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Abstract: We continue our investigation of Z-theory, the second double-copy component of open-string tree-level interactions besides super-Yang–Mills (sYM). We show that the amplitudes of the extended non-linear sigma model (NLSM) recently considered by Cachazo, Cha, and Mizera are reproduced by the leading α'-order of Z-theory amplitudes in the semi-abelian limit. The extension refers to a coupling of NLSM pions to bi-adjoint scalars, and the semi-abelian limit refers to a *partial* symmetrization over one of the color orderings that characterize the Z-theory amplitudes. Alternatively, the partial symmetrization corresponds to a mixed interaction among abelian and non-abelian states in the underlying open-superstring amplitude. We simplify these permutation sums via monodromy relations which greatly increase the efficiency in extracting the α'-expansion of these amplitudes. Their α'-corrections encode higher-derivative interactions between NLSM pions and bi-colored scalars all of which obey the duality between color and kinematics. Through double-copy, these results can be used to generate the predictions of supersymmetric Dirac–Born–Infeld–Volkov–Akulov theory coupled with sYM as well as a complete tower of higher-order α'-corrections.

Contents

1 Introduction

Z-theory [1, 2] refers to the α' dependent theory of bi-colored[1] scalars whose double copy [3–5] with maximally supersymmetric Yang–Mills theory (sYM) [6] generates the tree-level scattering predictions of the open superstring. Z-theory was originally defined by taking its amplitudes to be the set of doubly-ordered functions $Z_\sigma(\tau)$ of ref. [7] – iterated integrals over the boundary of a worldsheet of disk topology – which arise in the tree-level amplitudes of the open superstring [8, 9]. The complete α'-expansion of the non-linear Z-theory equations of motion is pinpointed in ref. [2].

To translate these doubly-ordered $Z_\sigma(\tau)$-functions to field-theory scattering amplitudes, one dresses the permutation $\sigma \in S_n$ encoding the integration domain with the Chan–Paton (CP) factors associated with open-string endpoints. Depending on whether the CP factors are entirely non-abelian or abelian, the low-energy limits of the corresponding Z-theory amplitudes reproduce the tree-level interactions of either bi-adjoint scalar particles [10, 11], or non-linear sigma model (NLSM) pions[2] [1]. Z-theory amplitudes offer a fascinating laboratory to study stringy emergence in a new and technically much simpler context. From a double-copy perspective they isolate, in a *scalar* field theory, what is ultra UV-soft in higher-derivative tree-level predictions of the open superstring.

The main result of this work concerns the semi-abelian version of Z-amplitudes – those involving a mixture of abelian and non-abelian CP factors. Their low-energy theory will be identified with interactions among NLSM pions and bi-adjoint scalar particles (NLSM+ϕ^3). Amplitudes in this theory have been recently studied [21] in a Cachazo–He–Yuan (CHY) representation [5, 11, 22]. To be concrete, we will generalize the emergence of color-stripped NLSM amplitudes from completely abelianized disk integrals or abelian Z-theory amplitudes [1],

$$A^{\rm NLSM}(1,2,\ldots,n) = \lim_{\alpha'\to 0} \frac{1}{n\alpha'} \int_{\mathbb{R}^n} \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\ldots\,\mathrm{d}z_n}{\mathrm{vol}(SL(2,\mathbb{R}))} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' k_i\cdot k_j}}{z_{12}z_{23}\ldots z_{n-1,n}z_{n1}} \,, \tag{1.1}$$

where $z_{ij} \equiv z_i - z_j$. In close analogy to (1.1), we will identify the doubly-stripped amplitudes of the (NLSM+ϕ^3) theory in the low-energy limit of semi-abelian Z-amplitudes (3.2),

$$A^{\mathrm{NLSM}+\phi^3}(1,2,\ldots,r|\tau(1,2,\ldots,n)) = \lim_{\alpha'\to 0} \alpha'^{\,r-3-\delta} \int\limits_{-\infty\leq z_1\leq z_2\leq\ldots\leq z_r\leq\infty} \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\ldots\,\mathrm{d}z_n}{\mathrm{vol}(SL(2,\mathbb{R}))} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' k_i\cdot k_j}}{\tau(z_{12}z_{23}\ldots z_{n-1,n}z_{n1})} \,, \tag{1.2}$$

with $2\leq r\leq n$ external bi-adjoint scalars. The power of α' is $\delta=0$ and $\delta=1$ for even and odd numbers of pions $n-r$, respectively, and the second ordering referring to the integrand is governed by a permutation $\tau \in S_n$.

[1] It may be tempting to refer to the Z-theory scalars as "bi-adjoint", since in the low-energy ($\alpha' \to 0$) limit, non-abelian Z-theory becomes bi-adjoint ϕ^3. We make a different choice here to emphasize the following important point: the α'-corrections imply that Z-theory scalars are not trivially Lie-algebra valued w.r.t. to one of the gauge groups. Charges under the gauge groups are referred to as "color" throughout this work which may equivalently be replaced by "flavour".

[2] See [12–16] and [17–20] for earlier references on the NLSM and its tree level amplitudes, respectively.

As first realized in ref. [23], the NLSM double copies with sYM to generate predictions in Dirac–Born–Infeld–Volkov–Akulov theory (DBIVA) — the supersymmetric completion of Born–Infeld (see e.g. [24]). The abelianized open string, as an all-order double copy of abelian Z-theory with sYM, provides α'-corrections to DBIVA [1]. Similarly, as the NLSM+ϕ^3 double copies with sYM to generate predictions in DBIVA coupled with sYM, the semi-abelian open string provides an all-order α'-completion to DBIVA+sYM. One can either use the field-theory ($\alpha' \to 0$) Kawai–Lewellen–Tye (KLT) relations at tree level [3] to double copy ordered amplitudes, or solve for Jacobi-satisfying numerators and take the double copy graph by graph, following the duality between color and kinematics due to Bern, Johansson and one of the present authors (BCJ) [4, 25]. As sYM amplitudes are by now standard textbook material [26], the new ingredient which we provide is the understanding of how to generate the various α'-components of semi-abelian Z-theory. Additionally, with these amplitudes in hand, one can even consider the double copy of semi-abelian Z-theory with itself, which results in a set of higher-derivative corrections to the theory of special Galileons coupled with NLSM+ϕ^3 discussed in ref. [21].

It should be noted that the type of non-linear symmetry at work in effective field theories like the NLSM and (through double copy with sYM) in DBIVA, has recently garnered some attention from applications to cosmology. Volkov–Akulov-type constrained $\mathcal{N} = 1$ superfields allow for technically simple inflationary models [27–30] and descriptions of dark energy [31–34]. This, as well as independent advances in the notion of a soft bootstrap, has motivated renewed interest in understanding the effect of such non-linear symmetries on the S-matrix, with special attention to its soft limits[3], see e.g. [37–40] and references therein. It should be interesting to discover what symmetries survive, and indeed emerge from, the higher-order string-theory type completion encoded in the Z-theory amplitudes presented in [1] and here.

To understand the functions at work we will recall in Section 2 the definition of the disk integrals or Z-amplitudes at the heart of the CP-stripped open string. As explained in section 3, the semi-abelian limit requires partial symmetrizations over CP orderings which are simplified using monodromy relations [41–43]. These techniques for evaluating (1.2) together with the Berends–Giele recursion for non-abelian Z-theory amplitudes [2] give efficient access to the higher-derivative interactions between pions and bi-colored scalars. Integration-by-parts relations among the disk integrals guarantee that the BCJ duality between color and kinematics [4, 25] holds to all orders in α'.

In the low-energy limit of semi-abelian Z-amplitudes (1.2) detailed in section 4 we will make contact with recent results involving NLSM+ϕ^3 [21]. Z-theory finds exact agreement with the tree amplitudes of ref. [21] for an even number of pions, while yielding additional couplings for odd numbers of pions. Our low-energy results reveal novel amplitude relations between the extended NLSM and pure ϕ^3 theory and imply simplifications of their CHY description [21, 23].

[3]The ability of soft limits of a theory's S-matrix to encode its symmetries has long been appreciated, from the conception of what became known as Adler zeros [35], to the imprint of coset symmetry on double-soft limits [36].

2 Review

He we provide a lightening overview of doubly-ordered Z-theory amplitudes so as to set up the main results. We refer the reader to [1, 2] for detailed reviews of Z-theory as well as properties of color-kinematics and the double copy.

As discussed in [1], it is possible, and indeed quite intriguing, to interpret the iterated disk integrals of the CP stripped open-string amplitude as predictions in an effective field theory. First we define the *doubly-ordered* Z-functions [7],

$$Z_{\sigma(1,2,\ldots,n)}(\tau(1,2,\ldots,n)) \equiv \alpha'^{n-3} \int\limits_{-\infty \le z_{\sigma(1)} \le z_{\sigma(2)} \le \ldots \le z_{\sigma(n)} \le \infty} \frac{\mathrm{d}z_1 \, \mathrm{d}z_2 \, \cdots \, \mathrm{d}z_n}{\mathrm{vol}(SL(2,\mathbb{R}))} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{\tau(z_{12} z_{23} \ldots z_{n-1,n} z_{n1})} \,, \tag{2.1}$$

with permutations $\sigma, \tau \in S_n$. The field-theory ordering τ determines the cyclic product of inverse $z_{ij} \equiv z_i - z_j$ in the integrand, and integration-by-parts manipulations imply [7] that different choices of τ are related by the BCJ relations [4]

$$\sum_{j=2}^{n-1} (k_1 \cdot k_{23\ldots j}) \, Z_{\sigma(1,2,\ldots,n)}(2,3,\ldots,j,1,j{+}1,\ldots,n) = 0 \tag{2.2}$$

at fixed σ. The CP-ordering σ, on the other hand, constrains the domain of integration such that $z_{\sigma(i)} \le z_{\sigma(i+1)}$ for $i = 1,2,\ldots,n{-}1$, where $Z_{12\ldots n}(\ldots)$ is cyclically equivalent to $Z_{2\ldots n1}(\ldots)$. The monodromy relations [41, 42]

$$\sum_{j=2}^{n-1} e^{i\pi\alpha' k_1 \cdot k_{23\ldots j}} \, Z_{23\ldots j,1,j+1\ldots n}(\tau(1,2,\ldots,n)) = 0 \tag{2.3}$$

intertwine the contributions from different integration domains resulting ultimately in an $(n{-}3)!$-basis at fixed integrand ordering τ. Accordingly, the σ-ordering in (2.1) will also be referred to as the monodromy ordering. Note that our conventions for Mandelstam invariants in (2.1) and multiparticle momenta in (2.3) are fixed by

$$k_{12\ldots p} \equiv k_1 + k_2 + \ldots + k_p \,, \qquad s_{12\ldots p} \equiv \frac{1}{2} \, k_{12\ldots p}^2 = \sum_{i<j}^{p} k_i \cdot k_j \,. \tag{2.4}$$

The prefactor α'^{n-3} in (2.1) is designed to obtain the doubly-partial amplitudes of the bi-adjoint scalar theory $m[\cdot|\cdot]$ in the limit [11]

$$\lim_{\alpha' \to 0} Z_{\sigma(1,2,\ldots,n)}(\tau(1,2,\ldots,n)) = m[\sigma(1,2,\ldots,n) \,|\, \tau(1,2,\ldots,n)] \,, \tag{2.5}$$

see [44] and [2] for Berends–Giele recursions for the field-theory amplitudes $m[\cdot|\cdot]$ and the full-fledged disk integrals (2.1), respectively.

Perhaps the most natural way to think about Z-theory as an effective field theory is as a doubly-colored scalar theory where one color (corresponding to color order σ, whose generators we will annotate with t^a) is provided by the stringy[4] CP factors. The CP color mixes with all higher-order kinetic terms[5] depending on $\alpha' k_i \cdot k_j$. The other color (corresponding to color order τ, whose generators we will annotate with T^a) represents a familiar field-theory non-abelian color dressing.

As mentioned in the introduction, to achieve familiar color-ordered amplitudes we must dress the doubly-ordered $Z_\sigma(\tau)$ along one of their orderings. Dressing σ with the CP factors leaves us with a manifestly factorizable theory whose amplitudes obey the standard field-theory BCJ relations (2.2). Explicitly, we sum (2.1) over all distinct σ orders, weighting each $Z_\sigma(\tau)$ with the σ-ordered CP trace:

$$\mathcal{Z}\left(\tau(1,2,\ldots,n)\right) \equiv \sum_{\sigma\in S_{n-1}} \mathrm{Tr}(t^1 t^{\sigma(2)}\cdots t^{\sigma(n)}) Z_{1,\sigma(2,\ldots,n)}(\tau(1,2,\ldots,n)) \,. \tag{2.6}$$

Starting from the CP-dressed Z-theory amplitude (2.6), the color-dressed open-string amplitude [8] can be written in the form [7]

$$\begin{aligned} M_n^{\mathrm{open}} = \sum_{\tau,\rho\in S_{n-3}} &\mathcal{Z}\left(1,\tau(2,\ldots,n{-}2),n,n{-}1\right) \\ &\times S[\tau(23\ldots n{-}2)\,|\,\rho(23\ldots n{-}2)]_1\, A^{\mathrm{YM}}(1,\rho(2,\ldots,n{-}2),n{-}1,n) \end{aligned} \tag{2.7}$$

of the KLT relations for supergravity amplitudes [3, 45]. The matrix $S[\cdot|\cdot]_1$ is known as the field-theory momentum kernel [46] and allows for the recursive representation [1],

$$S[A,j\,|\,B,j,C]_i = (k_{iB}\cdot k_j) S[A\,|\,B,C]_i \;, \qquad S[\emptyset\,|\,\emptyset]_i \equiv 1 \;, \tag{2.8}$$

with multiparticle labels such as $B=(b_1,\ldots,b_p)$ and $C=(c_1,\ldots,c_q)$, multiparticle momentum $k_{iB}\equiv k_i+k_{b_1}+\cdots+k_{b_p}$ and composite label $B,C=(b_1,\ldots,b_p,c_1,\ldots,c_q)$. In the next section, we will derive simplified representations for the CP dressed Z-amplitudes (2.6) when some of the generators t^a are abelian. In this semi-abelian limit, the open-string amplitudes (2.7) encode a UV completion of supersymmetric DBIVA coupled with sYM [47], and our subsequent results on $\mathcal{Z}(\ldots)$ should offer insight into the structure of its tree-level S-matrix.

3 Semi-abelian Z-theory amplitudes

3.1 A structural perspective

In the case of some abelian CP-charged particles where $t^a \to \mathbf{1}$, the traces in (2.6) reduce to only the relevant non-abelian generators. If there are r non-abelian charged particles with

[4]Of course there is nothing stringy about the CP factors themselves, rather the doubly-ordered amplitude obeys the string monodromy relations on the order dressed by the CP factors.

[5]Indeed these higher-derivative terms are responsible for the CP ordering satisfying monodromy relations as opposed to the field-theory relations of the field-theory color-ordering.

labels $1,2,\ldots,r$ and $n{-}r$ abelian particles, the color-ordered CP-dressed $\mathcal{Z}(\tau)$ amplitude (2.6) can be written as

$$\mathcal{Z}\left(\tau(1,2,\ldots,n)\right)\big|_{t^{r+1},\ldots,t^n\to\mathbf{1}} = \sum_{\sigma\in S_{r-1}} \mathrm{Tr}(t^1 t^{\sigma(2)}\cdots t^{\sigma(r)}) Z_{1,\sigma(2,3,\ldots,r)}(\tau(1,2,\ldots,n))\,. \tag{3.1}$$

In the notation $\Sigma(1,2,\ldots,r)\equiv\{1,\sigma(2,3,\ldots,r)\}$ for their integration domain, the semi-abelianized doubly-ordered $Z_\Sigma(\tau)$-amplitudes with $r\leq n$ are given as

$$Z_{\Sigma(12\ldots r)}\left(\tau(1,2,\ldots,n)\right)\equiv {\alpha'}^{n-3}\int\limits_{-\infty\leq z_{\Sigma(1)}\leq z_{\Sigma(2)}\leq\ldots\leq z_{\Sigma(r)}\leq\infty} \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\ldots\,\mathrm{d}z_n}{\mathrm{vol}(SL(2,\mathbb{R}))}\frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{\tau(z_{12}z_{23}\ldots z_{n-1,n}z_{n1})}\ , \tag{3.2}$$

where the punctures $z_{r+1},\ldots,z_n$ are understood to be integrated over the range $(z_{\Sigma(1)},\infty)$. Note that we have $\Sigma\equiv\{\emptyset\}$ for the abelianized Z-theory introduced in [1], and so what would putatively be a doubly-ordered integral becomes the only single-ordered integral relevant to the theory at a given multiplicity of $\tau_n\equiv\tau(1,2,\ldots,n)$.

Both the monodromy relations [41, 42] and the recent all-multiplicity developments on α'-expansions [2, 7, 48, 49] are tailored to non-abelian disk integrals $Z_\rho(\tau)$ in (2.1), where ρ and τ refer to all the n particles. In order to export these results to the semi-abelian disk integrals of (3.1), the latter need to be expressed in terms of their completely ordered counterparts $Z_\rho(\tau)$.

Of course, the inequalities among $z_1,z_2,\ldots,z_r$ imposed by the Σ-ordering in (3.2) can always be translated into a combination of n-particle orderings,

$$Z_{1,\sigma(2,3,\ldots,r)}(\tau_n) = \sum_{\substack{\rho(2,\ldots,n)\in\sigma(2,\ldots,r)\\ \shuffle\, r+1\,\shuffle\, r+2\,\shuffle\ldots\shuffle\, n}} Z_{1,\rho(2,3,\ldots,n)}(\tau_n)\ , \tag{3.3}$$

where the shuffle symbol acting on words $B=(b_1,\ldots,b_p)$ and $C=(c_1,\ldots,c_q)$ can be recursively defined by

$$\emptyset\shuffle B = B\shuffle\emptyset = B\ ,\qquad B\shuffle C\equiv b_1(b_2\ldots b_p\shuffle C)+c_1(c_2\ldots c_q\shuffle B)\,. \tag{3.4}$$

However, this "naive" expansion of semi-abelian Z-amplitudes $Z_{\Sigma(12\ldots r)}(\tau_n)=Z_{1,\sigma(2,\ldots,r)}(\tau_n)$ in terms of their non-abelian counterparts $Z_\rho(\tau)$ usually carries a lot of redundancies and obscures the leading low-energy order. Hence, we will be interested in a simplified representation in terms of $(n{-}2)!$ non-abelian orderings $\rho=\rho(2,3,\ldots,n{-}1)$ which is specified by an α'-dependent coefficient matrix $\mathcal{W}_{\alpha'}(\Sigma\,|\,\rho)$,

$$Z_\Sigma(\tau_n)\equiv\sum_{\rho\in S_{n-2}}\mathcal{W}_{\alpha'}(\Sigma\,|\,\rho(2,3,\ldots,n{-}1))\,Z_{1,\rho(2,3,\ldots,n-1),n}(\tau_n)\,, \tag{3.5}$$

where Σ might depart from permutations of the first legs $1,2,\ldots,r$. The expansion coefficients in the matrix $\mathcal{W}_{\alpha'}(\Sigma\,|\,\rho)$ will be identified as trigonometric functions of $\alpha' s_{ij}$ universal to all τ_n which clarify the first non-vanishing order of α'. This approach will be seen to yield particularly useful expressions for $Z_\Sigma(\tau)$ with a small number r of non-abelian CP factors, to expose their leading low-energy order, to simplify the identification of their field-theory limit and to render the computation of their α'-expansion more efficient.

The desired form (3.5) of semi-abelian Z-theory amplitudes can be achieved by exploiting the monodromy relations at the level of the CP-dressed integrals (2.6) [43],

$$\mathcal{Z}(\tau_n) = \sum_{\sigma\in S_{n-2}} \mathrm{Tr}([[\cdots[[t^1,t^{\sigma(2)}]_{\alpha'},t^{\sigma(3)}]_{\alpha'},\cdots],t^{\sigma(n-1)}]_{\alpha'}\ t^n)\, Z_{1,\sigma(2,3,\ldots,n-1),n}(\tau_n)\ . \tag{3.6}$$

In the context of the color-dressed open superstring (2.7), this can be viewed as a generalization of the Del-Duca–Dixon–Maltoni representation of color-dressed sYM amplitudes [50]. The complex phases seen in the monodromy relations (2.3) are absorbed into the symmetric version of the α'-weighted commutator of [43],

$$\begin{aligned}[][t^{i_1}t^{i_2}\ldots t^{i_p},t^{j_1}t^{j_2}\ldots t^{j_q}]_{\alpha'} \equiv{}& e^{ix_{i_1i_2\ldots i_p,j_1j_2\ldots j_q}}\,(t^{i_1}t^{i_2}\ldots t^{i_p})\,(t^{j_1}t^{j_2}\ldots t^{j_q}) \\ &- e^{-ix_{i_1i_2\ldots i_p,j_1j_2\ldots j_q}}\,(t^{j_1}t^{j_2}\ldots t^{j_q})\,(t^{i_1}t^{i_2}\ldots t^{i_p})\,,\end{aligned} \tag{3.7}$$

where the exponents are furnished by rescaled Mandelstam invariants (2.4)

$$x_{i_1i_2\ldots i_p,j_1j_2\ldots j_q} \equiv \frac{\pi\alpha'}{2}\,k_{i_1i_2\ldots i_p}\cdot k_{j_1j_2\ldots j_q}\ . \tag{3.8}$$

A simplified representation (3.5) of semi-abelian Z-theory amplitudes (3.2), in particular the explicit form of the coefficient matrix $\mathcal{W}_{\alpha'}(\Sigma\,|\,\rho)$ for $\Sigma\equiv\Sigma(12\ldots r)$, follows by isolating the coefficient of a given CP trace in (3.6) after abelianizing $t^{r+1},t^{r+2},\ldots,t^n\to\mathbf{1}$.

3.2 Simplified representation of abelian Z-theory amplitudes

Once we specialize (3.6) to abelian gauge bosons with $t^j\to\mathbf{1}$ for $j=1,2,\ldots,n$, the α'-weighted commutators (3.7) reduce to sine-functions and yield the following simplified expression for the abelian Z-theory amplitudes of [1],

$$\mathcal{Z}(\tau_n)\big|_{t^j\to\mathbf{1}} \equiv Z_\times(\tau_n) = (2i)^{n-2}\sum_{\rho\in S_{n-2}} Z_{1,\rho(2,3,\ldots,n-1),n}(\tau_n)\prod_{k=2}^{n-1}\sin(x_{1\rho(23\cdots(k-1)),\rho(k)})\,. \tag{3.9}$$

We continue to use the shorthand $\tau_n\equiv\tau(1,2,\ldots,n)$ for the integrands, and by the vanishing of odd-multiplicity instances $Z_\times(\tau_{2m-1})=0$, the multiplicity n is taken to be even, e.g.

$$Z_\times(\tau_4) = 4\sin^2\Big(\frac{\pi\alpha'}{2}\,s_{12}\Big)\,Z_{1234}(\tau_4) + 4\sin^2\Big(\frac{\pi\alpha'}{2}\,s_{13}\Big)\,Z_{1324}(\tau_4) \tag{3.10}$$

$$\begin{aligned}Z_\times(\tau_6) = 16\sum_{\rho\in S_4}&\sin\Big(\frac{\pi\alpha'}{2}\,s_{1\rho(2)}\Big)\,\sin\Big(\frac{\pi\alpha'}{2}\,(s_{1\rho(3)}+s_{\rho(23)})\Big) \\ &\times\,\sin\Big(\frac{\pi\alpha'}{2}\,(s_{\rho(45)}+s_{\rho(4)6})\Big)\,\sin\Big(\frac{\pi\alpha'}{2}\,s_{\rho(5)6}\Big)Z_{1\rho(2345)6}(\tau_6)\ .\end{aligned} \tag{3.11}$$

Given that each factor of

$$\sin(x_{12\ldots j-1,j}) = \sin\left(\frac{\pi\alpha'}{2}\,k_{12\ldots j-1}\cdot k_j\right) = \frac{\pi\alpha'}{2}\,k_{12\ldots j-1}\cdot k_j + \mathcal{O}(\alpha'^3) \tag{3.12}$$

introduces one power of $\pi\alpha' k^2$ into the low-energy limit, the leading behaviour of

$$Z_\times(\tau(1,2,\ldots,n)) = \mathcal{O}(\alpha'^{(n-2)}) \tag{3.13}$$

is manifest in (3.9), in lines with the identification of the NLSM amplitude in [1]. Hence, the sine-factors bypass the α'-expansion of disk integrals $Z_{1,\rho,n}(\tau)$ to the order α'^{n-2} when extracting the n-point NLSM amplitude – the field-theory limit (2.5) of $Z_{1,\rho,n}(\tau)$ is enough to obtain the leading order of (3.9) in α'. Moreover, one can identify the above sine-functions with the string-theory momentum kernel [46], defined recursively via [1]

$$\mathcal{S}_{\alpha'}[A,j\,|\,B,j,C]_i = \sin(\pi\alpha' k_{iB}\cdot k_j)\,\mathcal{S}_{\alpha'}[A\,|\,B,C]_i\ , \qquad \mathcal{S}_{\alpha'}[\emptyset\,|\,\emptyset]_i \equiv 1\ , \tag{3.14}$$

with the same notation as seen in its field-theory counterpart (2.8). More precisely, (3.9) can be rewritten in terms of its diagonal elements at rescaled value[6] $\alpha' \to \alpha'/2$

$$Z_\times(\tau_n) = (2i)^{n-2}\sum_{\rho\in S_{n-2}} \mathcal{S}_{\alpha'/2}[\rho(23\ldots n{-}1)\,|\,\rho(23\ldots n{-}1)]_1\, Z_{1,\rho(2,3,\ldots,n-1),n}(\tau_n)\ . \tag{3.15}$$

3.2.1 Deriving the BCJ numerators of the NLSM

Here we can resolve a mystery first identified in ref. [51], and made acute in ref. [1]. In the former reference it was shown that color-kinematic satisfying numerators can be written down for the NLSM as some sum over permuted entries of the KLT matrix (2.8). In ref. [1] it was realized that in fact one needed only the diagonal elements of the KLT matrix to construct the master numerators. The reason can be understood by recalling the emergence of NLSM amplitudes from abelian Z-theory [1],

$$A^{\rm NLSM}(\tau_n) = \lim_{\alpha'\to 0}\alpha'^{2-n} Z_\times(\tau_n)\ , \tag{3.16}$$

see (1.1), and inserting the $\alpha'\to 0$ limits of the two constituents in (3.15), namely (2.5) and

$$\mathcal{S}_{\alpha'/2}[\rho(2\ldots n{-}1)|\rho(2\ldots n{-}1)]_1 = \left(\frac{\pi\alpha'}{2}\right)^{n-2} S[\rho(2\ldots n{-}1)|\rho(2\ldots n{-}1)]_1 + \mathcal{O}(\alpha'^n)\,. \tag{3.17}$$

I.e. the string-theory KLT matrix (3.14) and the doubly-ordered Z-amplitudes (2.1) limit to the field-theory KLT matrix (2.8) and the doubly-stripped bi-adjoint scalar amplitude $m[\cdot|\cdot]$ [11], respectively. We therefore obtain the compact formula for NLSM master numerators proposed in [1]

$$A^{\rm NLSM}(\tau_n) = (\pi i)^{n-2}\sum_{\rho\in S_{n-2}} S[\rho(23\ldots n{-}1)\,|\,\rho(23\ldots n{-}1)]_1\, m[1,\rho(2,\ldots,n{-}1),n|\tau_n]\ , \tag{3.18}$$

[6] The rescaling stems from the present choice to incorporate the relative monodromy phase between the two color factors in the brackets $[t^i,t^j]_{\alpha'}$ of (3.7) via $e^{i\pi\alpha' s_{ij}/2}t^it^j - e^{-i\pi\alpha' s_{ij}/2}t^jt^i$ instead of the more conventional representation $t^it^j - e^{-i\pi\alpha' s_{ij}}t^jt^i$ underlying [43] and the original literature on monodromy relations [41, 42].

from the field-theory limit of (3.15). As firstly exploited implicitly in [52], color-kinematic satisfying master numerators enter the full amplitude through a sum over their product with the doubly-stripped partial amplitudes $m[\cdot|\cdot]$ of the bi-adjoint scalar theory. Hence, the role of the diagonal entries $S[\rho(23\ldots n{-}1)\,|\,\rho(23\ldots n{-}1)]_1$ in (3.18) identifies them as the master numerators of the NLSM [1].

We would be remiss if we did not refer to a remarkable recent result due to Cheung and Shen [53]. There an explicit cubic action was found for the NLSM which indeed generates exactly these color-dual kinematic numerators from application of naive Feynman rules.

3.3 Examples of semi-abelian Z-amplitudes

In this section, we extract the $\mathcal{W}_{\alpha'}$-matrices in (3.5) from the semi-abelian CP-dressed Z-amplitudes in their simplified representation (3.6) when a small number $r = 0, 1, \ldots, 5$ of CP factors is left non-abelian.

3.3.1 $r \leq 2$ non-abelian generators

Given the cyclic equivalence of integration domains $\Sigma(12\ldots r) \to \Sigma(2\ldots r1)$, we need a minimum of $r = 3$ non-trivial CP generators in (3.1) to deviate from the abelian disk integrals $Z_\times(\ldots)$: This can be immediately seen from the rearrangements of the integration region (3.3) following from the definition (3.2) of $Z_1(\tau_n)$ and $Z_{12}(\tau_n)$,

$$Z_1(\tau_n) = Z_{12}(\tau_n) = \sum_{\substack{\sigma(2,3,\ldots,n)\\ \in 2\shuffle 3\shuffle \ldots \shuffle n}} Z_{1,\sigma(2,3,\ldots,n)}(\tau_n) = \sum_{\sigma\in S_{n-1}} Z_{1,\sigma(2,3,\ldots,n)}(\tau_n) = Z_\times(\tau_n)\,. \tag{3.19}$$

Equivalently, one can check (3.19) by comparing the sine-functions in the trace (and its permutations in $2,3,\ldots,n{-}1$)

$$\mathrm{Tr}([\ldots[[t^1,t^2]_{\alpha'},t^3]_{\alpha'},\ldots,t^{n-1}]_{\alpha'}t^n)\,\big|_{t^2,\ldots,t^{n-1}=\mathbf{1}} = (2i)^{n-2}\mathrm{Tr}(t^1t^n)\prod_{k=2}^{n-1}\sin(x_{12\ldots k-1,k})\ , \tag{3.20}$$

with (3.9) after stripping off the trace $\mathrm{Tr}(t^1t^n)$ of the leftover non-abelian generators. Hence, the non-trivial semi-abelian disk integrals which are different from their abelian counterparts involve at least $r \geq 3$ non-abelian generators.

3.3.2 $r = 3$ non-abelian generators

For three non-trivial CP generators at positions i, j and n and all other generators abelian, $t^{\ell\neq i,j,n} \to \mathbf{1}$, the CP-dressed Z-amplitudes (3.6) boil down to traces of the form

$$\begin{aligned}&\mathrm{Tr}([\ldots[\ldots[\ldots[t^1,t^2]_{\alpha'},\ldots,t^i]_{\alpha'},\ldots,t^j]_{\alpha'},\ldots,t^{n-1}]_{\alpha'}t^n)\\ &\quad\to(2i)^{n-3}\prod_{\substack{k=2\\k\neq j}}^{n-1}\sin(x_{12\ldots k-1,k})\mathrm{Tr}(e^{ix_{12\ldots j-1,j}}t^it^jt^n - e^{-ix_{12\ldots j-1,j}}t^jt^it^n)\ ,\end{aligned} \tag{3.21}$$

where the coefficients of $\text{Tr}(t^{\Sigma(i)}t^{\Sigma(j)}t^n)$ in $\mathcal{Z}(\tau)$ determines the semi-abelian integrals $Z_{\Sigma(ij)n}(\tau)$. Since the latter are known to be real, we will only be interested in the real part of (3.21), e.g.

$$\text{Re}\Big[(2i)^{n-3}\prod_{\substack{k=2\\k\neq j}}^{n-1}\sin(x_{12\ldots k-1,k})e^{ix_{12\ldots j-1,j}}\Big] = (2i)^{n-3}\prod_{\substack{k=2\\k\neq j}}^{n-1}\sin(x_{12\ldots k-1,k})\begin{cases} i\sin(x_{12\ldots j-1,j}) \ : \ n \text{ even}\\ \cos(x_{12\ldots j-1,j}) \ \ : \ n \text{ odd}\end{cases} \tag{3.22}$$

along with $\text{Tr}(t^it^jt^n)$. Note that $\cos(x_{12\ldots j-1,j})$ enters with a different sign when considering $\text{Tr}(t^jt^it^n)$ instead of $\text{Tr}(t^it^jt^n)$. As such we arrive at the overall result

$$\begin{aligned}\mathcal{W}_{\alpha'}(\Sigma(ij)n\,|\,2\ldots i\ldots j\ldots n{-}1) &= (2i)^{n-3}\prod_{\substack{k=2\\k\neq j}}^{n-1}\sin(x_{12\ldots k-1,k})\\ &\quad\times\begin{cases} i\sin(x_{12\ldots j-1,j}) & : \ n \text{ even}\\ \text{sgn}(\Sigma(ij)|ij)\cos(x_{12\ldots j-1,j}) & : \ n \text{ odd}\end{cases}\end{aligned} \tag{3.23}$$

for the $\mathcal{W}_{\alpha'}$-matrix in (3.5), where $\text{sgn}(ij|ij)=1$ and $\text{sgn}(ji|ij)=-1$. For even multiplicity $n=2m$, one recovers half the result $\mathcal{W}_{\alpha'}(ij\,|\,23\ldots 2m{-}1) = (2i)^{n-2}\prod_{k=2}^{n-1}\sin(x_{12\ldots k-1,k})$ known from two non-abelian CP factors t^i and t^j. Hence, the semi-abelian disk integrals for three non-abelian CP factors and even n are again captured by their abelian counterparts[7],

$$Z_{1ij}(\tau_{2m}) = \frac{1}{2}\,Z_\times(\tau_{2m})\ . \tag{3.24}$$

The first novel expression for a semi-abelian integral (i.e. different from $Z_\times(\tau_n)$) can be found at five points with three non-abelian CP factors, where (3.23) implies that

$$\begin{aligned} Z_{345}(\tau_5) = 4\big[&\sin(x_{1,2})\sin(x_{12,4})\cos(x_{124,3})Z_{12435}(\tau_5) + \sin(x_{1,4})\sin(x_{14,2})\cos(x_{124,3})Z_{14235}(\tau_5)\\ &+\sin(x_{1,4})\sin(x_{134,2})\cos(x_{14,3})Z_{14325}(\tau_5) - \sin(x_{1,2})\sin(x_{12,3})\cos(x_{123,4})Z_{12345}(\tau_5)\\ &-\sin(x_{1,3})\sin(x_{13,2})\cos(x_{123,4})Z_{13245}(\tau_5) - \sin(x_{1,3})\sin(x_{134,2})\cos(x_{13,4})Z_{13425}(\tau_5)\big]\ .\end{aligned} \tag{3.25}$$

The two sine-factors in each term signal the leading low-energy order ${\alpha'}^2$ and lead to the α'-expansions

$$\begin{aligned} Z_{345}(1,2,3,4,5) &= (\pi\alpha')^2\left(1-\frac{s_{51}+s_{12}}{s_{34}}-\frac{s_{23}+s_{12}}{s_{45}}\right)+\frac{(\pi\alpha')^4}{12}\Big(2s_{12}s_{23}+2s_{12}^2+3s_{51}s_{23}\\ &\quad+2s_{51}s_{12}+s_{45}s_{23}-s_{45}s_{12}-2s_{45}s_{51}-2s_{34}s_{23}-s_{34}s_{12}+s_{34}s_{51}+2s_{34}s_{45}\\ &\quad-\frac{s_{12}^3+2s_{51}s_{12}^2+2s_{51}^2s_{12}+s_{51}^3}{s_{34}}-\frac{s_{12}^3+2s_{23}s_{12}^2+2s_{23}^2s_{12}+s_{23}^3}{s_{45}}\Big)+\mathcal{O}({\alpha'}^5)\\ Z_{235}(1,2,3,4,5) &= (\pi\alpha')^2\left(1-\frac{s_{45}+s_{51}}{s_{23}}\right)+\frac{(\pi\alpha')^4}{12}\Big(2s_{51}^2-2s_{51}s_{12}+s_{45}s_{12}+6s_{45}s_{51}+2s_{45}^2\\ &\quad+3s_{34}s_{12}+s_{34}s_{51}-2s_{34}s_{45}+2s_{23}s_{12}-3s_{23}s_{51}-3s_{23}s_{45}+2s_{23}s_{34}+2s_{23}^2\\ &\quad-\frac{s_{51}^3+2s_{45}s_{51}^2+2s_{45}^2s_{51}+s_{45}^3}{s_{23}}\Big)+\mathcal{O}({\alpha'}^5)\end{aligned} \tag{3.26}$$

[7] An alternative argument can be derived from reflection symmetry $Z_{123\ldots n}(\tau_n) = (-1)^n Z_{1,n\ldots 32}(\tau_n)$.

after appropriate relabelling in the second case. The non-abelian Z-amplitudes on the right hand side of (3.25) can for instance be evaluated through the Berends–Giele techniques of [2]. Seven-point examples of the low-energy limits at the order of $(\pi\alpha')^4$ to be found in (A.19) and (A.20) can be easily arrived at by inserting (3.23) into (3.5).

As will be detailed in section 4, the low-energy limits $\sim (\pi\alpha')^2$ of (3.26) tie in with the expressions in section 2.3 of [21],

$$\begin{aligned} A_5(1^\phi,2^\phi,3^\phi,4^\Sigma,5^\Sigma) &= \frac{s_{34}+s_{45}}{s_{12}} + \frac{s_{15}+s_{45}}{s_{23}} - 1 \\ A_5(1^\phi,2^\phi,3^\Sigma,4^\phi,5^\Sigma) &= \frac{s_{34}+s_{45}}{s_{12}} - 1 \ , \end{aligned} \tag{3.27}$$

which describe the doubly-ordered five-point amplitudes involving two NLSM pions Σ and three bi-colored ϕ^3 scalars.

3.3.3 $r=4$ and $r=5$ non-abelian generators

The $r=4$ analogue of the trace (3.21) with $t^{\ell\neq p,q,r,n} \rightarrow \mathbf{1}$ is given by

$$\begin{aligned} &\mathrm{Tr}([\ldots[\ldots[\ldots[\ldots[t^1,t^2]_{\alpha'},\ldots,t^p]_{\alpha'},\ldots,t^q]_{\alpha'},\ldots,t^r]_{\alpha'},\ldots,t^{n-1}]_{\alpha'}t^n) \\ &\quad\rightarrow (2i)^{n-4}\prod_{\substack{k=2\\k\neq q,r}}^{n-1}\sin(x_{12\ldots k-1,k})\mathrm{Tr}\Big(e^{ix_{12\ldots r-1,r}}(e^{ix_{12\ldots q-1,q}}t^pt^qt^rt^n - e^{-ix_{12\ldots q-1,q}}t^qt^pt^rt^n) \\ &\qquad\qquad - e^{-ix_{12\ldots r-1,r}}(e^{ix_{12\ldots q-1,q}}t^rt^pt^qt^n - e^{-ix_{12\ldots q-1,q}}t^rt^qt^pt^n)\Big) \\ &\quad= (2i)^{n-4}\prod_{\substack{k=2\\k\neq q,r}}^{n-1}\sin(x_{12\ldots k-1,k})\mathrm{Tr}\Big(\cos(x_{12\ldots q-1,q})\cos(x_{12\ldots r-1,r})[[t^p,t^q],t^r]t^n \\ &\qquad\qquad - \sin(x_{12\ldots q-1,q})\sin(x_{12\ldots r-1,r})\{\{t^p,t^q\},t^r\}t^n \\ &\qquad\qquad + i\cos(x_{12\ldots q-1,q})\sin(x_{12\ldots r-1,r})\{[t^p,t^q],t^r\}t^n \\ &\qquad\qquad + i\sin(x_{12\ldots q-1,q})\cos(x_{12\ldots r-1,r})[\{t^p,t^q\},t^r]t^n\Big) \ . \end{aligned} \tag{3.28}$$

Selecting the real part of (3.28) amounts to constraining the number of brackets accompanied by a sine-function, such that

$$\begin{aligned} &\mathcal{W}_{\alpha'}(\Sigma(pqr)n\,|\,23\ldots p\ldots q\ldots r\ldots n{-}1) = (2i)^{n-4}\prod_{\substack{k=2\\k\neq q,r}}^{n-1}\sin(x_{12\ldots k-1,k}) \\ &\quad\times\begin{cases} \cos(x_{12\ldots q-1,q})\cos(x_{12\ldots r-1,r})\mathrm{Tr}([[t^p,t^q],t^r]t^n) \\ -\sin(x_{12\ldots q-1,q})\sin(x_{12\ldots r-1,r})\mathrm{Tr}(\{\{t^p,t^q\},t^r\}t^n)\big|_{\mathrm{Tr}(t^{\Sigma(p)}t^{\Sigma(q)}t^{\Sigma(r)}t^n)} : \ n \text{ even} \\ \\ i\cos(x_{12\ldots q-1,q})\sin(x_{12\ldots r-1,r})\mathrm{Tr}(\{[t^p,t^q],t^r\}t^n) \\ +i\sin(x_{12\ldots q-1,q})\cos(x_{12\ldots r-1,r})\mathrm{Tr}([\{t^p,t^q\},t^r]t^n)\big|_{\mathrm{Tr}(t^{\Sigma(p)}t^{\Sigma(q)}t^{\Sigma(r)}t^n)} : \ n \text{ odd} \end{cases} . \end{aligned} \tag{3.29}$$

The notation $(Y)|_{\mathrm{Tr(X)}}$ instructs to select from the expression Y the coefficients of the CP trace $\mathrm{Tr}(X)$. The obvious $r=5$ counterpart of (3.28) with $t^{\ell\neq p,q,r,s,n}\to\mathbf{1}$ yields

$$\mathcal{W}_{\alpha'}(\Sigma(pqrs)n\,|\,23\ldots p\ldots q\ldots r\ldots s\ldots n{-}1) = (2i)^{n-5}\prod_{\substack{k=2\\k\neq q,r,s}}^{n-1}\sin(x_{12\ldots k-1,k}) \tag{3.30}$$

$$\times\begin{cases}\mathrm{Tr}(\cos(x_{12\ldots q-1,q})\cos(x_{12\ldots r-1,r})\cos(x_{12\ldots s-1,s})[[[t^p,t^q],t^r],t^s]t^n \\ -\cos(x_{12\ldots q-1,q})\sin(x_{12\ldots r-1,r})\sin(x_{12\ldots s-1,s})\{\{[t^p,t^q],t^r\},t^s\}t^n & :\ n\text{ odd}\\ -\sin(x_{12\ldots q-1,q})\cos(x_{12\ldots r-1,r})\sin(x_{12\ldots s-1,s})\{[\{t^p,t^q\},t^r],t^s\}t^n \\ -\sin(x_{12\ldots q-1,q})\sin(x_{12\ldots r-1,r})\cos(x_{12\ldots s-1,s})[\{\{t^p,t^q\},t^r\},t^s]t^n)\big|_{\mathrm{Tr}(t^{\Sigma(p)}t^{\Sigma(q)}t^{\Sigma(r)}t^{\Sigma(s)}t^n)} \\ \\ \mathrm{Tr}(-i\sin(x_{12\ldots q-1,q})\sin(x_{12\ldots r-1,r})\sin(x_{12\ldots s-1,s})\{\{\{t^p,t^q\},t^r\},t^s\}t^n \\ +i\sin(x_{12\ldots q-1,q})\cos(x_{12\ldots r-1,r})\cos(x_{12\ldots s-1,s})[[\{t^p,t^q\},t^r],t^s]t^n & :\ n\text{ even}\\ +i\cos(x_{12\ldots q-1,q})\sin(x_{12\ldots r-1,r})\cos(x_{12\ldots s-1,s})[\{[t^p,t^q],t^r\},t^s]t^n \\ +i\cos(x_{12\ldots q-1,q})\cos(x_{12\ldots r-1,r})\sin(x_{12\ldots s-1,s})\{[[t^p,t^q],t^r],t^s\}t^n)\big|_{\mathrm{Tr}(t^{\Sigma(p)}t^{\Sigma(q)}t^{\Sigma(r)}t^{\Sigma(s)}t^n)}\end{cases}$$

Examples for five- and six-point low-energy limits with $r=4$ can be found in (A.1) as well as (A.3) to (A.9), respectively. Moreover, the leading α'-orders of six-point integrals with $r=5$ are displayed in (A.11) to (A.18).

3.4 General form of the semi-abelian $\mathcal{W}_{\alpha'}$-matrix

To conjecture a form of the $\mathcal{W}_{\alpha'}$-matrix in (3.5) for an arbitrary number of non-trivial CP particles it is helpful to introduce a unifying notation relying on a set of binary vectors,

$$\mathrm{Bin}(a,b)\equiv\{v\in(\{0,1\})^a\ \text{s.t.}\ |v|^2\ \text{odd}\iff b\ \text{odd}\}\,. \tag{3.31}$$

$\mathrm{Bin}(a,b)$ is the set of binary vectors in an a-dimensional space whose magnitude squared is odd if and only if b is odd, e.g.

$$\mathrm{Bin}(3,1)=\mathrm{Bin}(3,3)=\{(1,0,0),\,(0,1,0),\,(0,0,1),\,(1,1,1)\} \tag{3.32}$$

$$\mathrm{Bin}(3,0)=\mathrm{Bin}(3,2)=\{(1,1,0),\,(1,0,1),\,(0,1,1),\,(0,0,0)\}\,. \tag{3.33}$$

Using the binary-vector notation, the above derivations are consistent with a $\mathcal{W}_{\alpha'}$ given as:

$$\mathcal{W}_{\alpha'}(\Sigma(p_1p_2\cdots p_{r-1})n\,|\,23\ldots p_1\ldots p_2\ldots\ldots p_{r-1}\ldots(n{-}1)) = (2i)^{n-r}\prod_{\substack{k=2\\k\neq p_2,\cdots,p_{r-1}}}^{n-1}\sin\left(x_{12\ldots(k-1),\,k}\right)$$

$$\times\sum_{v\in\mathrm{Bin}(r-2,n)}\mathrm{Tr}\left(\left[\left[\cdots\left[\left[p_1,\underline{p_2}\right]_{v_1},\underline{p_3}\right]_{v_2}\cdots\right]_{v_{r-3}},\underline{p_{r-1}}\right]_{v_{r-2}}t^n\right)\Bigg|_{\mathrm{Tr}\left(t^{\Sigma(p_1)}\ldots t^{\Sigma(p_{r-1})}\,t^n\right)}\,, \tag{3.34}$$

where without loss of generality we take the first leg to be CP-abelian, leg n to be CP-non-abelian, and the second entry of $\mathcal{W}_{\alpha'}$ to be canonically ordered. The binary commutators (whose trigonometric dressing singles out the underlined entry p_b) are defined as follows,

$$[p_a, \underline{p_b}\,]_0 \equiv (t^{p_a} t^{p_b} + t^{p_b} t^{p_a}) \times i \sin(x_{12\ldots(p_b-1),\, p_b})$$
$$[p_a, \underline{p_b}\,]_1 \equiv (t^{p_a} t^{p_b} - t^{p_b} t^{p_a}) \times \cos(x_{12\ldots(p_b-1),\, p_b})\,. \tag{3.35}$$

3.5 Structure of the low-energy expansion

In this subsection, we describe the implications of the representation (3.5) of semi-abelian disk integrals (3.2) for the structure of their low-energy expansion. As emphasized in (3.12), each sine-factor descending from the α'-weighted brackets in (3.6) contributes an overall factor of $\pi\alpha'$. For r non-abelian CP factors and n external legs, tracking the commutators $[\cdot,\cdot]_{\alpha'}$ with an identity matrix in one of their entries amounts to the lower bound $(\pi\alpha')^{n-r}$ on the leading low-energy order, see the examples in the previous section. Moreover, depending on $(2i)^{n-r}$ being real or imaginary, another sine factor with low-energy order $\pi\alpha'$ arises from the α'-weighted traces, leading to the refined lower bound

$$Z_{\Sigma(12\ldots r)}(\tau_n) = \left\{ \begin{array}{ll} \mathcal{O}(\pi\alpha'^{n-r}) & : \ n-r \text{ even}\,, \quad r\geq 2 \\ \mathcal{O}(\pi\alpha'^{n-r+1}) & : \ n-r \text{ odd}\,, \quad r\geq 3 \end{array} \right. . \tag{3.36}$$

For small n and r, (3.36) implies the following leading low-energy contributions for $Z_{\Sigma(12\ldots r)}(\tau_n)$,

n	$r\leq 2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
4	$\alpha'^2\zeta_2$	$\alpha'^2\zeta_2$	1	$\times$	$\times$	$\times$	$\times$
5	0	$\alpha'^2\zeta_2$	$\alpha'^2\zeta_2$	1	$\times$	$\times$	$\times$
6	$\alpha'^4\zeta_4$	$\alpha'^4\zeta_4$	$\alpha'^2\zeta_2$	$\alpha'^2\zeta_2$	1	$\times$	$\times$
7	0	$\alpha'^4\zeta_4$	$\alpha'^4\zeta_4$	$\alpha'^2\zeta_2$	$\alpha'^2\zeta_2$	1	$\times$
8	$\alpha'^6\zeta_6$	$\alpha'^6\zeta_6$	$\alpha'^4\zeta_4$	$\alpha'^4\zeta_4$	$\alpha'^2\zeta_2$	$\alpha'^2\zeta_2$	1
9	0	$\alpha'^6\zeta_6$	$\alpha'^6\zeta_6$	$\alpha'^4\zeta_4$	$\alpha'^4\zeta_4$	$\alpha'^2\zeta_2$	$\alpha'^2\zeta_2$

, (3.37)

unless special choices of Σ and τ_n lead to additional cancellations (see appendix A, in particular (A.7), (A.15) and (A.21) for examples). Smaller values of r than admitted in (3.36) are already accounted for by (3.19) and yield the abelian integrals $Z_\times(\tau_n) = \mathcal{O}(\alpha'^{n-2})$ in (3.9).

The even powers of π in the leading low-energy orders (3.36) can be obtained from rational multiples of Riemann zeta values ζ_{2k} in the α'-expansion of completely ordered disk integrals $Z_\sigma(\tau_n)$ in (2.1), e.g.

$$\zeta_2 = \frac{\pi^2}{6}\,, \qquad \zeta_4 = \frac{\pi^4}{90}\,, \qquad \zeta_6 = \frac{\pi^6}{945}\,, \qquad \ldots \qquad \zeta_{2k} = (-1)^{k-1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!}\,, \tag{3.38}$$

with B_{2k} denoting the Bernoulli numbers. When naively assembling their semi-abelian counterparts (3.2) from combinations of $Z_\sigma(\tau_n)$ via rearrangements (3.3) of the integration domain, the leading order of $(\pi\alpha')^{2k}$ reflects cancellations among the contributing $Z_\sigma(\tau_n)$ at all lower orders $\alpha'^{\leq 2k-1}$, see section 4.3 of [1].

As a major advantage of the trigonometric representation (3.5) of the semi-abelian disk integrals, their leading low-energy order (3.36) is determined from the field-theory limit (2.5) of completely ordered disk integrals $Z_\sigma(\tau)$. More generally, the $n-r$ and $n-r+1$ overall powers of α' in the $\mathcal{W}_{\alpha'}$-matrix for even and odd numbers of abelian CP factors, respectively, reduces the required order of α' in the low-energy expansion of $Z_\sigma(\tau)$ by the same amount when assembling the α'-expansion of semi-abelian disk integrals.

The α'-expansion of $Z_\sigma(\tau)$[8] involves multiple zeta values (MZVs)

$$\zeta_{n_1,n_2,\ldots,n_r} \equiv \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \ , \qquad n_r \geq 2 \tag{3.39}$$

in a uniformly transcendental pattern, i.e. the order of α'^w is accompanied by MZVs of transcendental weight $w = n_1 + n_2 + \ldots + n_r$ [9, 48, 60–63]. Uniform transcendentality is particularly transparent from the recursive method of [49] to obtain the α'-expansion of n-point integrals from the Drinfeld associator[9] acting on their $(n-1)$-point counterparts. Extending an alternative all-multiplicity technique based on polylogarithm integration [7], a Berends–Giele recursion for the α'-expansion of disk integrals $Z_\sigma(\tau)$ was given in [2] whose efficiency comes to maximal fruition at high multiplicity and fixed order in α'.

By (3.5), all these expansion-methods for $Z_\sigma(\tau)$ as well as the results available for download via [2, 65] can be neatly imported to infer the α'-dependence of semi-abelian disk integrals. The complete (conjectural) basis of MZVs over $\mathbb{Q}$ present in the α'-expansion of $Z_\sigma(\tau)$ generically enters their semi-abelian counterparts, accompanied by an appropriate global prefactor of $(\pi\alpha')^{2k}$ as determined by (3.36). The coefficient of each such basis MZV in the semi-abelian Z-amplitudes signals an independent effective higher-derivative interaction between NLSM pions and ϕ^3 scalars

4 NLSM coupled to bi-adjoint scalars in semi-abelian Z-theory

4.1 Summary and overview

In this section, we identify the low-energy limits of semi-abelian Z-theory amplitudes (3.2) with doubly partial amplitudes in a scalar field theory. We recall that the tree-level S-matrices of the bi-adjoint ϕ^3 theory and the NLSM emerge from the $(\alpha' \to 0)$-regime of completely ordered disk integrals (2.1) and their abelian limits (3.9), respectively. On these grounds, it is not at all surprising that the "interpolating" case of semi-abelian disk integrals incorporates couplings between NLSM pions and ϕ^3 scalars.

While the bi-colored ϕ^3 scalars are taken to be charged under two gauge groups with generators $t^a \otimes T^b$, the CP matrix t^a is absent in the color-dressing T^b of NLSM pions. In

[8] Both the initial studies of α'-expansions beyond four points [54–57] and powerful recent results on five-point integrals [58, 59] benefit from the connection with (multiple Gaussian) hypergeometric functions.

[9] Also see [64] for a connection with the pattern relating the appearance of MZVs $\zeta_{n_1,n_2,\ldots,n_r}$ of different depth r in open-superstring amplitudes [48].

any field theory with interaction vertices involving both species, the tree-level amplitudes of r bi-colored scalars and $n-r$ pions admit a color-decomposition

$$M^{\mathrm{NLSM}+\phi^3}_{r,n} = \sum_{\sigma\in S_{r-1}} \sum_{\tau\in S_{n-1}} \mathrm{Tr}(t^1 t^{\sigma(2)} t^{\sigma(3)} \cdots t^{\sigma(r)}) \mathrm{Tr}(T^1 T^{\tau(2)} T^{\tau(3)} \cdots T^{\tau(n)}) \\ \times A^{\mathrm{NLSM}+\phi^3}(1,\sigma(2,3,\ldots,r) \,|\, 1,\tau(2,3,\ldots,n)) \tag{4.1}$$

modulo multi-traces in the t^a due to the exchange of pions in the internal propagators. As a main result of this work, it is apparent that, for a suitable choice of NLSM+ϕ^3 couplings, the doubly-partial single-trace amplitudes $A^{\mathrm{NLSM}+\phi^3}(1,\sigma(2,\ldots,r)) \,|\, 1,\tau(2,\ldots,n))$ in (4.1) emerge from the low-energy limit of semi-abelian Z-theory amplitudes (3.2),

$$A^{\mathrm{NLSM}+\phi^3}(1,2,\ldots,r \,|\, \tau(1,2,\ldots,n)) = \lim_{\alpha'\to 0} (\alpha')^{-2\lceil \frac{n-r}{2} \rceil} Z_{12\ldots r}(\tau(1,2,\ldots,n)) \ , \tag{4.2}$$

which is equivalent to (1.2). The structure of a multi-trace completion of (4.1) as well as its tentative string-theory origin (see e.g. ref. [66]) is left as an interesting open problem for the future.

A specific set of couplings NLSM+ϕ^3 is singled out by the coefficients of the Adler zeros in the tree-level amplitudes of the NLSM [21]. In this reference, interactions of the NLSM with ϕ^3 scalars are inferred from a soft-limit extension of the NLSM, and the resulting single-trace doubly-partial amplitudes are represented in the CHY framework [5, 11, 22]. For an even number of pions, we claim that the tree amplitudes of the NLSM+ϕ^3 theory in [21] match the low-energy limits (4.2) of semi-abelian disk integrals, see (3.26) and (3.27) for five-point examples. For odd values of $n-r$, however, the NLSM+ϕ^3 amplitudes of [21] vanish and do not admit any non-trivial comparison with the leading α'-order of semi-abelian Z-theory.

From the incarnation of relevant double-copy structures in the CHY formalism [21], the NLSM+ϕ^3 theory under investigation is closely related to DBIVA coupled to sYM by dualizing the color-factors built from the field-theory T^a into kinematic factors of sYM. The DBIVA + sYM theory, in turn, appears in the low-energy limit of string theory with abelian and non-abelian CP factors in the tree amplitudes (2.7) and therefore supports the identification (4.2).

In the rest of this section, we discuss two implications of (4.2) for new representations of the tree-level S-matrix in the NLSM+ϕ^3 theory.

4.2 Amplitude relations: NLSM+ϕ^3 versus pure ϕ^3

The representation of semi-abelian disk integrals in (3.5) along with the low-energy limit of the $\mathcal{W}_{\alpha'}$-matrix therein reduces any doubly-partial amplitude of the coupled NLSM+ϕ^3 theory to those of pure ϕ^3,

$$A^{\mathrm{NLSM}+\phi^3}(1,2,\ldots,r \,|\, \tau_n) = \sum_{\rho\in S_{n-2}} W(12\ldots r \,|\, \rho(23\ldots n{-}1)) \, m[1,\rho(23\ldots n{-}1),n \,|\, \tau_n] \ , \tag{4.3}$$

where $\tau_n \equiv \tau(1,2,\ldots,n)$. The entries of the W-matrix are polynomials in the Mandelstam invariants,

$$W(12\ldots r\,|\,\rho(23\ldots n{-}1)) \equiv \lim_{\alpha'\to 0} (\alpha')^{-2\lceil \frac{n-r}{2}\rceil} \mathcal{W}_{\alpha'}(12\ldots r\,|\,\rho(23\ldots n{-}1)) \ , \tag{4.4}$$

which follow by replacing the trigonometric functions at the leading α'-order of $\mathcal{W}_{\alpha'}$ via $\sin(x_{A,B}) \to \frac{i\pi}{2} k_A\cdot k_B$ and $\cos(x_{A,B}) \to 1$. In the simplest case of $r\leq 2$ bi-adjoint scalars, they coincide with the diagonal entries of the field-theory KLT matrix (2.8),

$$\begin{aligned} W(\emptyset\,|\,23\ldots n{-}1) &= W(p\,|\,23\ldots n{-}1) = W(pq\,|\,23\ldots n{-}1) \\ &= (i\pi)^{n-2} S[23\ldots n{-}1\,|\,23\ldots n{-}1]_1 = (i\pi)^{n-2}\prod_{j=2}^{n-1} k_{12\ldots j-1}\cdot k_j \ , \end{aligned} \tag{4.5}$$

and do not depend on the legs p,q. The resulting amplitude relations connecting the bi-adjoint scalar theory with its coupling to the NLSM read

$$\begin{aligned} A^{\mathrm{NLSM}+\phi^3}(\emptyset\,|\,\tau_n) &= A^{\mathrm{NLSM}+\phi^3}(1\,|\,\tau_n) = A^{\mathrm{NLSM}+\phi^3}(1,2\,|\,\tau_n) \\ &= (i\pi)^{n-2}\sum_{\rho\in S_{n-2}} m[1,\rho(2,3,\ldots,n{-}1),n\,|\,\tau_n]\prod_{j=2}^{n-1}(k_{1\rho(23\ldots j-1)}\cdot k_{\rho(j)}) \ , \end{aligned} \tag{4.6}$$

and their general form (4.3) resembles recent relations [67–71] between Einstein–Yang–Mills amplitudes and those of pure Yang–Mills theory. We shall next elaborate on the cases with $r\geq 3$ bi-colored scalars and extract the field-theory limits (4.4) from the $\mathcal{W}_{\alpha'}$ matrices of the previous section via $\sin(x_{12\ldots j-1,j}) \to \frac{i\pi}{2}(k_{12\ldots j-1}\cdot k_j)$ and $\cos(x_{12\ldots j-1,j}) \to 1$.

4.2.1 $r=3,4$ bi-adjoint scalars

With three or four bi-adjoint scalars at legs n,p,q,r, (3.23) and (3.29) imply

$$W(\Sigma(pq)n\,|\,23\ldots p\ldots q\ldots n{-}1) = (i\pi)^{n-3}\prod_{\substack{j=2\\ j\neq q}}^{n-1} k_{12\ldots j-1}\cdot k_j \times \begin{cases} \frac{i\pi}{2}\, k_{12\ldots q-1}\cdot k_q \ : \ n \text{ even} \\ \mathrm{sgn}(\Sigma(pq)|pq) \ : \ n \text{ odd}\end{cases} \tag{4.7}$$

with $\mathrm{sgn}(pq|pq)=1$ and $\mathrm{sgn}(qp|pq)=-1$ as well as

$$\begin{aligned} W(\Sigma(pqr)n\,|\,23\ldots p\ldots q\ldots r\ldots n{-}1) &= (i\pi)^{n-4}\prod_{\substack{j=2\\ j\neq q,r}}^{n-1} k_{12\ldots j-1}\cdot k_j \\ &\times \begin{cases} \mathrm{Tr}([[t^p,t^q],t^r]t^n)\big|_{\mathrm{Tr}(t^{\Sigma(p)}t^{\Sigma(q)}t^{\Sigma(r)}t^n)} \ : \ n \text{ even} \\[2ex] \frac{i\pi}{2}\,(k_{12\ldots q-1}\cdot k_q)\mathrm{Tr}([\{t^p,t^q\},t^r]t^n) \\ +\frac{i\pi}{2}\,(k_{12\ldots r-1}\cdot k_r)\mathrm{Tr}(\{[t^p,t^q],t^r\}t^n)\big|_{\mathrm{Tr}(t^{\Sigma(p)}t^{\Sigma(q)}t^{\Sigma(r)}t^n)} \ : \ n \text{ odd}\end{cases} \end{aligned} \tag{4.8}$$

Note that cases with three bi-adjoint scalars at legs p,q,n and even multiplicity simplify to

$$W(\Sigma(pq)n\,|\,23\ldots p\ldots q\ldots n{-}1)\Big|_{n \text{ even}} = \frac{1}{2}\, W(\emptyset\,|\,23\ldots n{-}1) \tag{4.9}$$

by the first line of (4.7).

4.2.2 General form of the W-matrix in field theory

In the binary-vector representation of the $\mathcal{W}_{\alpha'}$-matrix given in section 3.4, words with large numbers of entries $v_j = 1$ dominate the sum in (3.34). For even numbers $n-r$ of pions, only the word $v = (1, 1, \ldots, 1)$ contributes and yields the simple result

$$W(\Sigma(p_1 p_2 \cdots p_{r-1})n \,|\, 23 \ldots p_1 \ldots p_2 \ldots \ldots p_{r-1} \ldots (n-1)) \big|_{n-r \text{ even}} = (i\pi)^{n-r} \tag{4.10}$$
$$\times \Bigg(\prod_{\substack{j=2 \\ j \neq p_2, \cdots, p_{r-1}}}^{n-1} k_{12\ldots(j-1)} \cdot k_j \Bigg) \, \mathrm{Tr}([[\ldots[[t^{p_1}, t^{p_2}], t^{p_3}], \ldots], t^{p_{r-1}}] t^n) \big|_{\mathrm{Tr}(t^{\Sigma(p_1)} t^{\Sigma(p_2)} \ldots t^{\Sigma(p_{r-1})} t^n)}$$

in terms of commutators. For odd values of $n-r$, on the other hand, the leading low-energy order of (3.34) stems from words with a single entry $v_\ell = 0$ such that the trace in

$$\begin{aligned} &W(\Sigma(p_1 p_2 \cdots p_{r-1})n \,|\, 23 \ldots p_1 \ldots p_2 \ldots \ldots p_{r-1} \ldots (n-1)) \big|_{n-r \text{ odd}} \\ &\quad = \frac{1}{2} (i\pi)^{n-r+1} \Bigg(\prod_{\substack{j=2 \\ j \neq p_2, \cdots, p_{r-1}}}^{n-1} k_{12\ldots(j-1)} \cdot k_j \Bigg) \sum_{\ell=2}^{r-1} (k_{12\ldots(p_\ell - 1)} \cdot k_{p_\ell}) \\ &\qquad \times \mathrm{Tr}([[\ldots[\{[\ldots[[t^{p_1}, t^{p_2}], t^{p_3}], \ldots], t^{p_\ell}\}, t^{p_{\ell+1}}], \ldots], t^{p_{r-1}}] t^n) \big|_{\mathrm{Tr}(t^{\Sigma(p_1)} t^{\Sigma(p_2)} \ldots t^{\Sigma(p_{r-1})} t^n)} \end{aligned} \tag{4.11}$$

exhibits one anti-commutator operation $\{\cdot, \cdot\}$ inside the nested commutators.

4.3 Comparison with CHY integrands

Recently the modern connected formalism of Cachazo, He and Yuan (CHY) [5, 11, 22] has given rise to all-multiplicity representations for NLSM amplitudes [23] and their (NLSM+ϕ^3) extensions [21]. These CHY representations arise from integrals over the moduli space of punctured Riemann spheres, where the integrands depend on both the external data $\{t^{a_i}, T^{b_i}, k_i\}$ of the NLSM- or ϕ^3 scalars and the punctures $z_i \in \mathbb{C}$ associated with the i^{th} leg. The punctures are constrained by the scattering equations

$$E_i \equiv \sum_{j \neq i}^{n} \frac{s_{ij}}{z_{ij}} = 0 \tag{4.12}$$

which mirror integration-by-parts relations in string theory and completely localize the integrals.

For any combination of the two species of scalars, the CHY integrands for NLSM+ϕ^3 amplitudes in [21] allow to factor out a universal piece, where n-particle Parke–Taylor factors $(z_{12} z_{23} \ldots z_{n,1})^{-1}$ are combined with traces $\mathrm{Tr}(T^{b_1} T^{b_2} \ldots T^{b_n})$ of the generators T^{b_i} of the common gauge group. The other factor of the integrand depending on the number of pions and ϕ^3 scalars is based on a matrix $A = A(\{k_i, z_i\})$ specified in [23]. For pure NLSM amplitudes, this non-universal piece of the CHY integrand is a reduced Pfaffian $(\mathrm{Pf}'A)^2$, where the prime refers to the deletion of two rows and columns each. For generic configurations of the two scalar species, the non-universal part of the integrand factorizes into an r-particle Parke–Taylor factor $(z_{12}z_{23}\ldots z_{r,1})^{-1}\mathrm{Tr}(t^{a_1}t^{a_2}\ldots t^{a_r})$ and a Pfaffian $(\mathrm{Pf}\,A_{r+1,\ldots,n})^2$ referring to the external pion legs $r{+}1,\ldots,n$. Accordingly, the integrands vanish for an odd number $n{-}r$ of pions.

In order to express the connected amplitudes in terms of doubly-partial amplitudes as done in (4.3), any z-dependence from $(\mathrm{Pf}'A)^2$ and $(z_{12}z_{23}\ldots z_{r,1})^{-1}(\mathrm{Pf}\,A_{r+1,\ldots,n})^2$ has to be reduced to Parke–Taylor factors – $(z_{12}z_{23}\ldots z_{n,1})^{-1}$ and permutations in $1,2,\ldots,n$. The naive evaluation of the Pfaffians, however, involves more diverse functions of z_j than captured by linear combinations of $\tau(z_{12}z_{23}\ldots z_{n,1})^{-1}$ with $\tau\in S_n$. The desired reduction to Parke–Taylor factors requires manifold applications of the scattering equations (4.12) and can in principle be addressed through the algorithms of [72, 73]. Still, the complexity of these manipulations grows rapidly with the multiplicity and has therefore obstructed a compact Parke–Taylor representation of the non-universal integrands with more than four legs.

From the amplitude relations (4.3) reducing the tree-level S-matrix of the NLSM+ϕ^3 theory to doubly-partial amplitudes, one can reverse-engineer a Parke–Taylor form of the underlying CHY integrands, valid on the support of the scattering equations (4.12). The simple form for the W-matrix of NLSM amplitudes in (4.5) translates into the following representation of the connected integrand,

$$\begin{aligned}(\mathrm{Pf}'A)^2 &= \sum_{\rho\in S_{n-2}} \frac{S[\rho(23\ldots n{-}1)|\rho(23\ldots n{-}1)]_1}{(1,\rho(2),\rho(3),\ldots,\rho(n{-}1),n)} \bmod E_i \\ &= \sum_{\rho\in S_{n-2}} \frac{\prod_{j=2}^{n-1}(k_{1\rho(23\ldots j-1)}\cdot k_{\rho(j)})}{(1,\rho(2),\rho(3),\ldots,\rho(n{-}1),n)} \bmod E_i \ , \end{aligned} \tag{4.13}$$

in terms of Parke–Taylor factors with

$$(1,2,3,\ldots,n{-}1,n) \equiv z_{12}z_{23}\ldots z_{n-1,n}z_{n,1} \ . \tag{4.14}$$

Similarly, (4.3) along with the explicit W-matrices given in the previous section yields a simplified form of the connected integrands for mixed amplitudes $A^{\mathrm{NLSM}+\phi^3}(\Sigma(p_1p_2\ldots p_{r-1})n\,|\,\ldots)$ with an even number of pions

$$\frac{(\mathrm{Pf}\,A_{\{12\ldots n-1\}\setminus\{p_1,p_2,\ldots,p_{r-1}\}})^2}{(\Sigma(p_1),\Sigma(p_2),\ldots,\Sigma(p_{r-1}),n)} = (i\pi)^{r-n}\sum_{\rho\in S_{n-2}} \frac{W(\Sigma(p_1p_2\ldots p_{r-1})n\,|\,\rho(23\ldots n{-}1))}{(1,\rho(2),\rho(3),\ldots,\rho(n{-}1),n)} \bmod E_i \ . \tag{4.15}$$

Note that one can use the expression (4.10) for the W-matrix, given the even values of $n{-}r$ in (4.15). The simplest instance beyond (4.13) involves two pions and three bi-adjoint scalars,

$$\begin{aligned}\frac{(\mathrm{Pf}\,A_{12})^2}{z_{34}z_{45}z_{53}} &= \frac{(k_1\cdot k_2)(k_{12}\cdot k_3)}{(1,2,3,4,5)} + \frac{(k_1\cdot k_3)(k_{13}\cdot k_2)}{(1,3,2,4,5)} + \frac{(k_1\cdot k_3)(k_{134}\cdot k_2)}{(1,3,4,2,5)} \\ &\quad - \frac{(k_1\cdot k_2)(k_{12}\cdot k_4)}{(1,2,4,3,5)} - \frac{(k_1\cdot k_4)(k_{14}\cdot k_2)}{(1,4,2,3,5)} - \frac{(k_1\cdot k_4)(k_{134}\cdot k_2)}{(1,4,3,2,5)} \bmod E_i \ , \end{aligned} \tag{4.16}$$

with the underlying W-matrix given in (4.7). While the number of terms in (4.15) and (4.16) generically grows when converting the rank-$(n-r)$ Pfaffians into sums over $(n-2)!$ permutations, our motivation for the rearrangement stems from the simplicity of the Parke–Taylor form (4.14) for the entire z-dependence.

5 Conclusions

Here we continue the program of understanding the predictions of color-stripped Z-theory as sYM-stripped open-superstring scattering. Unlike sYM where color and kinematics, along with their respective Lie-algebra structures, can be cleanly separated, the α'-dependent kinematic factors of color-stripped semi-abelian Z-theory involve functions of both CP traces and momenta. Each of these orders in α' can be understood as part of a successive set of color-kinematic satisfying effective field theories, whose culmination in Z-theory exhibits very soft UV behavior.

We find compact expressions for the doubly-ordered Z-amplitudes whose CP factors admit a mixture of both abelian and non-abelian generators. At leading order in α', these doubly-stripped amplitudes encode the predictions of a field theory of NLSM pions coupled to bi-adjoint scalars. Single-trace amplitudes in this theory were recently expressed in the CHY formalism by Cachazo, Cha, and Mizera [21]. The form of Z-theory's low-energy results presented here offer an efficient complementary representation. As color-kinematics is supported at every order in α' (as well as the resummation), the results presented here have applicability, through double copy, to a spectrum of theories including higher-derivative corrections to DBIVA+SYM of various supersymmetries, as well as higher-derivative corrections to scattering within the special-Galileon+NLSM+ϕ^3 theory.

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A Expansions of semi-abelian disk integrals

In this appendix, we gather examples for the low-energy expansion of semi-abelian Z-theory amplitudes (3.2). The expressions can be efficiently obtained by combining their simplified

$(n-2)!$ representation (3.5) with the Berends–Giele recursion for non-abelian Z-amplitudes [2] (using in particular the program `BGap` described in the reference).

Not all of the examples in this appendix follow the labelling of the $\mathcal{W}_{\alpha'}$-matrices given in sections 3.3 or 3.4 and might require some straightforward permutations of all the legs in $Z_\Sigma(\tau_n)$ and the momenta on the right hand side. Our subsequent choice of labels is tailored to reach all permutation-inequivalent arrangements of Σ and τ_n from the canonical field-theory ordering $\tau_n \to \mathbb{I}_n \equiv 1, 2, \ldots, n$.

A.1 semi-abelian five-point integrals

At five points, semi-abelian disk integrals $Z_\Sigma(\tau_5)$ with three non-abelian CP factors can be found in (3.26). Their counterparts with four non-abelian CP factors allow for three permutation-inequivalent arrangements of Σ and τ_5:

$$
\begin{aligned}
Z_{1234}(\mathbb{I}_5) &= \frac{(\pi\alpha')^2}{2}\Big(2 - \frac{s_{45}+s_{34}}{s_{12}} - \frac{s_{45}+s_{51}}{s_{23}} - \frac{s_{51}+s_{12}}{s_{34}}\Big) + \frac{(\pi\alpha')^4}{24}\Big(2s_{51}^2 + 2s_{45}^2 \\
&\quad + 4s_{34}s_{45} + 2s_{34}^2 + 2s_{23}s_{51} + 2s_{23}s_{45} + 4s_{12}s_{51} + 2s_{12}^2 - \frac{s_{45}^3 + 2s_{34}s_{45}^2 + 2s_{34}^2 s_{45} + s_{34}^3}{s_{12}} \\
&\quad - 2s_{12}s_{34} - \frac{s_{45}^3 + 2s_{51}s_{45}^2 + 2s_{51}^2 s_{45} + s_{51}^3}{s_{23}} - \frac{s_{12}^3 + 2s_{51}s_{12}^2 + 2s_{51}^2 s_{12} + s_{51}^3}{s_{34}}\Big) + \mathcal{O}(\alpha'^5) \\
Z_{1324}(\mathbb{I}_5) &= \frac{(\pi\alpha')^2}{2}\,\frac{s_{51}+s_{45}}{s_{23}} + \frac{(\pi\alpha')^4}{24}\Big(4s_{34}s_{45} + 2s_{34}^2 - 4s_{45}s_{51} - 2s_{34}s_{51} + 2s_{23}s_{51} \\
&\quad + 2s_{23}s_{45} + 4s_{12}s_{51} - 2s_{12}s_{45} - 4s_{12}s_{34} + 2s_{12}^2 + \frac{s_{51}^3 + 2s_{45}s_{51}^2 + 2s_{45}^2 s_{51} + s_{45}^3}{s_{23}}\Big) + \mathcal{O}(\alpha'^5) \\
Z_{1243}(\mathbb{I}_5) &= \frac{(\pi\alpha')^2}{2}\Big(\frac{s_{12}+s_{51}}{s_{34}} - \frac{s_{45}+s_{34}}{s_{12}}\Big) + \frac{(\pi\alpha')^4}{24}\Big(- \frac{s_{45}^3 + 2s_{34}s_{45}^2 + 2s_{34}^2 s_{45} + s_{34}^3}{s_{12}} \\
&\qquad + \frac{s_{51}^3 + 2s_{12}s_{51}^2 + 2s_{12}^2 s_{51} + s_{12}^3}{s_{34}} - 2s_{51}^2 + 2s_{45}^2 + 4s_{34}s_{51} + 4s_{34}s_{45} + 2s_{23}s_{51} \\
&\qquad - 2s_{23}s_{45} - 4s_{23}s_{34} - 4s_{12}s_{51} - 4s_{12}s_{45} + 4s_{12}s_{23}\Big) + \mathcal{O}(\alpha'^5)
\end{aligned} \tag{A.1}
$$

The field-theory amplitudes along with the low-energy limit vanish in the setup of [21] since the interaction vertices therein do not support any couplings to an odd number of pions.

A.2 semi-abelian six-point integrals

Among the semi-abelian six-point integrals, any instance with $r \leq 3$ non-abelian CP factors is proportional to the NLSM amplitude by (3.19) and (3.24), e.g.

$$
\begin{aligned}
Z_{ijk}(\mathbb{I}_6) = \frac{1}{2}\,Z_\times(1,2,\ldots,6) &= \frac{(\pi\alpha')^4}{2}\Big(\frac{(s_{12}+s_{23})(s_{45}+s_{56})}{s_{123}} + \frac{(s_{23}+s_{34})(s_{56}+s_{61})}{s_{234}} \\
&\quad + \frac{(s_{34}+s_{45})(s_{61}+s_{12})}{s_{345}} - s_{12} - s_{23} - s_{34} - s_{45} - s_{56} - s_{61}\Big) + \mathcal{O}(\alpha'^6)\,.
\end{aligned} \tag{A.2}
$$

Four non-abelian CP factors Starting with four non-abelian CP factors, one obtains inequivalent cases such as

$$
\begin{aligned}
Z_{1234}(\mathbb{I}_6) &= (\pi\alpha')^2\Big(\frac{1}{s_{12}}\Big[1-\frac{s_{45}+s_{56}}{s_{123}}\Big]+\frac{1}{s_{34}}\Big[1-\frac{s_{56}+s_{61}}{s_{234}}\Big] \\
&\qquad +\frac{1}{s_{23}}\Big[1-\frac{s_{45}+s_{56}}{s_{123}}-\frac{s_{56}+s_{61}}{s_{234}}\Big]-\frac{s_{345}+s_{56}}{s_{12}s_{34}}\Big)+\mathcal{O}(\alpha'^4) \qquad \text{(A.3)}\\
Z_{1324}(\mathbb{I}_6) &= \frac{(\pi\alpha')^2}{s_{23}}\Big(\frac{s_{45}+s_{56}}{s_{123}}+\frac{s_{56}+s_{61}}{s_{234}}-1\Big)+\mathcal{O}(\alpha'^4) \qquad \text{(A.4)}\\
Z_{1245}(\mathbb{I}_6) &= (\pi\alpha')^2\Big(\frac{1}{s_{12}}+\frac{1}{s_{45}}-\frac{s_{123}+s_{345}}{s_{12}s_{45}}\Big)+\mathcal{O}(\alpha'^4) \qquad \text{(A.5)}\\
Z_{1254}(\mathbb{I}_6) &= (\pi\alpha')^2\Big(\frac{s_{123}+s_{345}}{s_{12}s_{45}}-\frac{1}{s_{12}}-\frac{1}{s_{45}}\Big)+\mathcal{O}(\alpha'^4) \qquad \text{(A.6)}\\
Z_{1425}(\mathbb{I}_6) &= \frac{(\pi\alpha')^4}{12}\Big(3\,\frac{(s_{23}{+}s_{34})(s_{56}{+}s_{61})}{s_{234}}+2s_{36}\Big)+\mathcal{O}(\alpha'^5) \qquad \text{(A.7)}\\
Z_{1235}(\mathbb{I}_6) &= (\pi\alpha')^2\Big(\frac{1}{s_{12}}+\frac{1}{s_{23}}\Big)\Big(1-\frac{s_{45}+s_{56}}{s_{123}}\Big)+\mathcal{O}(\alpha'^4) \qquad \text{(A.8)}\\
Z_{1325}(\mathbb{I}_6) &= \frac{(\pi\alpha')^2}{s_{23}}\Big(\frac{s_{45}+s_{56}}{s_{123}}-1\Big)+\mathcal{O}(\alpha'^4)\ . \qquad \text{(A.9)}
\end{aligned}
$$

Apart from the exception (A.7) mentioned in section 3.5, the leading low-energy behaviour ties in with table (3.37), and a cyclic permutation of (A.3) matches the doubly partial amplitude

$$
\begin{aligned}
A_6(1^\Sigma,2^\Sigma,3^\phi,4^\phi,5^\phi,6^\phi) &= \frac{1}{s_{56}}\Big(1-\frac{s_{12}+s_{23}}{s_{123}}\Big)+\frac{1}{s_{34}}\Big(1-\frac{s_{12}+s_{16}}{s_{612}}\Big) \\
&\quad -\frac{s_{12}+s_{234}}{s_{34}s_{56}}+\frac{1}{s_{45}}\Big(1-\frac{s_{12}+s_{23}}{s_{123}}-\frac{s_{12}+s_{16}}{s_{612}}\Big) \qquad \text{(A.10)}
\end{aligned}
$$

involving two pions Σ and four bi-adjoint scalars ϕ^3 in section 3.2 of [21]. Note that $Z_{1245}(\tau_6)$ differs from $-Z_{1254}(\tau_6)$ at the orders $\alpha'^{\geq 4}$ suppressed in (A.5) and (A.6).

Five non-abelian CP factors With five non-abelian CP factors, we have eight inequivalent six-point cases

$$
\begin{aligned}
Z_{12345}(\mathbb{I}_6) &= \frac{(\pi\alpha')^2}{2}\Big(\frac{1}{s_{12}}\Big[2-\frac{s_{45}+s_{56}}{s_{456}}\Big]+\frac{1}{s_{23}}\Big[2-\frac{s_{45}+s_{56}}{s_{456}}-\frac{s_{56}+s_{61}}{s_{561}}\Big] \\
&\qquad +\frac{1}{s_{34}}\Big[2-\frac{s_{56}+s_{61}}{s_{561}}-\frac{s_{61}+s_{12}}{s_{612}}\Big]+\frac{1}{s_{45}}\Big[2-\frac{s_{61}+s_{12}}{s_{612}}\Big] \\
&\qquad -\frac{s_{345}+s_{56}}{s_{12}s_{34}}-\frac{s_{61}+s_{123}}{s_{23}s_{45}}-\frac{s_{123}+s_{345}}{s_{12}s_{45}}\Big)+\mathcal{O}(\alpha'^4) \qquad \text{(A.11)}\\
Z_{12435}(\mathbb{I}_6) &= \frac{(\pi\alpha')^2}{2}\Big(\frac{s_{345}+s_{56}}{s_{12}s_{34}}+\frac{1}{s_{34}}\Big[\frac{s_{56}+s_{61}}{s_{561}}+\frac{s_{61}+s_{12}}{s_{612}}-2\Big]\Big)+\mathcal{O}(\alpha'^4) \qquad \text{(A.12)}\\
Z_{13425}(\mathbb{I}_6) &= \frac{(\pi\alpha')^2}{2}\,\frac{s_{56}+s_{61}}{s_{34}s_{561}}+\mathcal{O}(\alpha'^4) \qquad \text{(A.13)}\\
Z_{14325}(\mathbb{I}_6) &= -\frac{(\pi\alpha')^2}{2}\Big(\frac{1}{s_{23}}+\frac{1}{s_{34}}\Big)\frac{s_{56}+s_{61}}{s_{561}}+\mathcal{O}(\alpha'^4) \qquad \text{(A.14)}
\end{aligned}
$$

$$Z_{13524}(\mathbb{I}_6) = \frac{(\pi\alpha')^4}{12}\,(2s_{45}+2s_{12}+s_{56}+s_{61}-2s_{123}-2s_{345}) + \mathcal{O}(\alpha'^5) \tag{A.15}$$

$$Z_{13254}(\mathbb{I}_6) = \frac{(\pi\alpha')^2}{2}\Big(\frac{s_{45}+s_{56}}{s_{23}s_{123}} - \frac{s_{61}+s_{123}}{s_{23}s_{45}}\Big) + \mathcal{O}(\alpha'^4) \tag{A.16}$$

$$Z_{12453}(\mathbb{I}_6) = \frac{(\pi\alpha')^2}{2}\Big(\frac{s_{45}+s_{56}}{s_{12}s_{123}} + \frac{s_{61}+s_{12}}{s_{45}s_{345}} - \frac{s_{123}+s_{345}}{s_{12}s_{45}}\Big) + \mathcal{O}(\alpha'^4) \tag{A.17}$$

$$\begin{aligned} Z_{12354}(\mathbb{I}_6) = \frac{(\pi\alpha')^2}{2}\Big(&\frac{1}{s_{45}}\Big[\frac{s_{61}+s_{12}}{s_{345}} + \frac{s_{123}+s_{345}}{s_{12}} + \frac{s_{61}+s_{123}}{s_{23}} - 2\Big] \\ &- \frac{s_{45}+s_{56}}{s_{123}}\Big[\frac{1}{s_{12}} + \frac{1}{s_{23}}\Big]\Big) + \mathcal{O}(\alpha'^4)\ , \end{aligned} \tag{A.18}$$

where also (A.15) starts at higher order $(\pi\alpha')^4$ as compared to the generic expectation from table (3.37). Again, the low-energy limits do not have any counterparts in [21] by the odd number of NLSM pions.

A.3 semi-abelian seven-point integrals

The low-energy limit of the seven-point integral with $r=3$ non-abelian CP factors

$$\begin{aligned} Z_{123}(\mathbb{I}_7) = -(\pi\alpha')^4\Big(&\frac{1}{s_{12}}\Big[\frac{(s_{34}+s_{45})(s_{67}+s_{712})}{s_{345}} + \frac{(s_{45}+s_{56})(s_{712}+s_{123})}{s_{456}} \\ &\quad + \frac{(s_{56}+s_{67})(s_{123}+s_{34})}{s_{567}} - s_{34}-s_{45}-s_{56}-s_{67}-s_{712}-s_{123}\Big] \\ &+ \frac{1}{s_{23}}\Big[\frac{(s_{45}+s_{56})(s_{71}+s_{123})}{s_{456}} + \frac{(s_{56}+s_{67})(s_{123}+s_{234})}{s_{567}} \\ &\quad + \frac{(s_{67}+s_{71})(s_{234}+s_{45})}{s_{671}} - s_{45}-s_{56}-s_{67}-s_{71}-s_{123}-s_{234}\Big] \\ &+ \frac{(s_{67}+s_{71})(s_{34}+s_{45})}{s_{345}s_{671}} - \frac{s_{34}+s_{45}}{s_{345}} - \frac{s_{45}+s_{56}}{s_{456}} - \frac{s_{56}+s_{67}}{s_{567}} - \frac{s_{67}+s_{71}}{s_{671}} + 2\Big) + \mathcal{O}(\alpha'^6)\ . \end{aligned} \tag{A.19}$$

agrees with the expression for $A_7(1^\phi,2^\phi,3^\phi,4^\Sigma,5^\Sigma,6^\Sigma,7^\Sigma)$ given in (2.25) of [21]. Moreover, there are three additional cases which cannot be obtained from relabellings of (A.19):

$$\begin{aligned} Z_{124}(\mathbb{I}_7) = (\pi\alpha')^4\Big(&\frac{1}{s_{12}}\Big[\frac{(s_{34}+s_{45})(s_{67}+s_{712})}{s_{345}} + \frac{(s_{45}+s_{56})(s_{712}+s_{123})}{s_{456}} \\ &\quad + \frac{(s_{56}+s_{67})(s_{123}+s_{34})}{s_{567}} - s_{34}-s_{45}-s_{56}-s_{67}-s_{712}-s_{123}\Big] \\ &+ \frac{(s_{67}+s_{71})(s_{34}+s_{45})}{s_{345}s_{671}} - \frac{s_{34}+s_{45}}{s_{345}} - \frac{s_{45}+s_{56}}{s_{456}} - \frac{s_{56}+s_{67}}{s_{567}} - \frac{s_{67}+s_{71}}{s_{671}} + 2\Big) + \mathcal{O}(\alpha'^6) \\ Z_{125}(\mathbb{I}_7) = (\pi\alpha')^4\Big(&\frac{1}{s_{12}}\Big[\frac{(s_{34}+s_{45})(s_{67}+s_{712})}{s_{345}} + \frac{(s_{45}+s_{56})(s_{712}+s_{123})}{s_{456}} \\ &\quad + \frac{(s_{56}+s_{67})(s_{123}+s_{34})}{s_{567}} - s_{34}-s_{45}-s_{56}-s_{67}-s_{712}-s_{123}\Big] \\ &+ \frac{(s_{67}+s_{71})(s_{34}+s_{45})}{s_{345}s_{671}} + \frac{(s_{67}+s_{71})(s_{23}+s_{34})}{s_{234}s_{671}} + \frac{(s_{23}+s_{34})(s_{56}+s_{67})}{s_{234}s_{567}} \end{aligned} \tag{A.20}$$

$$
\begin{aligned}
&- \frac{s_{23}+s_{34}}{s_{234}} - \frac{s_{34}+s_{45}}{s_{345}} - \frac{s_{45}+s_{56}}{s_{456}} - \frac{s_{56}+s_{67}}{s_{567}} - \frac{s_{67}+s_{71}}{s_{671}} + 2\Big) + \mathcal{O}(\alpha'^6) \\
Z_{135}(\mathbb{I}_7) &= (\pi\alpha')^4 \Big(2 - \frac{s_{23}+s_{34}}{s_{234}} - \frac{s_{45}+s_{56}}{s_{456}} - \frac{s_{56}+s_{67}}{s_{567}} - \frac{s_{67}+s_{71}}{s_{671}} - \frac{s_{71}+s_{12}}{s_{712}} \\
&+ \frac{(s_{67}+s_{71})(s_{23}+s_{34})}{s_{234}s_{671}} + \frac{(s_{23}+s_{34})(s_{56}+s_{67})}{s_{234}s_{567}} + \frac{(s_{45}+s_{56})(s_{71}+s_{12})}{s_{456}s_{712}} \Big) + \mathcal{O}(\alpha'^6) \, .
\end{aligned}
$$

For selected examples with $r \geq 4$ non-abelian CP factors, the low-energy expansion starts at higher orders as compared to table (3.37) such as

$$
\begin{aligned}
Z_{14725}(\mathbb{I}_7) &= -\pi^2 \alpha'^5 \zeta_3 s_{36} + \mathcal{O}(\alpha'^6) \\
Z_{135724}(\mathbb{I}_7) &= -\frac{(\pi\alpha')^4}{12} \frac{s_{56}+s_{67}}{s_{567}} + \pi^2 \alpha'^5 \zeta_3 (s_{56}+s_{67}) \\
&\quad - \frac{\pi^2 \alpha'^5 \zeta_3}{2 s_{567}} (s_{67}+s_{56})(s_{234}+s_{123}+s_{12}+s_{23}+s_{34}) + \mathcal{O}(\alpha'^6) \, .
\end{aligned} \tag{A.21}
$$

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