(t,q) Q-systems, DAHA and quantum toroidal algebras via generalized Macdonald operators

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(t, q)-DEFORMED Q-SYSTEMS, DAHA AND QUANTUM TOROIDAL ALGEBRAS VIA GENERALIZED MACDONALD OPERATORS

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Abstract. We introduce difference operators on the space of symmetric functions which are a natural generalization of the (q,t)-Macdonald operators. In the $t \to \infty$ limit, they satisfy the A_{N-1} quantum Q-system [DFK14, DFK15]. We identify the elements in the spherical A_{N-1} DAHA [Che05] which are represented by these operators, as well as within the quantum toroidal algebra of $\mathfrak{gl}_1$ [FJMM12] and the elliptic Hall algebra [SV11, Sch12]. We present a plethystic, or bosonic, formulation of the generating functions for the generalized Macdonald operators, which we relate to recent work of Bergeron et al [BGLX14]. Finally we derive constant term identities for the current that allow to interpret them in terms of shuffle products [Neg14]. In particular we obtain in the $t \to \infty$ limit a shuffle presentation of the quantum Q-system relations.

Contents

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1. Introduction: generalized Macdonald operators

1.1. **Introduction.** The aim in this paper is to make explicit the relation between the algebra satisfied by the generators of the A_{N-1} quantum Q-system, or more precisely, their t-deformation [DFK16], and the following algebras, whose relation to each other is better known: the A_{N-1} spherical Double Affine Hecke Algebra (sDAHA) [Che05], the level-0 quantum toroidal algebra of $\widehat{\mathfrak{gl}}_1$ [Mik07, FJMM12], the shuffle algebra [FO98, Neg14] and the Elliptic Hall Algebra (EHA) [SV11].

Quantum Q-systems arise naturally as the quantization [BZ05] of the cluster algebras [FZ02] associated with the classical Q-system [Ked08, DFK11]. The latter is a recursion relation for characters of Kirillov-Reshetikhin modules [KR87] of quantum affine algebras, and it is directly connected [HKO+99, DFK08] with fermionic formulas for the characters of the tensor products of KR-modules. The quantum Q-system is similarly connected with the graded characters of the Feigin-Loktev fusion product [FL99] of the same spaces [DFK14], where the quantum parameter q is associated with the homogeneous grading.

Graded characters of tensor products of KR-modules can be written as iterated, ordered products of difference operators, which satisfy the quantum Q-system, acting on the polynomial 1. That is, there is a representation of the algebra satisfied by the generators of the quantum Q-system in terms of difference operators acting on the space of symmetric functions [DFK14, DFK15].

In Ref. [DFK16], we gave a relation between this algebra and a non-standard level 0 representation of the Drinfeld currents in the quantum affine $\widehat{\mathfrak{sl}}_2$ algebra, or rather, a rank-dependent quotient of the algebra. The difference operators introduced in [DFK15] were identified as a generalization of the the dual Whittaker limit $t \to \infty$ of the celebrated Macdonald difference operators [Mac95]. This connection naturally leads [DFK16] to the introduction of a t-deformation of these operators.

The present paper, a continuation of [DFK16], gives the algebraic framework of this deformation. To do this, we define a new set of difference operators.

1.2. **Generalized Macdonald operators.** Let $\mathbb{C}_{q,t} := \mathbb{C}(q,t)$ be the field of rational functions in the formal variables q and t. We denote by $\mathcal{F}_N$ the space of rational symmetric functions of N variables over $\mathbb{C}_{q,t}$:

$$\mathcal{F}_N := \mathbb{C}_{q,t}(x_1, x_2, ..., x_N)^{S_N}. \tag{1.1}$$

It is a subpace of $\mathcal{S}_N$, symmetric functions of N variables over the same field. Here, the symmetric group S_N acts naturally on $\{x_1, ..., x_N\}$ by permutations.

Given any rational function $f(x_1, x_2, ..., x_N)$, let $\mathrm{Sym}(f)$ denote its symmetrization, and let Γ_i be the multiplicative shift operator, acting on functions in $\mathcal{S}_N$ as follows:

$$\Gamma_i\, f(x_1, ..., x_N) = f(x_1, ..., x_{i-1}, qx_i, x_{i+1}, ..., x_N) \qquad (1 \le i \le N). \tag{1.2}$$

Alternatively, $\Gamma_i = q^{\delta_i}$ where δ_i is the additive shift operator, $\delta_i:\ x_j \mapsto x_j + \delta_{i,j}$.

Consider the following q-difference operator, which is a generalization of the Macdonald operator, acting on $\mathcal{F}_N$:

Definition 1.1. *Let $\alpha \in [1, N]$ and $P \in \mathcal{F}_\alpha$. Define the associated generalized Macdonald operator to be the difference operator $\mathcal{D}_\alpha(P)$ acting on $\mathcal{F}_N$ as*

$$\mathcal{D}_\alpha(P) := \frac{1}{\alpha!\,(N-\alpha)!}\mathrm{Sym}\left(P(x_1, x_2, ..., x_\alpha) \prod_{1\le i\le\alpha<j\le N} \frac{tx_i - x_j}{x_i - x_j}\, \Gamma_1\Gamma_2\cdots\Gamma_\alpha\right). \tag{1.3}$$

The following special examples of the operator in Definition 1.1:

(1) The original Macdonald difference operators [Mac95] correspond to $P = 1 \in \mathcal{F}_\alpha$:

$$\widetilde{\mathcal{D}}_\alpha := \mathcal{D}_\alpha(1) = \sum_{\substack{I\subset[1,N]\\|I|=\alpha}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} \Gamma_i. \tag{1.4}$$

(2) The generalized Macdonald operators, introduced in [DFK16], correspond to the special case of $P = (x_1x_2\cdots x_\alpha)^n \in \mathcal{F}_\alpha$, $n \in \mathbb{Z}$:

$$\mathcal{M}_{\alpha;n} := \mathcal{D}_\alpha\Big((x_1x_2\cdots x_\alpha)^n\Big) = \sum_{\substack{I\subset[1,N]\\|I|=\alpha}} \prod_{i\in I}(x_i)^n \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} \Gamma_i. \tag{1.5}$$

These operators were inspired by the functional representation of the quantum Q-system. Note that $\mathcal{M}_{\alpha;0} = \widetilde{\mathcal{D}}_\alpha$ are the original Macdonald operators.

(3) More generally, Let P be a *generalized Schur function* $s_{a_1,\dots,a_\alpha} \in \mathcal{F}_\alpha$ with $a_i \in \mathbb{Z}$. This is the natural generalization of Schur polynomials to Schur Laurent polynomials:

Definition 1.2. *For any $(a_1, a_2, \dots, a_\alpha) \in \mathbb{Z}^\alpha$, define the generalized Schur function:*

$$s_{a_1,\dots,a_\alpha}(x_1, x_2, \dots, x_\alpha) := \frac{\det_{1\le i,j\le\alpha}\left(x_i^{a_j+\alpha-j}\right)}{\prod_{1\le i<j\le\alpha} x_i - x_j} = \mathrm{Sym}\left(\frac{\prod_{i=1}^\alpha x_i^{a_i+\alpha-i}}{\prod_{i<j} x_i - x_j}\right). \tag{1.6}$$

By construction, $s_{a_1,\dots,a_\alpha}(x_1, x_2, \dots, x_\alpha)$ is a Laurent polynomial in $\mathcal{F}_\alpha$. We denote the difference operators corresponding to generalized Schur functions by

$$\begin{aligned} \mathcal{M}_{a_1,a_2,\dots,a_\alpha} &:= \mathcal{D}_\alpha\Big(s_{a_1,\dots,a_\alpha}(x_1, x_2, \dots, x_\alpha)\Big) \\ &= \sum_{\substack{I\subset[1,N]\\ |I|=\alpha}} s_{a_1,\dots,a_\alpha}(\mathbf{x}_I) \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} \Gamma_i \end{aligned} \tag{1.7}$$

where $\mathbf{x}_I$ stands for the ordered collection of variables $(x_i)_{i\in I}$.

As a particular case, we recover $\mathcal{M}_{\alpha;n} = \mathcal{M}_{n,n,\dots,n}$, which when $n > 0$ corresponds to the usual Schur function $s_{n,n,\dots,n}(x_1, \dots, x_\alpha) = (x_1 x_2 \cdots x_\alpha)^n$, with rectangular Young diagram $\alpha \times n$.

1.3. **Main results and outline.** The paper is organized as follows.

Our first task is to show in Section 2 how the operators $\mathcal{M}_{\alpha;n}$ of (1.5) appear naturally in the context of the functional representation of the spherical DAHA. To this end, we introduce families of commuting operators $Y_{i,n} = (X_1X_2\cdots X_{i-1})^{-n}\, Y_i\, (X_1X_2\cdots X_i)^n$ in terms of the standard generators X_i, Y_i of the DAHA (see Sect. 2.1.1 for definitions). Next we show in Theorem 2.17 that the operators (1.5) correspond to their elementary symmetric functions, in the same way as usual Macdonald operators are obtained from elementary symmetric functions of the Y_i-generators of the DAHA. We find that $Y_{i,n}$ is proportional to the n-th iterate of the action on Y_i of one particular generator of the $SL_2(\mathbb{Z})$ action on DAHA. This clarifies the algebraic origin of the operators $\mathcal{M}_{\alpha;n}$.

In Section 3, we show (Theorem 3.3) that the generating functions for $\mathcal{M}_n = \mathcal{M}_{1;n}$, for all operators with $\alpha = 1$ is an element of the functional (level-0) representation of the quantum toroidal algebra of $\widehat{\mathfrak{gl}}_1$ [Mik99, FJMM12], or more precisely a particular quotient thereof that imposes the finite number N of variables (see (3.5) and (3.6)). We call these the "fundamental currents" for reasons which will become clear in Section 4. This section explains a posteriori one of the findings of Ref. [DFK16], which corresponds to the limit $t \to \infty$, where these generating functions are part of a non-standard representation of the quantum enveloping algebra of the affine algebra $\widehat{\mathfrak{sl}}_2$.

In Section 4, we give the plethystic formulation of the above currents, equivalent to their so-called bosonization. It describes their action on formal power series of the power sum functions $p_k = \sum x_i^k$ for $k \in \mathbb{Z}^*$. This generalizes a construction of Bergeron et al. [BGLX14].

In Section 5, we give an alternative definition $\mathcal{M}_\alpha(P)$ (see Definition 5.2) for the generalized Macdonald operators $\mathcal{D}_\alpha(P)$ of Definition 1.1, which expresses them as a multiple constant term involving products of the above fundamental currents. The coincidence of these two definitions is the subject of Theorem 5.3. The latter allows to write the multiple generating function for the operators $\mathcal{M}_{a_1,\ldots,a_\alpha}$ of (1.7) in terms solely of those of the $\mathcal{M}_n$'s (Theorem 5.7). We conjecture (Conjecture 5.9) that these may be reduced to polynomial expressions modulo the relations of the quantum toroidal algebra, and give the proof in the case $\alpha = 2$ (Theorem 5.10). Such polynomials play the role of $(q, t)-$determinants (see the expression (5.19) for $\alpha = 2$). In Section 5.3, we show that the definition of $\mathcal{M}_\alpha(P)$ is naturally compatible with a suitably defined non-commutative product $* : \mathcal{F}_\alpha \times \mathcal{F}_\beta \to \mathcal{F}_{\alpha+\beta}$, $(P, P') \mapsto P * P'$ (the shuffle product [FO98, Neg14]), which satisfies the morphism property: $\mathcal{M}_{\alpha+\beta}(P * P') = \mathcal{M}_\alpha(P)\mathcal{M}_\beta(P')$ (see Thorem 5.15). We may therefore translate relations between the generalized Macdonald operators into shuffle product identities, which sometimes are easier to prove (see examples in Sections 5.3.2-5.3.4).

Section 6 presents a functional representation of the EHA in terms of our generalized Macdonald operators. The established connection between spherical DAHA and EHA in the case of A_∞ (infinite number of variables) [SV11] extends to the quotient corresponding to A_{N-1} (finite number N of variables). This connection allows to derive new formulas for the operators $\mathcal{M}_{\alpha;n}$ of (1.5) as *polynomials* of the fundamental operators with $\alpha = 1$, thus proving Conjecture 5.9 for $a_1 = a_2 = \cdots = a_\alpha = n$ (Theorem 6.3).

In Section 7, we explore the dual Whittaker limit $t \to \infty$ of the constructions of this paper. In particular, we relate the finite t Macdonald operators $\mathcal{M}_n$ to their $t \to \infty$ limit M_n. We also find an explicit formula for the $t \to \infty$ limit of the operators $\mathcal{M}_{a_1,a_2,\ldots,a_\alpha}$ as a quantum determinant, which involves a summation over $\alpha \times \alpha$ Alternating Sign Matrices (Theorem 7.3). By considering the $t \to \infty$ limit of the shuffle product, we find an alternative shuffle expression for the quantum cluster algebra relations (Theorem 7.5).

Section 8 gathers a few concluding remarks on the (q, t)-determinant that expresses the operator $\mathcal{M}_{a_1,\ldots,a_\alpha}$ as a polynomial of the $\mathcal{M}_n$'s, and suggest that the A_{N-1} EHA quotient corresponding to a finite number N of variables is the natural t-deformation of the quantum Q-system algebra.

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2. Formulation of the generalized Macdonald operators in DAHA

In this section, we find the explicit elements in the spherical double affine Hecke algebra (sDAHA) whose functional representation are the generalized Macdonald operators $\mathcal{M}_{\alpha;n}$ of Equation (1.5). The standard definitions and properties in this section can be found in Cherednik's book [Che05].

2.1. The A_{N-1} DAHA: Definition and relations.

2.1.1. *Generators and relations.* Let q and θ be indeterminates, where $\theta = t^{\frac{1}{2}}$. The A_{N-1} double affine Hecke algebra is the algebra generated over $\mathbb{C}(q,t)$ by the generators $\{X_i, Y_i, T_j : i \in [1,N], j \in [1,N-1]\}$, subject to the following relations:

$$\begin{aligned}
T_i\,T_{i+1}\,T_i &= T_{i+1}\,T_i\,T_{i+1}; \qquad (2.1)\\
(T_i-\theta)(T_i+\theta^{-1}) &= 0;\\
T_i\,X_i\,T_i &= X_{i+1}; \qquad T_i^{-1}\,Y_i\,T_i^{-1} = Y_{i+1}, \qquad (1\leq i\leq N-1); \qquad (2.2)\\
T_i\,X_j &= X_j\,T_i; \qquad T_i\,Y_j = Y_j\,T_i, \quad (j\neq i,i+1);\\
X_1\,Y_2 &= Y_2\,T_1^2\,X_1;\\
X_iX_j &= X_jX_i; \qquad Y_iY_j = Y_jY_i;\\
Y_1\cdots Y_N\,X_j &= q\,X_j\,Y_1\cdots Y_N;\\
X_1\cdots X_N\,Y_j &= q^{-1}\,Y_j\,X_1\cdots X_N. \qquad (2.3)
\end{aligned}$$

2.1.2. *Other useful relations.* We list here several useful relations among the generators of the DAHA which follow from the definition above. Some of them can be found in [Che05] (see Sect. **Some relations** on pp 103-104, eqs. (1.4.64-69)).

The following relations give a way to reorder X_i and Y_j:

$$\begin{aligned}
X_i\,Y_{i+1} &= Y_{i+1}\,T_i^2\,X_i = Y_{i+1}\,T_i\,X_{i+1}\,T_i^{-1} = T_i^{-1}\,Y_i\,T_i\,X_i, \qquad (i=1,2,...,N-1);\\
Y_i\,X_{i+1} &= X_{i+1}\,T_i^{-2}\,Y_i = X_{i+1}\,T_i^{-1}\,Y_{i+1}\,T_i = T_i\,X_i\,T_i^{-1}\,Y_i, \qquad (i=1,2,...,N-1).
\end{aligned}$$

More generally, let $1\leq i\leq j\leq N$. Then

$$\begin{aligned}
X_i\,Y_{j+1} &= Y_{j+1}(T_jT_{j-1}\cdots T_{i+1}T_i^2T_{i+1}^{-1}T_{i+2}^{-1}\cdots T_j^{-1})X_i\\
&= Y_{j+1}(T_jT_{j-1}\cdots T_i)X_{i+1}(T_i^{-1}T_{i+1}^{-1}\cdots T_j^{-1});\\
Y_i\,X_{j+1} &= X_{j+1}(T_j^{-1}T_{j-1}^{-1}\cdots T_{i+1}^{-1}T_i^{-2}T_{i+1}T_{i+2}\cdots T_j)Y_i\\
&= X_{j+1}(T_j^{-1}T_{j-1}^{-1}\cdots T_i^{-1})Y_{i+1}(T_iT_{i+1}\cdots T_j).
\end{aligned}$$

Moreover,

$$\begin{aligned}(X_jX_{j-1}\cdots X_i)\,Y_{j+1} &= Y_{j+1}(T_jT_{j-1}\cdots T_{i+1}T_i^2T_{i+1}T_{i+2}\cdots T_j)(X_jX_{j-1}\cdots X_i)\\ &= Y_{j+1}(T_jT_{j-1}\cdots T_i)(X_{j+1}X_j\cdots X_{i+1})(T_i^{-1}T_{i+1}^{-1}\cdots T_j^{-1}).\end{aligned}$$

This equation can be iterated to obtain

$$(X_jX_{j-1}\cdots X_i)^n\,Y_{j+1} = Y_{j+1}(T_jT_{j-1}\cdots T_i)(X_{j+1}X_j\cdots X_{i+1})^n(T_i^{-1}T_{i+1}^{-1}\cdots T_j^{-1}). \tag{2.4}$$

Lemma 2.1. *For all* $1\leq i\leq j\leq N$,

$$(X_iX_{i+1}\cdots X_j)(T_i^{-1}T_{i+1}^{-1}\cdots T_{j-1}^{-1}) = (T_i^{-1}T_{i+1}^{-1}\cdots T_{j-1}^{-1})(X_iX_{i+1}\cdots X_j).$$

Proof. For all $1\leq i\leq j\leq N$, we have:

$$\begin{aligned}(X_iX_{i+1}\cdots X_j)(T_i^{-1}T_{i+1}^{-1}\cdots T_{j-1}^{-1}) &= (X_iX_{i+1}\cdots X_{j-1})(T_i^{-1}T_{i+1}^{-1}\cdots T_{j-2}^{-1})T_{j-1}X_{j-1}\\ &= X_i(T_iX_i\cdots T_{j-2}X_{j-2}T_{j-1}X_{j-1})\\ &= T_i^{-1}X_{i+1}X_i(T_{i+1}X_{i+1}\cdots T_{j-1}X_{j-1})\\ &= T_i^{-1}X_{i+1}(T_{i+1}X_{i+1}\cdots T_{j-1}X_{j-1})X_i\\ &= T_i^{-1}T_{i+1}^{-1}X_{i+2}(T_{i+2}X_{i+1}\cdots T_{j-1}X_{j-1})X_iX_{i+1}\\ &= (T_i^{-1}T_{i+1}^{-1}\cdots T_{j-1}^{-1})(X_iX_{i+1}\cdots X_j).\end{aligned}$$

The lemma follows. □

2.1.3. *The generator* π. Define

$$\pi = Y_1^{-1}T_1T_2\cdots T_{N-1}.$$

Using $Y_{i+1}=T_i^{-1}\,Y_i\,T_i^{-1}$, we may express each of the Y_is as:

$$Y_i = T_iT_{i+1}\cdots T_{N-1}\pi^{-1}T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1} \qquad (i=1,2...,N).$$

In other words, we may express π in N different manners:

$$\pi = T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1}Y_i^{-1}T_iT_{i+1}\cdots T_{N-1} \qquad (i=1,2,...,N). \tag{2.5}$$

The following two Lemmas show that π acts as a translation operator on T_is and X_is:

Lemma 2.2.

$$\pi T_i = T_{i+1}\pi \qquad (i=1,2,...,N-2)$$

Proof. Using the ith expression for π (2.5), we compute:

$$\begin{aligned}\pi\,T_i &= (T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1})Y_i^{-1}(T_iT_{i+1}T_i)(T_{i+2}\cdots T_{N-1})\\ &= (T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1})Y_i^{-1}T_{i+1}(T_iT_{i+1}\cdots T_{N-1})\\ &= T_{i+1}(T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1})Y_i^{-1}(T_iT_{i+1}\cdots T_{N-1}) = T_{i+1}\pi\end{aligned}$$

where we have first used the braid relations (2.1) and the commutation relations(2.2). □

Lemma 2.3.

$$\pi X_i = X_{i+1}\pi \qquad (i=1,2,...,N-1) \quad \text{and} \quad \pi X_N = q^{-1} X_1 \pi$$

Proof. Using $Y_i X_{i+1} = X_{i+1} T_i^{-2} Y_i$, wich implies $Y_i^{-1} X_{i+1} T_i^{-2} = X_{i+1} Y_i^{-1}$, and the expression (2.5) for π, we compute:

$$\begin{aligned}
\pi X_i &= (T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1})Y_i^{-1}(T_iT_{i+1}\cdots T_{N-1})X_i = (T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1})Y_i^{-1}T_iX_i(T_{i+1}\cdots T_{N-1})\\
&= (T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1})Y_i^{-1}X_{i+1}T_i^{-1}(T_{i+1}\cdots T_{N-1})\\
&= X_{i+1}(T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1})Y_i^{-1}T_i^{-1}(T_{i+1}\cdots T_{N-1}) = X_{i+1}\pi
\end{aligned}$$

The last relation is obtained by using

$$\pi X_1 \cdots X_N = q^{-1} X_1 \cdots X_N \pi,$$

obtained from the relation (2.3). □

2.2. **The functional representation of the DAHA.** Since the variables $X_1,...,X_N$ commute among themselves, we can define the functional representation ρ of the DAHA acting on $V = C_{q,t}(x_1,...,x_N)$ as follows:

$$\begin{aligned}
\rho(X_i)\, f(x_1,x_2,...,x_N) &= x_i\, f(x_1,x_2,...,x_N), \quad f\in V;\\
\rho(s_i)\, f(x_1,...,x_i,x_{i+1},...,x_N) &= f(x_1,...,x_{i+1},x_i,...,x_N), \quad f\in V;\\
\rho(T_i) &= \theta\rho(s_i) + \frac{\theta-\theta^{-1}}{x_ix_{i+1}^{-1}-1}(\rho(s_i)-1)\\
&= \frac{\theta x_i - \theta^{-1}x_{i+1}}{x_i-x_{i+1}}\rho(s_i) - x_{i+1}\frac{\theta-\theta^{-1}}{x_i-x_{i+1}};\\
\rho(T_i^{-1}) &= \rho(T_i) - \theta + \theta^{-1};\\
&= \frac{\theta x_i - \theta^{-1}x_{i+1}}{x_i-x_{i+1}}\rho(s_i) - x_i\frac{\theta-\theta^{-1}}{x_i-x_{i+1}};\\
\rho(\pi)\, f(x_1,x_2,...,x_N) &= f(x_2,x_3,...,x_N,q^{-1}x_1), \quad f\in V;\\
\rho(Y_i) &= \rho(T_iT_{i+1}\cdots T_{N-1}\pi^{-1}T_1^{-1}T_2^{-1}\cdots T_{i-1}^{-1}) \qquad (i=1,2,...,N).
\end{aligned}$$

We see that the q-shift operators Γ_i of (1.2) are a representation of the following element in the DAHA:

$$\Gamma_i = \rho(s_is_{i+1}\cdots s_{N-2}s_{N-1}\pi^{-1}s_1s_2\cdots s_{i-1}), \quad i=1,2,...,N. \tag{2.6}$$

2.3. **Macdonald difference operators.** The operators $Y_1,...,Y_N$ commute among themselves. Therefore one can define the elementary symmetric functions $e_m(Y_1,...,Y_N)$ unambiguously.

Definition 2.4. *The Macdonald operators are*

$$D_\alpha := e_\alpha(Y_1, ..., Y_N), \qquad (\alpha = 0, 1, 2, ..., N). \tag{2.7}$$

Equivalently one can write $\sum_{\alpha=0}^N z^\alpha D_\alpha = \prod_{i=1}^N (1 + zY_i)$.

It is well-known that the operators $\rho(D_\alpha)$ act on the space $\mathcal{S}_N$ of symmetric *functions* in the variables $x_1, ..., x_N$. The following is a standard result of Macdonald theory:

Theorem 2.5. *The restriction of the operators $\rho(D_\alpha)$ to $\mathcal{S}_N$ is*

$$\rho(D_\alpha)|_{\mathcal{S}_N} =: \mathcal{D}_\alpha = \theta^{-\alpha(N-\alpha)} \widetilde{\mathcal{D}}_\alpha$$

where $\widetilde{\mathcal{D}}_\alpha$ are the Macdonald operators (1.4).

2.4. **More commuting operators.** In this section, we introduce families of commuting operators $\{Y_{i,n}\}_{i\in[1,N]}$ for each $n \in \mathbb{Z}$. These are related to Cherednik's $SL_2(\mathbb{Z})$ action on Y_i by n iterations of the generator τ_+.

2.4.1. *Definition and commutation.*

Definition 2.6. *We introduce the family of operators:*

$$Y_{i,n} = (X_1 X_2 \cdots X_{i-1})^{-n}\, Y_i\, (X_1 X_2 \cdots X_i)^n, \qquad (i = 1, 2, ..., N; n \in \mathbb{Z}).$$

In particular, $Y_{i,0} = Y_i$, and $Y_{1,n} = Y_1 X_1^n$. We also see that

$$Y_{N,n} = (X_1 \cdots X_{N-1})^{-n} Y_N\, (X_1 \cdots X_N)^n = q^n X_N^n Y_N.$$

Lemma 2.7. *For fixed $n \in \mathbb{Z}$, the elements $\{Y_{i,n} : i \in [1, N]\}$ commute among themselves:*

$$Y_{i,n}\, Y_{j,n} = Y_{j,n}\, Y_{i,n} \qquad \forall\, i, j \in [1, N].$$

Proof. Writing $j = i + k$, $k > 0$, we have:

$$\begin{aligned}
&(X_1 \cdots X_{i-1})^n Y_{i,n} Y_{i+k,n} (X_1 \cdots X_i)^{-n} \\
&\qquad = Y_i (X_{i+1} \cdots X_{i+k-1})^{-n} Y_{i+k} (X_{i+1} \cdots X_{i+k})^n \\
&\qquad = Y_{i+k} Y_i (T_{i+k-1} \cdots T_{i+1})(X_{i+2} \cdots X_{i+k})^{-n} (T_{i+1}^{-1} \cdots T_{i+k-1}^{-1})(X_{i+1} \cdots X_{i+k})^n \\
&\qquad = Y_{i+k} (T_{i+k-1} \cdots T_{i+1})(Y_i X_{i+1}^n)(T_{i+1}^{-1} \cdots T_{i+k-1}^{-1}) \\
&\qquad = Y_{i+k} (T_{i+k-1} \cdots T_i) X_i^n (T_i^{-1} \cdots T_{i+k-1}^{-1}) Y_i
\end{aligned}$$

where we have first used the relation (2.4):

$$(X_{i+1} \cdots X_{i+k-1})^{-n} Y_{i+k} = Y_{i+k} (T_{i+k-1} \cdots T_{i+1})(X_{i+2} \cdots X_{i+k})^{-n} (T_{i+1}^{-1} \cdots T_{i+k-1}^{-1})$$

then Lemma 2.1:

$$(T_{i+1}^{-1} \cdots T_{i+k-1}^{-1})(X_{i+1} \cdots X_{i+k}) = (X_{i+1} \cdots X_{i+k})(T_{i+1}^{-1} \cdots T_{i+k-1}^{-1})$$

and finally $Y_i X_{i+1}^n = T_i X_i^n T_i^{-1} Y_i$ by iteration of (2.4). Likewise, we have:

$$\begin{aligned}
&(X_1 \cdots X_{i-1})^n Y_{i+k,n} Y_{i,n} (X_1 \cdots X_i)^{-n} \\
&\qquad = (X_i \cdots X_{i+k-1})^{-n} Y_{i+k} (X_i \cdots X_{i+k})^n Y_i \\
&\qquad = Y_{i+k} (T_{i+k-1} \cdots T_i)(X_{i+1} \cdots X_{i+k})^{-n} (T_i^{-1} \cdots T_{i+k-1}^{-1})(X_i \cdots X_{i+k})^n Y_i \\
&\qquad = Y_{i+k} (T_{i+k-1} \cdots T_i) X_i^n (T_i^{-1} \cdots T_{i+k-1}^{-1}) Y_i
\end{aligned}$$

by use of the relations

$$\begin{aligned}
(X_i \cdots X_{i+k-1})^{-n} Y_{i+k} &= Y_{i+k} (T_{i+k-1} \cdots T_i)(X_{i+1} \cdots X_{i+k})^{-n} (T_i^{-1} \cdots T_{i+k-1}^{-1}) \\
(T_i^{-1} \cdots T_{i+k-1}^{-1})(X_i \cdots X_{i+k}) &= (X_i \cdots X_{i+k})(T_i^{-1} \cdots T_{i+k-1}^{-1})
\end{aligned}$$

The lemma follows. □

2.4.2. *Expression in the functional representation.* From this point on, we will work in the functional representation ρ of Section 2.2. We introduce the following symmetric function of $X_1, ..., X_N$:

$$\gamma := \exp \left\{ \sum_{i=1}^{N} \frac{\mathrm{Log}(X_i)^2}{2\mathrm{Log}(q)} \right\} \tag{2.8}$$

This element does not belong to the DAHA, but to a suitable completion (see [Che05]). Nevertheless, it has some useful commutation relations with elements of the DAHA.

Lemma 2.8.

$$\pi^{-1}\,\gamma = q^{\frac{1}{2}} X_N\,\gamma\,\pi^{-1}$$

Proof. Using $\pi^{-1} X_i = X_{i-1} \pi^{-1}$, for $i \in [2, N]$ and $\pi^{-1} X_1 = q X_N \pi^{-1}$, we compute

$$\pi^{-1} \left(\sum_{j=1}^{r+1} \frac{\mathrm{Log}(X_j)^2}{2\mathrm{Log}(q)} \right) = \left(\sum_{j=1}^{r+1} \frac{\mathrm{Log}(X_j)^2}{2\mathrm{Log}(q)} + \mathrm{Log}(X_{r+1}) + \frac{\mathrm{Log}(q)}{2} \right) \pi^{-1}$$

and the Lemma follows. □

Lemma 2.9.

$$\Gamma_i\,\rho(\gamma) = q^{\frac{1}{2}} x_i\,\rho(\gamma)\,\Gamma_i$$

Proof. We simply note that

$$\Gamma_i \left(\sum_{j=1}^{r+1} \frac{\mathrm{Log}(x_j)^2}{2\mathrm{Log}(q)} \right) = \left(\sum_{j=1}^{r+1} \frac{\mathrm{Log}(x_j)^2}{2\mathrm{Log}(q)} + \mathrm{Log}(x_i) + \frac{\mathrm{Log}(q)}{2} \right) \Gamma_i$$

□

Theorem 2.10. *For all $n \in \mathbb{Z}$, we have:*

$$Y_{i,n} = q^{\frac{n}{2}}\,\gamma^{-n}\,Y_i\,\gamma^n.$$

Proof. As γ is a symmetric function of the X_i's, it commutes with s_i, and with all the T_i in the functional representation. We compute:

$$\begin{aligned}
\gamma^{-n}\,Y_i\,\gamma^n &= \gamma^{-n}\,T_i...T_{N-1}\pi^{-1}T_1^{-1}...T_{i-1}^{-1}\,\gamma^n \\
&= T_i...T_{N-1}\,\gamma^{-n}\,\pi^{-1}\,\gamma^n\,T_1^{-1}...T_{i-1}^{-1} \\
&= q^{\frac{n}{2}}T_i...T_{N-1}\,X_N^n\,\pi^{-1}\,T_1^{-1}...T_{i-1}^{-1} \\
&= q^{-\frac{n}{2}}T_i...T_{N-1}\,(X_1...X_{i-1})^{-n}(X_1...X_{i-1})^n\,\pi^{-1}\,X_1^nT_1^{-1}...T_{i-1}^{-1} \\
&= q^{-\frac{n}{2}}(X_1...X_{i-1})^{-n}T_i...T_{N-1}\,\pi^{-1}(X_1...X_i)^nT_1^{-1}...T_{i-1}^{-1} \\
&= q^{-\frac{n}{2}}(X_1...X_{i-1})^{-n}Y_i(X_1...X_i)^n = q^{-\frac{n}{2}}\,Y_{i,n}.
\end{aligned}$$

where we have first used the fact that γ is a symmetric function of $X_1,...,X_N$ and therefore commutes with T_j for all j, then we have used Lemma 2.8, and finally the commutations between the X's and the T's, in particular that the symmetric function $X_1...X_i$ of the variables $X_1,...,X_i$ commutes with T_j for $j = 1,2,...,i-1$, and also that $X_1...X_{i-1}$ commutes with T_j for $j = i, i+1,...,N-1$. ◻

Remark 2.11. *Theorem 2.10 above implies immediately the commutation of the operators $Y_{i,n}$ for any fixed n. However, the element γ (2.8) only belongs to a completion of the DAHA, as it involves infinite power series of the generators X_i. The direct proof of Lemma 2.7 bypasses this complication.*

2.4.3. *Comparison with the standard $SL(2,\mathbb{Z})$ action on DAHA.* Theorem 2.10 allows to identify the conjugation w.r.t. γ^{-1} as the action of the generator τ_+ of the standard $SL(2,\mathbb{Z})$ action on DAHA [Che05]. Indeed, using the definition[1]:

$$\tau_+(X_i) = X_i, \quad \tau_+(T_i) = T_i, \quad \tau_+(q) = q, \quad \tau_+(t) = t,$$
$$\tau_+(Y_1Y_2\cdots Y_i) = q^{-i/2}\,(Y_1Y_2\cdots Y_i)(X_1X_2\cdots X_i)$$

This leads to the expression $Y_{i,n} = q^{n/2}\,\tau_+^n(Y_i)$, which allows to finally identify:

Lemma 2.12. *The generator τ_+ of the standard $SL(2,\mathbb{Z})$ action on DAHA reads:*

$$\tau_+ = \mathrm{ad}_{\gamma^{-1}}$$

namely it acts by conjugation w.r.t. γ^{-1} of (2.8).

The second generator τ_- of the standard $SL(2,\mathbb{Z})$ action on DAHA is obtained by use of the anti-involution ϵ of the DAHA acting on generators and parameters as:

$$\epsilon:\ (X_i, Y_i, T_i, q, t) \mapsto (Y_i, X_i, T_i^{-1}, q^{-1}, t^{-1})$$

[1]This definition is in fact dual to that of [Che05], and corresponds to the definitions of Chapter 1.

and such that

$$\tau_- = \epsilon\, \tau_+\, \epsilon$$

This leads to the following:

Lemma 2.13. *The generator τ_- corresponds to the conjugation w.r.t. the element η^{-1}, where:*

$$\eta := \exp\left\{ - \sum_{i=1}^N \frac{\mathrm{Log}(Y_i)^2}{2\mathrm{Log}(q)} \right\} \tag{2.9}$$

namely

$$\tau_- = \mathrm{ad}_{\eta^{-1}}$$

Proof. Apply the anti-involution ϵ to γ, and note that $\epsilon(\gamma) = \eta$. □

Remark 2.14. *The quantity η^{-1} is very similar to the nabla operator ∇ of* [BG99] *in a version suitable for the case of N variables. To avoid confusion, we write $\eta^{-1} = \nabla^{(N)}$. It is known* [Che05] *that the Macdonald polynomial $P_\lambda(x_1, ..., x_N)$ for any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$ (or equivalently Young diagram with λ_i boxes in row i), is an eigenvector in the functional representation of any symmetric function $f(\{Y_i\}_{i=1}^N)$, with eigenvalue $f(\{t^{\frac{N+1}{2}-i} q^{\lambda_i}\}_{i=1}^N)$. In particular, this holds for $\nabla^{(N)}$, with the result:*

$$\begin{aligned}
\nabla^{(N)} P_\lambda &= \exp\left\{ \frac{1}{2\,\mathrm{Log}(q)} \sum_{i=1}^N \left((\frac{N+1}{2} - i)\,\mathrm{Log}(t) + \lambda_i\,\mathrm{Log}(q) \right)^2 \right\} P_\lambda \\
&= \left(C_N \prod_{i=1}^N q^{\frac{\lambda_i^2}{2}}\, t^{(\frac{N+1}{2}-i)\lambda_i} \right) P_\lambda = C_N\, u_\lambda\, P_\lambda
\end{aligned}$$

where $\mathrm{Log}(C_N) = \frac{N(N^2-1)}{24} \frac{\mathrm{Log}(t)^2}{\mathrm{Log}(q)}$ and $u_\lambda = t^{\frac{N-1}{2}|\lambda| - n(\lambda)} q^{\frac{1}{2}|\lambda| + n(\lambda')}$, where $n(\lambda) = \sum_i (i-1)\lambda_i$, and λ' is the usual reflected diagram, with 1 box in the bottommost $\lambda_1 - \lambda_2$ rows, 2 boxes in the next $\lambda_2 - \lambda_3$ rows, etc, such that $n(\lambda') = \sum_i \lambda_i(\lambda_i - 1)/2$. In [BG99]*, the ∇ operator is defined to have eigenvalue $t^{n(\lambda)} q^{n(\lambda')}$ on the* modified *Macdonald polynomials $\widetilde{H}_\lambda$, obtained from P_λ by a certain transformation. We see that $\nabla^{(N)}$ is an analogue of the operator ∇, acting instead on the P_λ.*

2.5. **Generalized Macdonald difference operators.**

Definition 2.15. *We define operators:*

$$D_{\alpha;n} \equiv D_{\alpha;n}^{q,t} := q^{-\alpha n} \sum_{1 \leq i_1 < i_2 < \cdots < i_\alpha \leq N} Y_{i_1,n} Y_{i_2,n} \cdots Y_{i_\alpha,n} \qquad (\alpha = 0, 1, ..., N) \tag{2.10}$$

Equivalently, we have: $\sum_{\alpha=0}^{r+1} z^\alpha q^{n\alpha}\, D_{\alpha;n} = \prod_{i=1}^N (1 + zY_{i,n})$.

Theorem 2.10 allows to rewrite immediately:

Lemma 2.16. *We have the following identity in the A_{N-1} DAHA functional representation:*

$$\rho(D_{\alpha;n}) = q^{-\frac{\alpha n}{2}} \rho(\gamma)^{-n} \rho(D_{\alpha,0}) \rho(\gamma)^n \ ,$$

where $D_{\alpha,0} \equiv D_\alpha$ are given by (2.7).

Theorem 2.17. *The operators $\rho(D_{\alpha;n})$ leave the space $\mathcal{S}_N$ of symmetric functions of the x's invariant. They take the following form:*

$$\rho(D_{\alpha;n})|_{\mathcal{S}_N} =: \mathcal{D}_{\alpha;n} = \theta^{-\alpha(N-\alpha)}\, \mathcal{M}_{\alpha;n} \tag{2.11}$$

where $\mathcal{M}_{\alpha;n}$ are the generalized Macdonald operators (1.5)*, and $\mathcal{D}_{\alpha;n}$ their slightly renormalized version:*

$$\mathcal{D}_{\alpha;n} = \sum_{|I|=\alpha,\, I\subset[1,N]} (x_I)^n \prod_{\substack{i\in I\\ j\notin I}} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j}\, \Gamma_I \tag{2.12}$$

where $x_I = \prod_{i\in I} x_i$ and $\Gamma_I = \prod_{i\in I} \Gamma_i$, with $\mathcal{D}_{\alpha;0} = \widetilde{\mathcal{D}}_\alpha$.

Proof. We use Lemma 2.9 to write for any index subset I of cardinality α:

$$\Gamma_I\, \rho(\gamma) = (\prod_{i\in I} \Gamma_i)\, \rho(\gamma) = q^{\frac{\alpha}{2}} x_I\, \rho(\gamma)\, \Gamma_I$$

Starting from the expression of Lemma 2.16, we may now conjugate the formula of Theorem 1.4 for $\rho(\widetilde{\mathcal{D}}_\alpha) = \rho(\mathcal{D}_{\alpha;0})$ with $\rho(\gamma)^n$ as follows:

$$\begin{aligned} \rho(\mathcal{D}_{\alpha;n}) = q^{-\frac{\alpha n}{2}} \rho(\gamma)^{-n} \rho(\widetilde{\mathcal{D}}_\alpha) \rho(\gamma)^n &= q^{-\frac{\alpha n}{2}} \sum_{|I|=\alpha,\, I\subset[1,N]} \prod_{\substack{i\in I\\ j\notin I}} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j}\, \rho(\gamma)^{-n} \Gamma_I \rho(\gamma)^n \\ &= \sum_{|I|=\alpha,\, I\subset[1,N]} (x_I)^n \prod_{\substack{i\in I\\ j\notin I}} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j}\, \Gamma_I \end{aligned}$$

where we have used Lemma 2.9. The Theorem follows. □

3. Interpretation via quantum toroidal algebra

3.1. **Definitions and results.** In this section, we show that the generalized Macdonald operators $\{D_{1,n} : n \in \mathbb{Z}\}$ satisfy the relations of the quantum toroidal algebra [Mik07, FJMM12] at level 0. We call the generating functions of these generators the fundamental currents, and in Section 6 (Theorem 6.3) we show, by use of the elliptic Hall algebra, that the generators $D_{\alpha,n}$ are polynomials in the fundamental generators $\{D_{1,m}\}$.

3.1.1. *Level* 0 *quantum toroidal* $\widehat{\mathfrak{gl}}_1$. For generic parameters $q, t \in \mathbb{C}^\star$ let us introduce the function:

$$g(z,w) := (z-qw)(z-t^{-1}w)(z-q^{-1}tw) \tag{3.1}$$

We will use the formal delta function

$$\delta(u) = \sum_{n\in\mathbb{Z}} u^n \tag{3.2}$$

with the property that $\delta(u)f(u) = \delta(u)f(1)$ for any function f which is non-singular at $u = 1$.

The quantum toroidal algebra of $\widehat{\mathfrak{gl}}_1$ is defined by generators and relations. The generators are $\{e_n, f_n, \psi^\pm_m : n \in \mathbb{Z}, m \in \mathbb{Z}_+\}$, and the relations are most easily expressed in terms of generating functions:

$$\mathfrak{e}(z) = \sum_{n\in\mathbb{Z}} z^n\, e_n\ , \quad \mathfrak{f}(z) = \sum_{n\in\mathbb{Z}} z^n\, f_n\ , \quad \psi^\pm(z) = \sum_{n\geq 0} z^{\pm n}\, \psi^\pm_n,$$

where z is a formal variable. In general, the algebra has a central charge c explicitly appearing in the defining relations. In the functional representation which appears in our context, $c = 0$.

Definition 3.1. *The* $c = 0$ *quantum toroidal* $\widehat{\mathfrak{gl}}_1$ *is the algebra generated by* $\{e_n, f_n, \psi^\pm_m : n \in \mathbb{Z}\, m \in \mathbb{Z}_+\}$ *with relations:*

$$\begin{aligned}
&g(z,w)\mathfrak{e}(z)\mathfrak{e}(w) + g(w,z)\mathfrak{e}(w)\mathfrak{e}(z) = 0,\\
&g(w,z)\mathfrak{f}(z)\mathfrak{f}(w) + g(z,w)\mathfrak{f}(w)\mathfrak{f}(z) = 0,\\
&[\mathfrak{e}(z), \mathfrak{f}(w)] = \frac{\delta(z/w)}{g(1,1)}\,(\psi^+(z) - \psi^-(z)),\\
&g(z,w)\psi^\pm(z)\,\mathfrak{e}(w) + g(w,z)\mathfrak{e}(w)\,\psi^\pm(z) = 0\ ,\\
&g(w,z)\psi^\pm(z)\,\mathfrak{f}(w) + g(z,w)\mathfrak{f}(w)\,\psi^\pm(z) = 0,\\
&\mathrm{Sym}_{z_1,z_2,z_3}\left(\frac{z_2}{z_3}\Big[\mathfrak{e}(z_1), [\mathfrak{e}(z_2), \mathfrak{e}(z_3)]\Big]\right) = 0,\\
&\mathrm{Sym}_{z_1,z_2,z_3}\left(\frac{z_2}{z_3}\Big[\mathfrak{f}(z_1), [\mathfrak{f}(z_2), \mathfrak{f}(z_3)]\Big]\right) = 0,
\end{aligned}$$

with $\psi^\pm_0$ invertible *and* $\psi^\pm_n$ *mutually commuting for* $n \in \mathbb{Z}_+$.

Note that when $t \to \infty$, we may set $g_0(z,w) = \lim_{t\to\infty} t^{-1} g(z,w) = z - qw$, and the first relations become relations in the quantum affine algebra of $\mathfrak{sl}_2$ in the Drinfeld presentation (with a non-standard deformation parameter $\sqrt{q}$, as in the Hall algbra of [Kap97]). The last two identities are Serre-type relations and distinguish the quantum toroidal algebra from the original Ding-Iohara algebra [DI97].

3.1.2. *Macdonald currents.* We claim that the generalized Macdonald operators introduced in Equation (2.12) are elements of a functional representation of a quotient of the quantum toroidal algebra. In order to make the comparison explicit, we first define generating currents for the generalized Macdonald operators (2.12) for $\alpha = 1, 2, ..., N$:

$$\mathfrak{e}_\alpha(z) := \frac{q^{\frac{\alpha}{2}}}{(1-q)^\alpha} \sum_{n\in\mathbb{Z}} q^{n\alpha/2}\, z^n\, \mathcal{D}^{q,t}_{\alpha;n}, \qquad \mathfrak{f}_\alpha(z) := \frac{q^{-\frac{\alpha}{2}}}{(1-q^{-1})^\alpha} \sum_{n\in\mathbb{Z}} q^{-n\alpha/2}\, z^n\, \mathcal{D}^{q^{-1},t^{-1}}_{\alpha;n}. \tag{3.3}$$

where we have indicated the q, t dependence as superscripts, so that:

$$\mathcal{D}^{q^{-1},t^{-1}}_{\alpha;n} = \sum_{|I|=\alpha,\, I\subset[1,N]} (x_I)^n \prod_{\substack{i\in I\\ j\notin I}} \frac{\theta^{-1}x_i - \theta x_j}{x_i - x_j}\, \Gamma_I^{-1} \tag{3.4}$$

and Γ_i^{-1} acts on functions of $x_1, ..., x_N$ by multiplying the i-th variable x_i by q^{-1}.

Remark 3.2. *Note also that if S denotes the involution acting on functions of $(x_1, ..., x_N)$ by sending $x_i \mapsto x_i^{-1}$ for all i, then we have $S\Gamma_I S = \Gamma_I^{-1}$ and $\mathcal{D}^{q^{-1},t^{-1}}_{\alpha;-n} = S\mathcal{D}^{q,t}_{\alpha;n}S$, so that*

$$S\, \mathfrak{e}_\alpha(z)\, S = (-1)^\alpha\, \mathfrak{f}_\alpha(z^{-1}) \ .$$

The currents (3.3) can be explicitly expressed as

$$\begin{aligned}
\mathfrak{e}_\alpha(z) &= \frac{q^{\frac{\alpha}{2}}}{(1-q)^\alpha} \sum_{|I|=\alpha,\, I\subset[1,N]} \delta(q^{\alpha/2} z x_I) \prod_{\substack{i\in I\\ j\notin I}} \frac{\theta x_i - \theta^{-1}x_j}{x_i - x_j}\, \Gamma_I, \\
\mathfrak{f}_\alpha(z) &= \frac{q^{-\frac{\alpha}{2}}}{(1-q^{-1})^\alpha} \sum_{|I|=\alpha,\, I\subset[1,N]} \delta(q^{-\alpha/2} z x_I) \prod_{\substack{i\in I\\ j\notin I}} \frac{\theta^{-1}x_i - \theta x_j}{x_i - x_j}\, \Gamma_I^{-1},
\end{aligned}$$

by use of the formal δ function (3.2).

Note that the finite number N of variables implies the vanishing relations:

$$\mathfrak{e}_{N+1}(z) = 0 \quad \text{and} \quad \mathfrak{f}_{N+1}(z) = 0. \tag{3.5}$$

3.1.3. *Main result.* We call $\mathfrak{e}(z) := \mathfrak{e}_1(z)$ and $\mathfrak{f}(z) := \mathfrak{f}_1(z)$ the fundamental currents. Our main result is that these satisfy relations in the quantum toroidal algebra, as suggested by the notation. We will see later that the vanishing condition (3.5) implies a certain quotient of the algebra by polynomials of degree $N+1$ in the fundamental generators.

Theorem 3.3. *The Macdonald currents $\mathfrak{e}(z) := \mathfrak{e}_1(z)$ and $\mathfrak{f}(z) := \mathfrak{f}_1(z)$ (3.3), together with the series:*

$$\psi^\pm(z) := \prod_{i=1}^N \frac{(1-q^{-\frac{1}{2}}t(zx_i)^{\pm1})(1-q^{\frac{1}{2}}t^{-1}(zx_i)^{\pm1})}{(1-q^{-\frac{1}{2}}(zx_i)^{\pm1})(1-q^{\frac{1}{2}}(zx_i)^{\pm1})} \in \mathbb{C}[[z^{\pm1}]] \tag{3.6}$$

satisfy the $c=0$ quantum toroidal $\widehat{\mathfrak{gl}}_1$ algebra relations of Def. 3.1.

We will prove this theorem in several steps in the following sections.

Note that $\psi_n^{\pm}$ are explicit symmetric polynomials of $x_1, ..., x_N$, with $\psi_0^+ = \psi_0^- = 1$, and in particular

$$\begin{aligned} \psi_1^+ &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(q^{\frac{1}{2}} t^{-\frac{1}{2}} - q^{-\frac{1}{2}} t^{\frac{1}{2}})(x_1 + x_2 + \cdots + x_N) \\ \psi_1^- &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(q^{\frac{1}{2}} t^{-\frac{1}{2}} - q^{-\frac{1}{2}} t^{\frac{1}{2}})(x_1^{-1} + x_2^{-1} + \cdots + x_N^{-1}) \end{aligned} \tag{3.7}$$

The finite number N of variables clearly imposes algebraic relations between the coefficients of the series $\psi^{\pm}$. The Macdonald currents generate a quotient of the quantum toroidal algebra by the relations (3.5) and those relations.

3.1.4. *The dual Whittaker limit $t \to \infty$.* Before proceeding to the proof of Theorem 3.3, we make the connection with our previous work in [DFK16] regarding the quantum Q-system generators. In that paper, we showed that the A_{N-1} quantum Q-system algebra can be realized as a quotient, depending on N, of the algebra of nilpotent currents in the quantum enveloping algebra of $\widehat{\mathfrak{sl}}_2$, $U_{\sqrt{q}}(\mathfrak{n}[u, u^{-1}])$. Furthermore, we were able to define currents $\mathfrak{e}$, $\mathfrak{f}$ in terms of fundamental quantum Q-system solutions which satisfied non-standard $U_{\sqrt{q}}(\widehat{\mathfrak{sl}}_2)$ with truncated Cartan currents. The functional representation of these was constructed in terms of the degenerate generalized Macdonald operators, corresponding to the so-called dual Whittaker ($t \to \infty$) limit of the t-deformation considered in this paper.

These observations become more transparent when we take the $t \to \infty$ limit of Theorem 3.3. Note that it is crucial that N be finite in order for this limit to make sense. Defining the limits of the renormalized currents:

$$\mathfrak{e}^{(\infty)}(u) := \lim_{t\to\infty} t^{-\frac{N-1}{2}}\, \mathfrak{e}(u), \quad \mathfrak{f}^{(\infty)}(u) := \lim_{t\to\infty} t^{-\frac{N-1}{2}}\, \mathfrak{f}(u), \quad \psi^{\pm(\infty)}(z) := \lim_{t\to\infty} t^{-N} \psi^{\pm}(z), \tag{3.8}$$

and $g^{(\infty)}(z, w) := \lim_{t\to\infty} \frac{-qt^{-1}}{zw}\, g(z, w) = z - qw$, we find that the limiting Cartan currents become:

$$\psi^{\pm(\infty)}(z) = (-q^{-\frac{1}{2}} z^{\pm 1})^N \prod_{i=1}^{N} \frac{(x_i)^{\pm 1}}{(1 - q^{-\frac{1}{2}}(zx_i)^{\pm 1})(1 - q^{\frac{1}{2}}(zx_i)^{\pm 1})}, \tag{3.9}$$

whereas the other relations reduce to those of $U_{\sqrt{q}}(\widehat{\mathfrak{sl}}_2)$ (the Serre relation is automatically satisfied). Note that the Cartan currents have a non-standard valuation of $z^{\pm N}$ in this limit, and vanish when $N \to \infty$, whereas in the Drinfeld presentation of the quantum affine algebra, the zero modes of the Cartan currents are invertible elements.

3.2. **Proof of Theorem 3.3.** For readability, we decompose the proof of the Theorem into several steps, Theorems 3.4, 3.5, 3.6 and 3.7 below.

Theorem 3.4. *We have the following relations:*

$$\begin{aligned} g(z,w)\mathfrak{e}(z)\mathfrak{e}(w)+g(w,z)\mathfrak{e}(w)\mathfrak{e}(z) &= 0\\ g(w,z)\mathfrak{f}(z)\mathfrak{f}(w)+g(z,w)\mathfrak{f}(w)\mathfrak{f}(z) &= 0\end{aligned}$$

Proof. We start with the computation of

$$\begin{aligned} \frac{(1-q)^2}{q}\,\mathfrak{e}(z)\mathfrak{e}(w) &= \sum_{i=1}^{r+1}\delta(q^{\frac12}zx_i)\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\Gamma_i\sum_{j=1}^{r+1}\delta(q^{\frac12}wx_j)\prod_{k\neq j}\frac{\theta x_j-\theta^{-1}x_k}{x_j-x_k}\Gamma_j\\ &= \sum_{1\leq i\neq j\leq r+1}\delta(q^{\frac12}zx_i)\delta(q^{\frac12}wx_j)\frac{\theta x_i-\theta^{-1}x_j}{x_i-x_j}\frac{\theta x_j-q\theta^{-1}x_i}{x_j-qx_i}\\ &\qquad\times\prod_{k\neq i,j}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\frac{\theta x_j-\theta^{-1}x_k}{x_j-x_k}\Gamma_i\Gamma_j\\ &\quad+\sum_{i=1}^{r+1}\delta(q^{\frac12}zx_i)\delta(q^{\frac32}wx_i)\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\frac{q\theta x_i-\theta^{-1}x_k}{qx_i-x_k}\Gamma_i^2.\end{aligned}$$

The second term is proportional to $\delta(z/(qw))$, and since $(z-qw)\delta(z/(qw))=0$, we have:

$$\begin{aligned} \frac{(1-q)^2}{q}\,(z-qw)\mathfrak{e}(z)\mathfrak{e}(w) &= \sum_{1\leq i\neq j\leq r+1}\delta(q^{\frac12}zx_i)\delta(q^{\frac12}wx_j)\times\\ &\times\frac{(z-qw)(\theta x_i-\theta^{-1}x_j)(\theta x_j-q\theta^{-1}x_i)}{(x_i-x_j)(x_j-qx_i)}\prod_{k\neq i,j}\frac{(\theta x_i-\theta^{-1}x_k)(\theta x_j-\theta^{-1}x_k)}{(x_i-x_k)(x_j-x_k)}\Gamma_i\Gamma_j\\ &= \frac{(z-tw)(z-qt^{-1}w)}{z-w}\sum_{1\leq i<j\leq r+1}(\delta(q^{\frac12}zx_i)\delta(q^{\frac12}wx_j)+\delta(q^{\frac12}zx_j)\delta(q^{\frac12}wx_i))\times\\ &\times\prod_{k\neq i,j}\frac{(\theta x_i-\theta^{-1}x_k)(\theta x_j-\theta^{-1}x_k)}{(x_i-x_k)(x_j-x_k)}\Gamma_i\Gamma_j \qquad (3.10)\end{aligned}$$

This makes the quantity $(z-qw)(z-t^{-1}w)(z-tq^{-1}w)\mathfrak{e}(z)\mathfrak{e}(w)$ manifestly skew-symmetric under the interchange $z\leftrightarrow w$. The relation for $\mathfrak{f}$ follows by taking $(q,t)\to(q^{-1},t^{-1})$, under which $g(z,w)\to g(w,z)$. □

This shows that the fundamental generalized Macdonald currents satisfy the first two relations of the quantum toroidal algebra. We now consider the third relation.

Theorem 3.5. *We have the commutation relation between Macdonald currents:*

$$[\mathfrak{e}(z),\mathfrak{f}(w)]=\frac{\delta(z/w)}{g(1,1)}\,(\psi^+(z)-\psi^-(z))$$

where $\psi^{\pm}(z)$ *are the power series* (3.6) *of* $z^{\pm 1}$.

Proof. Let us first compute

$$\begin{aligned}
(1-q)(1-q^{-1})\mathfrak{e}(z)\mathfrak{f}(w) &= \sum_{i=1}^{r+1}\delta(q^{\frac{1}{2}}zx_i)\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\Gamma_i\sum_{j=1}^{r+1}\delta(q^{-\frac{1}{2}}wx_j)\prod_{k\neq j}\frac{\theta^{-1}x_j-\theta x_k}{x_j-x_k}\Gamma_j^{-1}\\
&= \sum_{1\leq i\neq j\leq r+1}\delta(q^{\frac{1}{2}}zx_i)\delta(q^{-\frac{1}{2}}wx_j)\frac{\theta x_i-\theta^{-1}x_j}{x_i-x_j}\frac{\theta^{-1}x_j-q\theta x_i}{x_j-qx_i}\\
&\qquad\times\prod_{k\neq i,j}\frac{(\theta x_i-\theta^{-1}x_k)(\theta^{-1}x_j-\theta x_k)}{(x_i-x_k)(x_j-x_k)}\Gamma_i\Gamma_j^{-1}\\
&\qquad+\sum_{i=1}^{r+1}\delta(q^{\frac{1}{2}}zx_i)\delta(q^{\frac{1}{2}}wx_i)\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\frac{\theta^{-1}qx_i-\theta x_k}{qx_i-x_k},
\end{aligned}$$

and

$$\begin{aligned}
(1-q)(1-q^{-1})\mathfrak{f}(w)\mathfrak{e}(z) &= \sum_{j=1}^{r+1}\delta(q^{-\frac{1}{2}}wx_j)\prod_{k\neq j}\frac{\theta^{-1}x_j-\theta x_k}{x_j-x_k}\Gamma_j^{-1}\sum_{i=1}^{r+1}\delta(q^{\frac{1}{2}}zx_i)\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\Gamma_i\\
&= \sum_{1\leq i\neq j\leq r+1}\delta(q^{\frac{1}{2}}zx_i)\delta(q^{-\frac{1}{2}}wx_j)\frac{\theta^{-1}x_j-\theta x_i}{x_j-x_i}\frac{\theta x_i-q^{-1}\theta^{-1}x_j}{x_i-q^{-1}x_j}\\
&\qquad\times\prod_{k\neq i,j}\frac{(\theta x_i-\theta^{-1}x_k)(\theta^{-1}x_j-\theta x_k)}{(x_i-x_k)(x_j-x_k)}\Gamma_i\Gamma_j^{-1}\\
&\qquad+\sum_{i=1}^{r+1}\delta(q^{-\frac{1}{2}}zx_i)\delta(q^{-\frac{1}{2}}wx_i)\prod_{k\neq i}\frac{q^{-1}\theta x_i-\theta^{-1}x_k}{q^{-1}x_i-x_k}\frac{\theta^{-1}x_i-\theta x_k}{x_i-x_k}.
\end{aligned}$$

In the commutator $[\mathfrak{e}(z),\mathfrak{f}(w)]$, the terms corresponding to $i\neq j$ cancel out, while the remaining terms are proportional to $\delta(z/w)$, so that we are left with:

$$\begin{aligned}
[\mathfrak{e}(z),\mathfrak{f}(w)] &= \frac{\delta(z/w)}{(1-q)(1-q^{-1})}\sum_{i=1}^{r+1}\Bigg\{\delta(q^{\frac{1}{2}}zx_i)\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\frac{\theta^{-1}qx_i-\theta x_k}{qx_i-x_k}\\
&\qquad-\delta(q^{-\frac{1}{2}}zx_i)\prod_{k\neq i}\frac{q^{-1}\theta x_i-\theta^{-1}x_k}{q^{-1}x_i-x_k}\frac{\theta^{-1}x_i-\theta x_k}{x_i-x_k}\Bigg\}.
\end{aligned}$$

Recalling that $\delta(z)=\sum_{n\in\mathbb{Z}} z^n$, this is in agreement with the partial fraction decomposition of (3.6), which reads:

$$\psi^+(z)=1+\frac{g(1,1)}{(1-q)(1-q^{-1})}\sum_{i=1}^{r+1}\left\{\frac{1}{1-q^{\frac12}zx_i}\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\frac{\theta^{-1}qx_i-\theta x_k}{qx_i-x_k}\right.$$
$$\left.-\frac{1}{1-q^{-\frac12}zx_i}\prod_{k\neq i}\frac{q^{-1}\theta x_i-\theta^{-1}x_k}{q^{-1}x_i-x_k}\frac{\theta^{-1}x_i-\theta x_k}{x_i-x_k}\right\}, \tag{3.11}$$

$$\psi^-(z)=1-\frac{g(1,1)}{(1-q)(1-q^{-1})}\sum_{i=1}^{r+1}\left\{\frac{1}{1-q^{-\frac12}z^{-1}x_i^{-1}}\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\frac{\theta^{-1}qx_i-\theta x_k}{qx_i-x_k}\right.$$
$$\left.-\frac{1}{1-q^{\frac12}z^{-1}x_i^{-1}}\prod_{k\neq i}\frac{q^{-1}\theta x_i-\theta^{-1}x_k}{q^{-1}x_i-x_k}\frac{\theta^{-1}x_i-\theta x_k}{x_i-x_k}\right\}, \tag{3.12}$$

as power series of z and z^{-1}, respectively. Here, we have identified the constant $g(1,1)/((1-q)(1-q^{-1}))=(1-t^{-1})(1-tq^{-1})/(1-q^{-1})$. □

We proceed with the next two relations of the quantum toroidal algebra.

Theorem 3.6. *We have the following relations between the Macdonald currents and the series* (3.6)*:*

$$\begin{aligned} g(z,w)\psi^{\pm}(z)\,\mathfrak{e}(w)+g(w,z)\mathfrak{e}(w)\,\psi^{\pm}(z) &= 0\\ g(w,z)\psi^{\pm}(z)\,\mathfrak{f}(w)+g(z,w)\mathfrak{f}(w)\,\psi^{\pm}(z) &= 0\end{aligned}$$

Proof. By explicit calculation,

$$\begin{aligned}\frac{1-q}{q^{\frac12}}\psi^+(z)\,\mathfrak{e}(w) &= \prod_{j=1}^{r+1}\frac{(1-q^{-\frac12}\theta^2zx_j)(1-q^{\frac12}\theta^{-2}zx_j)}{(1-q^{-\frac12}zx_j)(1-q^{\frac12}zx_j)}\sum_{i=1}^{r+1}\delta(q^{\frac12}wx_i)\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\Gamma_i\\ &= \sum_{i=1}^{r+1}\delta(q^{\frac12}wx_i)\frac{(w-q^{-1}tz)(w-t^{-1}z)}{(w-q^{-1}z)(w-z)}\\ &\qquad\times\prod_{k\neq i}\frac{(1-q^{-\frac12}\theta^2zx_k)(1-q^{\frac12}\theta^{-2}zx_k)}{(1-q^{-\frac12}zx_k)(1-q^{\frac12}zx_k)}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\Gamma_i,\end{aligned}$$

$$\begin{aligned}\frac{1-q}{q^{\frac12}}\mathfrak{e}(w)\,\psi^+(z) &= \sum_{i=1}^{r+1}\delta(q^{\frac12}wx_i)\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\Gamma_i\prod_{j=1}^{r+1}\frac{(1-q^{-\frac12}\theta^2 zx_j)(1-q^{\frac12}\theta^{-2}zx_j)}{(1-q^{-\frac12}zx_j)(1-q^{\frac12}zx_j)}\\ &= \sum_{i=1}^{r+1}\delta(q^{\frac12}wx_i)\frac{(w-tz)(w-qt^{-1}z)}{(w-z)(w-qz)}\\ &\qquad\times\prod_{k\neq i}\frac{\theta x_i-\theta^{-1}x_k}{x_i-x_k}\frac{(1-q^{-\frac12}\theta^2 zx_k)(1-q^{\frac12}\theta^{-2}zx_k)}{(1-q^{-\frac12}zx_k)(1-q^{\frac12}zx_k)}\Gamma_i.\end{aligned}$$

Using

$$g(z,w)\frac{(w-q^{-1}tz)(w-t^{-1}z)}{(w-q^{-1}z)(w-z)}=-g(w,z)\frac{(w-tz)(w-qt^{-1}z)}{(w-z)(w-qz)}$$

we see that $g(z,w)\psi^+(z)\,\mathfrak{e}(w)+g(w,z)\mathfrak{e}(w)\,\psi^+(z)=0$. The derivation of the ψ^- equation follows a similar calculation. □

The final two relations are the cubic, Serre-type relations for the fundamental currents.

Theorem 3.7. *The fundamental currents satisfy the cubic relations*

$$\mathrm{Sym}_{z_1,z_2,z_3}\left(\frac{z_2}{z_3}\Big[\mathfrak{e}(z_1),[\mathfrak{e}(z_2),\mathfrak{e}(z_3)]\Big]\right) = 0, \tag{3.13}$$

$$\mathrm{Sym}_{z_1,z_2,z_3}\left(\frac{z_2}{z_3}\Big[\mathfrak{f}(z_1),[\mathfrak{f}(z_2),\mathfrak{f}(z_3)]\Big]\right) = 0. \tag{3.14}$$

Proof. It is sufficient to prove the statement of the theorem for $\mathfrak{e}$, eq. (3.13), as that for $\mathfrak{f}$ follows by the substitution $(q,t)\to(q^{-1},t^{-1})$. The proof of (3.13) is straightforward but extremely tedious. We prefer to postpone it to Section 5.3, where it reduces to a suffle product identity (Theorem 5.18) which is considerably easier to prove. □

We conclude that the currents $\mathfrak{e}(z)$, $\mathfrak{f}(z)$, $\psi^\pm(z)$ satisfy all the relations of the $c=0$ quantum toroidal algebra $\mathfrak{g}l_1$, and Theorem 3.3 follows.

4. Plethysms and bosonization

In this section, we show that the action of the currents $\mathfrak{e}(z),\mathfrak{f}(z)$ on the space of symmetric functions in N variables can be expressed in terms of plethysms, which act naturally on functions expressed in terms of the power sum symmetric functions. In the infinite rank limit $N\to\infty$, the power sum symmetric functions become algebraically independent, and together with derivatives with respect to these functions, they generate an infinite-dimensional Heisenberg algebra. Thus the plethystic expression becomes, in this limit, a formula for the bosonization of the action of the currents on the space of symmetric functions at level zero. This formulation is to be compared to recent work of Bergeron et al [BGLX14].

4.1. **Power sum action and plethysms.** Let $\{p_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ be an infinite set of algebraically independent variables, and consider the space $\mathcal{P}$ of functions that can be expressed as formal power series of the p_k's. For any collection X of variables, e.g. $X=(x_1,x_2,...,x_N)$, we may evaluate functions in $\mathcal{P}$ by using the substitution $p_k\mapsto p_k[X]:=\sum_{i=1}^N x_i^k$, namely by interpreting the p_k's as power sums of the x's, for $k\in\mathbb{Z}^*$, in which case any $F\in\mathcal{P}$ may be evaluated as $F[X]$ by taking all $p_k=p_k[X]$ in its power series. If the number of variables N is finite, the resulting space is a quotient of the original space as the functions $p_k[X]$s are not algebraically independent. Note that $F[X]$ is a symmetric function.

Given the collection X, we consider the action of the generalized Macdonald operators $\mathcal{D}_{1;n}^{q,t}$ and $\mathcal{D}_{1;n}^{q^{-1},t^{-1}}$ on the space of functions of the form $F[X]$, for $F\in\mathcal{P}$, which we denote by $\mathcal{P}[X]$. First, we present some commutation relations which hold for arbitrary N.

Lemma 4.1. *For any N and $i\leq N$, and given the collection $X=(x_1,x_2,...,x_N)$, the following commutation relations hold:*

$$[p_k[X],\mathcal{D}_{1;n}^{q,t}]=(1-q^k)\,\mathcal{D}_{1;n+k}^{q,t},\quad\text{and}\quad [p_k[X],\mathcal{D}_{1;n}^{q^{-1},t^{-1}}]=(1-q^{-k})\,\mathcal{D}_{1;n+k}^{q^{-1},t^{-1}}.$$

Proof. By direct computation, using $[p_k[X],\Gamma_i]=(1-q^k)\,x_i^k\,\Gamma_i$, with Γ_i as in (1.2). ☐

Corollary 4.2. *For any N, $k\in\mathbb{Z}^*$, and X as above,*

$$\begin{aligned}\mathfrak{e}(z)\,p_k[X] &= \left(p_k[X]+\frac{q^{k/2}-q^{-k/2}}{z^k}\right)\mathfrak{e}(z),\\ \mathfrak{f}(z)\,p_k[X] &= \left(p_k[X]-\frac{q^{k/2}-q^{-k/2}}{z^k}\right)\mathfrak{f}(z).\end{aligned}$$

Therefore, up to a scalar multiple coming from the action of the currents on the constant function 1, the action of the currents $\mathfrak{e}$, $\mathfrak{f}$ on $\mathcal{P}[X]$ is by substitutions on all the power sums of the form $p_k[X]\mapsto p_k[X]+\mu_k$ for some specific sequences μ_k.

Such substitutions are *plethysms* (see the notes [Hai99] for a detailed exposition). In λ-ring notations, one writes as above $X=(x_1,x_2,...)$ for the collection of variables (an alphabet), for which power sums are $p_k[X]=\sum_i x_i^k$. For two alphabets X, Y, the sum $X+Y$ refers to the concatenation of the two alphabets, with power sums $p_k[X+Y]=p_k[X]+p_k[Y]$, while for any scalar λ we have $p_k[\lambda X]=\lambda^k p_k[X]$. Note finally that for a single variable alphabet μ, we have $p_k[X+\mu]=p_k[X]+\mu^k$.

In the plethystic notation, Corollary 4.2 can be written as

$$\mathfrak{e}(z)\,F[X]=F\left[X+\frac{q^{1/2}-q^{-1/2}}{z}\right]\mathfrak{e}(z),\qquad \mathfrak{f}(z)\,F[X]=F\left[X-\frac{q^{1/2}-q^{-1/2}}{z}\right]\mathfrak{f}(z)\tag{4.1}$$

for any $F\in\mathcal{P}$, where the alphabet X is extended by two variables.

Define the two functions in $\mathcal{P}[X]$

$$C(z) = \prod_{i=1}^{N}(1 - z\, x_i) = e^{-\sum_{k=1}^{\infty} p_k[X]\frac{z^k}{k}}, \tag{4.2}$$

$$\widetilde{C}(z) = \prod_{i=1}^{N}(1 - z\, x_i^{-1}) = e^{-\sum_{k=1}^{\infty} p_{-k}[X]\frac{z^k}{k}}. \tag{4.3}$$

In terms of these functions we can write

Theorem 4.3. *The currents $\mathfrak{e}$ and $\mathfrak{f}$ act as the following plethysms on symmetric functions $F[X] \in \mathcal{P}[X]$:*

$$\mathfrak{e}(z)\, F[X] = \frac{q^{1/2}}{1-q}\frac{t^{-1/2}}{1-t^{-1}}\left\{t^{\frac{N}{2}}\frac{C(q^{1/2}t^{-1}z)}{C(q^{1/2}z)} - t^{-\frac{N}{2}}\frac{\widetilde{C}(q^{-1/2}tz^{-1})}{\widetilde{C}(q^{-1/2}z^{-1})}\right\} F\left[X + \frac{q^{1/2}-q^{-1/2}}{z}\right],$$

$$\mathfrak{f}(z)\, F[X] = \frac{q^{-1/2}}{1-q^{-1}}\frac{t^{1/2}}{1-t}\left\{t^{-\frac{N}{2}}\frac{C(q^{-1/2}tz)}{C(q^{-1/2}z)} - t^{\frac{N}{2}}\frac{\widetilde{C}(q^{1/2}t^{-1}z^{-1})}{\widetilde{C}(q^{1/2}z^{-1})}\right\} F\left[X - \frac{q^{1/2}-q^{-1/2}}{z}\right].$$

Proof. The plethystic part of the action was derived in (4.1). To fix the overall factors, apply the currents to the constant $F[X] = 1$. For example,

$$\begin{aligned}
\mathfrak{e}(z)\cdot 1 &= \frac{q^{1/2}}{1-q}\sum_{i=1}^{N}\delta(q^{1/2}zx_i)\prod_{j\neq i}\frac{\theta x_i - \theta^{-1}x_j}{x_i - x_j}\\
&= \frac{q^{1/2}}{1-q}\sum_{i=1}^{N}\left\{t^{\frac{N-1}{2}}\frac{1}{1-q^{1/2}zx_i}\prod_{j\neq i}\frac{x_i - t^{-1}x_j}{x_i - x_j} + t^{-\frac{N-1}{2}}\frac{1}{1-q^{-1/2}z^{-1}x_i^{-1}}\prod_{j\neq i}\frac{tx_i - x_j}{x_i - x_j}\right.\\
&\qquad\qquad\left. - \prod_{j\neq i}\frac{\theta x_i - \theta^{-1}x_j}{x_i - x_j}\right\}.
\end{aligned}$$

First, we have

$$\sum_{i=1}^{N}\prod_{j\neq i}\frac{\theta x_i - \theta^{-1}x_j}{x_i - x_j} = \frac{t^{\frac{N-1}{2}} - t^{-\frac{N+1}{2}}}{1-t^{-1}} = -\left\{\frac{t^{-\frac{N+1}{2}}}{1-t^{-1}} + \frac{t^{\frac{N+1}{2}}}{1-t}\right\}, \tag{4.4}$$

by noting that the left hand side of this equation is a symmetric rational function, whose common denominator is the Vandermonde determinant $\prod_{i<j} x_i - x_j$, and which has total degree 0, hence must be a constant c_N. The constant can be found, for example, by taking the limit $x_1 \to \infty$:

$$c_N = \theta^{N-1} + \theta^{-1}\sum_{i=2}^{N}\prod_{\substack{j\neq i\\ j>1}}\frac{\theta x_i - \theta^{-1}x_j}{x_i - x_j} = \theta^{N-1} + \theta^{-1}c_{N-1},$$

which, together with $c_0=0$, yields $c_N=\frac{\theta^N-\theta^{-N}}{\theta-\theta^{-1}}$, and (4.4) follows.

Next, by simple fraction decomposition,

$$\frac{C(q^{1/2}t^{-1}z)}{C(q^{1/2}z)}=\prod_{i=1}^N\frac{1-q^{1/2}t^{-1}zx_i}{1-q^{1/2}zx_i}=t^{-N}+(1-t^{-1})\sum_{i=1}^N\frac{1}{1-q^{1/2}zx_i}\prod_{j\neq i}\frac{x_i-t^{-1}x_j}{x_i-x_j}.$$

Therefore

$$\frac{t^{\frac{N-1}{2}}}{1-t^{-1}}\frac{C(q^{1/2}t^{-1}z)}{C(q^{1/2}z)}=\frac{t^{-\frac{N+1}{2}}}{1-t^{-1}}+t^{\frac{N-1}{2}}\sum_{i=1}^N\frac{1}{1-q^{1/2}zx_i}\prod_{j\neq i}\frac{x_i-t^{-1}x_j}{x_i-x_j}.$$

By using the change of variables $(q,t,z,x_i)\to(q^{-1},t^{-1},z^{-1},x_i^{-1})$ we get:

$$\frac{t^{-\frac{N-1}{2}}}{1-t}\frac{\widetilde{C}(q^{-1/2}tz^{-1})}{\widetilde{C}(q^{-1/2}z^{-1})}=\frac{t^{\frac{N+1}{2}}}{1-t}+t^{-\frac{N-1}{2}}\sum_{i=1}^N\frac{1}{1-q^{-1/2}z^{-1}x_i^{-1}}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}.$$

Summing the two above contributions yields the full action of $\mathfrak{e}(z)$ on 1. That of $\mathfrak{f}(z)$ follows immediately by sending $(q,t)\to(q^{-1},t^{-1})$. $\square$

Moreover, the Cartan currents act as scalars on symmetric functions of $x_1,x_2,...,x_N$:

$$\psi^+(z)=\frac{C(q^{-1/2}tz)C(q^{1/2}t^{-1}z)}{C(q^{-1/2}z)C(q^{1/2}z)},\quad \psi^-(z)=\frac{\widetilde{C}(q^{-1/2}tz^{-1})\widetilde{C}(q^{1/2}t^{-1}z^{-1})}{\widetilde{C}(q^{-1/2}z^{-1})\widetilde{C}(q^{1/2}z^{-1})}\ .$$

Remark 4.4. *The plethystic formulas above for $\mathfrak{e}$ and $-\mathfrak{f}$, as well as ψ^+ and ψ^-, are exchanged under the involution $x_i\mapsto 1/x_i$ for all i, and $z\mapsto z^{-1}$, under which $C(z)\mapsto\widetilde{C}(z^{-1})$, and the one-variable plethysm $[X+\mu]\mapsto[X+\mu^{-1}]$. This is in agreement with Remark 3.2 for $\alpha=1$.*

4.2. **Bosonization formulas in the $N\to\infty$ limit.** As mentioned at the beginning of the section, in the infinite rank $N\to\infty$ limit the power sum symmetric functions p_k become algebraically independent for all $k\in\mathbb{Z}\setminus\{0\}$, and together with the derivatives with respect to these functions, form a Heisenberg algebra.

4.2.1. *Plethystic formulas for the $N\to\infty$ limit.* The limit of an infinite number of variables is obtained by taking $N\to\infty$ (we assume $|t|>1$), whereas the collection $X=(x_1,x_2,...)$ becomes infinite. Accordingly, we define the limiting functions:

$$C_\infty(z):=\prod_{i=1}^\infty(1-zx_i),\qquad \widetilde{C}_\infty(z):=\prod_{i=1}^\infty(1-zx_i^{-1}).$$

In view of the plethystic formulas of Theorem 4.3, let us define: $\mathfrak{e}_\infty(z):=\lim_{N\to\infty}t^{\frac{1-N}{2}}\,\mathfrak{e}(z)$, and $\mathfrak{f}_\infty(z):=\lim_{N\to\infty}t^{\frac{1-N}{2}}\,\mathfrak{f}(z)$. By taking the $N\to\infty$ limit of the result of Theorem 4.3, we obtain the following:

Theorem 4.5. *This limiting currents $\mathfrak{e}_\infty(z), \mathfrak{f}_\infty(z)$ act on functions $F[X]$ as:*

$$\begin{aligned}\mathfrak{e}_\infty(z)\, F[X] &= \frac{q^{1/2}}{(1-q)(1-t^{-1})} \frac{C_\infty(q^{1/2}t^{-1}z)}{C_\infty(q^{1/2}z)}\, F\left[X + \frac{q^{1/2}-q^{-1/2}}{z}\right],\\ \mathfrak{f}_\infty(z)\, F[X] &= \frac{q^{-1/2}}{(1-q^{-1})(1-t^{-1})} \frac{\widetilde{C}_\infty(q^{1/2}t^{-1}z^{-1})}{\widetilde{C}_\infty(q^{1/2}z^{-1})}\, F\left[X - \frac{q^{1/2}-q^{-1/2}}{z}\right].\end{aligned}$$

4.2.2. *Comparison with the plethystic operators of Bergeron et al* [BGLX14]. To simplify the comparison with the operators of [BGLX14], let us introduce the generating functions:

$$\mathfrak{m}(z) := \sum_{n\in\mathbb{Z}} z^n\, \mathcal{M}_n = \frac{1-q}{q^{1/2}}\, t^{\frac{N-1}{2}}\, \mathfrak{e}(q^{-1/2}z), \tag{4.5}$$

$$\widetilde{\mathfrak{m}}(z) := \mathfrak{m}(qz)|_{q\to q^{-1},t\to t^{-1}} = \frac{1-q^{-1}}{q^{-1/2}}\, t^{-\frac{N-1}{2}}\, \mathfrak{f}(q^{-1/2}z). \tag{4.6}$$

In the infinite rank limit, we write

$$\mathfrak{m}_\infty(z) := \lim_{N\to\infty} t^{1-N}\, \mathfrak{m}(z), \quad \widetilde{\mathfrak{m}}_\infty(z) = \mathfrak{m}_\infty(qz)|_{q\to q^{-1},t\to t^{-1}}.$$

In this limit, the currents $\mathfrak{m}_\infty(z)$ and $\widetilde{\mathfrak{m}}_\infty$ act on symmetric functions as follows:

$$\mathfrak{m}_\infty(z)\, F[X] = \frac{1}{1-t^{-1}} \frac{C_\infty(t^{-1}z)}{C_\infty(z)} F\left[X + \frac{q-1}{z}\right], \tag{4.7}$$

$$\widetilde{\mathfrak{m}}_\infty(z)\, F[X] = \frac{1}{1-t} \frac{C_\infty(tq^{-1}z)}{C_\infty(q^{-1}z)} F\left[X - \frac{q-1}{z}\right]. \tag{4.8}$$

Note that as the prefactor only involves the power sums p_k with $k>0$ (via the function C_∞), we may restrict the action to power series $F\in\mathcal{P}_+$, where $\mathcal{P}_+$ is the space of formal power series of the $\{p_k\}_{k\in\mathbb{Z}_{>0}}$.

The action (4.7-4.8) can now be compared with the definition of the difference operators D_k, D_k^* of [BGLX14]. These were defined for $k\in\mathbb{Z}_+$ only, via a plethystic formulation. However, it is easy to extend the definition to all $k\in\mathbb{Z}$, by considering the generating currents $D(z) := \sum_{k\in\mathbb{Z}} z^k D_k$ and $D^*(z) := \sum_{k\in\mathbb{Z}} z^k D_k^*$. Their action on symmetric functions $F[X]\in\mathcal{P}_+[X]$, X an alphabet of infinitely many variables $x_1, x_2, \ldots$, is:

$$D(z)\, F[X] = C_\infty(z)\, F\left[X + \frac{(1-t)(1-q)}{z}\right], \tag{4.9}$$

$$D^*(z)\, F[X] = \frac{1}{C_\infty(z)}\, F\left[X - \frac{(1-t^{-1})(1-q^{-1})}{z}\right]. \tag{4.10}$$

In [BGLX14], the commutation relations between the D_k's and the D_k^*'s are derived. They can be rephrased as the following commutator of currents:

$$[D(z), D^*(\frac{w}{qt})] = \frac{(1-t)(1-q)}{qt-1}\delta(z/w)\frac{C_\infty(z)}{C_\infty(\frac{z}{qt})} \tag{4.11}$$

Theorem 4.6. *Let Σ be the operator acting by the plethysm $[X] \mapsto [X/(t-1)]$ (accordingly Σ^{-1} acts by $[X] \mapsto [X(t-1)]$). Then we have the following identities between operators acting on $\mathcal{P}_+[X]$:*

$$\begin{aligned} \mathfrak{m}_\infty(z) &= \frac{1}{1-t^{-1}}\left(\Sigma^{-1}D(z)\Sigma\right)\Big|_{t\to t^{-1}}, \\ \widetilde{\mathfrak{m}}_\infty(z) &= \frac{1}{1-t}\left(\Sigma^{-1}D^*\left(\frac{z}{qt}\right)\Sigma\right)\Big|_{t\to t^{-1}}. \end{aligned}$$

Proof. The bosonized expressions are matched using (4.9) and (4.10) and writing $F = \Sigma G$. Then, we have:

$$F\left[X \pm \frac{(1-t)(1-q)}{u}\right] = G\left[\frac{X}{t-1} \pm \frac{q-1}{u}\right],$$

while

$$\Sigma^{-1}\, C_\infty(u)\, \Sigma = \frac{C_\infty(tu)}{C_\infty(u)}, \quad \Sigma^{-1}\, \frac{1}{C_\infty(\frac{u}{qt})}\, \Sigma = \frac{C_\infty(t^{-1}q^{-1}u)}{C_\infty(q^{-1}u)}.$$

Hence

$$\begin{aligned} \Sigma^{-1}D(u)\Sigma\, G[X] &= \Sigma^{-1}D(u)F[X] = \frac{C_\infty(tu)}{C_\infty(u)}\, G\left[X + \frac{q-1}{u}\right], \\ \Sigma^{-1}D^*(\frac{u}{qt})\Sigma\, G[X] &= \Sigma^{-1}D^*(\frac{u}{qt})F[X] = \frac{C_\infty(t^{-1}q^{-1}u)}{C_\infty(q^{-1}u)}\, G\left[X - \frac{q-1}{u}\right], \end{aligned}$$

where the action of Σ^{-1} on $C_\infty(u)^{\pm 1} G\left[\frac{X}{t-1} \pm \frac{q-1}{u}\right]$ is expressed by considering the latter as a function of $[X]$. The Theorem follows by taking $t \to t^{-1}$ in the above, and comparing with eqns. (4.7-4.8). $\square$

Note that the commutation relation (4.11) is equivalent to the $N \to \infty$ limit of the commutation relation of Theorem 3.5, once expressed in terms of the $\mathfrak{m}$ currents.

Remark 4.7. *Theorem 4.6 allows to refine Remark 2.14 as follows. The transformation relating the Macdonald polynomials P_λ to the modified Macdonald polynomials $\widetilde{H}_\lambda$ is the following* [Mac95]*:*

$$\widetilde{H}_\lambda[X] = \phi_\lambda(t)\, P_\lambda\left[\frac{X}{t-1}\right]\Big|_{t\to t^{-1}} = \phi_\lambda(t)\, (\Sigma\, P_\lambda)|_{t\to t^{-1}},$$

where $\phi_\lambda(t)$ is a normalization factor independent of the x_i's. The ∇ operator of [BG99] *has eigenvectors $\widetilde{H}_\lambda$ with eigenvalues $T_\lambda := t^{n(\lambda)}q^{n(\lambda')}$, i.e. $\nabla\,\widetilde{H}_\lambda = T_\lambda\,\widetilde{H}_\lambda$. We deduce that*

$$(\Sigma^{-1}\,\nabla\,\Sigma)|_{t\to t^{-1}}\,P_\lambda = T_\lambda|_{t\to t^{-1}}\,P_\lambda = t^{-n(\lambda)}q^{n(\lambda')}\,P_\lambda.$$

We finally identify

$$\eta^{-1} = \nabla^{(N)} = C_N\,(t^{\frac{N-1}{2}}q^{\frac{1}{2}})^d\,(\Sigma^{-1}\,\nabla\,\Sigma)|_{t\to t^{-1}}, \tag{4.12}$$

where d acts on Macdonald polynomials as $d\,P_\lambda = |\lambda|\,P_\lambda$, C_N as in Remark 2.14, and ∇ is restricted to act on symmetric functions of $x_1, x_2, ..., x_N$. The element η in the completion of the DAHA was defined in Equation (2.9).

4.2.3. *Bosonization formulas for $\mathfrak{e}_\infty(z), \mathfrak{f}_\infty(z)$ in the $N\to\infty$ limit.* The so-called bosonization of the currents uses the operators p_k and $\frac{\partial}{\partial p_k}$, $k\in\mathbb{Z}^*$. They obey the following commutation relations

$$\left[\frac{\partial}{\partial p_j}, p_k\right] = \delta_{jk}$$

interpreted as independent harmonic oscillator relations. These relations still hold when evaluated on the infinite collection $X = (x_1, x_2, ...)$, as the $p_k[X]$ remain independent variables. We call the expression of the currents in terms of these elements evaluated on the infinite collection X, a bosonization.

As $\{p_k[X], k\in\mathbb{Z}^*\}$ are independent, a plethysm which adds one additional variable to the alphabet can be written as

$$F[X+\mu] = \exp\left\{\sum_{k\neq 0}\mu^k\frac{\partial}{\partial p_k[X]}\right\}F[X],$$

as readily seen from multiple Taylor expansion. This allows to rewrite Theorem 4.5 as

Theorem 4.8. *The limiting Macdonald currents $\mathfrak{e}_\infty(z)$, $\mathfrak{f}_\infty(z)\in\mathcal{P}[X]$ can be expressed in terms of the Heisenberg algebra generators as follows:*

$$\begin{aligned}
\mathfrak{e}_\infty(z) &= \frac{q^{1/2}}{(1-q)(1-t^{-1})}\,e^{\sum_{k>0}p_k[X]\frac{q^{k/2}(1-t^{-k})z^k}{k}}\,e^{\sum_{k\neq 0}\frac{q^{k/2}-q^{-k/2}}{z^k}\frac{\partial}{\partial p_k[X]}},\\
\mathfrak{f}_\infty(z) &= \frac{q^{-1/2}}{(1-q^{-1})(1-t^{-1})}e^{\sum_{k>0}p_{-k}[X]\frac{q^{k/2}(1-t^{-k})z^{-k}}{k}}\,e^{-\sum_{k\neq 0}\frac{q^{k/2}-q^{-k/2}}{z^k}\frac{\partial}{\partial p_k[X]}}.
\end{aligned}$$

4.3. **Plethystic formulas in the $t\to\infty$ limit.** We now investigate the dual Whittaker limit $t\to\infty$ of the plethystic expressions for the currents. To this end, we use the definition of the limiting currents $\mathfrak{e}^{(\infty)}, \mathfrak{f}^{(\infty)}, \psi^{\pm(\infty)}$ as in (3.8).

First, using (3.9), we are led to introduce the notation A for the following symmetric function:

$$A := x_1x_2\cdots x_N.$$

The function $\mathrm{Log}\, A$ can be understood as a renormalized version of the power sum $p_0[X]$, namely:

$$\mathrm{Log}\, A = \lim_{\epsilon\to 0} \frac{p_\epsilon[X]-N}{\epsilon}.$$

By a slight abuse of notation we shall write $A = e^{p_0}$. In the dual Whittaker limit,

$$\begin{aligned}
\psi^{+(\infty)}(z) &= \frac{(-q^{-1/2}z)^N\, A}{C(q^{-1/2}z)C(q^{1/2}z)} = (-q^{-1/2}z)^N\, e^{p_0+\sum_{k>0} p_k (q^{k/2}+q^{-k/2})\frac{z^k}{k}},\\
\psi^{-(\infty)}(z) &= \frac{(-q^{-1/2}z^{-1})^N\, A^{-1}}{\widetilde{C}(q^{-1/2}z)\widetilde{C}(q^{1/2}z)} = (-q^{-1/2}z^{-1})^N\, e^{-p_0+\sum_{k>0} p_{-k} (q^{k/2}+q^{-k/2})\frac{z^{-k}}{k}}.
\end{aligned}$$

The necessity of the introduction of the quantity A (or p_0) imposes on us to consider it as an independent variable, so that plethysms will act on the space $\mathcal{P}_0$ of functions that are power series of all p_k, $k\in\mathbb{Z}$ (including $k=0$). The plethysms must therefore acquire a p_0 dependence as well. More precisely, we must take into account an extra commutation relation in addition to those of Lemma 4.1.

Lemma 4.9. *We have the relations:*

$$\mathcal{D}^{q,t}_{1;n}\, p_0[X] = (p_0[X]+\mathrm{Log}\, q)\, \mathcal{D}^{q,t}_{1;n}, \quad \text{and} \quad \mathcal{D}^{q^{-1},t^{-1}}_{1;n}\, p_0[X] = (p_0[X]-\mathrm{Log}\, q)\, \mathcal{D}^{q^{-1},t^{-1}}_{1;n}. \tag{4.13}$$

Proof. We use the q-commutation relations:

$$\mathcal{D}^{q,t}_{1;n}\, A = q\, A\, \mathcal{D}^{q,t}_{1;n}, \quad \text{and} \quad \mathcal{D}^{q^{-1},t^{-1}}_{1;n}\, A = q^{-1} A\, \mathcal{D}^{q^{-1},t^{-1}}_{1;n}, \tag{4.14}$$

due to $\Gamma_i^{\pm 1}\, A = q^{\pm 1} A\, \Gamma_i$. □

We now extend plethysms to functions of the p_k, $k\in\mathbb{Z}$ so that, in the case of addition of one variable, we have:

$$p_k[X+\mu] = \begin{cases} p_k[X]+\mu^k & \text{if } k\neq 0\\ p_0[X]+\mathrm{Log}\,\mu & \text{if } k=0\end{cases}.$$

As a check,

$$p_k\left[X+\frac{q^{1/2}-q^{-1/2}}{z}\right] = \begin{cases} p_k[X]+\frac{q^{k/2}-q^{-k/2}}{z^k} & \text{if } k\neq 0\\ p_0[X]+\mathrm{Log}\, q & \text{if } k=0\end{cases}.$$

Note that the notation $[X+\mu]$ now refers to the full plethysm involving all the p_k, $k\in\mathbb{Z}$. With this notation, we easily get the action of the currents on functions $F[X]\in\mathcal{P}_0[X]$, i.e. formal power series of all $p_k[X]$, $k\in\mathbb{Z}$:

$$\begin{aligned}
\mathfrak{e}^{(\infty)}(z)\, F[X] &= \frac{q^{1/2}}{1-q}\left\{\frac{1}{C(q^{1/2}z)} - \frac{(-q^{-1/2}z^{-1})^N A^{-1}}{\widetilde{C}(q^{1/2}z^{-1})}\right\} F\left[X+\frac{q^{1/2}-q^{-1/2}}{z}\right],\\
\mathfrak{f}^{(\infty)}(z)\, F[X] &= \frac{q^{-1/2}}{1-q^{-1}}\left\{\frac{1}{\widetilde{C}(q^{1/2}z^{-1})} - \frac{(-q^{-1/2}z)^N A}{C(q^{-1/2}z)}\right\} F\left[X-\frac{q^{1/2}-q^{-1/2}}{z}\right].
\end{aligned}$$

5. Constant terms and shuffle product

5.1. **A constant term identity for generalized Macdonald operators.** In this section, we study the Macdonald and related currents defined above in the light of constant term identities. These allow in particular to reformulate current identities and relations in terms of shuffle products.

5.1.1. *Constant term identities and generating currents.* In this section we use the slightly different generating currents $\mathfrak{m}_\alpha(u)$ for the generalized Macdonald operators (1.5):

$$\mathfrak{m}_\alpha(u) := \sum_{n\in\mathbb{Z}} u^n\, \mathcal{M}_{\alpha;n} = \sum_{\substack{I\subset[1,N]\\ |I|=\alpha}} \delta(ux_I) \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j}\, \Gamma_I \tag{5.1}$$

The currents $\mathfrak{m}_\alpha(u)$ and $\mathfrak{e}_\alpha(u)$ are related via

$$\mathfrak{e}_\alpha(u) = \frac{q^{\frac{\alpha}{2}} t^{-\frac{\alpha(N-\alpha)}{2}}}{(1-q)^\alpha}\, \mathfrak{m}_\alpha(q^{\alpha/2}u) \tag{5.2}$$

Note that $\mathfrak{m}_1(u) = \mathfrak{m}(u)$ of (4.5). As a consequence of Theorem 3.4 and the above relation (5.2), the currents $\mathfrak{m}(u)$ satisfy in particular the following important exchange relation:

$$g(u,v)\,\mathfrak{m}(u)\,\mathfrak{m}(v) + g(v,u)\,\mathfrak{m}(v)\,\mathfrak{m}(u) = 0 \tag{5.3}$$

where g is as in (3.1). Equivalently, expressing the coefficient of $u^a v^b$ in (5.3), we get explicit quadratic relations between the $\mathcal{M}$'s, namely for all $a,b\in\mathbb{Z}$:

$$\mu_{a,b} := qt\mathcal{M}_{a-3}\mathcal{M}_b - (t^2+q^2t+q)\mathcal{M}_{a-2}\mathcal{M}_{b-1} + (qt^2+t+q^2)\mathcal{M}_{a-1}\mathcal{M}_{b-2} - qt\mathcal{M}_a\mathcal{M}_{b-3} = -\mu_{b,a} \tag{5.4}$$

Definition 5.1. *We define the multiple constant term of any rational symmetric function $f(u_1,...,u_\alpha)\in\mathcal{F}_\alpha$ as:*

$$CT_{u_1,...,u_\alpha}\left(f(u_1,...,u_\alpha)\right) := \prod_{i=1}^{\alpha} \oint \frac{du_i}{2i\pi\, u_i} f(u_1,...,u_\alpha) \tag{5.5}$$

where the contour integral picks up the residues at $u_i=0$.

In particular, we have $CT_v(f(v)\delta(u/v)) = f(u)$.

Definition 5.2. *To each symmetric rational function $P(x_1,x_2,...,x_\alpha)\in\mathcal{F}_\alpha$ we associate the difference operator $\mathcal{M}_\alpha(P)$:*

$$\mathcal{M}_\alpha(P) := \frac{1}{\alpha!} CT_{\mathbf{u}}\left(P(u_1^{-1},u_2^{-1},...,u_\alpha^{-1}) \prod_{1\le i<j\le\alpha} \frac{(u_i-u_j)(u_i-qu_j)}{(u_i-tu_j)(tu_i-qu_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(u_i)\right) \tag{5.6}$$

We have the following remarkable result:

Theorem 5.3. *For any symmetric rational function $P(x_1,...,x_\alpha)\in\mathcal{F}_\alpha$, $1\leq\alpha\leq N$, we have the identity:*

$$\mathcal{M}_\alpha(P)=\mathcal{D}_\alpha(P) \tag{5.7}$$

with $\mathcal{D}_\alpha(P)$ as in Definition 1.1.

Proof. Let us compute:

$$\begin{aligned}
&\prod_{i<j}\frac{u_i-u_j}{u_i-tu_j}\frac{u_i-qu_j}{tu_i-qu_j}\prod_{i=1}^{\alpha}\mathfrak{m}(u_i)\\
=&\prod_{i<j}\frac{u_i-u_j}{u_i-tu_j}\frac{u_i-qu_j}{tu_i-qu_j}\sum_{i_1,i_2,...,i_\alpha}\prod_{k=1}^{\alpha}\left(\delta(u_kx_{i_k})\prod_{j_k\neq i_k}\frac{tx_{i_k}-x_{j_k}}{x_{i_k}-x_{j_k}}\Gamma_{i_k}\right)\\
=&\prod_{i<j}\frac{u_i-u_j}{u_i-tu_j}\frac{u_i-qu_j}{tu_i-qu_j}\times\\
&\qquad\times\sum_{i_1\neq i_2\neq...\neq i_\alpha}\prod_{j=1}^{\alpha}\delta(u_jx_{i_j})\prod_{k<\ell}\frac{tx_{i_k}-x_{i_\ell}}{x_{i_k}-x_{i_\ell}}\frac{tx_{i_\ell}-qx_{i_k}}{x_{i_\ell}-qx_{i_k}}\prod_{j=1}^{\alpha}\prod_{i\neq i_1,...,i_\alpha}\frac{tx_{i_j}-x_i}{x_{i_j}-x_i}\,\Gamma_{i_1}\cdots\Gamma_{i_\alpha}\\
=&\sum_{i_1\neq i_2\neq...\neq i_\alpha}\prod_{j=1}^{\alpha}\delta(u_jx_{i_j})\prod_{i\neq i_1,...,i_\alpha}\frac{tx_{i_k}-x_i}{x_{i_k}-x_i}\,\Gamma_{i_1}\cdots\Gamma_{i_\alpha}\\
=&\ \alpha!\sum_{i_1<i_2<...<i_\alpha}\prod_{j=1}^{\alpha}\delta(u_jx_{i_j})\prod_{i\neq i_1,...,i_\alpha}\frac{tx_{i_k}-x_i}{x_{i_k}-x_i}\,\Gamma_{i_1}\cdots\Gamma_{i_\alpha}\\
=&\ \alpha!\,\frac{1}{\alpha!\,(N-\alpha)!}\mathrm{Sym}\left(\prod_{k=1}^{\alpha}\delta(u_kx_k)\prod_{1\leq i\leq\alpha<j\leq N}\frac{tx_i-x_j}{x_i-x_j}\,\Gamma_1\cdots\Gamma_\alpha\right)
\end{aligned}$$

We have first noted that terms with any two identical $i_k=i_\ell$, $k<\ell$, in the sum must vanish. This is due to the prefactor (u_k-qu_ℓ) which when multiplying the delta function $\delta(u_kx_{i_k})\delta(u_\ell qx_{i_k})$ yields a zero result. Comparing with Definition 1.3, the constant term (5.7) follows immediately. □

As a by-product of the proof of Theorem 5.3 above, we note that if $\alpha>N$, then there are no terms in which all i_k are distinct (as there are at most N of them), thus causing the result to vanish. We deduce the following:

Corollary 5.4. *For any symmetric rational function $P(x_1,...,x_\alpha)\in\mathcal{F}_\alpha$, $\alpha>N$, we have:*

$$\mathcal{M}_\alpha(P)=\mathcal{D}_\alpha(P)=0\qquad\forall\ \alpha>N. \tag{5.8}$$

This implies in particular that

$$\mathcal{M}_{N+1;n}=0\quad\forall\ n\in\mathbb{Z}. \tag{5.9}$$

in agreement with (3.5).

Recalling the definition (1.7), Theorem 5.3 has also the following immediate application to $P = s_{a_1,\dots,a_\alpha}(x_1,\dots,x_\alpha)$:

Corollary 5.5. *We have:*

$$\mathcal{M}_{a_1,\dots,a_\alpha} = \frac{1}{\alpha!} CT_{\mathbf{u}}\left(s_{a_1,\dots,a_\alpha}(\mathbf{u}^{-1}) \prod_{1\leq i<j\leq\alpha} \frac{(u_i-u_j)(u_i-qu_j)}{(u_i-tu_j)(tu_i-qu_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(u_i)\right) \tag{5.10}$$

In particular, this implies:

$$\mathcal{M}_{\alpha;n} = \frac{1}{\alpha!} CT_{\mathbf{u}}\left((u_1u_2\cdots u_\alpha)^{-n} \prod_{1\leq i<j\leq\alpha} \frac{(u_i-u_j)(u_i-qu_j)}{(u_i-tu_j)(tu_i-qu_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(u_i)\right) \tag{5.11}$$

or equivalently in terms of the currents $\mathfrak{m}_\alpha(z)$ or $\mathfrak{e}_\alpha(z)$ of (3.3):

Corollary 5.6. *We have the following constant term identities:*

$$\mathfrak{m}_\alpha(z) = \frac{1}{\alpha!} CT_{\mathbf{u}}\left(\delta(u_1u_2\cdots u_\alpha/z) \prod_{1\leq i<j\leq\alpha} \frac{(u_i-u_j)(u_i-qu_j)}{(u_i-tu_j)(tu_i-qu_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(u_i)\right) \tag{5.12}$$

$$\mathfrak{e}_\alpha(z) = \frac{1}{\alpha!} CT_{\mathbf{u}}\left(\delta(u_1u_2\cdots u_\alpha/z) \prod_{1\leq i<j\leq\alpha} \frac{(u_i-u_j)(u_i-qu_j)}{(u_i-tu_j)(u_i-q/tu_j)} \prod_{i=1}^{\alpha} \mathfrak{e}(u_i)\right) \tag{5.13}$$

Proof. The first equation is the current form of (5.11) obtained by multiplying by z^n and summing over $n\in\mathbb{Z}$. The second equation results from the change of variables $u_i\mapsto q^{1/2}u_i$ for all i and $z\mapsto q^{\alpha/2}\,z$ in the previous multiple constant term residue integral, while using the relation (5.2). □

The result of Corollary 5.5 may be rephrased in terms of generating currents as follows. We consider the generating multi-current with argument $\mathbf{v}=(v_1,v_2,\dots,v_\alpha)$:

(5.14)

$$\mathfrak{M}_\alpha(\mathbf{v}) := \sum_{a_1,\dots,a_\alpha\in\mathbb{Z}} \mathcal{M}_{a_1,\dots,a_\alpha}\, v_1^{a_1}v_2^{a_2}\cdots v_\alpha^{a_\alpha} = \frac{1}{\prod_{j=1}^{\alpha} v_j^{\alpha-j}} \mathcal{D}_\alpha\left(\frac{\det\left((\delta(x_i\,v_j))_{1\leq i,j\leq\alpha}\right)}{\prod_{1\leq i<j\leq\alpha}(x_i-x_j)}\right)$$

Theorem 5.7. *The generating current for the generalized Macdonald operators* (1.7) *reads:*

$$\mathfrak{M}_\alpha(\mathbf{v}) = \prod_{1\leq i<j\leq\alpha} \frac{(v_i-qv_j)}{(t-v_iv_j^{-1})(tv_i-qv_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(v_i) \tag{5.15}$$

Proof. Using the identity (5.10), and the expression (1.6) for the generalized Schur function, we compute:

$$\begin{aligned}
\mathfrak{M}_\alpha(\mathbf{v}) &= \frac{1}{\alpha!} CT_{\mathbf{u}} \left(\sum_{a_1,\ldots,a_\alpha\in\mathbb{Z}} \det\left(u_i^{-a_j-\alpha+j}\right) v_1^{a_1} v_2^{a_2}\cdots v_\alpha^{a_\alpha} \times \right. \\
&\qquad \left. \times \prod_{1\le i<j\le\alpha} \frac{(u_i - q u_j)}{(t u_i^{-1} - u_j^{-1})(t u_i - q u_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(u_i) \right) \\
&= \frac{1}{\alpha!} CT_{\mathbf{u}} \left(\det\left(\delta(v_j/u_i)\, v_j^{j-\alpha}\right) \prod_{1\le i<j\le\alpha} \frac{(u_i - q u_j)}{(t u_i^{-1} - u_j^{-1})(t u_i - q u_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(u_i) \right) \\
&= \frac{1}{\alpha!} CT_{\mathbf{u}} \left(\det\left(\delta(v_j/u_i)\right) \prod_{1\le i<j\le\alpha} \frac{(u_i - q u_j)}{(t - u_i u_j^{-1})(t u_i - q u_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(u_i) \right) \\
&= \prod_{1\le i<j\le\alpha} \frac{(v_i - q v_j)}{(t - v_i v_j^{-1})(t v_i - q v_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(v_i)
\end{aligned}$$

where in the last step we have used the skew-symmetry of both the determinant and the quantity next to it, to see that each of the $\alpha!$ terms in the expansion of the determinant contributes the same as the diagonal term $\prod \delta(v_i/u_i) u_i^{i-\alpha}$. ∎

Corollary 5.8. *We have the following alternative expression for the generalized Macdonald operators* (1.7)*:*

$$\mathcal{M}_{a_1,\ldots,a_\alpha} = CT_{\mathbf{u}} \left(\prod_{i=1}^{\alpha} u_i^{-a_i} \prod_{1\le i<j\le\alpha} \frac{(u_i - q u_j)}{(t - u_i u_j^{-1})(t u_i - q u_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(u_i) \right) \tag{5.16}$$

Proof. The constant term (5.16) picks up the coefficient of $u_1^{a_1} u_2^{a_2} \cdots u_\alpha^{a_\alpha}$ in $\mathfrak{M}_\alpha(\mathbf{u})$. ∎

We also have the corresponding current version, when $a_1 = a_2 = \cdots = a_\alpha = n$:

$$\mathfrak{m}_\alpha(z) = CT_{\mathbf{u}} \left(\delta(u_1 u_2 \cdots u_\alpha/z) \prod_{1\le i<j\le\alpha} \frac{(u_i - q u_j)}{(t - u_i u_j^{-1})(t u_i - q u_j)} \prod_{i=1}^{\alpha} \mathfrak{m}(u_i) \right) \tag{5.17}$$

$$\mathfrak{e}_\alpha(z) = CT_{\mathbf{u}} \left(\delta(u_1 u_2 \cdots u_\alpha/z) \prod_{1\le i<j\le\alpha} \frac{(u_i - q u_j)}{(t - u_i u_j^{-1})(u_i - q/t u_j)} \prod_{i=1}^{\alpha} \mathfrak{e}(u_i) \right) \tag{5.18}$$

5.1.2. *Polynomiality and the (q,t)-determinant.* In this section, we state the following:

Conjecture 5.9. *The generalized Macdonald operators $\mathcal{M}_{a_1,\ldots,a_\alpha}$ may be expressed as polynomials of finitely many $\mathcal{M}_p$'s. These polynomials are t-deformation of the quantum determinant expression* (7.11) *below.*

Note that if they exist, such polynomials are not necessarily unique, as they can be modified by use of the exchange relations (5.4).

We first give the proof of the conjecture in the case $\alpha = 2$ for arbitrary $\mathcal{M}_{a,b}$, $a, b \in \mathbb{Z}$, by deriving an explicit polynomial expression (see Theorem 5.10 below), and the sketch of the proof in the case $\alpha = 3$ (Theorem 5.11). Further evidence will be derived in Section 6, where the connection to Elliptic Hall algebra leads naturally to a polynomial expression for $\mathcal{M}_{\alpha;n} = \mathcal{M}_{n,n,\ldots,n}$ for all α, n, as a function of solely $\mathcal{M}_n, \mathcal{M}_{n\pm1}$.

Theorem 5.10. *For all $a, b \in \mathbb{Z}$, the operator $\mathcal{M}_{a,b}$ can be expressed as an explicit quadratic polynomial of the $\mathcal{M}_n$'s, with coefficients in $\mathbb{C}(q,t)$. More precisely, we have:*

$$\mathcal{M}_{a,b} = \frac{q(q+t^2)\nu_{a,b} - t(1+q)\nu_{a+1,b-1} + (q+t^2)\nu_{b-1,a+1} - qt(1+q)\nu_{b-2,a+2}}{(q-1)(q^2-t^2)(1-t^2)} \tag{5.19}$$

where $\nu_{a,b}$ stands for the following "quantum determinant":

$$\nu_{a,b} = \begin{vmatrix} \mathcal{M}_a & \mathcal{M}_{b-1} \\ \mathcal{M}_{a+1} & \mathcal{M}_b \end{vmatrix}_q := \mathcal{M}_a\,\mathcal{M}_b - q\,\mathcal{M}_{a+1}\,\mathcal{M}_{b-1} \tag{5.20}$$

Proof. We start from the current relation of Theorem 5.7:

$$\mathfrak{M}_2(v_1, v_2) = \frac{(v_1 - qv_2)}{(t - v_1 v_2^{-1})(tv_1 - qv_2)}\,\mathfrak{m}(v_1)\,\mathfrak{m}(v_2) \tag{5.21}$$

We note that the exchange relation (5.3) can be rewritten as

$$\frac{(v_1 - qv_2)}{(tv_2 - v_1)(tv_1 - qv_2)}\,\mathfrak{m}(v_1)\,\mathfrak{m}(v_2) + \frac{(v_2 - qv_1)}{(tv_1 - v_2)(tv_2 - qv_1)}\,\mathfrak{m}(v_2)\,\mathfrak{m}(v_1) = 0$$

As a consequence, the renormalized double current $\mathfrak{N}_2(v_1, v_2) := \mathfrak{M}_2(v_1, v_2)/v_2$ is skew-symmetric, and we may rewrite (5.21) as:

$$\delta_{1,2}\,\mathfrak{N}_2(v_1, v_2) = \frac{v_1 - qv_2}{v_1 v_2}\,\mathfrak{m}(v_1)\,\mathfrak{m}(v_2) =: \mu_2(v_1, v_2) \tag{5.22}$$

where we use the notation

$$\delta_{i,j} := \left(t - \frac{v_i}{v_j}\right)\left(t - q\frac{v_j}{v_i}\right)$$

Using the skew-symmetry of $\mathfrak{N}_2$, let us apply the transposition (12) that interchanges $v_1 \leftrightarrow v_2$, with the result:

$$\delta_{2,1}\,\mathfrak{N}_2(v_1, v_2) = -\mu_2(v_2, v_1) \tag{5.23}$$

Next, we eliminate the prefactors by using the "inversion" relation:

$$\eta_{1,2}\delta_{1,2} - \theta_{1,2}\delta_{2,1} = 1, \quad \eta_{i,j} := \frac{q(q+t^2) - t(1+q)\frac{v_j}{v_i}}{(q-1)(1-t^2)(q^2-t^2)}, \quad \theta_{i,j} := \frac{(q+t^2) - qt(1+q)\frac{v_j}{v_i}}{(q-1)(1-t^2)(q^2-t^2)} \tag{5.24}$$

which is easily derived by decomposing $1/(\delta_{1,2}\delta_{2,1})$ into simple fractions. With the above choice, we conclude that:

$$\mathfrak{M}_2(v_1,v_2) = v_2\mathfrak{N}_2(v_1,v_2) = v_2(\eta_{1,2}\,\mu_2(v_1,v_2) + \theta_{1,2}\,\mu_2(v_2,v_1))$$

Defining $\nu_2(v_1,v_2) := v_2\mu_2(v_1,v_2) = \Big(1-q\frac{v_2}{v_1}\Big)\,\mathfrak{m}(v_1)\,\mathfrak{m}(v_2)$, we finally get:

$$\mathfrak{M}_2(v_1,v_2) = \eta_{1,2}\,\nu_2(v_1,v_2) + \frac{v_2}{v_1}\,\theta_{1,2}\,\nu_2(v_2,v_1) \tag{5.25}$$

Introducing the mode expansion: $\nu_2(v_1,v_2) = \sum_{a,b\in\mathbb{Z}} \nu_{a,b} v_1^a v_2^b$, with $\nu_{a,b}$ as in (5.20), the formula (5.19) follows from the mode expansion of (5.25). □

Note that the expression (5.19) expresses $\mathcal{M}_{a,b}$ in terms of more variables that just $\mathcal{M}_a, \mathcal{M}_{a+1}, \mathcal{M}_{b-1}, \mathcal{M}_b$. However, if we consider the limit when $t\to\infty$ of this expression, after defining $M_{a,b} = \lim_{t\to\infty} t^{-2(N-2)}\mathcal{M}_{a,b}$, $M_a = \lim_{t\to\infty} t^{-(N-1)}\mathcal{M}_a$, and $n_{a,b} = \lim_{t\to\infty} t^{-2(N-1)}\nu_{a,b}$, we obtain:

$$M_{a,b} = \frac{q\ n_{a,b} + n_{b-1,a+1}}{q-1} = n_{a,b} = M_a M_b - q M_{a+1} M_{b-1}$$

where we have used the $t\to\infty$ exchange relation. Indeed, for finite t the expression (5.19) is not unique: it is unique only up to the exchange relation (5.4) for the $\mathcal{M}_n$'s. Here is a simple example. Revisiting the proof of the Theorem, we note that there is another inverting pair $(\eta'_{1,2},\theta'_{1,2}) = (-\theta_{2,1},-\eta_{2,1})$ obtained by acting with the transposition (12) on the inversion relation (5.24), namely we also have:

$$\eta'_{1,2}\,\delta_{1,2} - \theta'_{1,2}\,\delta_{2,1} = 1$$

This choice leads to alternative expressions:

$$\begin{aligned}
\mathfrak{M}_2(v_1,v_2) &= v_2\mathfrak{N}_2(v_1,v_2) = -v_2\Big(\theta_{2,1}\,\mu_2(v_1,v_2) + \eta_{2,1}\,\mu_2(v_2,v_1)\Big)\\
&= -\theta_{2,1}\,\nu_2(v_1,v_2) - \frac{v_2}{v_1}\,\eta_{2,1}\,\nu_2(v_2,v_1)
\end{aligned}$$

$$\mathcal{M}_{a,b} = \frac{qt(1+q)\nu_{a-1,b+1} - (q+t^2)\nu_{a,b} + t(1+q)\nu_{b,a} - q(q+t^2)\nu_{b-1,a+1}}{(q-1)(q^2-t^2)(1-t^2)} \tag{5.26}$$

The difference between the two expressions (5.19) and (5.26) is proportional to:

$$(q+t^2)(\nu_{a,b} + \nu_{b-1,a+1}) - t(\nu_{a+1,b-1} + \nu_{b,a}) - qt(\nu_{a-1,b+1} + \nu_{b-2,a+2}) \tag{5.27}$$

Rewriting the exchange relation (5.4) as:

$$\varphi_{a,b} := qt\nu_{a-3,b} - (q+t^2)\nu_{a-2,b-1} + t\nu_{a-1,b-2} = -\varphi_{b,a} \tag{5.28}$$

we see that (5.27) is nothing but $-\varphi_{a+2,b+1} - \varphi_{b+1,a+2} = 0$, as a direct consequence of the exchange relation. This gives a simple example of equivalence of two polynomial expressions for $\mathcal{M}_{a,b}$ modulo the exchange relations of the algebra.

Theorem 5.11. *The conjecture 5.9 holds for $\alpha = 3$.*

Proof. Sketch of the proof. Use simple fraction decomposition of the quantity $1/(\delta_{1,2}\delta_{2,1}\delta_{1,3}\delta_{3,1}\delta_{2,3}\delta_{3,2})$ to obtain a relation of the form:

$$\sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)\, A_\sigma(v_1, v_2, v_3)\, \delta_{\sigma(1),\sigma(2)}\delta_{\sigma(1),\sigma(3)}\delta_{\sigma(2),\sigma(3)} = 1$$

with explicit Laurent polynomials $A_\sigma(v_1, ..., v_\alpha)$. This allows to express the skew-symmetric current $\mathfrak{N}_\alpha = \mathfrak{M}_\alpha/(v_2v_3^2)$ as a sum over the symmetric group S_3 of Laurent polynomials of the v's times permuted products of the fundamental currents $\mathfrak{m}(v_i)$'s. The polynomiality property of the coefficients follows. $\square$

More generally, one could try to generalize the above argument. Defining the skew-symmetric function $\mathfrak{N}_\alpha = \mathfrak{M}_\alpha/(v_2v_3^2 \cdots v_\alpha^{\alpha-1})$, we wish to invert the relation

$$\left(\prod_{1\le i<j\le\alpha} \delta_{i,j}\right) \mathfrak{N}_\alpha(v_1, ..., v_\alpha) = \prod_{1\le i<j\le\alpha} \frac{v_i - qv_j}{v_iv_j}\, \mathfrak{m}(v_1)\cdots\mathfrak{m}(v_\alpha) =: \mu_\alpha(v_1, ..., v_\alpha)$$

Acting with the permutation group of the v's, we have accordingly for all $\sigma \in S_\alpha$:

$$\prod_{1\le i<j\le\alpha} \delta_{\sigma(i),\sigma(j)}\, \mathfrak{N}_\alpha(v_1, ..., v_\alpha) = \mathrm{sgn}(\sigma)\, \mu_\alpha(v_{\sigma(1)}, ..., v_{\sigma(\alpha)})$$

Inverting the system could be done by looking for Laurent polynomials $A_\sigma(v_1, ..., v_\alpha)$ such that

$$\sum_{\sigma\in S_\alpha} \mathrm{sgn}(\sigma)\, A_\sigma(v_1, ..., v_\alpha) \prod_{1\le i<j\le\alpha} \delta_{\sigma(i),\sigma(j)} = 1$$

If such A_σ's existed, then we could write

$$\mathfrak{M}_\alpha = v_2v_3^2 \cdots v_\alpha^{\alpha-1} \sum_{\sigma\in S_\alpha} A_\sigma(v_1, ..., v_\alpha)\, \mu_\alpha(v_{\sigma(1)}, ..., v_{\sigma(\alpha)})$$

and polynomiality would follow.

5.2. **Plethystic formulation.** Using the plethystic formulas of Section 4, we derive a plethystic formula for the higher currents $\mathfrak{e}_\alpha$.

Theorem 5.12. *The current $\mathfrak{e}_\alpha(z)$ acts on functions $F[X]$ as follows:*

$$\begin{aligned}\mathfrak{e}_\alpha(z)\cdot F[X] \;&=\; \left(\frac{q^{1/2}t^{-1/2}}{(1-q)(1-t^{-1})}\right)^\alpha \mathrm{CT}_{\mathbf{u}}\Bigg(\prod_{1\le i<j\le\alpha}\frac{(u_i-u_j)^2}{(u_i-tu_j)(u_i-t^{-1}u_j)}\\ &\times\prod_{i=1}^{\alpha}\left\{t^{\frac{N}{2}}\frac{C(q^{1/2}t^{-1}u_i)}{C(q^{1/2}u_i)}-t^{-\frac{N}{2}}\frac{\tilde C(q^{-1/2}tu_i^{-1})}{\tilde C(q^{-1/2}u_i^{-1})}\right\}\\ &\qquad\delta(u_1u_2\cdots u_\alpha/z)\cdot F\left[X+(q^{1/2}-q^{-1/2})\sum_{i=1}^{\alpha}\frac{1}{u_i}\right]\Bigg)\end{aligned}$$

Proof. We start from the bosonized formula for $\mathfrak{e}(z)$ of Theorem 4.3, and substitute it into (5.18). We need to compute the action of the plethysm $[X+\frac{q^{1/2}-q^{-1/2}}{z}]$ on $\mathfrak{e}(w)$: it sends respectively

$$\begin{aligned}\frac{C(q^{1/2}t^{-1}w)}{C(q^{1/2}w)} &\mapsto \frac{(z-qt^{-1}w)(z-w)}{(z-t^{-1}w)(z-qw)}\frac{C(q^{1/2}t^{-1}w)}{C(q^{1/2}w)}\\ \frac{\tilde C(q^{-1/2}tw^{-1})}{\tilde C(q^{-1/2}w^{-1})} &\mapsto \frac{(w-q^{-1}tz)(w-z)}{(w-tz)(w-q^{-1}z)}\frac{\tilde C(q^{-1/2}tw^{-1})}{\tilde C(q^{-1/2}w^{-1})}\end{aligned}$$

hence both terms are mapped identically, and the theorem follows by repeated applications until all u_i's are exhausted. ☐

Similarly, we have the following bosonization of the multi-current $\mathfrak{M}_\alpha(\mathbf{v})$ of (5.14).

Theorem 5.13. *The multi-current $\mathfrak{M}_\alpha(\mathbf{v})$ for the generalized Macdonald operators acts on functions $F[X]$ as follows.*

$$\begin{aligned}\mathfrak{M}_\alpha(\mathbf{v})\cdot F[X] \;&=\; \left(\frac{t^{\frac{N}{2}}}{t-1}\right)^\alpha\prod_{1\le i<j\le\alpha}\frac{(v_i-v_j)}{(t-v_iv_j^{-1})(tv_i-v_j)}\times\\ &\times\prod_{i=1}^{\alpha}\left\{t^{\frac{N}{2}}\frac{C(t^{-1}v_i)}{C(v_i)}-t^{-\frac{N}{2}}\frac{\tilde C(tv_i^{-1})}{\tilde C(v_i^{-1})}\right\}\cdot F\left[X+(q-1)\sum_{i=1}^{\alpha}\frac{1}{v_i}\right]\end{aligned}$$

Proof. Starting from (5.14), we substitute $\mathfrak{m}(v_i)=q^{-1/2}(1-q)t^{\frac{N-1}{2}}\mathfrak{e}(q^{-1/2}v_i)$, and then use the proof of Theorem 5.12 to rearrange the prefactors of the plethysms. ☐

5.3. **Shuffle product.** The shuffle product is a non-commutative product $*:\mathcal{F}_\alpha\times\mathcal{F}_\beta\to\mathcal{F}_{\alpha+\beta}$, sending $(P,P')\mapsto Q=P*P'$. It allows to express the product of any two difference operators $\mathcal{D}_\alpha(P)$ and $\mathcal{D}_\beta(P')$ for arbitrary rational functions $P\in\mathcal{F}_\alpha$ and $P'\in\mathcal{F}_\beta$, as a difference operator of the form $\mathcal{D}_{\alpha+\beta}(Q)$, for some $Q\in\mathcal{F}_{\alpha+\beta}$. In the following sections, we first define the shuffle product, and then present various applications, among which alternative proofs of the exchange relation (5.3), and of Theorems 5.7, 3.7 and 5.10.

5.3.1. *Definition and main result.* Let us introduce the quantity:

$$\zeta(x) := \frac{1-tx}{1-x}\frac{t-qx}{1-qx}$$

Definition 5.14. *The shuffle product $P * P' \in \mathcal{F}_{\alpha+\beta}$ of $P \in \mathcal{F}_\alpha$ and $P' \in \mathcal{F}_\beta$ is defined as the symmetrized expression:*

$$P * P'(x_1,...,x_{\alpha+\beta}) := \frac{1}{\alpha!\,\beta!}\mathrm{Sym}\left(P(x_1,...,x_\alpha)P'(x_{\alpha+1},...,x_{\alpha+\beta})\prod_{1\le i\le\alpha<j\le\alpha+\beta}\zeta(x_i/x_j)\right) \tag{5.29}$$

where the symmetrization Sym *is over $x_1, x_2, ..., x_{\alpha+\beta}$.*

This product is in general non-commutative by construction, but associative.

Theorem 5.15. *For any rational functions $P \in \mathcal{F}_\alpha$ and $P' \in \mathcal{F}_\beta$, we have the relations*

$$\mathcal{D}_\alpha(P)\,\mathcal{D}_\beta(P') = \mathcal{D}_{\alpha+\beta}(P * P'), \qquad \mathcal{M}_\alpha(P)\,\mathcal{M}_\beta(P') = \mathcal{M}_{\alpha+\beta}(P * P') \tag{5.30}$$

with the shuffle product $P * P' \in \mathcal{F}_{\alpha+\beta}$ defined in (5.29)*.*

Proof. We use Theorem 5.3 and compute, for $\mathbf{u} = (u_1, u_2, ..., u_{\alpha+\beta})$:

$$\begin{aligned}
\mathcal{M}_\alpha(P)\,\mathcal{M}_\beta(P') &= CT_{\mathbf{u}}\Bigg(\frac{P(u_1^{-1},...,u_\alpha^{-1})P'(u_{\alpha+1}^{-1},...,u_{\alpha+\beta}^{-1})}{\alpha!\,\beta!}\times\\
&\qquad\times\prod_{i<j\,\in[1,\alpha]\text{ or }[\alpha+1,\alpha+\beta]}\zeta(u_j/u_i)^{-1}\prod_{k=1}^{\alpha+\beta}\mathfrak{m}(u_k)\Bigg)\\
&= CT_{\mathbf{u}}\Bigg(\frac{P(u_1^{-1},...,u_\alpha^{-1})P'(u_{\alpha+1}^{-1},...,u_{\alpha+\beta}^{-1})}{\alpha!\,\beta!}\times\\
&\qquad\times\prod_{1\le i\le\alpha<j\le\alpha+\beta}\zeta(u_j/u_i)\prod_{i<j\,\in[1,\alpha+\beta]}\zeta(u_j/u_i)^{-1}\prod_{k=1}^{\alpha+\beta}\mathfrak{m}(u_k)\Bigg)\\
&= \frac{1}{(\alpha+\beta)!}CT_{\mathbf{u}}\left(P * P'(u_1^{-1},...,u_{\alpha+\beta}^{-1})\prod_{i<j\,\in[1,\alpha+\beta]}\zeta(u_j/u_i)^{-1}\prod_{k=1}^{\alpha+\beta}\mathfrak{m}(u_k)\right)\\
&= \mathcal{M}_{\alpha+\beta}(P * P')
\end{aligned}$$

where we have used the symmetry of the last factor in the second line, and the fact that the constant term is preserved under symmetrization, namely $CT_{u_1,...,u_m}\big(\mathrm{Sym}(f(u_1,...,u_m))\big)/m! = CT_{u_1,...,u_m}\big(f(u_1,...,u_m)\big)$. The Theorem follows. $\square$

In the next subsections, we explore applications of Theorem 5.15.

5.3.2. *Application I: Macdonald current relations.* Note that

$$\mathfrak{m}(v) = \mathcal{D}_1(\delta(vx_1))$$

By iterated use of Theorem 5.15, we may express any product $\mathfrak{m}(v_1)\mathfrak{m}(v_2)\cdots\mathfrak{m}(v_\alpha)$ as:

$$\mathfrak{m}(v_1)\mathfrak{m}(v_2)\cdots\mathfrak{m}(v_\alpha) = \mathcal{D}_\alpha\Big(\delta(v_1x_1) * \delta(v_2x_1) * \cdots * \delta(v_\alpha x_1)\Big)$$

where we used associativity of the shuffle product to drop parentheses. In particular, the exchange relation (5.3) boils down to the following shuffle identity on $\mathcal{F}_2$.

Theorem 5.16. *We have the relation:*

$$g(u,v)\,\delta(ux_1) * \delta(vx_1) + g(v,u)\,\delta(vx_1) * \delta(ux_1) = 0$$

Proof. Use Definition 5.14 to compute:

$$\begin{aligned} g(u,v)\,\delta(ux_1) * \delta(vx_1) &= g(u,v)\mathrm{Sym}\left(\delta(ux_1)\,\delta(vx_2)\frac{(x_2-tx_1)(tx_2-qx_1)}{(x_2-x_1)(x_2-qx_1)}\right) \\ &= g(u,v)\frac{(u-tv)(tu-qv)}{(u-v)(u-qv)}\mathrm{Sym}\left(\delta(ux_1)\,\delta(vx_2)\right) \\ &= \frac{(v-tu)(u-tv)(tv-qu)(tu-qv)}{qt(u-v)}\left(\delta(ux_1)\,\delta(vx_2) + \delta(vx_1)\,\delta(ux_2)\right) \end{aligned}$$

which is manifestly skew-symmetric in (u,v). ◻

More generally, Theorem 5.7 translates into the following shuffle identity.

Theorem 5.17. *We have the relation:*

$$\frac{\det\Big(\left(\delta(x_iv_j)\right)_{1\le i,j\le\alpha}\Big)}{\prod_{1\le i<j\le\alpha}(x_i-x_j)} = \prod_{i=1}^{\alpha} v_j^{\alpha-1} \prod_{1\le i<j\le\alpha}\frac{v_i-qv_j}{(tv_j-v_i)(tv_i-qv_j)}\,\delta(v_1x_1)*\delta(v_2x_1)*\cdots*\delta(v_\alpha x_1)$$

Proof. We compute:

$$\delta(v_1x_1) * \delta(v_2x_1) * \cdots * \delta(v_\alpha x_1) = \mathrm{Sym}\left(\delta(v_1x_1)\delta(v_2x_2)\cdots\delta(v_\alpha x_\alpha)\prod_{1\le i<j\le\alpha}\frac{(x_j-tx_i)(tx_j-qx_i)}{(x_j-x_i)(x_j-qx_i)}\right)$$

$$= \frac{1}{(\prod_{i=1}^{\alpha} v_j)^{\alpha-1}}\prod_{1\le i<j\le\alpha}\frac{(tv_j-v_i)(tv_i-qv_j)}{(v_i-qv_j)}\mathrm{Sym}\left(\frac{\delta(v_1x_1)\delta(v_2x_2)\cdots\delta(v_\alpha x_\alpha)}{\prod_{1\le i<j\le\alpha}(x_i-x_j)}\right)$$

and the result follows, as the symmetrization produces the desired determinant. ◻

Finally let us revisit Theorem 3.7 for $\mathfrak{m}$, namely the identity

$$\mathrm{Sym}_{v_1,v_2,v_3}\left(\frac{v_2}{v_3}\Big[\mathfrak{m}(v_1),\big[\mathfrak{m}(v_2),\mathfrak{m}(v_3)\big]\Big]\right) = 0$$

The corresponding shuffle identity is the following.

Theorem 5.18. *We have the identity:*

$$\mathrm{Sym}_{v_1,v_2,v_3}\left\{\frac{v_2}{v_3}\Big(\delta(v_1x_1)*\big(\delta(v_2x_1)*\delta(v_3x_1)-\delta(v_3x_1)*\delta(v_2x_1)\big)\right.$$
$$\left.-\big(\delta(v_2x_1)*\delta(v_3x_1)-\delta(v_3x_1)*\delta(v_2x_1)\big)*\delta(v_1x_1)\Big)\right\}=0$$

Proof. Using the proof of Theorem 5.17 for $\alpha=3$, we compute:

$$\delta(v_1x_1)*\delta(v_2x_1)*\delta(v_3x_1)=\prod_{1\le i<j\le 3}\frac{(tv_j-v_i)(tv_i-qv_j)}{(v_i-qv_j)(v_j-v_i)}\mathrm{Sym}_{x_1,x_2,x_3}\left(\delta(v_1x_1)\delta(v_2x_2)\delta(v_3x_3)\right)$$

Noting that the symmetrized term is also symmetric in (v_1,v_2,v_3), the statement of the Theorem boils down to:

$$\mathrm{Sym}_{v_1,v_2,v_3}\left(\frac{v_2}{v_3}\big(1-(23)-(123)+(13)\big)\prod_{1\le i<j\le 3}\frac{(tv_j-v_i)(tv_i-qv_j)}{(v_i-qv_j)(v_j-v_i)}\right)=0$$

where the permutations on the left act by permuting the v's. This latter identity is easily checked. ☐

5.3.3. *Application II: commuting difference operators.* The Macdonald operators form a commuting family whose common eigenfunctions are the celebrated Macdonald polynomials [Mac95]. Their commutativity boils down via the relations (5.30) to the identity

$$1_\alpha*1_\beta=1_\beta*1_\alpha$$

where we denote by 1_α the constant function $1\in\mathcal{F}_\alpha$. This identity amounts to the following:

Lemma 5.19. *The following identity in $\mathcal{F}_{\alpha+\beta}$ holds.*

$$\mathrm{Sym}\left(\prod_{1\le i\le\alpha<j\le\alpha+\beta}\zeta(x_i/x_j)\right)=\mathrm{Sym}\left(\prod_{1\le i'\le\beta<j'\le\alpha+\beta}\zeta(x_{i'}/x_{j'})\right)\tag{5.31}$$

Proof. The symmetrized function is by definition invariant under any permutation of the x's. Consider the permutation $\sigma\in S_{\alpha+\beta}$ such that $\sigma(i)=\alpha+\beta+1-i$ for all $i\in\{1,2,...,\alpha+\beta\}$. The permutation σ clearly maps the range $1\le i\le\alpha<j\le\alpha+\beta$ to $1\le\sigma(j)\le\beta<\sigma(i)\le\alpha+\beta$, and the identity follows by noting that the result is invariant under $x_i\mapsto x_i^{-1}$ for all i. ☐

Likewise, using the definition $\mathcal{M}_{\alpha;n}=\mathcal{D}_\alpha((x_1\cdots x_\alpha)^n)$, the commutation of the operators $\mathcal{M}_{\alpha;n}$ for any fixed n and $\alpha=1,2,...,N$ is a consequence of the commutation:

$$(x_1\cdots x_\alpha)^n*(x_1\cdots x_\beta)^n=(x_1\cdots x_\beta)^n*(x_1\cdots x_\alpha)^n$$

The latter is a trivial consequence of Lemma 5.19, as the product $(x_1\cdots x_\alpha)^n(x_{\alpha+1}\cdots x_{\alpha+\beta})^n$ is symmetric in the x's, and therefore drops out of the symmetrization.

More generally, we could try to find commuting families of similar difference operators, by looking for families of shuffle-commuting rational functions. In fact, any factorized choice for $P(x_1,...,x_\alpha)=f(x_1)f(x_2)\cdots f(x_\alpha)$ and similarly for $P'(x_1,...,x_\beta)=f(x_1)f(x_2)\cdots f(x_\beta)$ will trivially lead to commuting operators from $P*P'=P'*P$, as the product $f(x_1)\cdots f(x_{\alpha+\beta})$ drops out of the symmetrization and leaves us with the identity of Lemma 5.19. The previous case corresponds to $f(x)=x^n$.

Consider now the function $f(x)=1+ax$ for some fixed arbitrary coefficient a. Note that

$$\prod_{i=1}^{\alpha} f(x_i)=\sum_{k=0}^{\alpha} a^k s_{1^k,0^{\alpha-k}}(x_1,...,x_\alpha)$$

where the notation $1^k0^{\alpha-k}$ stands for k 1's followed by $\alpha-k$ 0's. We deduce that

$$\mathcal{D}_\alpha(f(x_1)\cdots f(x_\alpha))=\sum_{k=0}^{\alpha} a^k \mathcal{M}_{1^k,0^{\alpha-k}}$$

Expressing the commutation relation $[\mathcal{D}_\alpha(f(x_1)\cdots f(x_\alpha)),\mathcal{D}_\beta(f(x_1)\cdots f(x_\beta))]=0$ and identifying the coefficient of a^k yields the following relations between the $\mathcal{M}$'s:

$$\sum_{\ell=\mathrm{Max}(0,k-\beta)}^{\mathrm{Min}(k,\alpha)} [\mathcal{M}_{1^\ell,0^{\alpha-\ell}},\mathcal{M}_{1^{k-\ell},0^{\beta-k+\ell}}]=0 \qquad (0\leq k\leq \alpha+\beta)$$

For $\alpha=1$, $\beta=2$, this reduces for instance to:

$$[\mathcal{M}_0,\mathcal{M}_{0,0}]=0,\ \ [\mathcal{M}_0,\mathcal{M}_{1,0}]+[\mathcal{M}_1,\mathcal{M}_{0,0}]=0,\ \ [\mathcal{M}_0,\mathcal{M}_{1,1}]+[\mathcal{M}_1,\mathcal{M}_{1,0}]=0,\ \ [\mathcal{M}_1,\mathcal{M}_{1,1}]=0\ .$$

5.3.4. *Application III: proving relations between the difference operators.* Another application of Theorem 5.15 consists in using shuffle identities to prove identities between difference operators, the former being much simpler than the latter. Let us illustrate this on the following examples, which allow to express any operator $\mathcal{M}_{n,p}$ as a quadratic polynomial of the $\mathcal{M}_n$'s (an alternative expression to that of Theorem 5.10). We denote by $[x,y]_q=xy-qyx$ the q-commutator of x,y.

We shall proceed in several steps. First we note that, due to the skew-symmetry of the determinantal definition of the generalized Schur function (1.6), we have $s_{n,p}=-s_{p-1,n+1}$, and therefore $\mathcal{M}_{n,p}=-\mathcal{M}_{p-1,n+1}$. This allows to restrict ourselves to $n\geq p$, as $\mathcal{M}_{n,n+1}=0$. In the two following Theorems 5.20 and 5.21, we find explicit expressions for $\mathcal{M}_{n+k,n}$, respectively for $k>0$ odd and even. Finally, we express $\mathcal{M}_{n,n}$ in Lemma 5.37, which allows to complete the Theorem, with explicit quadratic expressions for all $\mathcal{M}_{n,p}$.

Theorem 5.20. *We have the relation:*

$$(1-q)t\,\mathcal{M}_{n+2k+1,n}=\sum_{\ell=0}^{k}\nu_{n+2\ell,n+2k-2\ell+1}=\sum_{\ell=0}^{k}[\mathcal{M}_{n+2\ell},\mathcal{M}_{n+2k-2\ell+1}]_q \tag{5.32}$$

Proof. The shuffle relation corresponding to (5.32) reads, in $\mathcal{F}_2$:

$$\begin{aligned}\sum_{\ell=0}^{k}(x_1)^{n+2k-2\ell} * (x_1)^{n+2\ell+1} - q(x_1)^{n+2k-2\ell+1} * (x_1)^{n+2\ell} &= (1-q)t\, s_{n+2k+1,n}(x_1,x_2)\\ &= (1-q)t\,(x_1x_2)^n\, s_{2k+1,0}(x_1,x_2)\end{aligned}$$

Explicitly, we write

$$\begin{aligned}&\mathrm{Sym}\left(\Big(\sum_{\ell=0}^{k}(x_1)^{n+2k-2\ell}(x_2)^{n+2\ell+1} - q(x_1)^{n+2k-2\ell+1}(x_2)^{n+2\ell}\Big)\frac{(x_2-tx_1)(tx_2-qx_1)}{(x_2-x_1)(x_2-qx_1)}\right)\\ &\quad= (x_1x_2)^n\mathrm{Sym}\left(\frac{(x_2^2)^{k+1}-(x_1^2)^{k+1}}{x_2^2-x_1^2}\frac{(x_2-tx_1)(tx_2-qx_1)}{x_2-x_1}\right)\\ &\quad= \frac{(x_1x_2)^n}{(x_2-x_1)(x_2^2-x_1^2)}\mathrm{Sym}\left(((x_2)^{2k+2}-(x_1)^{2k+2})(x_2-tx_1)(tx_2-qx_1)\right)\\ &\quad= (1-q)t\,(x_1x_2)^n\, s_{2k+1,0}(x_1,x_2)\end{aligned}$$

where we identified $s_{a,0}(x_1,x_2)=(x_1^{a+1}-x_2^{a+1})/(x_1-x_2)$. The Theorem follows. □

This gives an explicit expression for $\mathcal{M}_{n+2k+1,n}$ for $k\geq 0$.

Theorem 5.21. *We have the relations, for all $k\geq 0$:*

$$\begin{aligned}(1-q)t\,(\mathcal{M}_{n+4k,n}-\mathcal{M}_{n+2k,n+2k}) &= \sum_{\ell=0}^{k-1}\nu_{n+2\ell,n+4k-2\ell}+\nu_{n+4k-1-2\ell,n+2\ell+1}\\ &= \sum_{\ell=0}^{k-1}[\mathcal{M}_{n+2\ell},\mathcal{M}_{n+4k-2\ell}]_q+[\mathcal{M}_{n+4k-1-2\ell},\mathcal{M}_{n+2\ell+1}]_q \qquad (5.33)\\ (1-q)t\,(\mathcal{M}_{n+4k+2,n}+\mathcal{M}_{n+2k+1,n+2k+1}) &= \sum_{\ell=0}^{k}\nu_{n+2\ell,n+4k+2-2\ell}+\nu_{n+4k+1-2\ell,n+2\ell+1}\\ &= \sum_{\ell=0}^{k}[\mathcal{M}_{n+2\ell},\mathcal{M}_{n+4k+2-2\ell}]_q+[\mathcal{M}_{n+4k+1-2\ell},\mathcal{M}_{n+2\ell+1}]_q \qquad (5.34)\end{aligned}$$

Proof. The shuffle relations corresponding to (5.33) and (5.34) read respectively in $\mathcal{F}_2$:

$$\begin{aligned}
&\sum_{\ell=0}^{k-1}\big\{(x_1)^{n+4k-1-2\ell}*(x_1)^{n+2\ell+1}-q(x_1)^{n+4k-2\ell}*(x_1)^{n+2\ell}\\
&\qquad +(x_1)^{n+2\ell}*(x_1)^{n+4k-2\ell}-q(x_1)^{n+2\ell+1}*(x_1)^{n+4k-1-2\ell}\big\}\\
&\qquad\qquad =(1-q)t\big(s_{n+4k,n}(x_1,x_2)-s_{n+2k,n+2k}(x_1,x_2)\big)\\
&\sum_{\ell=0}^{k}\big\{(x_1)^{n+4k+1-2\ell}*(x_1)^{n+2\ell+1}-q(x_1)^{n+4k+2-2\ell}*(x_1)^{n+2\ell}\\
&\qquad +(x_1)^{n+2\ell}*(x_1)^{n+4k+2-2\ell}-q(x_1)^{n+2\ell+1}*(x_1)^{n+4k+1-2\ell}\big\}\\
&\qquad\qquad =(1-q)t\big(s_{n+4k+2,n}(x_1,x_2)+s_{n+2k+1,n+2k+1}(x_1,x_2)\big)
\end{aligned}$$

Explicitly, we write:

$$\begin{aligned}
&\mathrm{Sym}\left(\Big(\sum_{\ell=0}^{k-1}(x_1)^{n+4k-1-2\ell}(x_2)^{n+2\ell}+(x_1)^{n+2\ell}(x_2)^{n+4k-1-2\ell}\Big)\frac{(x_2-tx_1)(tx_2-qx_1)}{x_2-x_1}\right)\\
&=\ t(1-q)(x_1+x_2)(x_1x_2)^n\Big(\sum_{\ell=0}^{k-1}(x_1)^{4k-1-2\ell}(x_2)^{2\ell}+(x_1)^{2\ell}(x_2)^{4k-1-2\ell}\Big)\\
&=\ t(1-q)(x_1x_2)^n\Big(\sum_{\ell=0}^{k-1}(x_1)^{4k-2\ell}(x_2)^{2\ell}+(x_1)^{2\ell}(x_2)^{4k-2\ell}\\
&\qquad\qquad +(x_1)^{2\ell+1}(x_2)^{4k-1-2\ell}+(x_1)^{4k-1-2\ell}(x_2)^{2\ell+1}\Big)\\
&\qquad =t(1-q)\,(x_1x_2)^n\Big(\sum_{\ell=0}^{4k}(x_1)^{4k-\ell}(x_2)^{\ell}-(x_1x_2)^{2k}\Big)\\
&\qquad =t(1-q)\,\Big(s_{n+4k,n}(x_1,x_2)-s_{n+2k,n+2k}(x_1,x_2)\Big)
\end{aligned}$$

Analogously, we have:

$$\begin{aligned}
&\mathrm{Sym}\left(\Big(\sum_{\ell=0}^{k}(x_1)^{n+4k+1-2\ell}(x_2)^{n+2\ell}+(x_1)^{n+2\ell}(x_2)^{n+4k+1-2\ell}\Big)\frac{(x_2-tx_1)(tx_2-qx_1)}{x_2-x_1}\right)\\
&= t(1-q)(x_1+x_2)(x_1x_2)^n\Big(\sum_{\ell=0}^{k}(x_1)^{4k+1-2\ell}(x_2)^{2\ell}+(x_1)^{2\ell}(x_2)^{4k+1-2\ell}\Big)\\
&= t(1-q)(x_1x_2)^n\Big(\sum_{\ell=0}^{k}(x_1)^{4k+2-2\ell}(x_2)^{2\ell}+(x_1)^{4k+1-2\ell}(x_2)^{2\ell+1}\\
&\qquad\qquad +(x_1)^{2\ell+1}(x_2)^{4k+1-2\ell}+(x_1)^{2\ell}(x_2)^{4k+2-2\ell}\Big)\\
&\qquad = t(1-q)\,(x_1x_2)^n\Big(\sum_{\ell=0}^{4k+2}(x_1)^{4k+2-\ell}(x_2)^{\ell}+(x_1x_2)^{2k+1}\Big)\\
&\qquad = t(1-q)\,\Big(s_{n+4k+2,n}(x_1,x_2)+s_{n+2k+1,n+2k+1}(x_1,x_2)\Big)
\end{aligned}$$

This completes the proof of the shuffle relations and the Theorem follows. $\square$

To get an expression for $\mathcal{M}_{n+2k,n}$ for all $k>0$, we also need to compute $\mathcal{M}_{n,n}$.

Lemma 5.22. *We have the relations:*

$$(\mathcal{M}_n)^2 - q\,\mathcal{M}_{n+1}\mathcal{M}_{n-1} = (q+t+t^2)\,\mathcal{M}_{n,n} - qt\,\mathcal{M}_{n+1,n-1} \tag{5.35}$$

$$(\mathcal{M}_n)^2 - q^{-1}\mathcal{M}_{n-1}\mathcal{M}_{n+1} = (1+t+q^{-1}t^2)\,\mathcal{M}_{n,n} - q^{-1}t\,\mathcal{M}_{n+1,n-1} \tag{5.36}$$

Equivalently, we have:

$$\mathcal{M}_{n,n} = \frac{(1-q^2)(\mathcal{M}_n)^2 + q\,[\mathcal{M}_{n-1},\mathcal{M}_{n+1}]}{(1-q)(1+t)(q+t)} \tag{5.37}$$

and

$$\mathcal{M}_{n+1,n-1} = \frac{(1-q)(q+t^2)(\mathcal{M}_n)^2 + (1+t)(q+t)\,[\mathcal{M}_{n-1},\mathcal{M}_{n+1}]_q - tq\,[\mathcal{M}_{n-1},\mathcal{M}_{n+1}]}{(1-q)(1+t)(q+t)} \tag{5.38}$$

Proof. The shuffle relations corresponding to (5.35) and (5.36) read in $\mathcal{F}_2$:

$$\begin{aligned}
(x_1)^n * (x_1)^n - q(x_1)^{n+1} * (x_1)^{n-1} &= (q+t+t^2)s_{n,n}(x_1,x_2) - qt\,s_{n+1,n-1}(x_1,x_2)\\
(x_1)^n * (x_1)^n - q(x_1)^{n-1} * (x_1)^{n+1} &= (1+t+q^{-1}t^2)s_{n,n}(x_1,x_2) - q^{-1}t\,s_{n+1,n-1}(x_1,x_2)
\end{aligned}$$

These read respectively:

$$\begin{aligned}\mathrm{Sym}\left((x_1^n x_2^n - q x_1^{n+1}x_2^{n-1})\frac{(x_2-tx_1)(tx_2-qx_1)}{(x_2-x_1)(x_2-qx_1)}\right)\\ = (x_1x_2)^{n-1}\mathrm{Sym}\left(\frac{x_1(x_2-tx_1)(tx_2-qx_1)}{x_2-x_1}\right)\\ = (x_1x_2)^{n-1}((q+t+t^2)x_1x_2 - qt(x_1^2+x_1x_2+x_2^2))\end{aligned}$$

and

$$\begin{aligned}\mathrm{Sym}\left((x_1^n x_2^n - q^{-1} x_1^{n-1}x_2^{n+1})\frac{(x_2-tx_1)(tx_2-qx_1)}{(x_2-x_1)(x_2-qx_1)}\right)\\ = -q^{-1}(x_1x_2)^{n-1}\mathrm{Sym}\left(\frac{x_2(x_2-tx_1)(tx_2-qx_1)}{x_2-x_1}\right)\\ = -q^{-1}(x_1x_2)^{n-1}(t(x_1^2+x_1x_2+x_2^2) - (q+qt+t^2)x_1x_2)\end{aligned}$$

and eqs. 5.35 and 5.36 follow from the values $s_{n,n}=(x_1x_2)^n$ and $s_{n+1,n-1}=(x_1x_2)^{n-1}(x_1^2+x_1x_2+x_2^2)$. Finally eq. (5.37) follows from the combination (5.35)$-q$ (5.36). □

Substitution of the expression (5.37) into the relations (5.33-5.34) yields alternative polynomial expressions to those of Theorem 5.10.

6. Elliptic Hall algebra and q,t-deformed Q-system relations

In this section, we use the known relation between DAHA and Elliptic Hall Algebra to obtain new relations between our generalized Macdonald operators.

6.1. Elliptic Hall Algebra: definition and isomorphisms. In this section, we recall known results from [Sch12]. Let

$$\alpha_k := \frac{1}{k}(1-q^k)(1-t^{-k})(1-q^{-k}t^k), \qquad (k\in\mathbb{Z}^*). \tag{6.1}$$

Definition 6.1. *The Elliptic Hall Algebra (EHA) has generators $u_{a,b},\theta_{a,b}$, $a,b\in\mathbb{Z}^*$, subject to the commutations:*

$$(E1)\quad [u_{c,d},u_{a,b}]=0 \qquad \text{if (0,0), (a,b), (c,d) are aligned}$$

and

$$(E2)\quad [u_{c,d},u_{a,b}]=\frac{\epsilon_{a,b;c,d}}{\alpha_1}\theta_{a+c,b+d},\quad \text{if } gcd(a,b)=1 \quad\text{and}\quad \Delta_{a,b;c,d}=\emptyset$$

where $\Delta_{a,b;c,d}$ is the intersection with $\mathbb{Z}^2$ of the strict interior of the triangle $(0,0),(a,b),(a+c,b+d)$, and $\epsilon_{a,b;c,d}=\mathrm{sgn}(ad-bc)$. The generators θ and u for non-coprime indices are *further related via:*

$$(E3) \quad 1+\sum_{n=1}^{\infty}\theta_{n(a,b)}z^n = e^{\sum_{k=1}^{\infty}\alpha_k u_{k(a,b)}z^k} \qquad (gcd(a,b)=1)$$

and in particular $\theta_{n,p}=\alpha_1\, u_{n,p}$ *whenever* $gcd(n,p)=1$*.*

The EHA is isomorphic to the quantum toroidal algebra of Section 3 via the following assignments:

$$e(z)=\sum_{n\in\mathbb{Z}}u_{1,n}\,z^n,\quad f(z)=\sum_{n\in\mathbb{Z}}u_{-1,n}\,z^n,\quad \psi_\pm(z)=1+\sum_{n\geq 1}\theta_{0,\pm n}\,z^{\pm n} \tag{6.2}$$

There is a natural action of $SL(2,\mathbb{Z})$ on the generators $u_{a,b}$ given by:

$$\begin{pmatrix} a_0 & a_1\\ a_2 & a_3\end{pmatrix}\cdot u_{a,b}=u_{a_0a+a_2b,a_1a+a_3b}$$

For further use, we single out the two generators:

$$T=\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad U=\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix} \tag{6.3}$$

acting on the EHA generators respectively as $T\,u_{a,b}=u_{a,a+b}$ and $U\,u_{a,b}=u_{a+b,b}$.

In [SV11], the authors constructed an isomorphism between the EHA and the Spherical DAHA (in infinitely many variables X_i, $i\in\mathbb{N}$). In particular, the natural $SL(2,\mathbb{Z})$ action on the EHA maps onto the natural $SL(2,\mathbb{Z})$ action on the DAHA [Che05], namely:

$$T\mapsto \tau_+=\mathrm{ad}_{\gamma^{-1}},\quad \text{and} \quad U\mapsto\tau_-=\mathrm{ad}_{\eta^{-1}}$$

with γ and η as in (2.8) and (2.9), and Lemmas 2.12 and 2.13 respectively.

The isomorphism maps the generators $u_{k,0}$ to the power sums $P_k:=\sum_i(Y_i)^k$ operators in the spherical version of DAHA, which is respected by the functional representation of Sect. 2.2 above.

6.2. **EHA representation via generalized Macdonald operators.** We are now ready to complete the identification of generators of the EHA in terms of generalized Macdonald operators.

Comparing (6.2) with the representation of Theorem 3.3 involving generalized Macdonald operators, we easily deduce the following representation of the EHA, or rather a quotient thereof corresponding to the finite number N of variables[2]:

$$u_{1,n}=\frac{q^{1/2}}{1-q}q^{\frac{n}{2}}\,\mathcal{D}_{1;n}^{q,t}=\frac{q^{1/2}}{1-q}q^{\frac{n}{2}}t^{\frac{1-N}{2}}\,\mathcal{M}_n,\qquad u_{-1,n}=\frac{q^{-1/2}}{1-q^{-1}}q^{-\frac{n}{2}}\,\mathcal{D}_{1;n}^{q^{-1},t^{-1}}, \tag{6.4}$$

[2]These assignments first appeared in [Mik07] Proposition 3.3, with different notations $(q,\gamma,y_i)\mapsto(\theta=t^{1/2},q/\theta,x_i)$.

whereas the relation:

$$1+\sum_{n\geq 1}\theta_{0,\pm n}\, z^{\pm n}=e^{\sum_{k\geq 1}\frac{z^{\pm k}}{k}p_{\pm k}q^{\frac{k}{2}}(1-t^{-k})(1-t^k q^{-k})}$$

fixes the values:

$$u_{0,\pm k}=\frac{q^{\frac{k}{2}}}{(1-q^k)}p_{\pm k}\qquad (k\in\mathbb{Z}_{>0}) \tag{6.5}$$

Moreover, from the spherical DAHA homomorphism, it is easy to identify the $u_{k,0}$ from the DAHA power sum operators, which in the functional representation are:

$$\mathcal{P}_k:=\sum_{i=1}^N (Y_i)^k|_{\mathcal{S}_N}$$

and are related for $k>0$ to the Macdonald operators $\mathcal{D}_\alpha\equiv\mathcal{D}_{\alpha;0}^{q,t}$ via:

$$\prod_{i=1}^N(1-zY_i)|_{\mathcal{S}_N}=\sum_{\alpha=0}^N(-z)^\alpha\mathcal{D}_\alpha=e^{-\sum_{k\geq 1}\mathcal{P}_k\frac{z^k}{k}} \tag{6.6}$$

while for $k<0$ they are expressed in terms of the dual operators $\bar{\mathcal{D}}_\alpha\equiv\mathcal{D}_{\alpha;0}^{q^{-1},t^{-1}}$ via:

$$\sum_{\alpha=0}^N(-z)^\alpha\bar{\mathcal{D}}_\alpha=e^{-\sum_{k\geq 1}\mathcal{P}_{-k}\frac{z^k}{k}} \tag{6.7}$$

Finally from the action of $-\epsilon$ which maps $u_{0,k}$ to $u_{k,0}$, we have the identification:

$$u_{\pm k,0}=\frac{q^{\frac{k}{2}}}{(1-q^k)}\mathcal{P}_{\pm k}\qquad (k\in\mathbb{Z}_{>0}) \tag{6.8}$$

Remark 6.2. *We continue the comparison with the work of* [BGLX14] *started in Remarks 2.14 and 4.7. In this paper a connection to the positive quadrant of the Elliptic Hall Algebra (with generators $u_{a,b}\to Q_{a,b}$ with $a,b\geq 0$) was found by setting*

$$Q_{1,k}=D_k\quad (k\geq 0),\quad \text{and}\quad Q_{0,1}=-e_1=-p_1.$$

for $k\geq 0$, where the D_k's are defined via their generating current in (4.9)*, and extending the identification via the $SL(2,Z)$ action on the EHA. By Theorem 4.6, we have the relations*

$$\begin{aligned}\lim_{N\to\infty} t^{1-N}\,\mathcal{M}_n &= \frac{1}{1-t^{-1}}(\Sigma^{-1}\,D_n\,\Sigma)|_{t\to t^{-1}}\\ \lim_{N\to\infty} -p_1(x_1,x_2,...,x_N) &= -p_1(x_1,x_2,...)\end{aligned}$$

Comparing with our functional representation of EHA in the limit of infinite number of variables of $x_1, x_2, \ldots$, we find:

$$\lim_{N\to\infty} t^{\frac{1-N}{2}} u_{1,k} = \frac{q^{\frac{k+1}{2}}}{(1-q)(1-t^{-1})} (\Sigma^{-1}\, Q_{1,k}\, \Sigma)|_{t\to t^{-1}}$$

and

$$\lim_{N\to\infty} u_{0,1} = \frac{q^{\frac{1}{2}}}{(1-q)(1-t^{-1})} (\Sigma^{-1}\, (-p_1)\, \Sigma)|_{t\to t^{-1}}$$

Moreover, by using (4.12)*, and* $\nabla^{(N)} u_{a,b} \nabla^{(N)^{-1}} = u_{a+b,b}$ *we have:*

$$\begin{aligned}
&\lim_{N\to\infty} t^{\frac{(1-N)(k+1)}{2}} \nabla^{(N)} u_{1,k} \nabla^{(N)^{-1}} = \lim_{N\to\infty} t^{\frac{(1-N)(k+1)}{2}} u_{k+1,k} \\
&= \lim_{N\to\infty} t^{\frac{(1-N)(k+1)}{2}} (t^{\frac{N-1}{2}} q^{\frac{1}{2}})^d \, (\Sigma^{-1}\, \nabla\, \Sigma)|_{t\to t^{-1}} u_{1,k} (\Sigma^{-1}\, \nabla^{-1}\, \Sigma)|_{t\to t^{-1}} \, (t^{\frac{N-1}{2}} q^{\frac{1}{2}})^{-d} \\
&= \frac{q^{\frac{k+1}{2}}}{(1-q)(1-t^{-1})} \lim_{N\to\infty} t^{\frac{(1-N)(k)}{2}} (t^{\frac{N-1}{2}} q^{\frac{1}{2}})^d \, (\Sigma^{-1}\, \nabla\, Q_{1,k}\, \nabla^{-1} \Sigma)|_{t\to t^{-1}} \, (t^{\frac{N-1}{2}} q^{\frac{1}{2}})^{-d}
\end{aligned}$$

Note that the action of $Q_{1,k}$ on $\tilde{H}_\lambda$ is a linear combination of terms in which k boxes are added to λ, and similarly for $u_{1,k}$ acting on P_λ. We deduce that the conjugation by $(t^{\frac{N-1}{2}} q^{\frac{1}{2}})^d$ amounts to a factor $(t^{\frac{N-1}{2}} q^{\frac{1}{2}})^k$, and using $\nabla Q_{a,b} \nabla^{-1} = Q_{a+b,b}$ (from [BGLX14]*) we finally get:*

$$\lim_{N\to\infty} t^{\frac{(1-N)(k+1)}{2}} u_{k+1,k} = \frac{q^{\frac{2k+1}{2}}}{(1-q)(1-t^{-1})} (\Sigma^{-1}\, Q_{k+1,k} \Sigma)|_{t\to t^{-1}}$$

Repeating the inductive proof of [BGLX14]*, we arrive in general at:*

$$\lim_{N\to\infty} t^{\frac{(1-N)a}{2}} u_{a,b} = \frac{q^{\frac{a+b}{2}}}{(1-q)(1-t^{-1})} (\Sigma^{-1}\, Q_{a,b} \Sigma)|_{t\to t^{-1}}$$

for any coprime (a,b).

6.3. **EHA and relations between generalized Macdonald operators.** The EHA representation provides us with an alternative way of finding relations between the generalized Macdonald operators. We first concentrate on an alternative version of Theorem 5.7, aimed at expressing the generalized Macdonald operator $\mathcal{M}_{\alpha;n}$ as an explicit polynomial of the $\mathcal{M}_i$'s.

Theorem 6.3. *The operator $\mathcal{D}_{\alpha;n}$ is expressible as a homogeneous polynomial of degree α in the variables $\mathcal{D}_{1;n}, \mathcal{D}_{1;n\pm1}$, with coefficients in $\mathbb{C}(q,t)$.*

Proof. Let us first use the definition of EHA to compute $\theta_{n,0}$ as an iterated commutator. We first note that

$$\begin{aligned} u_{n-1,1} &= [u_{n-2,1},u_{1,0}] = \cdots = \big[[\cdots[u_{1,1},u_{1,0}],u_{1,0}]\cdots,u_{1,0}\big] \\ &= \frac{q^{\frac{n}{2}}}{(1-q)^{n-1}}\big[[\cdots[\mathcal{D}_{1;1},\mathcal{D}_{1;0}],\mathcal{D}_{1;0}]\cdots,\mathcal{D}_{1;0}\big] \end{aligned}$$

Moreover, we have:

$$\theta_{n,0} = \alpha_1\,[u_{n-1,1},u_{1,-1}] = \frac{\alpha_1\,q^{\frac{n}{2}}}{(1-q)^n}\,\big[\cdots[\mathcal{D}_{1;1},\mathcal{D}_{1;0}],\mathcal{D}_{1;0}],\ldots,\mathcal{D}_{1;0}],\mathcal{D}_{1;-1}\big] \tag{6.9}$$

for $n \geq 2$, while $\theta_{1,0} = \alpha_1 u_{1,0} = \frac{\alpha_1\,q^{\frac{1}{2}}}{(1-q)}\,\mathcal{D}_{1;0}$. On the other hand, $\theta_{n,0}$ is related to the $u_{k,0}$ via the relation $(E3)$ of Definition 6.1 above:

$$1+\sum_{n>0}\theta_{n,0}z^n = e^{\sum_{k>0}\alpha_k u_{k,0}z^k} = e^{\sum_{k>0}q^{k/2}(1-t^{-k})(1-q^{-k}t^k)\mathcal{P}_k\frac{z^k}{k}} \tag{6.10}$$

Eliminating $\mathcal{P}_k$ between this and (6.6), we are left with an algebraic relation of the form:

$$\mathcal{D}_{\alpha;0} = \varphi(\theta_{1,0},\theta_{2,0},...,\theta_{\alpha,0})$$

where φ is a quasi-homogeneous polynomial of total degree α (assuming $\theta_{i,0}$ has degree i), with coefficients in $\mathbb{C}(q,t)$. Combining this with (6.9), we deduce that $\mathcal{D}_{\alpha;0}$ is a homogeneous polynomial of degree α in the three variables $(\mathcal{D}_{1;1},\mathcal{D}_{1;0},\mathcal{D}_{1;-1})$, with coefficients in $\mathbb{C}(q,t)$. This proves the Theorem for $n=0$. To get to arbitrary n, we simply have to iteratively conjugate by γ^{-1} and use $\mathcal{D}_{\alpha;n} = q^{\alpha n/2}\gamma^{-n}\mathcal{D}_{\alpha;0}\gamma^n$. $\square$

Corollary 6.4. *The operator $\mathcal{M}_{\alpha;n}$ is expressible as a homogeneous polynomial of degree α in the variables $\mathcal{M}_n,\mathcal{M}_{n\pm1}$, with coefficients in $\mathbb{C}(q,t)$.*

Proof. The result for $\mathcal{D}_{\alpha;n}$ of Theorem 6.3 immediately translates into that for $\mathcal{M}_{\alpha;n}$, due to the relations $\mathcal{D}_{\alpha;n} = t^{\alpha(\alpha-N)/2}\mathcal{M}_{\alpha;n}$. $\square$

Example 6.5. *Let us write the explicit polynomial relation of Theorem 6.3 above for $\alpha = 2,3$. We start by expressing $\mathcal{D}_{i;0}$'s in terms of $\mathcal{P}_i$'s via* (6.6)*, which gives:*

$$\mathcal{D}_{1;0} = \mathcal{P}_1, \quad \mathcal{D}_{2;0} = \frac{\mathcal{P}_1^2-\mathcal{P}_2}{2}, \quad \mathcal{D}_{3;0} = \frac{\mathcal{P}_1^3-3\mathcal{P}_1\mathcal{P}_2+2\mathcal{P}_3}{6}\ . \tag{6.11}$$

Next, we express the $\mathcal{P}_i$'s in terms of the $\theta_{i,0}$'s via (6.10)*, which gives:*

$$\mathcal{P}_1 = \frac{\theta_{1,0}}{q^{1/2}(1-t^{-1})(1-q^{-1}t)},\ \mathcal{P}_2 = \frac{2\theta_{2,0}-\theta_{1,0}^2}{q(1-t^{-2})(1-q^{-2}t^2)},\ \mathcal{P}_3 = \frac{3\theta_{3,0}-3\theta_{1,0}\theta_{2,0}+\theta_{1,0}^3}{q^{3/2}(1-t^{-3})(1-q^{-3}t^3)}\ .$$

These are reexpressed in terms of $\mathcal{D}_{1;1}, \mathcal{D}_{1;0}, \mathcal{D}_{1;-1}$ *by using:*

$$\theta_{1,0} = \alpha_1 \frac{q^{1/2}}{1-q}\mathcal{D}_{1;0}, \;\; \theta_{2,0} = \alpha_1 \frac{q}{(1-q)^2}[\mathcal{D}_{1;1}, \mathcal{D}_{1;-1}], \;\; \theta_{3,0} = \alpha_1 \frac{q^{3/2}}{(1-q)^3}\big[[\mathcal{D}_{1;1}, \mathcal{D}_{1;0}], \mathcal{D}_{1;-1}\big],$$

as:

$$\begin{aligned}
\mathcal{P}_1 &= \mathcal{D}_{1;0}\\
\mathcal{P}_2 &= \frac{\alpha_1}{(1-q)^2(1-t^{-2})(1-q^{-2}t^2)}\left(2[\mathcal{D}_{1;1}, \mathcal{D}_{1;-1}] - \alpha_1 \mathcal{D}_{1;0}^2\right)\\
\mathcal{P}_3 &= \frac{\alpha_1}{(1-q)^3(1-t^{-3})(1-q^{-3}t^3)}\left(3\big[[\mathcal{D}_{1;1}, \mathcal{D}_{1;0}], \mathcal{D}_{1;-1}\big] - 3\alpha_1 \mathcal{D}_{1;0}[\mathcal{D}_{1;1}, \mathcal{D}_{1;-1}] + \alpha_1^2 \mathcal{D}_{1;0}^3\right)
\end{aligned}$$

Substituting this into (6.11) *gives the polynomial relations:*

$$\begin{aligned}
\mathcal{D}_{2;0} &= \frac{t}{(1+t)(q+t)}\left\{(1+q)\mathcal{D}_{1;0}^2 - \frac{q}{1-q}\left[\mathcal{D}_{1;1}, \mathcal{D}_{1;-1}\right]\right\}\\
\mathcal{D}_{3;0} &= \frac{t^2}{(1+t)(q+t)(1+t+t^2)(q^2+qt+t^2)}\Big\{(q(q+t^2) + t(1+q+q^2+q^3))\mathcal{D}_{1;0}^3\\
&\quad + \frac{q((1+q)(1+t)(q+t) + t(1-q+q^2))}{1-q}\mathcal{D}_{1;0}[\mathcal{D}_{1;1}, \mathcal{D}_{1;-1}] + \frac{q^2}{(1-q)^2}\big[[\mathcal{D}_{1;1}, \mathcal{D}_{1;0}], \mathcal{D}_{1;-1}\big]\Big\}
\end{aligned}$$

Finally conjugating n *times w.r.t.* γ^{-1} *gives:*

$$\begin{aligned}
\mathcal{D}_{2;n} &= \frac{t}{(1+t)(q+t)}\left\{(1+q)\mathcal{D}_{1;n}^2 - \frac{q}{1-q}\left[\mathcal{D}_{1;n+1}, \mathcal{D}_{1;n-1}\right]\right\}\\
\mathcal{D}_{3;n} &= \frac{t^2}{(1+t)(q+t)(1+t+t^2)(q^2+qt+t^2)}\Big\{(q(q+t^2) + t(1+q+q^2+q^3))\mathcal{D}_{1;n}^3\\
&\quad + \frac{q((1+q)(1+t)(q+t) + t(1-q+q^2))}{1-q}\mathcal{D}_{1;n}[\mathcal{D}_{1;n+1}, \mathcal{D}_{1;n-1}] + \frac{q^2}{(1-q)^2}\big[[\mathcal{D}_{1;n+1}, \mathcal{D}_{1;n}], \mathcal{D}_{1;n-1}\big]\Big\}
\end{aligned}$$

Note that the expression for $\mathcal{D}_{2;n}$ *above is in agreement with* (5.37)*, upon identifying* $\mathcal{D}_{2;n} = t^{2-N}\mathcal{M}_{n,n}$ *and* $\mathcal{D}_{1;n} = t^{\frac{1-N}{2}}\mathcal{M}_n$*. Moreover, using also* $\mathcal{D}_{3;n} = t^{3(3-N)/2}\mathcal{M}_{n,n,n}$*, we get:*

$$\begin{aligned}
\mathcal{M}_{3;n} &= \mathcal{M}_{n,n,n} = \frac{t^{-1}}{(1+t)(q+t)(1+t+t^2)(q^2+qt+t^2)}\Big\{(q(q+t^2) + t(1+q+q^2+q^3))\mathcal{M}_n^3\\
&\quad + \frac{q((1+q)(1+t)(q+t) + t(1-q+q^2))}{1-q}\mathcal{M}_n[\mathcal{M}_{n+1}, \mathcal{M}_{n-1}] + \frac{q^2}{(1-q)^2}\big[[\mathcal{M}_{n+1}, \mathcal{M}_n], \mathcal{M}_{n-1}\big]\Big\}
\end{aligned}$$

7. The dual Whittaker limit $t \to \infty$

In this section, we describe the $t \to \infty$ limit of the constructions of this paper, in particular the degenerations of our generalized Macdonald operators, and of the shuffle product.

7.1. **Quantum M-system and quantum determinant.**

7.1.1. *Quantum M-system.* The generalized Macdonald operators $\mathcal{M}_{\alpha;n}$ (1.5) were introduced in [DFK15] as the natural t-deformation of the difference operators:

$$M_{\alpha;n} := \lim_{t\to\infty} t^{-\alpha(N-\alpha)}\,\mathcal{M}_{\alpha;n} = \sum_{\substack{I\subset[1,N]\\|I|=\alpha}} x_I^n \prod_{\substack{i\in I\\ j\notin I}} \frac{x_i}{x_i-x_j}\,\Gamma_I \tag{7.1}$$

The difference operators $M_{\alpha;n}$, together with the quantities $\Delta=\Gamma_1\Gamma_2\cdots\Gamma_N$ and $A=x_1x_2\cdots x_N$ satisfy the quantum M-system relations, inherited from the (graded) quantum cluster algebra associated to the A_{N-1} quantum Q-system with a coefficient [DFK15]. These relations read for $\alpha,\beta=1,2,...,N$:

$$\begin{aligned}
M_{\alpha;n}\,M_{\beta;p} &= q^{\min(\alpha,\beta)(p-n)}\,M_{\beta;p}\,M_{\alpha;n} \qquad (|n-p|\le|\alpha-\beta|+1)\\
q^{\alpha}\,M_{\alpha;n+1}\,M_{\alpha;n-1} &= (M_{\alpha;n})^2 - M_{\alpha+1;n}\,M_{\alpha-1;n} \qquad (1\le\alpha\le N)\\
M_{0;n} &= 1, \qquad M_{N+1;n}=0, \qquad M_{N,n}=A^n\,\Delta\\
\Delta\,M_{\alpha;n} &= q^{\alpha\,n}\,M_{\alpha;n}\,\Delta, \quad M_{\alpha;n}\,A=q^{\alpha}\,A\,M_{\alpha;n}, \quad \Delta\,A=q^N\,A\,\Delta
\end{aligned}$$

7.2. **Currents.**

There is a simple linear relation between the t-deformed $\mathcal{M}_{1;n}$'s and the $M_{1;n}$'s.

Theorem 7.1. *We have the relation:*

$$\mathcal{M}_{1;n} = \frac{t^N}{t-1}\sum_{j=0}^{N}(-t^{-1})^j\,e_j(x_1,...,x_N)\,M_{1;n-j} \tag{7.2}$$

where e_i are the elementary symmetric functions.

Proof. Evaluating

$$\prod_{i=1}^{N}(tx-x_i) = \sum_{j=0}^{N}(-1)^j t^{N-j}x^{N-j}e_j(x_1,...,x_N) \tag{7.3}$$

at $x=x_1$, we get:

$$\prod_{i=2}^{N}(tx_1-x_i) = \frac{t^N}{t-1}\sum_{j=0}^{N}(-t^{-1})^j x_1^{N-1-j}e_j(x_1,...,x_N) \tag{7.4}$$

We deduce that

$$x_1^n\prod_{i=2}^{N}\frac{tx_1-x_i}{x_1-x_i} = \frac{t^N}{t-1}\sum_{j=0}^{N}(-t^{-1})^j\,e_j(x_1,...,x_N)\,x_1^{n-j}\,\prod_{i=2}^{N}\frac{x_1}{x_1-x_i}, \tag{7.5}$$

and the lemma follows by substituting this and its images under the interchanges $x_1\leftrightarrow x_i$ for $i=2,3,...,N$ into the expression (1.5) for $\alpha=1$. $\square$

In terms of currents, defining $m(z) := \sum_{n\in\mathbb{Z}} z^n M_{1;n}$, Theorem 7.1 translates into the following relation.

Corollary 7.2. *The t-deformed Macdonald current $\mathfrak{m}(u)$ is expressed in terms of the $t\to\infty$ limit $m(u)$ as:*

$$\mathfrak{m}(u) = \frac{t^N}{t-1} C(t^{-1}u)\, m(u) \tag{7.6}$$

with C as in (4.2).

Proof. Note that by definition $C(t^{-1}u) = \sum_{j=0}^{N} (-t^{-1}u)^j e_j(x_1,...,x_N)$ and compute the generating currents for both sides of (7.2). □

With the obvious definition of limiting currents

$$m_\alpha(z) := \sum_{n\in\mathbb{Z}} z^n M_{\alpha;n} = \lim_{t\to\infty} t^{-\alpha(N-\alpha)} \mathfrak{m}_\alpha(z) \ ,$$

it is easy to recover from eq.(5.17) the following quantum determinant expression for $m_\alpha(z)$ (see [DFK16], Theorem 2.10):

$$m_\alpha(z) = \mathrm{CT}_{\mathbf{u}} \left(\delta(u_1\cdots u_\alpha/z) \left(\prod_{1\le i<j\le\alpha} 1 - q\frac{u_j}{u_i} \right) \prod_{i=1}^{\alpha} m(u_i) \right) \tag{7.7}$$

Note also that the limit of the result of Corollary 5.6 yields the following alternative formula:

$$m_\alpha(z) = \frac{1}{\alpha!} \mathrm{CT}_{\mathbf{u}} \left(\delta(u_1\cdots u_\alpha/z) \prod_{1\le i<j\le\alpha} (u_i^{-1} - u_j^{-1})(u_i - qu_j) \prod_{i=1}^{\alpha} m(u_i) \right)$$

Similarly, we may consider the limiting difference operators:

$$D_\alpha(P) := \lim_{t\to\infty} t^{-\alpha(N-\alpha)}\, \mathcal{D}_\alpha(P) = M_\alpha(P) := \lim_{t\to\infty} t^{-\alpha(N-\alpha)}\, \mathcal{M}_\alpha(P) \ , \tag{7.8}$$

where

$$D_\alpha(P) = \frac{1}{\alpha!(N-\alpha)!} \mathrm{Sym} \left(P(x_1,...,x_\alpha) \prod_{1\le i\le\alpha<j\le N} \frac{x_i}{x_i - x_j} \Gamma_1\cdots\Gamma_\alpha \right) \tag{7.9}$$

$$M_\alpha(P) = \frac{1}{\alpha!} \mathrm{CT}_{\mathbf{u}} \left(P(u_1^{-1},...,u_\alpha^{-1}) \prod_{1\le i<j\le\alpha} (u_i^{-1} - u_j^{-1})(u_i - qu_j) \prod_{i=1}^{\alpha} m(u_i) \right) \tag{7.10}$$

as well as

$$M_{a_1,...,a_\alpha} := \lim_{t\to\infty} t^{-\alpha(N-\alpha)} \mathcal{M}_{a_1,...,a_\alpha} = M_\alpha(s_{a_1,...,a_\alpha}) \ ,$$

which is a polynomial of degree α in the M_ℓ's, as a direct consequence of formula (7.10) and the fact that $s_{a_1,...,a_\alpha}$ is a Laurent polynomial of $x_1,...,x_\alpha$.

7.2.1. *Quantum determinants and Alternating Sign Matrices.* The formula of Corollary 5.16 also gives the following alternative "quantum determinant" expression:

$$M_{a_1,\ldots,a_\alpha} = CT_{\mathbf{u}}\left(\prod_{i=1}^{\alpha} u_i^{-a_i}\left(\prod_{1\le i<j\le\alpha} 1-q\frac{u_j}{u_i}\right)\prod_{i=1}^{\alpha} m(u_i)\right) =: \Big|\big(M_{a_j+i-j}\big)_{1\le i,j\le\alpha}\Big|_q \tag{7.11}$$

or equivalently the generating multi-current expression:

$$M_\alpha(v_1,\ldots,v_\alpha) := \sum_{a_1,\ldots,a_\alpha\in\mathbb{Z}} M_{a_1,\ldots,a_\alpha} v_1^{a_1}v_2^{a_2}\cdots v_\alpha^{a_\alpha} = \left(\prod_{1\le i<j\le\alpha} 1-q\frac{v_j}{v_i}\right)\prod_{i=1}^{\alpha} m(v_i) \tag{7.12}$$

There is a very nice expression of the quantum determinant (7.11), involving a sum over Alternating Sign Matrices. This is because the quantity $\prod_{i<j} v_i+\lambda v_j$ is the λ-determinant $\lambda\det(V_n)$ (as defined by Robbins and Rumsey [RR86]) of the Vandermonde matrix $V_n := (v_i^{n-j})_{1\le i,j\le n}$. Recall that an $n\times n$ ASM A has elements $a_{i,j}\in\{0,1,-1\}$ such that each row and column sum is 1, and the non-zero entries alternate in sign along each row and column. We denote by ASM_n the set of such matrices. We need a few more definitions. The inversion number of an ASM is the quantity $I(A)=\sum_{i>k,j<\ell} A_{i,j}A_{k,\ell}$. We also count its number of -1's, which we call the -1 number, denoted by $N(A)$. Let us also define the column vector $v=(n-1,n-2,..,1,0)^t$, and for each ASM A we denote by $m_i(A) := (Av)_i$. Then we have the explicit formula, obtained by taking $\lambda=-q$ for the λ-determinant of the $\alpha\times\alpha$ Vandermonde matrix V_α:

$$\prod_{1\le i<j\le\alpha} v_i - qv_j = \sum_{A\in ASM_n} (-q)^{I(A)-N(A)}(1-q)^{N(A)}\prod_{i=1}^{n} v_i^{m_i(A)}$$

Combining this with (7.11), we deduce the following compact expression for the quantum determinant:

Theorem 7.3. *The quantum determinant of the matrix* $\big(M_{a_j+i-j}\big)_{1\le i,j\le\alpha}$ *reads:*

$$\Big|\big(M_{a_j+i-j}\big)_{1\le i,j\le\alpha}\Big|_q = \sum_{A\in ASM_\alpha} (-q)^{I(A)-N(A)}(1-q)^{N(A)}\prod_{i=1}^{\alpha} M_{a_i+\alpha-i-m_i(A)} \tag{7.13}$$

Example 7.4. *For $\alpha=2$, we have two ASMs:*

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

with respective inversion and -1 numbers $I(A)=0,1$ and $N(A)=0,0$, and with $(m_1(A),m_2(A)) = (1,0),(0,1)$. The formula (7.13) *gives:*

$$M_{a_1,a_2} = \begin{vmatrix} M_{a_1} & M_{a_2-1}\\ M_{a_1+1} & M_{a_2}\end{vmatrix}_q := M_{a_1}M_{a_2} - qM_{a_1+1}M_{a_2-1}$$

For $\alpha=3$, we have seven ASMs:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

with respective inversion and -1 numbers $I(A)=0,1,1,3,2,2,2$ and $N(A)=0,0,0,0,0,0,1$, and $(m_1(A),m_2(A),m_3(A)) = (2,1,0),(1,2,0),(2,0,1),(0,1,2),(0,2,1),(1,0,2),(1,1,1)$. The formula (7.13) gives:

$$M_{a_1,a_2,a_3}=\begin{vmatrix}M_{a_1}&M_{a_2-1}&M_{a_3-2}\\M_{a_1+1}&M_{a_2}&M_{a_3-1}\\M_{a_1+2}&M_{a_2+1}&M_{a_3}\end{vmatrix}_q := M_{a_1}M_{a_2}M_{a_3}-qM_{a_1+1}M_{a_2-1}M_{a_3}-qM_{a_1}M_{a_2+1}M_{a_3-1}$$
$$-q^3M_{a_1+2}M_{a_2}M_{a_3-2}+q^2M_{a_1+2}M_{a_2-1}M_{a_3-1}+q^2M_{a_1+1}M_{a_2+1}M_{a_3-2}-q(1-q)M_{a_1+1}M_{a_2}M_{a_3-1}$$

7.3. **The $t\to\infty$ limit of the shuffle product and the quantum M-system.** It is instructive to consider the limiting definition $\star$ of the shuffle product $*$, compatible with the limiting difference operators (7.8).

Define for $(P,P')\in\mathcal{F}_\alpha\times\mathcal{F}_\beta$ the product:

$$P\star P'(x_1,...,x_{\alpha+\beta}):=\frac{1}{\alpha!\,\beta!}\mathrm{Sym}\left(\frac{P(x_1,...,x_\alpha)P'(x_{\alpha+1},...,x_{\alpha+\beta})}{\prod_{1\le i\le\alpha<j\le\alpha+\beta}(x_j^{-1}-x_i^{-1})(x_j-qx_i)}\right)$$

It is given by the limit

$$P\star P'(x_1,...,x_{\alpha+\beta})=\lim_{t\to\infty}t^{-2\alpha\beta}\,P*P'(x_1,...,x_{\alpha+\beta})$$

The compatibility:

$$D_\alpha(P)D_\beta(P')=D_{\alpha+\beta}(P\star P')$$

simply follows by computing the limit

$$\lim_{t\to\infty}t^{-\alpha(N-\alpha)}\,\mathcal{D}_\alpha(P)\,t^{-\beta(N-\beta)}\,\mathcal{D}_\beta(P')=\lim_{t\to\infty}t^{-(\alpha+\beta)(N-\alpha-\beta)}\,\mathcal{D}_{\alpha+\beta}(t^{-2\alpha\beta}\,P*P')$$

.

Recall that $M_{\alpha;n}=D_\alpha((x_1x_2\cdots x_\alpha)^n)$. The quantum M-system relations boil down to relations in the $\star$ shuffle algebra, namely:

Theorem 7.5. *We have the following relations:*

$$\begin{aligned}&(x_1x_2\cdots x_\alpha)^n\star(x_1x_2\cdots x_\beta)^p\\&\qquad=q^{Min(\alpha,\beta)(p-n)}(x_1x_2\cdots x_\beta)^p\star(x_1x_2\cdots x_\alpha)^n\quad(n,p\in\mathbb{Z},\ |p-n|\le|\alpha-\beta|+1)\\&q^\alpha(x_1x_2\cdots x_\alpha)^{n+1}\star(x_1x_2\cdots x_\alpha)^{n-1}\\&\qquad=(x_1x_2\cdots x_\alpha)^n\star(x_1x_2\cdots x_\alpha)^n-(x_1x_2\cdots x_{\alpha+1})^n\star(x_1x_2\cdots x_{\alpha-1})^n\quad(n\in\mathbb{Z})\end{aligned}$$

which hold respectively in $\mathcal{F}_{\alpha+\beta}$ and $\mathcal{F}_{2\alpha}$, with $\alpha,\beta\in[1,N-1]$.

Example 7.6. *For $\alpha=\beta=1$, we have in $\mathcal{F}_2$:*

$$x_1^n \star x_1^{n+1} = q\, x_1^{n+1} \star x_1^n, \quad (x_1x_2)^n = x_1^n \star x_1^n - q\, x_1^{n+1} \star x_1^{n-1}$$

These follow from respectively

$$\begin{aligned}\mathrm{Sym}\left(\frac{(x_1x_2)^n}{(x_2^{-1}-x_1^{-1})}\right) = (x_1x_2)^{n+1}\,\mathrm{Sym}\left(\frac{1}{(x_1-x_2)}\right) &= 0\\ \mathrm{Sym}\left(\frac{x_1(x_1x_2)^{n-1}}{(x_2^{-1}-x_1^{-1})}\right) = (x_1x_2)^n\,\mathrm{Sym}\left(\frac{x_1}{(x_1-x_2)}\right) &= (x_1x_2)^n\end{aligned}$$

where Sym *denotes the symmetrization in x_1, x_2. The "higher" identities:*

$$\mathrm{Sym}\left(\frac{x_1^{k+1}(x_1x_2)^{n-1}}{(x_2^{-1}-x_1^{-1})}\right) = (x_1x_2)^n\,\mathrm{Sym}\left(\frac{x_1^{k+1}}{(x_1-x_2)}\right) = (x_1x_2)^n\, s_{k,0}(x_1,x_2) = s_{n+k,n}(x_1,x_2)$$

amount to:

$$s_{n+k,n}(x_1,x_2) = x_1^{k+n} \star x_1^n - q\, x_1^{k+n+1} \star x_1^{n-1}$$

which amounts to the relation

$$M_{n+k,n} = M_{n+k}\,M_n - q\,M_{n+k+1}\,M_{n-1} = \begin{vmatrix} M_{n+k} & M_{n-1}\\ M_{n+k+1} & M_n\end{vmatrix}_q$$

This is nothing but the $v_1^{n+k}v_2^n$ coefficient of $M_2(v_1,v_2)$ (7.12).

8. Discussion/Conclusion

8.1. (q,t)**-quantum determinant.** One of the goals of this paper can be understood as the construction of a t-deformation of the quantum determinant of [DFK15] defined in the context of the quantum Q-system for A_{N-1}. Concretely, we conjecture that the quantity $\mathcal{M}_{a_1,a_2,\ldots,a_\alpha}$ is a polynomial of the $\mathcal{M}_n$'s, and proved the conjecture in the case $\alpha=2$ (Theorem 5.10), and in the case when $a_1=a_2=\cdots=a_\alpha=n$ (Theorem 6.3, and Corollary 6.4). The latter case uses explicitly the relations in the Elliptic Hall Algebra. We would like to interpret this polynomial the (q,t)-determinant of the matrix $\left(\mathcal{M}_{a_j+i-j}\right)_{1\le i,j\le\alpha}$.

The main difference with the usual and the quantum determinant is that, in both of the cases above, the polynomial expression for $\mathcal{M}_{a_1,a_2,\ldots,a_\alpha}$ depends, in general, on more than just the matrix elements $\{\mathcal{M}_{a_j+i-j},\ 1\le i,j\le\alpha\}$ (on which it depends in the quantum determinant case). This polynomial is unique modulo the relations of the quantum toroidal algebra, namely the exchange relation (5.3) and the Serre relations (3.13), expressed respectively as quadratic and cubic relations between the $\mathcal{M}_n$'s. In that respect, the expressions (5.32) of Theorem 5.20 for $k=0$, (5.37) derived from Lemma 5.22, and the alternative expression (5.26) for $a = n+2$, $b = n$, all have this desired property, as we may write for them:

$$\begin{aligned}
\mathcal{M}_{n+1,n} &= \begin{vmatrix} \mathcal{M}_n & \mathcal{M}_n \\ \mathcal{M}_{n+1} & \mathcal{M}_{n+1} \end{vmatrix}_{q,t} = \frac{1}{(1-q)t}\,[\mathcal{M}_n, \mathcal{M}_{n+1}]_q \\
\mathcal{M}_{n,n} &= \begin{vmatrix} \mathcal{M}_n & \mathcal{M}_{n-1} \\ \mathcal{M}_{n+1} & \mathcal{M}_n \end{vmatrix}_{q,t} = \frac{1}{(1-q)(1+t)(q+t)}\left((q^{-1}-q)\mathcal{M}_n^2 + [\mathcal{M}_{n-1}, \mathcal{M}_{n+1}]\right) \\
\mathcal{M}_{n+2,n} &= \begin{vmatrix} \mathcal{M}_{n+2} & \mathcal{M}_{n-1} \\ \mathcal{M}_{n+3} & \mathcal{M}_n \end{vmatrix}_{q,t} = \frac{1}{(q-1)(1-t^2)(q^2-t^2)} \times \\
&\quad \times \Big(t(1+q)\,[\mathcal{M}_n, \mathcal{M}_{n+2}]_{q^2} - (q+t^2)\,[\mathcal{M}_{n+2}, \mathcal{M}_n]_{q^2} - q(q+t^2)\,[\mathcal{M}_{n-1}, \mathcal{M}_{n+3}]\Big)
\end{aligned}$$

The property still holds for $\mathcal{M}_{n+3,n}$, but breaks down for $\mathcal{M}_{n+4,n}$ which can at best be expressed as a polynomial of $\mathcal{M}_{n+4}, \mathcal{M}_n, \mathcal{M}_{n+5}, \mathcal{M}_{n-1}, \mathcal{M}_{n+3}, \mathcal{M}_{n+1}$.

The (q,t)-determinant is therefore a subtle deformation of the quantum determinant. However, it is possible that one of the expressions for this quantity has a nice combinatorial expression, generalizing eq. (7.13) of Theorem 7.3.

8.2. **EHA as t-deformed quantum cluster algebra.** In this paper, we have constructed a representation of the EHA for finitely many variables $x_1, x_2, ..., x_N$. The algebra itself admits a quotient by the ideal generated by the relations $\mathcal{M}_{N+1;n} = 0$, $n \in \mathbb{Z}$, and the relations expressing that $\psi^{\pm} \propto \sum_{k>0} u_{0,\pm k} z^{\pm k}$ are series expansions of finite products. In particular, the condition $\mathcal{M}_{N+1;n} = 0$ expresses that a degree $N+1$ polynomial of $\mathcal{M}_n, \mathcal{M}_{n\pm 1}$ must vanish (by use of Theorem 6.3 and Corollary 6.4).

This allows to view the EHA as a natural t-deformation of the quantum A_{N-1} M-system algebra, which corresponds to the limit $t \to \infty$. We expect the defining relations of the M-system to be explicitly t-deformed. For instance, the first q-commutation relation:

$$M_n M_{n+1} - q M_{n+1} M_n = 0 \tag{8.1}$$

is t-deformed into:

$$\mathcal{M}_n\,\mathcal{M}_{n+1} - q\,\mathcal{M}_{n+1}\,\mathcal{M}_n = t(1-q)\mathcal{M}_{n+1,n}\ ,$$

obtained from eq. (5.32) of Theorem 5.20 for $k = 0$. Indeed, as $\mathcal{M}_n \sim t^{N-1}$ and $\mathcal{M}_{n,p} \sim t^{2(N-2)}$ for large t, we write:

$$t^{1-N}\mathcal{M}_n\, t^{1-N}\mathcal{M}_{n+1} - q\, t^{1-N}\mathcal{M}_{n+1}\, t^{1-N}\mathcal{M}_n = t^{-1}(1-q)t^{2(N-2)}\mathcal{M}_{n+1,n}$$

which shows that the r.h.s. is subleading at $t \to \infty$, and we recover the M-system relation (8.1) in this limit.

Similarly, the M-system relation

$$M_{2;n} = M_n^2 - q\, M_{n+1} M_{n-1} \tag{8.2}$$

is deformed into (5.35), namely:

$$\mathcal{M}_n^2 - q\,\mathcal{M}_{n+1}\mathcal{M}_{n-1} = (q+t+t^2)\mathcal{M}_{2;n} - qt\mathcal{M}_{n+1,n-1}.$$

Repeating the scaling analysis, we get

$$t^{2(1-N)}\mathcal{M}_n^2 - q\,t^{1-N}\mathcal{M}_{n+1}t^{1-N}\mathcal{M}_{n-1} = (1+t^{-1}+qt^{-2})t^{2(2-N)}\mathcal{M}_{2;n} - qt^{-1}t^{2(2-N)}\mathcal{M}_{n+1,n-1}$$

and we recover (8.2) in the $t\to\infty$ limit, by neglecting the subleading terms $O(t^{-1}), O(t^{-2})$.

More generally, the quantum M-system relations were shown in [DFK16] to be solved by the quantum determinant (7.7) expressions for $M_{\alpha;n}$ as polynomials of $\{M_{1;n+i}\}_{|i|<\alpha}$, with the condition that $M_{N+1;n}=0$ for all $n\in\mathbb{Z}$. More generally, we may consider the quantum determinant expression (7.11) for $M_{a_1,a_2,\ldots,a_\alpha}$. Let us rewrite the (q,t) relation of Theorem 5.7 in the following manner:

(8.3)

$$\prod_{1\le i<j\le\alpha}\left(1-t^{-1}\frac{v_i}{v_j}\right)\left(1-t^{-1}q\frac{v_j}{v_i}\right)t^{-\alpha(N-\alpha)}\mathfrak{M}_\alpha(\mathbf{v}) = \prod_{1\le i<j\le\alpha}\left(1-q\frac{v_j}{v_i}\right)\prod_{i=1}^{\alpha}t^{-(N-1)}\mathfrak{m}(v_i)$$

where $\lim_{t\to\infty}t^{-\alpha(N-\alpha)}\mathfrak{M}_\alpha(\mathbf{v}) = M_\alpha(\mathbf{v})$, and accordingly $\lim_{t\to\infty}t^{-(N-1)}\mathfrak{m}(v)=m(v)$. The r.h.s. of (8.3) is simply the generating function for the quantum determinants $|(t^{1-N}\mathcal{M}_{a_j+j-i})_{1\le i,j\le\alpha}|_q$ which tend to $|(M_{a_j+j-i})_{1\le i,j\le\alpha}|_q$ when $t\to\infty$. Expanding the l.h.s. in powers of t^{-1} at large t, we see that the dominant term is $t^{-\alpha(N-\alpha)}\mathfrak{M}_\alpha(\mathbf{v})$ and all other terms are of stricly smaller order. This displays explicitly in which sense this relation is a t-deformation of the quantum determinant relation (7.11).

Example 8.1. *For $\alpha=2$, (8.3) gives in components:*

$$t^{2(2-N)}\left((1+qt^{-2})\mathcal{M}_{a_1,a_2} - t^{-1}\mathcal{M}_{a_1-1,a_2+1} - t^{-1}q\mathcal{M}_{a_1+1,a_2-1}\right) = t^{2(1-N)}\left(\mathcal{M}_{a_1}\mathcal{M}_{a_2} - q\mathcal{M}_{a_1+1}\mathcal{M}_{a_2-1}\right).$$

8.3. **Relation to graded characters.** The difference operators $M_{\alpha;n} = \lim_{t\to\infty}t^{\alpha(\alpha-N)}\mathcal{M}_{\alpha;n}$ were introduced in [DFK15] to generate graded characters of tensor products of Kirillov-Reshetikhin modules by iterated action on the constant function 1. In particular, any expression of the form $\prod_{i=k}^{1}\prod_{\alpha=1}^{N-1}(M_{\alpha;i})^{n_{\alpha,i}}\cdot 1$ for $n_{\alpha,i}\in\mathbb{Z}_+$ is Schur positive, namely decomposes onto Schur functions with graded multiplicities in $\mathbb{Z}_+[q]$. This is not the case for the t-deformed version. As an example, it is easy to see that $\mathcal{M}_2\cdot 1 = t^{N-1}e_2 - t^{N-2}e_1$, where the e's are the elementary symmetric functions of $x_1,\ldots,x_N$, so Schur positivity is lost. It would be interesting to understand the geometric or representation-theoretical meaning of this t-deformation of the q-graded characters.

8.4. **Possible generalizations.** The objects and structures of this paper are all linked to the type A groups/algebras. However, many of them can be extended to other types: on the one hand, DAHA was defined for other types [Che05]; on the other hand (quantum) cluster algebras and Q-sytems have been defined for other types as well. Preliminary results indicate that similar constructions to those of this paper should exist for the other types.

Another interesting direction, even in the A_{N-1} case, is to try to understand the meaning of the other cluster variables (not of the form $M_{\alpha;n}$) in the quantum cluster algebra of the Q-system. Preliminary explorations show that those other variables are also difference operators. Understanding these could be a step in the direction of fully comprehending the t-deformation of the quantum cluster algebra.

References

[BG99] F. Bergeron and A. M. Garsia. Science fiction and Macdonald's polynomials. In *Algebraic methods and q-special functions (Montréal, QC, 1996)*, volume 22 of *CRM Proc. Lecture Notes*, pages 1–52. Amer. Math. Soc., Providence, RI, 1999.

[BGLX14] F. Bergeron, A. Garsia, E. Leven, and G. Xin. Some remarkable new plethystic operators in the theory of macdonald polynomials. *arXiv:1405.0316 [math.CO]*, 2014.

[BZ05] Arkady Berenstein and Andrei Zelevinsky. Quantum cluster algebras. *Adv. Math.*, 195(2):405–455, 2005.

[Che05] Ivan Cherednik. *Double affine Hecke algebras*, volume 319 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[DFK08] Philippe Di Francesco and Rinat Kedem. Proof of the combinatorial Kirillov-Reshetikhin conjecture. *Int. Math. Res. Not. IMRN*, (7):Art. ID rnn006, 57, 2008.

[DFK11] Philippe Di Francesco and Rinat Kedem. Non-commutative integrability, paths and quasi-determinants. *Adv. Math.*, 228(1):97–152, 2011.

[DFK14] Philippe Di Francesco and Rinat Kedem. Quantum cluster algebras and fusion products. *Int. Math. Res. Not. IMRN*, (10):2593–2642, 2014.

[DFK15] Philippe Di Francesco and Rinat Kedem. Difference equations for graded characters from quantum cluster algebra. *arXiv:1505.01657 [math.RT]*, 2015.

[DFK16] Philippe Di Francesco and Rinat Kedem. Quantum q systems: From cluster algebras to quantum current algebras. *arXiv:1606.09052 [math.QA]*, 2016.

[DI97] Jintai Ding and Kenji Iohara. Generalization of Drinfeld quantum affine algebras. *Lett. Math. Phys.*, 41(2):181–193, 1997.

[FJMM12] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin. Quantum toroidal $\mathfrak{gl}_1$-algebra: plane partitions. *Kyoto J. Math.*, 52(3):621–659, 2012.

[FL99] Boris Feigin and Sergey Loktev. On generalized Kostka polynomials and the quantum Verlinde rule. In *Differential topology, infinite-dimensional Lie algebras, and applications*, volume 194 of *Amer. Math. Soc. Transl. Ser. 2*, pages 61–79. Amer. Math. Soc., Providence, RI, 1999.

[FO98] B. L. Feĭgin and A. V. Odesskiĭ. Vector bundles on an elliptic curve and Sklyanin algebras. In *Topics in quantum groups and finite-type invariants*, volume 185 of *Amer. Math. Soc. Transl. Ser. 2*, pages 65–84. Amer. Math. Soc., Providence, RI, 1998.

[FZ02] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[Hai99] Mark Haiman. Macdonald polynomials and geometry. In *New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97)*, volume 38 of *Math. Sci. Res. Inst. Publ.*, pages 207–254. Cambridge Univ. Press, Cambridge, 1999.

[HKO+99] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Remarks on fermionic formula. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 243–291. Amer. Math. Soc., Providence, RI, 1999.

[Kap97] M. M. Kapranov. Eisenstein series and quantum affine algebras. *J. Math. Sci. (New York)*, 84(5):1311–1360, 1997. Algebraic geometry, 7.

[Ked08] Rinat Kedem. Q-systems as cluster algebras. *J. Phys. A*, 41(19):194011, 14, 2008.

[KR87] A. N. Kirillov and N. Yu. Reshetikhin. Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 160(Anal. Teor. Chisel i Teor. Funktsii. 8):211–221, 301, 1987.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Mik99] Kei Miki. Toroidal and level 0 $U_q'(\widehat{\mathrm{sl}_{n+1}})$ actions on $U_q(\widehat{\mathrm{gl}_{n+1}})$ modules. *J. Math. Phys.*, 40(6):3191–3210, 1999.

[Mik07] Kei Miki. A (q,γ) analog of the $W_{1+\infty}$ algebra. *J. Math. Phys.*, 48(12):123520, 35, 2007.

[Neg14] Andrei Negut. The shuffle algebra revisited. *Int. Math. Res. Not. IMRN*, (22):6242–6275, 2014.

[RR86] David P. Robbins and Howard Rumsey, Jr. Determinants and alternating sign matrices. *Adv. in Math.*, 62(2):169–184, 1986.

[Sch12] Olivier Schiffmann. Drinfeld realization of the elliptic Hall algebra. *J. Algebraic Combin.*, 35(2):237–262, 2012.

[SV11] O. Schiffmann and E. Vasserot. The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials. *Compos. Math.*, 147(1):188–234, 2011.

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