R}$ by:

$$\mathbf{R} = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}, \quad \text{with:} \quad aa^* + bb^* = 1. \tag{48}$$

We obtain then that the triples:

$$\tau_x = \begin{bmatrix} -ab - a^*b^* & a^2 - b^{*2} \\ a^{*2} - b^2 & ab + a^*b^* \end{bmatrix}, \quad \tau_y = \imath \begin{bmatrix} ab - a^*b^* & -a^2 - b^{*2} \\ a^{*2} + b^2 & a^*b^* - ab \end{bmatrix}, \quad \tau_z = \begin{bmatrix} aa^* - bb^* & 2ab^* \\ 2a^*b & bb^* - aa^* \end{bmatrix}, \tag{49}$$

with $(a,b) \in \mathbb{C}^2$ and $aa^* + bb^* = 1$, are all alternative choices for representation matrices satisfying the six conditions $\tau_\mu\tau_\nu + \tau_\nu\tau_\mu = 2\delta_{\mu\nu}\mathbb{1}$. The parameters $(a,b) \in \mathbb{C}^2$ and $aa^* + bb^* = 1$ explore the three-dimensional manifold SU(2).

2.6.2 Construction based on SL(2,$\mathbb{C}$)

The argument used in paragraph 2.6.1 shows that we can actually use any matrix $\mathbf{S}$ with an inverse $\mathbf{S}^{-1}$ instead of $\mathbf{R}$ to obtain an equivalent basis of representation matrices. Let us consider such a matrix $\mathbf{S}$:

$$\mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{with:} \quad (a,b,c,d) \in \mathbb{C}^4 \;\&\; D = \det(\mathbf{S}) = ad - bc \neq 0. \tag{50}$$

The inverse matrix is then:

$$\mathbf{S}^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{51}$$

Put now: $\mathbf{L} = \frac{1}{D^{\frac{1}{2}}}\mathbf{S}$. Here both square roots of D can be used. We will then have $\mathbf{L}^{-1} = D^{\frac{1}{2}}\mathbf{S}^{-1}$, $\det(\mathbf{L}) = \det(\mathbf{L}^{-1}) = 1$, and:

$$\mathbf{L}\sigma_\mu\mathbf{L}^{-1} = \mathbf{S}\sigma_\mu\mathbf{S}^{-1}. \tag{52}$$

This shows that we obtain already all possible new sets of representation matrices τ_μ by taking $ad - bc = 1$. Matrices with $ad - bc = D \neq 0$, and $D \neq 1$ do not add new possible solutions. The set of matrices:

$$\mathbf{L} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{with:} \quad (a,b,c,d) \in \mathbb{C}^4 \;\&\; ad - bc = 1, \tag{53}$$

is called SL(2,$\mathbb{C}$) and is shown in reference [4] to built a representation for the homogeneous Lorentz group. In this representation, four-vector matrices:

$$\mathbf{V} = v_t\mathbb{1} + \mathbf{v}{\boldsymbol\cdot}\boldsymbol{\sigma} = \begin{bmatrix} v_t + v_z & v_x - \imath v_y \\ v_x + \imath v_y & v_t - v_z \end{bmatrix}, \tag{54}$$

representing the four-vector $(v_t, \mathbf{v})$, transform under a Lorentz transformation represented by the matrix $\mathbf{L}$ in Eq. 53 according to: $\mathbf{V} \rightarrow \mathbf{L}\mathbf{V}\mathbf{L}^\dagger$, where *a priori* $\mathbf{L}^\dagger \neq \mathbf{L}^{-1}$. The rotation matrices $\mathbf{R}$ from SU(2) are a subset of SL(2,$\mathbb{C}$). The most general form of an alternative set of representation matrices we obtain by this method is thus:

$$\tau_x = \begin{bmatrix} bd - ac & a^2 - b^2 \\ d^2 - c^2 & ac - bd \end{bmatrix}, \quad \tau_y = \imath \begin{bmatrix} ac + bd & -a^2 - b^2 \\ c^2 + d^2 & -ac - bd \end{bmatrix}, \quad \tau_z = \begin{bmatrix} ad + bc & -2ab \\ 2cd & -ad - bc \end{bmatrix}, \tag{55}$$

where $(a, b, c, d) \in \mathbb{C}^4$ and $ad - bc = 1$. These parameters explore now the six-dimensional manifold SL(2,$\mathbb{C}$).

2.6.3 Absence of a fourth Pauli matrix

Given the three Pauli matrices $\sigma_x, \sigma_y, \sigma_z$, one can not find a fourth Pauli matrix σ_v for which $\sigma_v^2 = \mathbb{1}$ and $\sigma_v\sigma_\mu + \sigma_\mu\sigma_v = \mathbf{0}, \forall\mu \in [1, 3] \cap \mathbb{N}$. This is shown by choosing

$$\sigma_v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{with} \quad (a, b, c, d) \in \mathbb{C}^4, \tag{56}$$

and expressing the various conditions. The condition $\sigma_v\sigma_x + \sigma_x\sigma_v = 0$ leads to: $c = -b$, $d = -a$. Applying further the consition $\sigma_v\sigma_y + \sigma_y\sigma_v = 0$ leads then to $b = 0$, such that $\sigma_v = a\sigma_z$. Such a matrix cannot satisfy further the condition $\sigma_v\sigma_z + \sigma_z\sigma_v = 0$ unless we take $a = 0$, such that $\sigma_v = \mathbf{0}$. Obviously, σ_v does then not satisfy $\sigma_v^2 = \mathbb{1}$.

One may argue that perhaps, if we had made some other choice of matrices τ_μ to satisfy the six conditions $\tau_\mu\tau_\nu + \tau_\nu\tau_\mu = 2\delta_{\mu\nu}\mathbb{1}$, we might have managed to find a fourth matrix τ_v, satisfying $\tau_v^2 = \mathbb{1}$ and $\tau_v\tau_\mu + \tau_\mu\tau_v = \mathbf{0}, \forall\mu \in [1, 3] \cap \mathbb{N}$. Showing that this is impossible is rather tedious. First of all, assume:

$$\tau_\mu = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \tau_\mu^2 = \begin{bmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{bmatrix} = \mathbb{1}. \tag{57}$$

If we assume $a + d \neq 0$ we must have $b = c = 0$, such $a^2 = d^2 = 1$. As we have assumed $a + d \neq 0$, the only solutions are $a = d = 1$ and $a = d = -1$. In both cases τ_μ would commute with all 2×2 matrices such that it cannot be τ_μ. The assumption $a + d \neq 0$ leads thus to a contradiction. We must thus assume $a + d = 0$. We find then that $a^2 + bc = 1$. This is equivalent to the condition: $-\det(\tau_\mu) = bc - ad = 1$. The eigenvalue equation of τ_μ is $\lambda^2 - (a+d)\lambda + ad - bc = 0$ and becomes this way $\lambda^2 - 1 = 0$, such that the eigenvalues are 1 and -1. The eigenvalues of $\sigma_x, \sigma_y, \sigma_z$ are also 1 and -1. There exists thus a similarity transformation $\mathbf{L}\tau_x\mathbf{L}^{-1} = \sigma_x = \eta_x$. Under the same similarity transformation we obtain $\mathbf{L}\tau_y\mathbf{L}^{-1} = \eta_y$ and $\mathbf{L}\tau_z\mathbf{L}^{-1} = \eta_z$, whereby nothing warrants for the moment that $\eta_y = \sigma_y$ and $\eta_z = \sigma_z$ would be true, unless we had $\mathbf{L} = \mathbb{1}$. It must be reminded that such a similarity transformation is not necessarily a Lorentz transformation. Lorentz transformations transform vectors according to $\mathbf{V} \rightarrow \mathbf{L}\mathbf{V}\mathbf{L}^\dagger$, not according to $\mathbf{V} \rightarrow \mathbf{L}\mathbf{V}\mathbf{L}^{-1}$, and in general $\mathbf{L}^\dagger \neq \mathbf{L}^{-1}$. We only have $\mathbf{L}^\dagger = \mathbf{L}^{-1}$ when $\mathbf{L}$ is a rotation matrix $\mathbf{L} = \mathbf{R}$. The matrices η_μ will now still satisfy $\eta_\mu\eta_\nu + \eta_\nu\eta_\mu = 2\delta_{\mu\nu}\mathbb{1}$. As $\eta_y^2 = \mathbb{1}$ and $\eta_z^2 = \mathbb{1}$, the matrices η_y and η_z must be both of the form:

$$\eta_\mu = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \tag{58}$$

because $a + d = 0$, while the condition $\eta_x\eta_\mu + \eta_\mu\eta_x = \mathbf{0}$ leads to $b + c = 0$. The matrices η_y and η_z are thus both of the form:

$$\eta_\mu = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}, \quad \text{with:} \quad \det(\eta_\mu) = b^2 - a^2 = -1. \tag{59}$$

We can thus put:

$$\eta_y = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}, \text{ with: } \det(\eta_y) = b^2 - a^2 = -1, \quad \eta_z = \begin{bmatrix} A & B \\ -B & -A \end{bmatrix}, \text{ with: } \det(\eta_z) = B^2 - A^2 = -1. \tag{60}$$

Expressing $\eta_y\eta_z + \eta_z\eta_y = \mathbf{0}$ leads to $aA = bB$. Putting $A = kb$, one obtains then $B = ka$, such that:

$$\eta_z = k \begin{bmatrix} b & a \\ -a & -b \end{bmatrix} \quad \Rightarrow \det(\eta_z) = k^2(-b^2 + a^2) = k^2. \tag{61}$$

As $\det(\eta_z) = -1$, we obtain $k^2 = -1$ such that $k \in \{-\imath, \imath\}$. We have thus:

$$\eta_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \eta_y = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}, \quad \eta_z = \pm\imath \begin{bmatrix} b & a \\ -a & -b \end{bmatrix}. \tag{62}$$

One can then show that also here, there is no fourth Pauli matrix. In fact, η_v will need to have the same type of structure as η_y and η_z and moreover satisfy $\eta_v\eta_\mu + \eta_\mu\eta_v = \mathbf{0}$. Let us thus put:

$$\eta_v = \begin{bmatrix} A & B \\ -B & -A \end{bmatrix}, \tag{63}$$

where A, B no longer have the meaning given to them in Eq. 60. This leads to: $Aa = bB$ and $Ab = aB$. Put $A = kb$. We obtain then from $Aa = bB$ that $B = ka$. And from $Ab = aB$ that: $kb^2 = aB = ka^2$ such that $b^2 = a^2$. For $a = b$ this yields:

$$\eta_v = \begin{bmatrix} kb & ka \\ -ka & -kb \end{bmatrix} = ka \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \quad \det(\eta_v) = 0, \tag{64}$$

such that this is not a solution, because we must have $\det(\eta_v) = -1$. For $a = -b$ this yields:

$$\eta_v = \begin{bmatrix} kb & ka \\ -ka & -kb \end{bmatrix} = ka \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \det(\eta_v) = 0, \tag{65}$$

such that this is again not a solution, because we must have $\det(\eta_v) = -1$. If there had existed a solution $\tau_x, \tau_y, \tau_z, \tau_v$ then it would have been carried over into a solution $\eta_x, \eta_y, \eta_z, \eta_v$ by the similarity transformation.

2.6.4 Full set of possible choices for the Pauli-like matrices

The fact that $\eta_y = \sigma_y$ and $\eta_z = \sigma_z$ are not necessarily true, as transpires from Eq. 62, shows that even the similarity transformations based on $\mathbf{L} \in \mathrm{SL}(2,\mathbb{C})$ do no exhaust all possibilities for obtaining alternative sets of Pauli-like representation matrices. In fact, one must first determine the continuum of solutions η_y and η_z that are compatible with the choice $\eta_x = \sigma_x$. From these, we can generate all possible solutions by similarity transformations based on matrices of $\mathrm{SL}(2,\mathbb{C})$. The full parameter space is thus seven-dimensional.

2.7 Justifying the introduction of a Clifford algebra

The author has figured out the whole contents of the present paper from scratch because he found the textbook presentations impenetrable. The author has also not studied books on Clifford algebra in depth, such that some works may well provide the motivation we will try to give here, and which we were not able to spot in textbooks. Our criticism is based on the observation that very often mathematical objects that algebraically look identical are in reality entirely different geometrical objects. We have seen that we can introduce representations $[\,\mathbf{v}\cdot\boldsymbol{\sigma}\,]$ for vectors $\mathbf{v} \in \mathbb{R}^3$ into the formalism by extrapolating the meaning of the algebra of the representations $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ of reflection operators $A \in \mathrm{L}(\mathbb{R}^3, \mathbb{R}^3)$. We have seen in Footnote 18 how confusing $A \in \mathrm{L}(\mathbb{R}^3, \mathbb{R}^3)$ and $\mathbf{a} \in \mathbb{R}^3$ through the algebraic identity of their representation matrices $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ can trap us into a conceptual impasse of trying to give geometrical meaning to mindless algebra. This is not the end of the story. Whereas it is meaningful in the group theory to consider the product $R = BA$ of two reflections B and A and the corresponding representation matrix $\mathbf{R} = [\,\mathbf{b}\cdot\boldsymbol{\sigma}\,]\,[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$, it is *a priori* not defined what the purely formal product of two vectors $\mathbf{v}_1$ and $\mathbf{v}_1$ defined by $[\,\mathbf{v}_2\cdot\boldsymbol{\sigma}\,]\,[\,\mathbf{v}_1\cdot\boldsymbol{\sigma}\,]$ is supposed to mean. Here again, entirely different geometrical objects are represented by identical algebraic expressions. We have learned definitions for $\mathbf{v}_1\cdot\mathbf{v}_2$ and for $\mathbf{v}_1 \wedge \mathbf{v}_2$, but not for $[\,\mathbf{v}_2\cdot\boldsymbol{\sigma}\,]\,[\,\mathbf{v}_1\cdot\boldsymbol{\sigma}\,]$. But inspection of the algebra reveals that:

$$[\,\mathbf{v}_2\cdot\boldsymbol{\sigma}\,]\,[\,\mathbf{v}_1\cdot\boldsymbol{\sigma}\,] = (\mathbf{v}_1\cdot\mathbf{v}_2)\,\mathbb{1} + \imath\,[\,(\mathbf{v}_2 \wedge \mathbf{v}_1)\cdot\boldsymbol{\sigma}\,], \tag{66}$$

an algebraic identity we used in deriving Eq. 8. We recognize here the familiar quantities $\mathbf{v}_1\cdot\mathbf{v}_2$ and $\mathbf{v}_1 \wedge \mathbf{v}_2$. Whereas this kind of algebra is meaningful for reflection matrices, it is *a priori* not meaningful for vectors. It can be given a meaning *a posteriori* in terms of vectors, at the risk of introducing confusion by ignoring the fact that the vector formalism is a parallel formalism, as we clearly outlined from the outset. Based on this confusion one can obtain then a formalism whereby one sums quantities that are not of the same type, by writing expressions of the type:

$$\mathbf{v}_2 \vee \mathbf{v}_1 = \mathbf{v}_2 \cdot \mathbf{v}_1 + \mathbf{v}_2 \wedge \mathbf{v}_1, \tag{67}$$

as a shorthand for Eq. 66. What Clifford algebra does is defining *mano militare* that such expressions are meaningful as an algebra on multi-vectors. In general such a definition is introduced out of the blue. By focussing on the purely algebraic part of the formalism, it is possible to confuse the vectors $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ and the reflection matrices $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$. Erroneously identifying vectors and reflections based on their algebraic identity is part of the perceived magic we alluded to in Footnote 13 above.[20]

This has several inconveniences. First of all, it is puzzling for the reader to understand where this idea comes from, because the algebra adds quantities of different symmetries and dimensions. All at once one teaches him that from now on one can add kiwis and bananas, while one has told him before during his whole life that this is not feasible. Moreover, this is done tacitly, as though this would not be problem at all. Nothing is done to ease away the bewilderment of a critical reader. One only laconically teaches him how to to get used to it without asking further questions. One just rolls out the algebra such that reader can learn to imitate it mindlessly. As this is rather easy the reader will quickly become acquainted with it such that the justified initial questions will be silenced. But it takes a algebraic shortcut to the full geometrical explanation by exploiting algebraic coincidences.

The second problem is that after the introduction of the definition of the Clifford algebra with its cuisine of adding kiwis and bananas, all the geometry of the rotations seems to follow effortless from this definition in an extremely elegant way. This gives the impression that everything is derived by magic from thin air, which really leaves one left wondering. In fact, the only vital ingredient needed to obtain this powerful and elegant formalism seems to be the impenetrable slight of hand of adding kiwis and bananas. In certain presentations, one uses an equation of the type of Eq. 34 to introduce the rotation, stipulating that $\mathbf{e}_j \wedge \mathbf{e}_k$, with $j \neq k$ are infinitesimal generators. But to stipulate this one has to derive first the Lie algebra from the Lie group representation theory. This is nothing more than a cheat if one does not provide the justification for these procedures as given in the present presentation.

For sure, our presentation looks somewhat more cumbersome and less elegant than the approach where one takes off from the definition of the Clifford algebra in grand style. But that elegant grand style is only a short-cut to the detailed explanation, and is obtained by sweeping some more tedious parts under the carpet. The strong point of our approach is that it provides the detailed geometrical motivation for the complete Clifford algebra. An interesting feature that also exists in our approach is that we can consider all kinds of products:

$$[\,\mathbf{a}_1\cdot\boldsymbol{\sigma}\,]\,[\,\mathbf{a}_2\cdot\boldsymbol{\sigma}\,]\,\cdots[\,\mathbf{a}_m\cdot\boldsymbol{\sigma}\,] = \mathbf{a}_1 \vee \mathbf{a}_2 \vee \cdots \vee \mathbf{a}_m. \tag{68}$$

The worked-out expressions correspond contain hyper-parallelopepids of various dimensions (that can be symmetrical or anti-symmterical). The symmetry is signaled by the presence or absence of a factor $\imath$. These quantities transform under a rotation $\mathbf{R}$ to:

$$\mathbf{R}[\,\mathbf{a}_1\cdot\boldsymbol{\sigma}\,]\mathbf{R}^{-1}\mathbf{R}\,[\,\mathbf{a}_2\cdot\boldsymbol{\sigma}\,]\,\mathbf{R}^{-1}\cdots\mathbf{R}\,[\,\mathbf{a}_m\cdot\boldsymbol{\sigma}\,]\,\mathbf{R}^{-1} = \mathbf{R}\,[\,\mathbf{a}_1\cdot\boldsymbol{\sigma}\,]\,[\,\mathbf{a}_2\cdot\boldsymbol{\sigma}\,]\,\cdots[\,\mathbf{a}_m\cdot\boldsymbol{\sigma}\,]\,\mathbf{R}^{-1}. \tag{69}$$

That we can rotate all these quantities within a unique formalism is thus not an asset of Clifford algebra that would not exist in our exploratory approach.[21]

We see that by formalizing the algebra for the sake of elegance we can obtain a very abstract formulation whereby we loose completely sight of the clear geometrical ideas. Mathematicians would argue that this does not matter. But the problem is that now confusion reigns. And when the cat is away, the mice will play. The abstraction eases extrapolating the algebra in a meaningless way beyond the limits defined by its geometrical meaning, e.g. by introducing linear combinations of spinors. From that point on the framework may now contain some well hidden logical nonsense, as taking linear combinations of spinors is not a granted procedure. The structure that results from this transgression

[20] What we can perhaps do to interpret Eqs. 66-67 is to state that a rotation in SU(2) is defined by a set of parameters, that contain one scalar and one pseudo-vector. This gives then meaning to the vector space of 2^n functions of $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$. In fact, we will see below that we can introduce a basis of 2^n multi-vectors that may occur in parameter sets that have specific symmetries and can be used to define group actions on multi-vectors. A reflection A is e.g. a function $A \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^n)$ whose parameter set is a vector with vector symmetry. Other functions can be defined by parameter sets which are multi-vectors with other tensor symmetries.

[21] My monograph "From Spinors to quantum mechanics" was rejected by Springer on the basis of an anonymous referee report that argued that the book was old-fashioned as it lacked the modern insights of Clifford algebra and did not present the subject matter in its elegant grand style. It argued that the Rodrigues equation would be *obscene*. It also suggested that I should rewrite the whole book in the language of Clifford algebra. If I had followed that advice, it would have destroyed all the intuition I wanted to convey to the readers about the geometrical meaning of spinors. Such an understanding is essential for understanding quantum mechanics. Without it quantum mechanics becomes a complete mystery. In other words, following the recommendations of the referee would just have destroyed the book.

is the very elegant Hilbert space formalism of quantum mechanics. This is now highly abstract and any obvious link with the original geometrical meaning has been completely flushed. This favours an attitude where calculating becomes much more important than thinking. As matter of fact, in quantum mechanics we are told to "shut up and calculate". And after hiding away this way all the geometrical meaning of the formalism, a physicist may enter the scene and ask: I have a formalism that grinds out theoretical predictions which agree with the experimental data to unprecedented precision, but I just cannot figure out what it means. Can you solve the conundrum what this all means?

2.8 Intermezzo

2.8.1 Summary of the relations in SU(2)

It is perhaps worthwhile to highlight the beautiful pattern of the delicate interconnections we have found in the combined formalism for rotations and vectors in SU(2). Let us introduce the notation $E = \mathbb{C}^2$ for the representation space of SU(2), and note the basis vectors of E as the column vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_1$.

[1] There is a one-to-one correspondence between a rotation and a *Vielbein.*

[2] There is a one-to-one correspondence between the *Vielbein* and an isotropic vector.

[3] The column matrices $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ can be used to constitute a vector basis $\boldsymbol{\eta}_1 \otimes \boldsymbol{\eta}_2^\dagger$ for the vector space of linear mappings $\mathrm{L}(E, E) = \mathrm{L}(\mathbb{C}^2, \mathbb{C}^2)$. The manifold of the rotation group and the manifold of the group generated by the reflections are (only) algebraically embedded in $\mathrm{L}(\mathbb{C}^2, \mathbb{C}^2)$.

[4] Because their representation matrices $\mathbf{M}$ have $\det \mathbf{M} = 0$, the isotropic vectors can be written under the form $\mathbf{M} = \boldsymbol{\chi} \otimes \dot{\boldsymbol{\psi}}^\dagger = \boldsymbol{\eta}_1 \otimes \boldsymbol{\eta}_2^\dagger$.

[5] An alternative vector basis for the function space $\mathrm{L}(E, E)$ is constituted by the representation matrices for the multi-vectors $\mathbb{1}$, $[\,\mathbf{e}_1{\cdot}\boldsymbol{\sigma}\,]$, $[\,\mathbf{e}_2{\cdot}\boldsymbol{\sigma}\,]$, and $[\,(\,\mathbf{e}_1 \wedge \mathbf{e}_2\,){\cdot}\boldsymbol{\sigma}\,]$ in the parallel formalism.

[6] Spinors are a stenographic notation for the rotation matrices. They can be expressed in the vector basis $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$ of E.

[7] We may note that for the moment, the column vectors $\boldsymbol{\eta}_j \in E$ do not have a clear physical meaning. The entire vector space E is disconnected from any visual clues and all its vectors are rather abstract quantities. It is just that the manifold of the rotation group and the manifold of the group generated by the reflections can be algebraically embedded in $\mathrm{L}(E, E)$. All this is due to the fact that we have defined the group elements and the vector representation matrices directly as 2×2 matrices *en bloc* without defining a basis for E.

2.8.2 What is really going on behind the scenes in SU(2)

In summary we see that the basic idea of SU(2) is to represent a group by its automorphism group, and that the automorphisms can be coded by coding the complete information content of a basis. We treat the rotation group as the group generated by the reflections. We will now generalize this idea to rotations in $\mathbb{R}^n$. Again we will first do this for a specific case and then generalize the procedures.

2.8.3 What can be extrapolated to SO(n)?

When we will move on to vector spaces $\mathbb{R}^n$, with $n \geq 4$, some of these interconnections will be disrupted. This reveals that some of the the interconnections observed in $\mathbb{R}^3$ are accidental and thus not conceptual. Such accidental interconnections are a source of confusion. When quantities are accidentally identical in a lower-dimensional space and become different in a higher-dimensional space we may make the error of extrapolating the identity wrongly to the higher-dimensional space, by trusting the illusion created by the accidental identity in the lower-dimensional space. We have already seen an example of such a possible confusion in the relation between the generators of the rotation group SU(2) and the infinitesimal generators of the rotation group SU(2). We can thus improve our understanding of SU(2) by also studying SO(n), with $n > 3$. Let us thus point out which identities are accidental and which ones are true.

[1] Property [1] will remain true.

[2] Property [2] can be generalized. When we will generalize our ideas to $\mathbb{R}^n = \mathbb{R}^{2\nu}$ or $\mathbb{R}^n = \mathbb{R}^{2\nu+1}$, we will no longer have to code just one but ν isotropic vectors $\mathbf{e}_{2k+1} + \imath\mathbf{e}_{2k+2}$ (with $k \in [0, \nu-1] \cap \mathbb{Z}$), constructed from the basis vectors $\mathbf{e}_j$. The ν isotropic vectors $\mathbf{e}_{2k+1} - \imath\mathbf{e}_{2k+2}$ (with $k \in [0, \nu-1] \cap \mathbb{Z}$) complete the basis. Again, when n is odd, such that $n = 2\nu+1$, we will not need to code the last basis vector $\mathbf{e}_{2\nu+1}$ of $\mathbb{R}^{2\nu+1}$ as it will be already completely defined by all the other basis vectors. In the representation SO(n) the real rotation matrices will transform $\mathbf{e}_j$ to $\mathbf{e}'_j$ and will transform thus each isotropic vector $\mathbf{e}_{2k+1} + \imath\mathbf{e}_{2k+2}$ (with $k \in [0, \nu-1] \cap \mathbb{Z}$) into an isotropic vector $\mathbf{e}'_{2k+1} + \imath\mathbf{e}'_{2k+2}$ (with $k \in [0, \nu-1] \cap \mathbb{Z}$). From the rotated isotropic vectors $\mathbf{e}'_{2k+1} + \imath\mathbf{e}'_{2k+2}$ (with $k \in [0, \nu-1] \cap \mathbb{Z}$), one can recover the full rotated basis $\mathbf{e}'_j$, $j \in [1, n] \cap \mathbb{N}$ by taking the real and imaginary parts. Call B the set $\{\mathbf{e}_{2k+1} + \imath\mathbf{e}_{2k+2} \in \mathbb{C}^\nu \parallel k \in [0, \nu-1] \cap \mathbb{Z}\}$, and $\mathscr{I} = \{R(B) \parallel R \in \text{SO}(n)\}$. $\mathscr{I}$ is an ideal because the group SO(n) is closed under the operation of composition of rotations.

Let us now consider the case $n = 2\nu$. We can also construct by the Gramm-Schmidt procedure an orthogonal isotropic basis I for $\mathbb{C}^\nu$, consisting of ν isotropic vectors $\boldsymbol{\varepsilon}_j, j \in [1, \nu] \cap \mathbb{N}$, whereby $\forall j \in [1, \nu] \cap \mathbb{N}, \forall k \in [1, \nu] \cap \mathbb{N}, \boldsymbol{\varepsilon}_j \cdot \boldsymbol{\varepsilon}^*_k = \delta_{jk}$. We can construct now the set $\mathscr{J}$ of all such isotropic bases $\mathscr{J} = \{(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \cdots \boldsymbol{\varepsilon}_j \cdots \boldsymbol{\varepsilon}_\nu) \in (\mathbb{C}^\nu)^\nu \parallel (\forall j \in [1, \nu] \cap \mathbb{N})(\forall k \in [1, \nu] \cap \mathbb{N})(\boldsymbol{\varepsilon}_j \cdot \boldsymbol{\varepsilon}^*_k = \delta_{jk})\}$. We can also construct I^* and $\mathscr{J}^*$. The problem is now that the ideal $\mathscr{I}$ is not equal to the set $\mathscr{J}$, but only a subset of it: $\mathscr{I} \subset \mathscr{J}$. In fact, $\Re(\boldsymbol{\varepsilon}_j)$ and $\Im(\boldsymbol{\varepsilon}_j)$ will not necessarily define a normalized orthogonal basis for $\mathbb{R}^{2n}$, because orthogonality $\boldsymbol{\varepsilon}_j \cdot \boldsymbol{\varepsilon}^*_k = 0$ does not imply all four conditions $\Re(\boldsymbol{\varepsilon}_j) \cdot \Re(\boldsymbol{\varepsilon}_k) = 0$ & $\Re(\boldsymbol{\varepsilon}_j) \cdot \Im(\boldsymbol{\varepsilon}_k) = 0$ & $\Im(\boldsymbol{\varepsilon}_j) \cdot \Re(\boldsymbol{\varepsilon}_k) = 0$ & $\Im(\boldsymbol{\varepsilon}_j) \cdot \Im(\boldsymbol{\varepsilon}_k) = 0$.[22] This is a remarkable difference between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, even though initially we wanted $\mathbb{C}^n \equiv \mathbb{R}^{2n}$. It is the Hermitian norm function that introduces the difference. The reason is that it defines the norm of the quantity $x + \imath y$ considered *en bloc*.

There is another example of a surprising consequence of defining $x + \imath y$ *en bloc*. It occurs when we want to define under which conditions a function is differentiable in $\mathbb{C}$. When we define a function $f \in F(\mathbb{R}^2, \mathbb{R}^2) : (x, y) \to f(x, y) = (u(x, y), v(x, y))$ then the condition for it being differentiable is that $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ must exist. The function f is then $\mathbb{R}$-differentiable. However, when we define a function $f \in F(\mathbb{C}, \mathbb{C}) : z = x + \imath y \to f(z)$, we find that not only $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ must exist, but that the partial derivatives must also satisfy the conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. This is a necessary and sufficient condition. From this it can be shown that u and v must both satisfy the harmonic conditions $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. The function f is now $\mathbb{C}$-differentiable. The reason for this difference is that in the differentials we calculate limits $\lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x}$, $\lim_{\Delta y \to 0} \frac{v(y+\Delta y) - v(y)}{\Delta y}$ in $\mathbb{R}^2$, while we calculate limits $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ in $\mathbb{C}$. In other words, in $\mathbb{R}^2$ we are approaching $(x, y) \in \mathbb{R}^2$ only from two directions, while in $\mathbb{C}$ we are approaching $z \in \mathbb{C}$ from all directions. Moreover, all these directional derivatives must be equal. If a function $f \in F(\mathbb{R}^2, \mathbb{R}^2)$ has rotational symmetry it must thus be $\mathbb{C}$-differentiable in its formulation $f \in F(\mathbb{C}, \mathbb{C})$ unless it is not differentiable in any direction at all. In $\mathbb{C}$ the definition *en bloc* of $x + \imath y$ results thus in coordinates that are much better adapted to describe rotational symmetry than the coordinates (x, y) of $\mathbb{R}^2$ which were designed to describe positions and displacements, and thus translational symmetry. The rotational invariance can be expressed as a "translational invariance" along the unit circle described by $e^{\imath\varphi}$ in $\mathbb{C}$. This is why we have symmetry-adapted functions $e^{-\imath k_\varphi \varphi}$ for rotational symmetry. By describing the geometry in $\mathbb{R}^2$ we allow for true translational invariance with symmetry-adapted functions $e^{-\imath k_x x} e^{-\imath k_y y}$. We see thus that defining $x + \imath y$ *en bloc* is more appropriate for describing rotational invariance and the isotropy of $\mathbb{R}^n$, while defining them separately as (x, y) is more appropriate for describing translational invariance and the homogeneity of $\mathbb{R}^n$. This is thus the reason why quantum mechanics strays into Hilbert space whereby we should nevertheless not treat spinors like vectors. In the case of rotations the group is a curved manifold acting on an curved ideal, in the case of translations the group is a vector space acting on a vector space. Of course, we can only be aware of the rotations by the displacements they cause in space, such that it remains interesting to know how the functions act on $\mathbb{R}^{2\nu}$, even when we describe them on $\mathbb{C}^\nu$.

When we want to describe rotational invariance in $\mathbb{R}^4$ it is thus better to use coordinates $(z_1, z_2) \in \mathbb{C}^2$ than $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ but even then this is not optimal, because to account for the rotational invariance it would be better to treat also $(z_1, z_2) \in \mathbb{C}^2$ *en bloc*. We will have to introduce additional constraints if we do not describe the points $(z_1, z_2) \in \mathbb{C}^2$ *en bloc*. We should thus rather define $\mathbb{C}^2$ *en bloc*. But this is exactly what the graded algebra is doing! In fact we have an isomorphism between $\mathbb{C}$ and 2×2 matrices:

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[22] A counter-example showing this is: $\mathbf{a}_1 = \frac{1}{2}(1, 1, 1, 1)$, $\mathbf{a}_2 = \frac{1}{2}(1, -1, 1, -1)$, $\mathbf{a}_3 = \frac{1}{2}(1, -1, 1, -1)$, $\mathbf{a}_4 = \frac{1}{2}(1, 1, 1, 1)$. From this we can construct $\boldsymbol{\varepsilon}_1 = \frac{1}{\sqrt{2}}(\mathbf{a}_1 + \imath\mathbf{a}_2) = \frac{1}{2\sqrt{2}}(1+\imath, 1-\imath, 1+\imath, 1-\imath)$, and $\boldsymbol{\varepsilon}_2 = \frac{1}{\sqrt{2}}(\mathbf{a}_3 + \imath\mathbf{a}_4) = \frac{1}{2\sqrt{2}}(1+\imath, -1+\imath, 1+\imath, -1+\imath)$. We have then: $\boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}_1 = 0$, $\boldsymbol{\varepsilon}_2 \cdot \boldsymbol{\varepsilon}_2 = 0$, $\boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}^*_1 = 1$, $\boldsymbol{\varepsilon}_2 \cdot \boldsymbol{\varepsilon}^*_2 = 1$, and $\boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}^*_2 = 0$, such that $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ are isotropic vectors of $\mathbb{R}^4$, which are of "zero length" according to the (invalid) extrapolation of the Euclidean norm function from $\mathbb{R}^4$ to $\mathbb{C}^2$. In reality they are of length 1 in $\mathbb{C}^2$ according to the Hermitian norm function and mutually orthogonal according to the Hermitian norm function, such that they are perfect orthogonal basis vectors for $\mathbb{C}^2$. But they do not yield perfect orthogonal basis vectors for $\mathbb{R}^4$ after decomposition. In fact, whereas we have $\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$, $\mathbf{a}_1 \cdot \mathbf{a}_3 = 0$, in $\mathbb{R}^4$, we no longer have $\mathbf{a}_1 \cdot \mathbf{a}_4 \neq 0$. Similarly, we have $\mathbf{a}_2 \cdot \mathbf{a}_1 = 0$, $\mathbf{a}_2 \cdot \mathbf{a}_4 = 0$, but no longer $\mathbf{a}_2 \cdot \mathbf{a}_3 \neq 0$.

$$x + \imath y \leftrightarrow \left[\begin{array}{cc} x & -y \\ y & x \end{array} \right], \quad \text{with:} \quad 1 \leftrightarrow \left[\begin{array}{cc} 1 & \\ & 1 \end{array} \right], \quad \text{with:} \quad \imath \leftrightarrow \left[\begin{array}{cc} & -1 \\ 1 & \end{array} \right], \tag{70}$$

for multiplications. We can use this as a source of inspiration for defining coordinates *en bloc*. To export this idea to $\mathbb{R}^4$ and to be able to define $\mathbb{C}^2$ en bloc we can thus think of introducing a super-$\imath$:

$$I \leftrightarrow \left[\begin{array}{cc} & -\mathbb{1} \\ \mathbb{1} & \end{array} \right], \tag{71}$$

But as in addition the groups beyond SO(2) are non-abelian, we truly need non-commuting matrices. As it is useless for describing rotational symmetry to separate our first coordinates $(x, y) \in \mathbb{C}$ into x and y, we can say that we have introduced a useless separation of the space into two dimensions. For the next two coordinates we can say the same. We have thus a useless combined separation into 4 parts. We can this way understand why we need the graded algebra. There will thus be in SO(2ν) a useless separation into 2^ν parts, and these parts are reproduced in the dimension of the grading. To highlight the interconnections we may note here that the harmonic polynomials are used as representations of SO(3) and that they are obtained as tensor products of spinors as discussed in [1,4]. The harmonic polynomials obey the Laplace equation in analogy with the equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ for F($\mathbb{C}, \mathbb{C}$).

To impede any temptation to introduce special directions in space we could pinch out the space of the displacements. This is in a sense what we are doing when we are using the coordinates $(x, y, z) = \mathbf{e}_1 + \imath \mathbf{e}_2$ of an isotropic vector in SU(2). The set of all isotropic vectors of $\mathbb{C}^3$ is the isotropic cone $\mathscr{C} \subset \mathbb{C}^3$. Biedenharn [15] states that a spinor corresponds to an isotropic vector $(x, y, z) \in \mathscr{C}$. There is only one element $(x, y, z) = (0, 0, 0) \in \mathscr{C}$ that belongs to real space $\mathbb{R}^3$. Biedenharn concludes from this observation that spinors can certainly not be objects that rotate in physical space. This is of course not correct. Spinors are not position coordinates. Moreover, the point $(0, 0, 0)$ does not belong conceptually to the isotropic cone $\mathscr{C}$ in the way we introduced it, because contrary to all other points of $\mathscr{C}$ it cannot be used to describe a reference frame. Many physicists argue, following Biedenharn, that a "spin operator" like σ_z must operate on an abstract internal space like the isospin operator I_z. But this is self-contradictory because they identify later on the index z of the "spin operator" σ_z with the z-coordinate or the z-axis in physical space. As we may note, σ_z belongs to the realm of Euclidean geometry such that it certainly does not contain the electron spin.

To construct the ideal $\mathscr{I}$ with respect to the vector space wherein we describe displacements, we must in the Gramm-Schmidt procedure add $\boldsymbol{\varepsilon}_j$, in such a way that not only $\boldsymbol{\varepsilon}_j \perp \boldsymbol{\varepsilon}_k, \forall k < j$, but also $\boldsymbol{\varepsilon}_j^* \perp \boldsymbol{\varepsilon}_k, \forall k < j$. Hence: $\mathscr{I} = \{(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \cdots \boldsymbol{\varepsilon}_j \cdots \boldsymbol{\varepsilon}_\nu) \in (\mathbb{C}^\nu)^\nu \parallel (\forall j \in [1, \nu] \cap \mathbb{N})(\forall k \in [1, \nu] \cap \mathbb{N})(\boldsymbol{\varepsilon}_j \cdot \boldsymbol{\varepsilon}_k^* = \delta_{jk} \,\&\, \boldsymbol{\varepsilon}_j \cdot \boldsymbol{\varepsilon}_k = \delta_{jk})\}$. We can construct two ideals $\mathscr{I}$ and $\mathscr{I}^*$ and what determines to which ideal a basis B belongs depends on the handedness of the basis in $\mathbb{R}^{2\nu}$ we can derive from it. The ideals serve as elements of a language that fully accounts for the rotational symmetry we want to describe.

This illustrates once more that isolated isotropic column vectors in $\mathbb{C}^\nu$ are not very informative about the rotation. The ideal consists of multi-column quantities, which are subject to a compatibility constraint. Geometrically one must work with the ideal $\mathscr{I}$, whereby a rotation R corresponds in a one-to-one fashion to a full set of normalized isotropic vectors $\mathbf{e}'_{2k+1} + \imath \mathbf{e}'_{2k+2}, k \in [0, \nu - 1] \cap \mathbb{Z}$ obtained by the rotation R from the isotropic vectors $\mathbf{e}_{2k+1} + \imath \mathbf{e}_{2k+2}, k \in [0, \nu - 1] \cap \mathbb{Z}$ of the reference basis. The full set $\mathbf{e}_{2k+1} + \imath \mathbf{e}_{2k+2}, k \in [0, \nu - 1] \cap \mathbb{Z}$ of the reference basis is bodily rotated by the rotation and an isolated isotropic vector that belongs to the set takes part in the global rotation. The *Vielbein* can be put in one-to-one correspondence with a *Vielbein* of ν isotropic basis vectors of the ideal $\mathscr{I}$.

This discussion drops us a hint that we could define spinors and the representation in terms of isotropic vectors. The problem is that the isotropic vectors are an ideal in the representation SO(n) but not in the representation based on $\mathscr{S} \subset \mathbb{C}^{2^\nu}$ (as defined in Footnote 3), because, like all vectors, the isotropic vectors transform quadratically under rotations. They are therefore tensor products. This fact renders the approach based on isotropic vectors more complicated than the one based on rotations, which transform linearly. In building the representation theory by starting conceptually from the ideal $\mathscr{I}$ of normalized isotropic vectors, we end up with the riddle that a spinor would be a kind of square root of an isotropic vector, which is an unnecessary conceptual complication. Moreover an isolated isotropic vector does not contain the full information about the rotation, such that it is more complicated to interpret. On the other hand, the full set $\mathbf{e}'_{2k+1} + \imath \mathbf{e}'_{2k+2}, k \in [0, \nu - 1] \cap \mathbb{Z}$ does.[23]

One may feel puzzled about what happens when we move from $\mathbb{R}^{2\nu}$ to $\mathbb{R}^{2\nu+1}$. In $\mathbb{R}^2$ the rotations have two eigenvectors, the two isotropic vectors, with eigenvalues $e^{-\imath\varphi}$ and $e^{\imath\varphi}$. When we make a rotation of an isotropic vector,

[23] This seems to indicate how Cartan discovered spinors. He may have started from the observation that the isotropic vectors are the eigenvectors of the rotations in SO(2) and then have realized that in the generalization to SO(3) these isotropic vectors no longer remain eigenvectors, but are constituting an ideal. He may then have followed this track further. What pleads for this idea is that the first two chapters of Cartan's monograph are dealing with isotropic vectors, and that he defined a spinor as a kind of isotropic vector. (However, isotropic vectors play an important in the interpretation of the formalism as we will see in Section 6). This may also indicate that he had not realized that there exists a simpler approach based on the idea that we can represent a group by its automorphism group and that spinors are then just images of rotations.

it remains in its vector space. It is only multiplied by a phase vector. But when we move to $\mathbb{R}^3$ the isotropic vectors can be tilted out of the plane. Under the action of the group each of them builds then an ideal. However, the rotated isotropic vector still codes the full basis. We can thus reconstruct $\mathbf{e}'_x = \mathbf{a}$ and $\mathbf{e}'_y = \mathbf{b}$ from the rotated isotropic vector by taking the real and imaginary parts and then calculate $\mathbf{e}'_z$ from:

$$\mathbf{e}'_z = \mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}, \tag{72}$$

which is the vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. In fact, for each vector $\mathbf{u} = (u_x, u_y, u_z)$, the scalar product $\mathbf{u}\cdot\mathbf{e}'_z$ can be obtained by replacing $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ by u_x, u_y, u_z in Eq. 72. Consider now the vector space $\mathbb{C}^\nu$ with the isotropic vectors $\mathbf{e}_{2k+1} + \imath\mathbf{e}_{2k+2}, k \in [0, \nu-1] \cap \mathbb{N}$ as its basis. All linear combinations of these isotropic vectors will be isotropic vectors. It is thus an isotropic subspace $\mathbb{C}^\nu$. As a rotation is a linear transformation, a rotation of SO(2ν) will also transform $\mathbb{C}^\nu$ into itself, just like a rotation of an isotropic vector in $\mathbb{C}$. The picture is thus analogous. But the rotations of SO(2ν) will not generate the full space by acting on the isotropic basis. They will build an ideal. Also here we have two subspaces obtained one from another by complex conjugation. When we add the dimension $2\nu+1$ the two ideals will be tilted out of $\mathbb{R}^{2\nu}$ and become ideals whose points are elements of $\mathbb{C}^{2\nu+1}$ rather than of $\mathbb{C}^{2\nu}$, and again the isotropic basis with its coordinates in $\mathbb{C}^{2\nu+1}$ will continue to contain enough information to reconstitute the full rotated *Vielbein*. We can find the information by decomposing it into its real and imaginary parts to find $\mathbf{e}'_1, \mathbf{e}'_2, \cdots \mathbf{e}'_{2\nu} = (\mathbf{a}, \mathbf{b}, \cdots \mathbf{s})$. Again, the last vector $\mathbf{e}'_{2\nu+1}$, which is perpendicular to all the vectors $\mathbf{e}'_1, \mathbf{e}'_2, \cdots \mathbf{e}'_{2\nu}$ is obtained from:

$$\mathbf{e}'_{2\nu+1} = \mathbf{a} \wedge \mathbf{b} \wedge \cdots \wedge \mathbf{s} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_{2\nu+1} \\ a_1 & a_2 & \cdots & a_{2\nu+1} \\ b_1 & b_2 & \cdots & b_{2\nu+1} \\ \vdots & \vdots & & \vdots \\ s_1 & s_2 & \cdots & s_{2\nu+1} \end{vmatrix}. \tag{73}$$

[3] Property [3] will be preserved provided we disconnect $\boldsymbol{\eta}_j$ from their meaning of spinors that would represent the full information about a group element as is the case in SU(2). This is discussed under [6]. The linear transformations $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ of the representation space form a vector space of dimension $2^{2\nu}$. The most trivial basis for the vector space $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ is the set of matrices:

$$\text{position } j \rightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & \cdots & \underset{\substack{\uparrow \\ \text{position } k}}{1} & \cdots & 0 & 0 \end{bmatrix} = \boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k^\dagger. \tag{74}$$

These basis vectors correspond to linear mappings $f_{jk} \in \mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu}) \parallel (\forall \ell \in [1, 2^\nu] \cap \mathbb{N})(f_{jk}(\boldsymbol{\eta}_\ell) = \delta_{j\ell}\,\boldsymbol{\eta}_k)$, which are not bijective. They transform the vector the basis vector $\boldsymbol{\eta}_j$ into $\boldsymbol{\eta}_k$ and all other basis vectors $\boldsymbol{\eta}_\ell$ into $\mathbf{0}$. We can decompose this way $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ in a basis of vectors $\boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k^\dagger$. The most obvious basis for bijective linear mappings $\mathrm{L}(\mathbb{R}^{2^\nu}, \mathbb{R}^{2^\nu})$ would be the one of the reflection matrices of the vector space $\mathbb{R}^{2^\nu}$. These would be the mappings $f_{jk} \in \mathrm{L}(\mathbb{R}^{2^\nu}, \mathbb{R}^{2^\nu})$, that transform $\boldsymbol{\eta}_j$ into $\boldsymbol{\eta}_k$, $\boldsymbol{\eta}_k$ into $\boldsymbol{\eta}_j$ and leave all other basis vectors $\boldsymbol{\eta}_\ell$ unchanged: $(\forall \ell \in [1, 2^\nu] \cap \mathbb{N})(f_{jk}(\boldsymbol{\eta}_\ell) = \delta_{j\ell}\boldsymbol{\eta}_k + \delta_{k\ell}\boldsymbol{\eta}_j + (1-\delta_{j\ell})(1-\delta_{k\ell})\boldsymbol{\eta}_\ell)$. In $\mathbb{C}^{2^\nu}$ we can use the same canonical basis vectors $\boldsymbol{\eta}_j$ as in $\mathbb{R}^{2^\nu}$. Only the coefficients c_j in $\sum_j c_j \boldsymbol{\eta}_j$ change from real to complex in going from $\mathbb{R}^{2^\nu}$ to $\mathbb{C}^{2^\nu}$.

[4] The generalization of property [4] will generate complications. We have already pointed out this difficulty in Remark 2 of paragraph 2.5.1. We will still have $\det \mathbf{M} = 0$, but we will no longer have $\mathbf{M} = \boldsymbol{\chi} \otimes \dot{\boldsymbol{\psi}}^\dagger$ because only two columns of $\mathbf{M}$ need to be proportional to obtain $\det \mathbf{M} = 0$. We will rather have a sum of $\nu/2$ terms, $\mathbf{M} = \sum_{\nu/2\,\text{terms}} \boldsymbol{\chi}_j \otimes \dot{\boldsymbol{\psi}}_k^\dagger$.

[5] Property [5] will remain intact.

[6] Property [6] will no longer be true. Rotation matrices will no longer have a stenographic notation in terms of column matrices. We have already pointed out in Remark 3 of Subsection 2.1 that the column vectors $\boldsymbol{\eta}_j$ will no longer represent the full information about a rotation in SO(n). In fact, as spinors do not build a vector space, we cannot see a spinor as a meaningful linear combination:

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_j \\ \vdots \\ \psi_{2^\nu} \end{bmatrix} = \psi_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \psi_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \psi_j \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots \psi_{2^\nu} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix} \cong \sum_{j=1}^{k} \psi_j \mathbf{e}_j, \tag{75}$$

of basis spinors $\mathbf{e}_j \in \mathbb{C}^{2^\nu}$. The column spinor is just not decomposable into a linear combination of other spinors where the rows would correspond to the components ψ_j. The spinor is not splittable into rows that have a meaning in the rotation group. In this respect, the notation $\mathscr{S} \subset \mathbb{C}^{2^\nu}$ introduced in Footnote 3 is not rigorous. We should rather state $\mathscr{S} \cong S \subset \mathbb{C}^{2^\nu}$, to indicate that the set of spinors has the algebraic appearance of a subset $S \subset \mathbb{C}^{2^\nu}$. Now the columns of a square matrix are representing the same vector decomposition as the rows. A rotation matrix can therefore also not be split meaningfully into columns. It is for this reason that defining spinors as column matrices is in general a tricky idea. Only in the exceptional case of SU(2) does this have meaning. It is in this respect also not rigorous to write that $\mathbf{R} \in \mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ for the rotation matrices $\mathbf{R}$. The reflexes we have inherited from linear algebra are here our worst enemy. We are not dealing with vector spaces describing translational symmetry. We are dealing with manifolds describing rotational symmetry. When we discuss spinors as "square roots of vectors" we get trapped in this linear-algebra reflex, splitting what in reality is futile to split for the description of the symmetry. We will nevertheless for convenience use these non-rigorous notations $\mathscr{S} \subset \mathbb{C}^{2^\nu}$ and $\mathbf{R} \in \mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ throughout the paper.

[7] Property [7] refers to the fact that we have defined the group elements and the multi-vectors *en bloc*. We will continue to define the group elements and the multi-vectors *en bloc* in the following. Because the entire vector space E seems to be disconnected from visual geometrical clues it seems to be more logical to stop associating spinors with single column matrices $\boldsymbol{\chi}_j \in E$, $\boldsymbol{\psi}_k \in E$ or $\boldsymbol{\eta}_j \in E$. The most logical meaningful generalization of the spinor concept will then be to associate spinors with the rotation matrices in $\mathrm{L}(E, E)$ themselves, or with the rotated images in SO(n) of the isotropic *Vielbein* of the isotropic vector space $\mathscr{I}_1$. The sets of rotation matrices and the sets of rotated isotropic *Vielbeins* will then be ideals. We will discuss in Subsection 4.5 a connection between the geometrically meaningful quantities with their visual clues and E in the form of an alternative multi-vector basis for $\mathrm{L}(E, E)$. The natural basis vectors $\boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k$ for $\mathrm{L}(E, E)$ can be expressed in this multi-vector basis.

3 Construction of a basis of gamma matrices for $\mathbb{R}^4$ and $\mathbb{R}^5$

We now want to generalize these methods to SO(n), with $n > 3$. In a first step we will do this for SO(4) and SO(5). We do not develop here how this can be generalized to groups SO(k, p) of transformations that preserve a pseudo-Euclidean metric $ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_k^2 - dx_{k+1}^2 - dx_{k+2}^2 \cdots - dx_{k+p}^2$ with a metric tensor $g_{\mu\nu}$ rather than a Euclidean metric $ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_k^2 + dx_{k+1}^2 + dx_{k+2}^2 \cdots + dx_{k+p}^2$ with a metric tensor $\delta_{\mu\nu}$. The generalization is trivial, because it suffices to replace γ_{k+j} by $\imath\gamma_{k+j}$ to obtain the change of sign in the metric.

We start from $\mathbb{R}^3$ and the 2×2 Pauli matrices. They satisfy $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\,\mathbb{1}$. We now have one or two more basis vectors, and we need thus more square matrices satisfying $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}\,\mathbb{1}$ in order to be able to represent all vectors of $\mathbb{R}^4$ and $\mathbb{R}^5$. There are no more such matrices in the set of complex 2×2 matrices $\mathrm{L}(\mathbb{C}^2, \mathbb{C}^2)$ as shown in paragraph 2.6.3. We therefore consider 4×4 representation matrices. The reader will notice that the procedure echoes the procedure for Pauli matrices at the block level. The way we proceed will not be unique. The situation is here comparable to what we discovered in Subsection 2.6. There are loads of alternative choices. Our present choice will be dictated by a concern of simplicity. With other equivalent choices for the representation matrices, the proofs may be much less simple and elegant. We keep using the Pauli matrices and embed the 2×2 formalism inside the 4×4 formalism as follows:

$$\gamma_j = \begin{bmatrix} \sigma_j & \\ & -\sigma_j \end{bmatrix} = \sigma_j \otimes \sigma_z. \tag{76}$$

As $\sigma_j^\dagger = \sigma_j$, it follows that also $\gamma_j^\dagger = \gamma_j$. The calculation of $\gamma_j\gamma_k + \gamma_k\gamma_j$ yields:

$$\begin{bmatrix} \sigma_j & \\ & -\sigma_j \end{bmatrix}\begin{bmatrix} \sigma_k & \\ & -\sigma_k \end{bmatrix} + \begin{bmatrix} \sigma_k & \\ & -\sigma_k \end{bmatrix}\begin{bmatrix} \sigma_j & \\ & -\sigma_j \end{bmatrix} = \begin{bmatrix} \sigma_j\sigma_k & \\ & \sigma_j\sigma_k \end{bmatrix} + \begin{bmatrix} \sigma_k\sigma_j & \\ & \sigma_k\sigma_j \end{bmatrix} =$$

$$\begin{bmatrix} \sigma_j\sigma_k + \sigma_k\sigma_j & \\ & \sigma_j\sigma_k + \sigma_k\sigma_j \end{bmatrix} = \begin{bmatrix} 2\delta_{jk}\,\mathbb{1} & \\ & 2\delta_{jk}\,\mathbb{1} \end{bmatrix} = 2\delta_{jk}\,\mathbb{1}_{4\times 4}. \tag{77}$$

This shows how the basis vectors of 2×2 formalism carry over to the 4×4 formalism. Now we add:

$$\gamma_4 = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \mathbb{1}\otimes\sigma_x. \tag{78}$$

The matrix γ_4 is Hermitian: $\gamma_4^\dagger = \gamma_4$. We can check that:

$$(\,\gamma_4\,)^2 = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix}\begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \\ & \mathbb{1} \end{bmatrix} = \mathbb{1}_{4\times 4}. \tag{79}$$

$$\begin{aligned}\gamma_4\gamma_j + \gamma_j\gamma_4 &= \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix}\begin{bmatrix} \sigma_j & \\ & -\sigma_j \end{bmatrix} + \begin{bmatrix} \sigma_j & \\ & -\sigma_j \end{bmatrix}\begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} \\ &= \begin{bmatrix} & -\sigma_j + \sigma_j \\ \sigma_j - \sigma_j & \end{bmatrix} = 0. \end{aligned} \tag{80}$$

Finally we add:

$$\gamma_5 = \begin{bmatrix} & -\imath\,\mathbb{1} \\ \imath\,\mathbb{1} & \end{bmatrix} = \mathbb{1}\otimes\sigma_y. \tag{81}$$

The matrix γ_5 is obviously Hermitian: $\gamma_5^\dagger = \gamma_5$. We have $\gamma_5^2 = \mathbb{1}_{4\times 4}$. We also have $\gamma_4\gamma_5 + \gamma_5\gamma_4 = 0$, due to $\sigma_x\sigma_y + \sigma_y\sigma_x = 0$. Finally:

$$\begin{aligned}\gamma_5\gamma_j + \gamma_j\gamma_5 &= \begin{bmatrix} & -\imath\,\mathbb{1} \\ \imath\,\mathbb{1} & \end{bmatrix}\begin{bmatrix} \sigma_j & \\ & -\sigma_j \end{bmatrix} + \begin{bmatrix} \sigma_j & \\ & -\sigma_j \end{bmatrix}\begin{bmatrix} & -\imath\,\mathbb{1} \\ \imath\,\mathbb{1} & \end{bmatrix} \\ &= \begin{bmatrix} & \imath\sigma_j - \imath\sigma_j \\ \imath\sigma_j - \imath\sigma_j & \end{bmatrix} = 0. \end{aligned} \tag{82}$$

It is easy to check that there is nothing more we can add. We cannot find a sixth matrix to represent a sixth basis vector, because there are only five 4×4 matrices that satisfy simultaneously all equations $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{kj}\mathbb{1}_{4\times 4}$. The procedure of construction we have used is not unique. But we have obtained this way a basis for $\mathbb{R}^5$. Any other choice of basis will be equivalent by similarity transformation. And if one could find a sixth unit vector via another procedure, using the inverse of the similarity transformation would enable us to find a sixth vector in the present procedure.

4 Construction of a basis for a representation of SO(n) by a Peano induction

4.1 The basis vectors

We will now generalize this procedure. The reader can check that it is obtained from the previous Section by substituting ℓ for 4, $\ell+1$ for 5, and ζ_j for σ_j. We have thus $\ell = 2\nu$. With $2^2\times 2^2$ matrices we have been able to define a basis for $\mathbb{R}^4$ and for $\mathbb{R}^5$. We will now set up a general procedure to define a basis for $\mathbb{R}^{2\nu}$ and $\mathbb{R}^{2\nu+1}$. We will proceed by Peano induction, assuming that we have already found the procedure to define a basis for $\mathbb{R}^{2\nu}$ and $\mathbb{R}^{2\nu+1}$ and prove how we can then construct a basis for $\mathbb{R}^{2\nu+2}$ and $\mathbb{R}^{2\nu+3}$. This way we will be able to obtain a basis for $\mathbb{R}^6$ and $\mathbb{R}^7$ with 8×8 matrices. Then we can use 16×16 matrices to set up basis vectors for $\mathbb{R}^8$ and $\mathbb{R}^9$. In general we will need $2^\nu\times 2^\nu$ matrices to set up basis vectors for $\mathbb{R}^{2\nu}$ and $\mathbb{R}^{2\nu+1}$. This explains why we introduced the definition $\nu = \lfloor \frac{n}{2} \rfloor$ in Footnote 3. In the following we will no longer take care of identifying the rank of the unit matrices like we did for $\mathbb{1}_{4\times 4}$. We will use the notation $\mathbb{1}$ for a unit matrix $\mathbb{1}_{\rho\times\rho}$ of arbitrary rank ρ.

Assume thus that we have $2\nu+1$ $2^\nu\times 2^\nu$ gamma matrices ζ_j that define a basis for $\mathbb{R}^{2\nu+1}$. We will show that we can construct $2\nu+3$ $2^{\nu+1}\times 2^{\nu+1}$ gamma matrices γ_j that define a basis for $\mathbb{R}^{2\nu+3}$. We keep using matrices ζ_j and embed the $2^\nu\times 2^\nu$ formalism inside the $2^{\nu+1}\times 2^{\nu+1}$ formalism as follows:

$$\gamma_j = \begin{bmatrix} \zeta_j & \\ & -\zeta_j \end{bmatrix} = \zeta_j \otimes \sigma_z. \tag{83}$$

As $\zeta_j^\dagger = \zeta_j$, it follows that also $\gamma_j^\dagger = \gamma_j$. The calculation of $\gamma_j\gamma_k + \gamma_k\gamma_j$ yields:

$$\begin{bmatrix} \zeta_j & \\ & -\zeta_j \end{bmatrix}\begin{bmatrix} \zeta_k & \\ & -\zeta_k \end{bmatrix} + \begin{bmatrix} \zeta_k & \\ & -\zeta_k \end{bmatrix}\begin{bmatrix} \zeta_j & \\ & -\zeta_j \end{bmatrix} = \begin{bmatrix} \zeta_j\zeta_k & \\ & \zeta_j\zeta_k \end{bmatrix} + \begin{bmatrix} \zeta_k\zeta_j & \\ & \zeta_k\zeta_j \end{bmatrix} =$$

$$\begin{bmatrix} \zeta_j\zeta_k + \zeta_k\zeta_j & \\ & \zeta_j\zeta_k + \zeta_k\zeta_j \end{bmatrix} = \begin{bmatrix} 2\delta_{jk}\,\mathbb{1} & \\ & 2\delta_{jk}\,\mathbb{1} \end{bmatrix} = 2\delta_{jk}\,\mathbb{1}. \tag{84}$$

We have used here the fact that we have assumed that ζ_j are gamma matrices for $\mathbb{R}^{2\nu+1}$, such that $\zeta_j\zeta_k + \zeta_k\zeta_j = 2\delta_{jk}\mathbb{1}$. This shows how the basis vectors of the $2^\nu \times 2^\nu$ formalism carry over to the $2^{\nu+1} \times 2^{\nu+1}$ formalism. We may note that γ_z is always diagonal, and that it contains the numbers 1 and -1 in equal amounts. Now we add the $2^{\nu+1} \times 2^{\nu+1}$ matrix:

$$\gamma_\ell = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \mathbb{1} \otimes \sigma_x. \tag{85}$$

The matrix γ_ℓ is Hermitian: $\gamma_\ell^\dagger = \gamma_\ell$. We can check that:

$$(\,\gamma_\ell\,)^2 = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix}\begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \\ & \mathbb{1} \end{bmatrix} = \mathbb{1}. \tag{86}$$

$$\gamma_\ell\gamma_j + \gamma_j\gamma_\ell = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix}\begin{bmatrix} \zeta_j & \\ & -\zeta_j \end{bmatrix} + \begin{bmatrix} \zeta_j & \\ & -\zeta_j \end{bmatrix}\begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix}$$

$$= \begin{bmatrix} & -\zeta_j + \zeta_j \\ \zeta_j - \zeta_j & \end{bmatrix} = 0. \tag{87}$$

Finally we add the $2^{\nu+1} \times 2^{\nu+1}$ matrix:

$$\gamma_{\ell+1} = \begin{bmatrix} & -\imath\,\mathbb{1} \\ \imath\,\mathbb{1} & \end{bmatrix} = \mathbb{1} \otimes \sigma_y. \tag{88}$$

The matrix $\gamma_{\ell+1}$ is obviously Hermitian: $\gamma_{\ell+1}^\dagger = \gamma_{\ell+1}$. We have $(\gamma_{\ell+1})^2 = \mathbb{1}$. We also have $\gamma_\ell\gamma_{\ell+1} + \gamma_{\ell+1}\gamma_\ell = 0$. The calculation mimics $\sigma_x\sigma_y + \sigma_y\sigma_x = 0$ in block form. Finally:

$$\gamma_{\ell+1}\gamma_j + \gamma_j\gamma_{\ell+1} = \begin{bmatrix} & -\imath\,\mathbb{1} \\ \imath\,\mathbb{1} & \end{bmatrix}\begin{bmatrix} \zeta_j & \\ & -\zeta_j \end{bmatrix} + \begin{bmatrix} \zeta_j & \\ & -\zeta_j \end{bmatrix}\begin{bmatrix} & -\imath\,\mathbb{1} \\ \imath\,\mathbb{1} & \end{bmatrix}$$

$$= \begin{bmatrix} & \imath\zeta_j - \imath\zeta_j \\ \imath\zeta_j - \imath\zeta_j & \end{bmatrix} = 0. \tag{89}$$

It is easy to check that there is nothing more we can add. We cannot find another matrix such that simultaneously all conditions $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}\mathbb{1}$ are satisfied and we could use this additional matrix to represent an additional basis vector. At each doubling of the rank $\rho = 2^\nu$ of the gamma matrices we can thus add just two basis vectors. This is the reason we introduced $\nu = \lfloor \frac{n}{2} \rfloor$. We may again observe that the procedure of construction is not unique. But we have obtained this way a basis for $\mathbb{R}^{2\nu+3}$. Any other choice of basis will be equivalent by similarity transformation to the one we introduce here. And if it were possible to find a $(2\nu+4)$-th matrix to represent an additional unit vector via another procedure, using the inverse of the similarity transformation would enable us to find a $(2\nu+4)$-th matrix in the present procedure. To represent the vector space $\mathbb{R}^n$, we need thus gamma matrices of the size $2^{\lfloor \frac{n}{2} \rfloor} \times 2^{\lfloor \frac{n}{2} \rfloor}$, i.e. $2^\nu \times 2^\nu$.

4.2 The eigenvalues of the basis vectors

The eigenvalues of all $2^\nu \times 2^\nu$ basis vectors are -1 ($2^{\nu-1}$ times) and 1 ($2^{\nu-1}$ times). Let us consider the successive definitions of γ_z. We have:

$$\sigma_z = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad \zeta_z = \begin{bmatrix} \sigma_z & \\ & -\sigma_z \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}, \tag{90}$$

$$\gamma_z = \begin{bmatrix} \zeta_z & \\ & -\zeta_z \end{bmatrix} = \begin{bmatrix} 1 & & & & & & & \\ & -1 & & & & & & \\ & & -1 & & & & & \\ & & & 1 & & & & \\ & & & & -1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \end{bmatrix}. \tag{91}$$

The successive γ_z matrices are thus diagonal and the diagonal is a Thue-Morse series of 1 and -1 entries. Let us now show that all other γ matrices can be linked to γ_z by a similarity transformation based on a rotation. We prove this by Peano induction. Let us first consider γ_j. If $\zeta_j = \mathbf{S}\,\zeta_z\,\mathbf{S}^{-1}$, then:

$$\gamma_j = \zeta_j \otimes \sigma_z = \begin{bmatrix} \zeta_j & \\ & -\zeta_j \end{bmatrix} = \begin{bmatrix} \mathbf{S} & \\ & \mathbf{S} \end{bmatrix} \begin{bmatrix} \zeta_z & \\ & -\zeta_z \end{bmatrix} \begin{bmatrix} \mathbf{S}^{-1} & \\ & \mathbf{S}^{-1} \end{bmatrix}, \tag{92}$$

whereby $\mathbf{S}$ is a rotation in $\mathbb{R}^{2\nu-1}$. Next we consider γ_ℓ. Let us first show how one can do it wrongly. By mimicry with σ_x, we have

$$\gamma_\ell = \mathbb{1} \otimes \sigma_x = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{bmatrix} \frac{1}{\sqrt{2}}. \tag{93}$$

Both $\mathbb{1} \otimes \sigma_z$ and $\zeta_z \otimes \sigma_z$ have both $2^{\nu-1}$ entries 1 and $2^{\nu-1}$ entries -1. We can now exchange an entry -1 with an adjacent entry 1 to move it down by a transposition as follows:

$$\begin{bmatrix} \ddots & & \\ & \boxed{\begin{matrix} -1 & \\ & +1 \end{matrix}} & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \boxed{\sigma_x} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & \boxed{\begin{matrix} +1 & \\ & -1 \end{matrix}} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & \boxed{\sigma_x} & \\ & & \ddots \end{bmatrix}. \tag{94}$$

The matrix σ_x used here is its own inverse such that this is a similarity transformation. We can move this way all the -1 entries from $\zeta_z \otimes \sigma_z$ down until we obtain $\mathbb{1} \otimes \sigma_z$. The number of transpositions (i.e. swaps between adjacent entries) needed can be pair such that the product of all the moves is a "rotation". It can also be odd, such the product is a reversal. But we can then add a dummy swap between two adjacent -1 entries to make it a "rotation". We put here the word "rotation" between quotes, because it is a rotation in $\mathbb{R}^{2^\nu}$ not in $\mathbb{R}^n$. The problem we encounter here is a direct consequence of the fact that we have not defined any meaning for the individual column vectors that occur in the representation matrices of the vectors. The vector representations have been defined for the columns *en bloc*. In fact, the matrices $\gamma_\ell = \mathbb{1} \otimes \sigma_x$ and $\gamma_{\ell+1} = \mathbb{1} \otimes \sigma_y$ are only duplicates of σ_x and σ_y where we have replaced one dimension by a whole set of dimensions. All other vectors are also inflated in the process. But this does not mean anything special. The true unit vectors are not multi-dimensional. Trying to find a meaning for the individual columns is therefore difficult. Despite these problems, this calculation shows that the two vector matrices are linked by a similarity transformation. The only problem is that we do not know whether the product of these transpositions is a pure rotation of $\mathbb{R}^n$.[24]

To make a better analysis, we must show that the unit vectors $\mathbf{e}_\ell$ is linked to $\mathbf{e}_z$ through a similarity transformation based on a true rotation of $\mathbb{R}^n$. The transformation of the columns will then also be done *en bloc* without any further consideration for the individual columns. Finding this rotation is actually not too difficult because we can easily

[24] With the interpretation given in Section 6 we will be able to give a meaning to these permutations.

visualize it in the plane spanned by the couple of vectors $(\mathbf{e}_\ell, \mathbf{e}_z)$. The rotation must only change vectors in this plane. All other unit vectors must remain invariant. They are spanning a subspace of $\mathbb{R}^n$ that remains invariant under the rotation, just like a rotation axis in $\mathbb{R}^3$ remains invariant under the rotation. This invariant subspace must be a subspace of the two hyperplanes of the reflections that define the rotation. The hyperplanes are the invariant subspaces of the reflections. Therefore the invariant subspace of the rotation must be a subspace of both hyperplanes. The normal unit vectors that define the hyperplanes must thus be orthogonal to $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_4$, $\cdots$ $\mathbf{e}_{\ell-1}$, $\mathbf{e}_{\ell+1}$. They must also be orthogonal to the intersections of their hyperplanes with the plane spanned by the vectors $(\mathbf{e}_\ell, \mathbf{e}_z)$. The hyperplanes intersect this plane in lines at angles $\pi/8$ and $3\pi/8$. The first normal vector is $\cos(5\pi/8)\mathbf{e}_\ell + \sin(5\pi/8)\mathbf{e}_z$. The second normal vector is $\cos(7\pi/8)\mathbf{e}_\ell + \sin(7\pi/8)\mathbf{e}_z$. We will then rotate from $\mathbf{e}_\ell$ to $\mathbf{e}_z$ along a great circle. The product matrix is: $\mathbf{R} = \cos(\pi/4)\mathbb{1} - \sin(\pi/4)\gamma_z\gamma_\ell = \frac{1}{\sqrt{2}}(\mathbb{1} - \gamma_z\gamma_\ell)$, its inverse is: $\mathbf{R}^{-1} = \frac{1}{\sqrt{2}}(\mathbb{1} + \gamma_z\gamma_\ell)$. We can check that $\mathbf{R}$ commutes with γ_x, γ_y, γ_4, $\cdots$ $\gamma_{\ell-1}$, $\gamma_{\ell+1}$. In fact, its part $\cos(\pi/4)\mathbb{1}$ is all-commuting. The part $\gamma_z\gamma_\ell$ commutes with all γ_j for $j \notin \{z, \ell\}$ because then $\gamma_j\gamma_z\gamma_\ell = -\gamma_z\gamma_j\gamma_\ell = \gamma_z\gamma_\ell\gamma_j$. This implies that $\mathbf{R}\gamma_j\mathbf{R}^{-1} = \gamma_j$ such that $\mathbf{e}_j$ really belongs to the invariant subspace. The only representation matrices that do not commute and therefore are affected by the rotation are γ_z and γ_ℓ. Now:

$$\mathbf{R} = \frac{1}{\sqrt{2}}(\mathbb{1} - \gamma_z\gamma_\ell) = \begin{bmatrix} \mathbb{1} & \zeta_z \\ -\zeta_z & \mathbb{1} \end{bmatrix}, \quad \mathbf{R}^{-1} = \begin{bmatrix} \mathbb{1} & -\zeta_z \\ \zeta_z & \mathbb{1} \end{bmatrix}. \tag{95}$$

We have thus:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} \mathbb{1} & \zeta_z \\ -\zeta_z & \mathbb{1} \end{bmatrix} \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} \begin{bmatrix} \mathbb{1} & -\zeta_z \\ \zeta_z & \mathbb{1} \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} \zeta_z & \\ & -\zeta_z \end{bmatrix}, \tag{96}$$

where we know that the transition matrix is a rotation matrix. Also $\gamma_{\ell+1} = \mathbb{1} \otimes \sigma_y$ can by mimicry with σ_y be linked to $\mathbb{1} \otimes \sigma_z$ through a similarity transformation.

$$\gamma_{\ell+1} = \mathbb{1} \otimes \sigma_y = \begin{bmatrix} & -\imath\mathbb{1} \\ \imath\mathbb{1} & \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbb{1} & \imath\mathbb{1} \\ \imath\mathbb{1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & -\imath\mathbb{1} \\ -\imath\mathbb{1} & \mathbb{1} \end{bmatrix} \frac{1}{\sqrt{2}}. \tag{97}$$

The quantity $\mathbb{1} \otimes \sigma_z$ can then again be transformed by a similarity transformation to $\gamma_z = \zeta_z \otimes \sigma_z$. This way, all gamma matrices can be transformed by a similarity transformation based on a "rotation" to γ_z. Here again we can find a pure rotation of $\mathbb{R}^n$ along a great circle directly. This time it is $\mathbf{R} = \cos(\pi/4)\mathbb{1} - \sin(\pi/4)\gamma_z\gamma_{\ell+1}$, $\mathbf{R}^{-1} = \cos(\pi/4)\mathbb{1} + \sin(\pi/4)\gamma_z\gamma_{\ell+1}$. This yields now:

$$\mathbf{R} = \frac{1}{\sqrt{2}}(\mathbb{1} - \gamma_z\gamma_\ell) = \begin{bmatrix} \mathbb{1} & -\imath\zeta_z \\ -\imath\zeta_z & \mathbb{1} \end{bmatrix}, \quad \mathbf{R}^{-1} = \begin{bmatrix} \mathbb{1} & \imath\zeta_z \\ \imath\zeta_z & \mathbb{1} \end{bmatrix}. \tag{98}$$

We have then:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} \mathbb{1} & -\imath\zeta_z \\ -\imath\zeta_z & \mathbb{1} \end{bmatrix} \begin{bmatrix} & -\imath\mathbb{1} \\ \imath\mathbb{1} & \end{bmatrix} \begin{bmatrix} \mathbb{1} & \imath\zeta_z \\ \imath\zeta_z & \mathbb{1} \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} \zeta_z & \\ & -\zeta_z \end{bmatrix}, \tag{99}$$

where we know again that the transition matrix is a rotation matrix. This concludes the proof by Peano induction.

4.3 Multi-vectors

The various multi-vectors of $\mathbb{R}^n$ are constructed from 0, 1, 2, $\cdots$ n vectors. The multi-vectors constructed from j vectors form a vector space of dimension $\begin{pmatrix} n \\ j \end{pmatrix}$. The direct sum of all these vector spaces is a vector space of dimension $\begin{pmatrix} n \\ 0 \end{pmatrix} + \begin{pmatrix} n \\ 1 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix} \cdots \begin{pmatrix} n \\ n \end{pmatrix} = 2^n$. From this we get the idea that the multi-vectors could be used as a basis for the space $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$. The representation matrices $[\,\mathbf{e}_j{\cdot}\boldsymbol{\gamma}\,]$ of the basis vectors $\mathbf{e}_j$ are actually the sum of 2^ν basis vectors $\boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k^\dagger$. In the Clifford algebra, all these multi-vectors are linearly independent. What we have to show is that their representations matrices are also linearly independent (see Subsection 4.5). We can then conclude that all linear transformations of $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ can be decomposed in a basis of multi-vectors.

4.4 A first possible definition of the spinors and its interpretation

In the following we will assume that $n = 2\nu+1$. If $n = 2\nu$ we can consider the embedding of $\mathbb{R}^{2\nu}$ in $\mathbb{R}^{2\nu+1}$. The reason why we do this is that we will need γ_z. The matrix γ_z is then of the form:

$$\gamma_z = \sum_{k\in\mathscr{S}_+} \boldsymbol{\eta}_k \otimes \boldsymbol{\eta}_k^\dagger - \sum_{k\in\mathscr{S}_-} \boldsymbol{\eta}_k \otimes \boldsymbol{\eta}_k^\dagger. \tag{100}$$

Here $\mathscr{S}_+ = \{k \in [1,2^\nu]\cap\mathbb{N} \parallel (\gamma_z)_{kk} = 1\}$ and $\mathscr{S}_- = \{k \in [1,2^\nu]\cap\mathbb{N} \parallel (\gamma_z)_{kk} = -1\}$. We could define $\boldsymbol{\eta}_k$ as the spinors. The matrices $\boldsymbol{\eta}_k \otimes \boldsymbol{\eta}_m^\dagger$ form a natural basis for the function space $\mathrm{L}(\mathbb{C}^{2^\nu},\mathbb{C}^{2^\nu})$. The column matrix $\mathbf{R}\boldsymbol{\eta}_k$ is the k-th column of $\mathbf{R}$. Under a rotation Eq. 100 transforms to:

$$\mathbf{R}\,\gamma_z\,\mathbf{R}^\dagger = \sum_{k\in\mathscr{S}_+} \boldsymbol{\chi}_k \otimes \boldsymbol{\chi}_k^\dagger - \sum_{k\in\mathscr{S}_-} \boldsymbol{\chi}_k \otimes \boldsymbol{\chi}_k^\dagger. \tag{101}$$

To distinguish between the two sets of indices, we can write this as:

$$\mathbf{R}\,\gamma_z\,\mathbf{R}^\dagger = \sum_{k\in\mathscr{S}_+} \boldsymbol{\chi}_k \otimes \boldsymbol{\chi}_k^\dagger - \sum_{k\in\mathscr{S}_-} \dot{\boldsymbol{\psi}}_k \otimes \dot{\boldsymbol{\psi}}_k^\dagger. \tag{102}$$

The spinors $\boldsymbol{\chi}_k$ that constitute the "square root" of a vector $\mathbf{a}$ represented by the matrix $[\,\mathbf{a}\cdot\boldsymbol{\gamma}\,]$ are thus columns of the matrix $\mathbf{R}$ of the rotation R along a great circle that maps $\mathbf{e}_z$ onto $\mathbf{a}$ in $\mathbb{R}^{2\nu+1}$. The conjugate spinors $\dot{\boldsymbol{\psi}}_k$ are equally columns of the matrix $\mathbf{R}$. The individual columns do not give much information about the rotation matrix. We further encountered the problem that we cannot interpret easily the individual columns when we tried to obtain the similarity transformations by transpositions in Subsection 4.2. It seems therefore for the moment more logical to define a spinor as the full rotation matrix rather than as a single column of it.

Cartan illustrated the square-root relationship for isotropic vectors. This approach is more complicated and conceals a part of the quadratic relation because the terms that occur are no longer of the truly quadratic type $\boldsymbol{\chi}\otimes\boldsymbol{\chi}^\dagger$ or $\dot{\boldsymbol{\psi}}\otimes\dot{\boldsymbol{\psi}}^\dagger$, but of the mixed type $\boldsymbol{\chi}\otimes\dot{\boldsymbol{\psi}}^\dagger$.

4.5 Can we find a meaning for the single columns?

We will give an easy geometrical interpretation for the single columns in Section 6. Here we can give a geometrical interpretation to the tensor products $\boldsymbol{\eta}_j\otimes\boldsymbol{\eta}_k^\dagger$. The interpretation is also validated by Peano induction. In SU(2), we have:

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \imath\sigma_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad -\imath\sigma_x\sigma_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{103}$$

We see thus that the full set of multi-vectors generated by the set of matrices $\mathbb{1},\sigma_x,\sigma_y$ introduced for $\mathbb{R}^2$ can be used to define a basis $\boldsymbol{\eta}_j\otimes\boldsymbol{\eta}_k^\dagger$ for the function space $\mathrm{L}(\mathbb{C}^2,\mathbb{C}^2)$:

$$\frac{1}{2}\,(\mathbb{1}-\imath\sigma_x\sigma_y) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \frac{1}{2}\,(\mathbb{1}+\imath\sigma_x\sigma_y) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \frac{1}{2}\,(\sigma_x+\imath\sigma_y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \frac{1}{2}\,(\sigma_x-\imath\sigma_y) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{104}$$

And of course this can be reverted: the basis $\boldsymbol{\eta}_j\otimes\boldsymbol{\eta}_k^\dagger \in \mathrm{L}(\mathbb{C}^2,\mathbb{C}^2)$ can serve as an alternative basis basis for the space of all multi-vectors. We can now prove the same thing for the basis we use for $\mathbb{R}^4$. We consider now the set $\mathbb{1}_{4\times4},\gamma_4$, γ_5. Its members can be used to obtain:

$$\mathbb{1}_{4\times4} = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix}, \quad \gamma_u = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}, \quad \imath\gamma_v = \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}, \quad -\imath\gamma_u\gamma_v = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \tag{105}$$

and:

$$\frac{1}{2}\,(\mathbb{1}-\imath\gamma_u\gamma_v) = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{bmatrix},\ \frac{1}{2}\,(\mathbb{1}+\imath\gamma_u\gamma_v) = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{bmatrix},\ \frac{1}{2}\,(\gamma_u+\imath\gamma_v) = \begin{bmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{bmatrix},\ \frac{1}{2}\,(\gamma_u-\imath\gamma_v) = \begin{bmatrix} 0 & 0 \\ \mathbb{1} & 0 \end{bmatrix}. \tag{106}$$

It is then easy to see that the full set of multi-vectors generated by the sets $\mathbb{1},\gamma_x,\gamma_y$ and $\mathbb{1},\gamma_u,\gamma_v$ can be used to form a basis for $\mathrm{L}(\mathbb{C}^4,\mathbb{C}^4)$. In fact, the 2×2 matrices in Eq. 104 occur (in certain cases up to a sign) twice on the diagonal in $L(\mathbb{C}^4, \mathbb{C}^4)$. We can thus obtain any element $\boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k^\dagger$ by multiplication. E.g. if we want just a non-zero entry on position $(2,3)$, we can multiply:

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \times & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & \times & 0 \end{array}\right] \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & \times & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{107}$$

Here the symbols $\times$ indicate that the entry is 1, perhaps up to a sign. It is then obvious how the generalization of this by Peano induction works. In fact, we must dissect the matrix into quadrants and select the operator that corresponds to the quadrant that contains the position (j,k). Then we must dissect this quadrant again into quadrants and select the quadrant that corresponds to the position (j,k), etc... Note that the number of multi-vectors of $\mathbb{R}^n$ (with $n = 2\nu$), is 2^n which is exactly the dimension of $L(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$. We leave thus systematically γ_z out of the set of generating basis vectors, and consider it as related to a bi-vector generated by multiplication. We see thus that we obtain an interpretation for the basis for the function space $L(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$, but not for the vector space $\mathbb{C}^{2^\nu}$. This should not surprise us. The multi-vectors have been defined without any relationship to a function space $L(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$, as the vector matrices have been defined *en bloc*. Their only relation with a function space is that they are (up to a sign convention) algebraically identical to reflection matrices. We have defined these reflection matrices also *en bloc* as group elements, i.e. functions, without bothering about the possible meaning of $\mathbb{C}^{2^\nu}$. We will solve the problem of the interpretation of the vector space $\mathbb{C}^{2^\nu}$ in terms of group elements in Section 6. We could then consider the relation between the multi-vectors and the group elements as an isomorphism.

4.6 Cartan's approach: Isotropic vectors

4.6.1 Isotropic basis

Let us consider the case $n = 2\nu$. We can then make the following change of basis:

$$\forall k \in [0, \nu - 1\,] \cap \mathbb{Z} : \begin{cases} \mathbf{e}'_{2k+1} = \frac{1}{\sqrt{2}}\,(\mathbf{e}_{2k+1} + \imath \mathbf{e}_{2k+2}) \\ \\ \mathbf{e}'_{2k+2} = \frac{1}{\sqrt{2}}\,(\mathbf{e}_{2k+1} - \imath \mathbf{e}_{2k+2}) \end{cases} \tag{108}$$

The pre-factor $\frac{1}{\sqrt{2}}$ ensures here that $|\mathbf{e}'_{2k+1}|_H = 1$, $|\mathbf{e}'_{2k+2}|_H = 1$. This corresponds to the coordinate transformation:

$$\begin{aligned} x'_{2k+1} &= \frac{1}{\sqrt{2}}\,(x_{2k+1} - \imath x_{2k+2}) \\ x'_{2k+2} &= \frac{1}{\sqrt{2}}\,(x_{2k+1} + \imath x_{2k+2}) \end{aligned} \tag{109}$$

The metric can then be expressed by:

$$\sum_{j=1}^{2\nu} x_j^2 = 2 \sum_{k=0}^{\nu-1} x'_{2k+1}\, x'_{2k+2}. \tag{110}$$

The isotropic vectors $\mathbf{e}'_{2k+1}$ span an isotropic subspace $\mathscr{I}_1$ of $\mathbb{C}^{2\nu}$:

$$\forall (\alpha_0, \alpha_1 \cdots \alpha_{\nu-1}) \in \mathbb{C}^\nu : \quad \left| \sum_{k=0}^{\nu-1} \alpha_{2k+1}\, \mathbf{e}'_{2k+1} \right|^2 = 0. \tag{111}$$

As a consequence of this, a linear combination of isotropic vectors from $\mathscr{I}_1$ is an isotropic vector. The same is true for the isotropic vectors $\mathbf{e}'_{2k+2} = \mathbf{e}'^{*}_{2k+1}$. They form the isotropic subspace $\mathscr{I}_2$. However, because $\mathbf{e}'_{2k+1}, \mathbf{e}'_{2k+2}, k \in [0, \nu-1\,] \cap \mathbb{Z}$, are a basis, any vector of $\mathbb{C}^n$ can be written as a linear combination of $\mathbf{e}'_{2k+1}, \mathbf{e}'_{2k+2}, k \in [0, \nu-1\,] \cap \mathbb{Z}$. This applies thus also for vectors which are not isotropic vectors. An arbitrary linear combination of the isotropic vectors is therefore not necessarily an isotropic vector. This is easily checked on an example. The sum of the two isotropic vectors $\mathbf{e}'_{2k+1}$ and $\mathbf{e}'_{2k+2}$ is $\sqrt{2}\,\mathbf{e}_{2k+1}$, which is not an isotropic vector, because $|\sqrt{2}\,\mathbf{e}_{2k+1}|^2 = 2$.

These isotropic basis vectors completely define the *Vielbein*. When $n = 2\nu + 1$ this remains true, because $\mathbf{e}_n$ is uniquely defined by $\mathbf{e}_1, \mathbf{e}_2, \cdots \mathbf{e}_{2\nu}$. We can thus leave $\mathbf{e}_n$ unchanged in the coordinate transformation. The metric is then expressed by:

$$\sum_{j=1}^{2\nu} x_j^2 = x_n^2 + 2\sum_{k=0}^{\nu-1} x'_{2k+1}\, x'_{2k+2}. \tag{112}$$

By neglecting the normalization and choosing a different change of basis:

$$\begin{aligned} x'_{2k+1} &= x_{2k+1} - \imath x_{2k+2}, \\ x'_{2k+2} &= x_{2k+1} + \imath x_{2k+2}, \end{aligned} \tag{113}$$

we can obtain the Euclidean distance also under the more symmetric form:

$$\sum_{j=1}^{2\nu} x_j^2 = x_n^2 + \sum_{k=0}^{\nu-1} x'_{2k+1}\, x'_{2k+2}. \tag{114}$$

4.6.2 The basic difficulty

We have already mentioned the basic difficulty in Remark 2 of paragraph 2.5.1. The representation matrices of the isotropic vectors have determinant zero. In SU(2) and SL(2,$\mathbb{C}$) (which is a representation of the homogeneous Lorentz group), an isotropic vector is represented by a 2×2 matrix with a zero determinant. This implies that the two rows are proportional and also that the two columns are proportional. The matrix can therefore be written as $\boldsymbol{\phi} \otimes \dot{\boldsymbol{\psi}}^{\dagger}$. An isotropic vector in $\mathbb{C}^n$ may well be represented by a matrix with zero determinant as well, but this does not imply that the matrix could be written as $\boldsymbol{\phi} \otimes \dot{\boldsymbol{\psi}}^{\dagger}$, because this would imply that all columns are proportional and all rows are proportional. In fact, it is only necessary that two rows are proportional for the determinant to be zero. The same applies for the columns. The problem is thus to find out what kind of structure the matrix that represents an isotropic vector in $\mathbb{C}^n$ will have. We will see that it will no longer be $\boldsymbol{\phi} \otimes \dot{\boldsymbol{\psi}}^{\dagger}$ but a sum of terms $\boldsymbol{\phi}_j \otimes \dot{\boldsymbol{\psi}}_k^{\dagger}$

4.6.3 Isotropic vectors render certain parts of the formalism clearer

By the special choice of our basis, we circumvent the problem of deriving the structure of the representation matrix $\mathbf{M}$ of an isotropic vector from the property $\det \mathbf{M} = 0$. In Cartan's approach based on isotropic vectors, the relationship between $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ and the multi-vectors is simpler than in the approach based on real unit-vectors. In fact, we have:

$$\begin{aligned} \sigma_+ = \frac{1}{2}(\sigma_x + \imath\sigma_y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \sigma_- = \frac{1}{2}(\sigma_x - \imath\sigma_y) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\ -\imath\sigma_x\sigma_y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \sigma_-\sigma_+ = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{115}$$

It it now obvious how me must continue this scheme by Peano induction. Each element $\boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k^{\dagger}$ is a product of operators σ_- and σ_+ in their various block forms. We have already explained after Eq. 107 how we can proceed by dissecting the matrix into quadrants and work this way backwards with ever smaller representation matrices. Each quadrant corresponds to an operator in Eq. 115 for a given size of the representation in the Peano induction scheme. E.g. the matrix $\boldsymbol{\eta}_2 \otimes \boldsymbol{\eta}_3^{\dagger}$ in Eq. 107 is obtained as $\sigma_-\tilde{\sigma}_+$, where $\tilde{\sigma}_+ = \mathbb{1} \otimes \sigma_+$ is the block form of σ_+. To summarize: We can write both unit vectors and isotropic vectors as sums of tensor products of spinors. The expression for the unit vectors leads to optimal clarity in revealing the relationship between spinors and vectors, while the use of isotropic vectors permits to write the basis vectors of $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ in a very simple way as a product of isotropic vectors.

5 Reflections and rotations

The reflection matrices $\mathbf{A}$ defined by the reflection normal $\mathbf{a}$ will be of the type $\mathbf{a} \in \mathbb{R}^n \leftrightarrow \mathbf{A} = \sum_{j=1}^n a_j\gamma_j$, because that will enable us to satisfy $\mathbf{A}^2 = \mathbb{1}$. The product of two reflections $\mathbf{A}$ and $\mathbf{B}$ will be a rotation:

$$\mathbf{AB} = [\sum_{j=1}^n a_j\gamma_j\,]\,[\sum_{k=1}^n b_k\gamma_k\,] = \sum_{j=1}^n a_jb_j\,\mathbb{1} + \sum_{j<k}(a_jb_k - a_kb_j)\gamma_j\gamma_k. \tag{116}$$

This just separates $\mathbf{AB}$ into a symmetric part $\frac{1}{2}(\mathbf{AB}+\mathbf{BA})$ and an antisymmetric part $\frac{1}{2}(\mathbf{AB}-\mathbf{BA})$. In Clifford algebra this is noted as $\mathbf{a}\vee\mathbf{b}=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}$. There are $n(n-1)/2$ different products $\gamma_j\gamma_k$, with $j<k$. The number $n(n-1)/2$ is the number of independent real parameters for a rotation. We can also obtain this number by calculating the number of parameters necessary to specify completely the *Vielbein* of mutually orthogonal unit vectors that define a basis, which is $(n-1)+(n-2)+\cdots+1=n(n-1)/2$. The first unit vector can be chosen at will, such that it is defined by n real parameters. However, it must be normalized to 1 such that there are only $n-1$ independent real parameters. The second vector can be taken at will, provided we make sure that it is orthogonal to the first one and normalized to 1, such that it is defined by $n-2$ additional independent real parameters. This way we can carry on, and we find in total $(n-1)+(n-2)+\cdots+1=n(n-1)/2$ independent real parameters. The equation $[\sum_{j=1}^{n} a_j\gamma_j]^2=\sum_1^n a_j^2\,\mathbb{1}$ is an obvious identity. Now $(a_j^2+a_k^2)(b_j^2+b_k^2)=a_j^2b_j^2+a_k^2b_k^2+a_j^2b_k^2+a_k^2b_j^2=(a_jb_j+a_kb_k)^2+(a_jb_k-a_kb_j)^2$. From this we can conclude that the squares of the norms of $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\wedge\mathbf{b}$ in their respective spaces are not independent and that they behave as $\cos^2\chi$ and $\sin^2\chi$. Let us finally note that $(\gamma_j-\gamma_k)(\gamma_j+\gamma_k)=\gamma_j\gamma_k-\gamma_k\gamma_j$. We can see this way that Clifford algebra is just a more formal and abstract way to write down the algebra we have developed. There is thus a natural motivation to introduce it. Once again, one could start the whole development by starting to define the Clifford algebra and deriving everything from it. But such a craze for abstraction would leave the reader guessing about the underlying motivation and leaving him with the impression that the definitions come out of the blue.

Let us note the quantities $\gamma_j\gamma_k$ as γ_{jk}. They serve as basis vectors for rotation matrices. In $\mathbb{R}^3$ we have $\sigma_x\sigma_y=\imath\sigma_z(cycl.)$. One can take $S_j=\frac{1}{2}\sigma_j$ as infinitesimal generators for SU(2). We have then $[S_x,S_y]=\imath S_z(cycl.)$. We see that we satisfy a condition for a Lie algebra as the commutators can be expressed in terms of the basis vectors. We pointed out in Remark 4 of Subsection 2.5 that this situation in $\mathbb{R}^3$ is exceptional, in that the reflection matrices σ_j occur also in the infinitesimal generators for the Lie algebra, which correspond to rotations. In SO(n) with $n>3$ there will be no such coincidence. In SO(n) there are n generators γ_j in the form of reflections (or reflection matrices), but there will be $n(n-1)/2$ infinitesimal generators $\gamma_j\gamma_k$ that can serve as basis vectors for the tangent space to the Lie manifold of the rotation group and build the Lie algebra. The fact that both γ_j and γ_{jk} can be expressed in terms of the same matrices σ_j is a unique feature of SO(3) and not general. In SO(n) the rôle of infinitesimal generators can be played by the matrices γ_{jk}. The commutators are then of the type $[\gamma_{jk},\gamma_{lm}]$, and they can indeed be expressed in terms of the basis vectors γ_{np}. In fact, first suppose that j,k,l,m are all different. We have then: $\gamma_{jk}\gamma_{lm}=\gamma_j\gamma_k\gamma_l\gamma_m=-\gamma_j\gamma_l\gamma_k\gamma_m=\gamma_l\gamma_j\gamma_k\gamma_m=-\gamma_l\gamma_j\gamma_m\gamma_k=\gamma_l\gamma_m\gamma_j\gamma_k=\gamma_{lm}\gamma_{jk}$, such that $[\gamma_{jk},\gamma_{lm}]=0$. The next possibility is that one of the indices occurs in both gamma matrices. The possibilities are: $\gamma_{jk}\gamma_{jm}$, $\gamma_{jk}\gamma_{lj}$, $\gamma_{jk}\gamma_{km}$, $\gamma_{jk}\gamma_{lk}$. In each of these possibilities we can assemble the two identical gamma matrices by permutations. E.g. $\gamma_j\gamma_k\gamma_j\gamma_m=-\gamma_j\gamma_j\gamma_k\gamma_m=-\gamma_k\gamma_m$. This proves that the commutators can all be expressed in terms of the basis vectors. In fact, it is not possible that more than two indices are equal. We obtain thus the structure constants for a rotation group by straightforward calculation.

6 An interpretation in terms of left-handed and right-handed visualizations of the identity element

The four natural basis vectors of $\mathrm{L}(\mathbb{C}^2,\mathbb{C}^2)$ are the transformations:

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1&0\end{bmatrix},\quad \begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0&1\end{bmatrix},$$
$$\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1&0\end{bmatrix},\quad \begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0&1\end{bmatrix}. \tag{117}$$

They are all of the type $\boldsymbol{\eta}_j\otimes\boldsymbol{\eta}_k^\dagger$. We could interpret $\boldsymbol{\eta}_k^\dagger$ as the operation of annihilating $\boldsymbol{\eta}_k$ and $\boldsymbol{\eta}_j$ as the operation of creating $\boldsymbol{\eta}_j$. This would mimic exactly what the matrix does $\boldsymbol{\eta}_j\otimes\boldsymbol{\eta}_k^\dagger$ does in operating on the column matrices. We must then consider:

$$\boldsymbol{\eta}_1=\begin{bmatrix}1\\0\end{bmatrix},\quad \boldsymbol{\eta}_2=\begin{bmatrix}0\\1\end{bmatrix}, \tag{118}$$

as the two only possible states of the system. We are talking here about group elements which can be visualized by a reference frame, e.g. a triad in $\mathbb{R}^3$. We could consider $\boldsymbol{\eta}_1$ as the identity element described in a right-handed reference frame and $\boldsymbol{\eta}_2$ as the identity element described in a left-handed frame. We must here remember that in our constructions γ_z was always considered as a bi-vector. The reflection γ_z was only added on[25] in increasing the

[25] There is absolutely no need to consider γ_z as truly representing $\mathbf{e}_z$. We only use the index z to indicate the similarity with the Pauli matrix σ_z. In going from $\mathbb{R}^2$ to $\mathbb{R}^4$ we can e.g. identify γ_ℓ with $\mathbf{e}_z$ and $\gamma_{\ell+1}$ with $\mathbf{e}_4$, reserving γ_z for $\mathbf{e}_5$.

dimension of the vector spaces from $\mathbb{R}^{2\nu}$ to $\mathbb{R}^{2\nu+1}$. The two fundamental states are thus already defined in SO(2). In the two-dimensional group, the states $\boldsymbol{\eta}_j$ can therefore e.g. not refer to the reflection $[\,\mathbf{e}_z{\cdot}\boldsymbol{\sigma}\,]$, because the notion of a third dimension still does not exist in SO(2). We should thus also not use triads, but two-dimensional frames to visualize the states. The states $\boldsymbol{\eta}_j$ could thus be visualized by a non-symmetrical figure and its mirror in SO(2). Or we could we could visualize them in terms of oriented planes. We could define the plane as right-handed if the angle of the counterclockwise rotation that maps $\mathbf{e}_x$ onto $\mathbf{e}_y$ is $\frac{\pi}{2}$ and left-handed if this angle is $3\pi/2 \equiv -\pi/2 \pmod{2\pi}$.

If we choose σ_x as the operator that changes the handedness of the frames we use for the visualization, the model works. It changes the right-handed $\boldsymbol{\eta}_1$ into the left-handed $\boldsymbol{\eta}_2$ and *vice versa*. We have therefore:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{119}$$

Actually a rotation in SO(2) changes $\boldsymbol{\eta}_1$ into $e^{-\imath\varphi/2}\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ into $e^{\imath\varphi/2}\boldsymbol{\eta}_2$. We can thus indeed consider $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ as right-handed and left-handed objects, whose handedness we can visualize by left-handed and right-handed frames. We may note that the isotropic vectors $\mathbf{e}_x + \imath\mathbf{e}_y$ and $\mathbf{e}_x - \imath\mathbf{e}_y$, which contain the two spinors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ in the simplest possible way, can also be considered as representing right- and left-handed frames. Such isotropic vectors are not reflected like vectors by a simple change of sign, but by conjugation.

But we must remain vigilant about the dangers of reasoning on frames, because frames are defined in terms of vectors, while we are trying to reason on group elements. E.g. a reflection matrix $\mathbf{A}$ will operate on the left of the spinors $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$ in the vector matrix $\mathbf{V} = \boldsymbol{\eta}_1 \otimes \boldsymbol{\eta}_1^\dagger - \boldsymbol{\eta}_2 \otimes \boldsymbol{\eta}_2^\dagger$ and simultaneously on the right of the conjugated spinors $\boldsymbol{\eta}_1^\dagger$, $\boldsymbol{\eta}_2^\dagger$, because we have $\mathbf{V} \to -\mathbf{AVA}$. Hence if we reason in terms of frames, we will miss the point as we are trying to make sense of the meaning of $\mathbf{G} \to \mathbf{AG}$. When we are reasoning on group elements, a reflection operator $\mathbf{A}$ in the operation $\mathbf{G} \to \mathbf{AG}$ changes the handedness of a group element $\mathbf{G}$, but when $\mathbf{G}$ is a reflection, the result will not necessarily be a new reflection. It is thus really better to discuss the situation in terms of right- and left-handed oriented planes rather than in terms of right- and left-handed two-dimensional reference frames. In this respect, the isotropic vectors are perhaps less misleading in representing oriented planes than real vectors, because they do not describe the changes along one direction but for the full plane. Even though they are two-dimensional, the oriented planes can be conveniently described three-dimensionally by us as three-dimensional beings watching what happens in the *Oxy* plane. We can picture the two spinors by an arrow $\mathbf{e}_z$ with a right or left hand around it. With our three-dimensional vision we can then interpret $e^{-\imath\varphi/2}\boldsymbol{\eta}_1$ as a spin-up state (counter-clockwise rotation) and $e^{\imath\varphi/2}\boldsymbol{\eta}_2$ as a spin-down state (clockwise rotation). The interpretation of the states is thus simple because it is purely two-dimensional. This is why $\mathbf{e}_z$ only appears as a bi-vector. It corresponds then not to a dimension for vectors in SO(2), but just to a matrix we can use to describe rotations.

In SU(2) we could define $\boldsymbol{\eta}_2$ also as the rotation around the y-axis over π because it the first column of this rotation matrix. This corresponds also to a change of handedness of the frame in SO(2). But in SO(2) the three-dimensional rotations around the x-axis and the y-axis do not exist, such that defining $\boldsymbol{\eta}_2$ as a rotation around the y-axis over π is not possible in SO(2).[26]

We add on the reflection operator σ_z only when we want to discuss also rotations in SO(3). We can now check what happens if we add $\mathbf{e}_z$ to the formalism and we perform a rotation in $\mathbb{R}^3$. According to Eq. 39, we obtain then e.g. spinors:

$$\begin{bmatrix} \cos(\theta/2) \\ -\imath\sin(\theta/2)e^{\imath(\phi+\pi/2)} \end{bmatrix}, \tag{120}$$

This is not the most general form because one entry is not complex. But what it illustrates is that this spinor is no longer purely $\boldsymbol{\eta}_1$ or $\boldsymbol{\eta}_2$. In fact the pure states are rotations in the plane. Because we move the rotation axis away from the z-axis, we obtain now a "mixed state". However, this is not a linear combination of $\boldsymbol{\eta}_1$ or $\boldsymbol{\eta}_2$ but a generalized state. The state is a pure rotational state because we know that the spinor in Eq. 120 is now the first column of the rotation matrix. We know also that rotations do not build a vector space but a manifold, such that we should not try to analyze this spinor in terms of linear combinations of $\boldsymbol{\eta}_1$ or $\boldsymbol{\eta}_2$. The spinor is not defined in terms of components but *en bloc*. It cannot be decomposed into $\boldsymbol{\eta}_1$ or $\boldsymbol{\eta}_2$ like a vector can be decomposed into components. Any reflex inspired by what we have learned about linear algebra for vector spaces is thus taboo.

In quantum mechanics, physicists transgress this taboo. Fortunately this can be justified *a posteriori* as explained in paragraph 2.3.2. As we have seen, we can also interpret the spinor wave function in Eq. 120 alternatively in terms of a statistical ensemble of spinors, i.e. a mixture whereby a fraction $\cos^2(\theta/2)$ are in the $\boldsymbol{\eta}_1$ state and $\sin^2(\theta/2)$ are in the $\boldsymbol{\eta}_2$ state. In quantum mechanics this statistical interpretation makes often a lot of sense. E.g. if we manage to remove all spinors in the $\boldsymbol{\eta}_2$ state from a statistical ensemble described by Eq. 120 we obtain a fraction of $\cos^2(\theta/2)$ spinors in the $\boldsymbol{\eta}_1$ state. This corresponds exactly to Malus' law in physics. But the fact that a state can be interpreted this way in

[26] However, we can use this in $\mathbb{R}^3$ to describe a clockwise rotation of a right-handed frame.

two different non-equivalent ways can also lead to confusion in quantum mechanics. In the Stern-Gerlach experiment we can turn individual spins, and we should not interpret such individual spins in terms of statistical ensembles. It is in this respect misleading to ask for a unique answer to the question which one of the two interpretations of a wave function is correct, the single-particle one or the statistical one. In fact, the answer will depend on the type of experiment one performs, such that the question is ill-conceived.

As in the generalization to $\mathbb{R}^{2\nu}$ by Peano induction we add two dimensions at each step, we can consider that we add each time an oriented plane. We have thus in total ν oriented planes. There are 2^ν possible combinations for the orientations of these ν planes. At each step we add "block-spinors":

$$\boldsymbol{\kappa}_1 = \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix}, \quad \boldsymbol{\kappa}_2 = \begin{bmatrix} 0 \\ \mathbb{1} \end{bmatrix}, \quad \text{where: } \mathbb{1} \in \mathrm{L}(\mathbb{C}^{2^{\nu-1}}, \mathbb{C}^{2^{\nu-1}}). \tag{121}$$

because at each step we can describe the added plane left-handedly or right-handedly. We can make combinations of the orientations of the planes described by these blocks and the orientations of the planes that were introduced on previous level of Peano induction. Under rotations the combined orientations of the planes will not change. It takes a reflection to change the combination of orientations. The two orientations of the plane in SO(2) are given by $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$. There are two possible operations we can carry out on the handedness. The operation $\mathbb{1}$ maintains the handedness, the operation σ_x inverts the handedness. By adding a second plane we extend the formalism from SO(2) to SO(4). The operations on the initial plane are now represented by $\mathbb{1} \otimes \sigma_z$, and $\sigma_x \otimes \sigma_z$ in SO(4). The two orientations of the new plane are presented by $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$. The analogous operators in SO(4) that change the handedness of this plan in SO(4) are now $\mathbb{1} \otimes \mathbb{1}$ and $\mathbb{1} \otimes \sigma_x$. The four possible combinations of handedness can now be represented by:

$$\begin{array}{ll} [\,\mathbb{1}\otimes\sigma_z\,][\,\mathbb{1}\otimes\mathbb{1}\,] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & [\,\mathbb{1}\otimes\sigma_z\,][\,\mathbb{1}\otimes\sigma_x\,] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \\ [\,\sigma_x\otimes\sigma_z\,][\,\mathbb{1}\otimes\mathbb{1}\,] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, & [\,\sigma_x\otimes\sigma_z\,][\,\mathbb{1}\otimes\sigma_x\,] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \end{array} \tag{122}$$

such that the 4 basis vectors $\boldsymbol{\eta}_j \in \mathbb{C}^4$ now represent the 4 possible combinations of handedness of the two oriented planes that intervene in the definition of the identity element of SO(4). Let us call such combinations constellations of handedness. By changing the order of the operators in the products, one can actually get rid of the minus signs. By Peano induction the 2^ν basis vectors $\boldsymbol{\eta}_j \in \mathbb{C}^{2^\nu}$ will represent the 2^ν possible constellations of handedness of the ν oriented planes that intervene in the definition of the identity element of SO(2ν). The block tensor products of the block spinors can be further decomposed into tensor products of these basis vectors, e.g.:

$$\boldsymbol{\kappa}_1 \boxtimes \boldsymbol{\kappa}_2^\dagger = \begin{bmatrix} \mathbb{1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \boxtimes \begin{bmatrix} 0 & 0 & \mathbb{1} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbb{1} \\ 0 \\ 0 \end{bmatrix} \boxtimes \begin{bmatrix} 0 & 0 & 0 & \mathbb{1} \end{bmatrix}, \quad \text{where: } \mathbb{1} \in \mathrm{L}(\mathbb{C}^{2^{\nu-2}}, \mathbb{C}^{2^{\nu-2}}). \tag{123}$$

Here, the notation $\boxtimes$ serves to indicate the block tensor product. One could also try to interpret the formalism in terms of the two choices one has in order to define the orientation of each coordinate axis of the *Vielbein* in $\mathbb{R}^{2\nu}$. The corresponding swaps of orientation are generalized parity transformations, like P and T in the homogeneous Lorentz group. As we now are considering vectors, we must apply the reflection operators to the left and the right. The transformation space $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ has $2^\nu \times 2^\nu = 2^{2\nu}$ entries $\boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k^\dagger$. As we have seen, all basis vectors can be written under the form of Eq. 102, such that they are Hermitian. When they contain a non-diagonal entry $\boldsymbol{\eta}_j \otimes \boldsymbol{\eta}_k^\dagger$, with $j \neq k$, they must thus also contain $\boldsymbol{\eta}_k \otimes \boldsymbol{\eta}_j^\dagger$. The $2^{2\nu}$ basis vectors of $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ can thus not visualize directly all possible swaps of the configuration of parities within the *Vielbein* without a prior change of basis. The representation matrices of the isotropic vectors are on the contrary not Hermitian. Hermitian conjugation swaps $(\mathbf{e}_{2k+1} + \imath\mathbf{e}_{2k+2})\cdot\boldsymbol{\gamma}$ and $(\mathbf{e}_{2k+1} - \imath\mathbf{e}_{2k+2})\cdot\boldsymbol{\gamma}$. The prior change of basis we could use is thus given by Eq. 108. All basis vectors of $\mathrm{L}(\mathbb{C}^{2^\nu}, \mathbb{C}^{2^\nu})$ can be obtained as products of isotropic vectors as shown in Eq. 115. The isotropic vectors are thus more appropriate to visualize the constellations of handedness of all the planes, as we already anticipated above. They are themselves a basis for the linear space of multi-vectors. Under the action of a rotation R, we transform $\boldsymbol{\eta}_j \to \boldsymbol{\eta}'_j = \mathbf{R}\boldsymbol{\eta}_j$ where $\boldsymbol{\eta}'_j$ are now the spinors in their general form. They correspond to the columns of the rotation matrix, which express it for all different possible choices for the constellation of handedness. They are representations of different handedness of the rotation, which contain *a priori* not the complete information about the rotation and are therefore not faithful. In SU(2) we can obtain the second column from the first column by a simple prescription, but this is exceptional. In general, the columns $\boldsymbol{\eta}'_j$ of $\mathbf{R}$ cannot be obtained from $\boldsymbol{\eta}'_1$ by some simple rule. In fact, the required transformation is $\boldsymbol{\eta}_j = \mathbf{T}_j\boldsymbol{\eta}_1$, with $(\mathbf{T}_j)_{k\ell} = \delta_{jk}\delta_{\ell 1}$, such that $\boldsymbol{\eta}'_j = \mathbf{R}\mathbf{T}_j\boldsymbol{\eta}_1$. Here $\mathbf{T}_j$ is "trapped" between $\mathbf{R}$ and $\boldsymbol{\eta}_1$, from where it cannot be trivially moved out to the left and not at all to the right. To fully characterize $\mathbf{R}$ we need thus *a priori* all its columns.

One may wonder why we construct the representation plane by plane, why we do not care about the individual parity transformations in the planes, and only consider the handedness of the planes. The answer is of course given by the considerations in point [2] of paragraph 2.8.3. The individual handedness of all these planes do actually matter little. Only the global handedness is important to distinguish a left-handed from a right-handed basis and proper rotations from reversals. This may give some intuition for the formalism.

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