Longest interval between zeros of the tied-down random walk, the Brownian bridge and related renewal processes

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Abstract. The probability distribution of the longest interval between two zeros of a simple random walk starting and ending at the origin, and of its continuum limit, the Brownian bridge, was analyzed in the past by Rosén and Wendel, then extended by the latter to stable processes. We recover and extend these results using simple concepts of renewal theory, which allows to revisit past or recent works of the physics literature. We also discuss related problems and open questions.

1. Introduction

Problems that can be recast in the language of renewal processes appear recurrently in a number of studies of statistical physics without necessarily being recognized as such. It is therefore useful to have access to this body of knowledge in simple terms. This is one of the aims of the present study, where we revisit the question, investigated in the past by Wendel [1], of the longest interval between zeros of the Brownian bridge, seen as the continuum limit of the tied-down random walk (the simple random walk starting and ending at the origin), and of its generalization by a self-similar process with index $0 < \theta < 1$ (the Brownian bridge corresponding to $\theta = 1/2$). We recover his results using simple methods systematically developed in former studies of renewal processes [2, 3]. We then extend this study in new directions and tackle open problems. Finally we use this knowledge to put several related works [3, 4, 5] in perspective.

The detailed content of the paper is as follows. We start by recalling Rosén's results for the tied-down random walk and its continuum limit, the Brownian bridge, as reported in [1]. We detail the derivation of this continuum limit and analyze the rescaled distributions of the longest interval and of its inverse, as well as their averages. We extend this study to the computation of the number of intervals between zeros of the walk and of the probability of record breaking, that is, the probability that the last interval be the longest (section 2). In section 3 we give a reminder on renewal processes. We then define the tied-down renewal process, which is a generalization of the tied-down random walk, or its continuum limit, the Brownian bridge (section 4). When the tail index of the distribution of intervals is less than 1, this process corresponds to the tied-down semi-stable process considered in [1]. We analyze the rescaled distributions of the longest interval and of its inverse, as well as other characteristics of this process, such as the number of events, the statistics of a single interval and the probability of

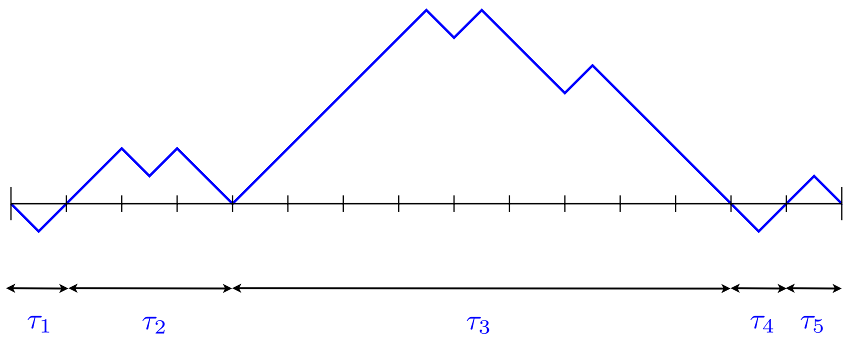

Figure 1. A tied-down random walk made of $2N = 30$ steps, with $M_{15} = 5$ intervals between zeros, $\tau_1, \ldots, \tau_5$, taking the values $2, 6, 18, 2, 2$, respectively. In this example, the longest interval $I_{15} \equiv \tau_3$. The ticks on the $x-$axis correspond to two time-steps.

record breaking. We finally consider the cases of a narrow distribution of intervals or a broad distribution with tail index $\theta > 1$. We close by revisiting past or recent relevant studies [3, 4, 5]. Some definitions and derivations are relegated to the appendices.

2. The tied-down random walk

2.1. Rosén's results

We first recall Rosén's results for the tied-down random walk as reported by Wendel in [1], keeping his notations. Consider the sum S_n of n independent random variables taking the values ± 1, with probabilities $1/2$. The random walk S_n is conditioned to be 'tied down' at time $2N$, i.e., to return to the origin at that time. Let I_N be the longest interval between zeros of S_n, during time $0 \le n \le 2N$. Define the joint probability

$$v_{N,k} = \mathrm{Prob}(I_N \le 2k, S_{2N} = 0), \qquad v_{0,k} = 1. \tag{2.1}$$

The quantity of interest is the conditional probability

$$\mathrm{Prob}(I_N \le 2k | S_{2N} = 0) = \frac{v_{N,k}}{u_N}, \tag{2.2}$$

where u_N is the probability of return of the walk at time $2N$ (see Appendix B)

$$u_N = \mathrm{Prob}(S_{2N} = 0) = v_{N,\infty}. \tag{2.3}$$

The joint probability $v_{N,k}$ satisfies the renewal equation

$$v_{N,k} = \sum_{n=1}^{k} f_n \, v_{N-n,k}, \tag{2.4}$$

where f_n is the probability of first return to zero at time $2n$ (see Appendix B). From (2.4) we deduce that the generating functions

$$\tilde{v}_k(z) = \sum_{N \ge 0} v_{N,k} z^N, \qquad \tilde{f}_k(z) = \sum_{n=1}^{k} f_n z^n \tag{2.5}$$

are related by

$$\tilde{v}_k(z) = \frac{1}{1 - \tilde{f}_k(z)}. \tag{2.6}$$

The result (2.10) below, due to Rosén, as stated in [1], gives the continuum limit, at large times $2N$, of the conditional probability (2.2), using (2.6). This conditional probability reads, using a star to indicate the tied-down condition,

$$F_R^\star(r) = \lim_{N\to\infty} \mathrm{Prob}(R_N \le r | S_{2N} = 0) = \lim_{N\to\infty} \frac{v_{N,k=Nr}}{u_N}, \tag{2.7}$$

also equal to

$$\overline{F}_V^\star(v) = \lim_{N\to\infty} \mathrm{Prob}(V_N > v | S_{2N} = 0), \tag{2.8}$$

the bar referring to the complementary distribution function, with the following notations:

$$\begin{aligned} R_N &= \frac{I_N}{2N}, \qquad R^\star = \lim_{N\to\infty} R_N, \qquad r = \frac{k}{N}. \\ V_N &= \frac{1}{R_N}, \qquad V^\star = \frac{1}{R^\star}, \qquad v = \frac{N}{k} = \frac{1}{r}, \end{aligned} \tag{2.9}$$

and where r and v are real variables, with $0 < r < 1$ and $v > 1$. According to [1], we have

$$F_R^\star(r) = \overline{F}_V^\star(v) = \pi\sqrt{v} \sum_{k=-\infty}^{\infty} (-2x_k)\mathrm{e}^{x_k(1+v)}, \tag{2.10}$$

$$= \pi\sqrt{v}\, f_V^{\mathrm{II}}(v), \tag{2.11}$$

denoting by $f_V^{\mathrm{II}}(v)$ the sum on the right side of (2.10), and where the x_k are the zeros of the function

$$D(x) = 1 + \sqrt{\pi x}\,\mathrm{e}^x \,\mathrm{erf}\,\sqrt{x}. \tag{2.12}$$

These zeros have all negative real parts, for instance

$$x_0 = -0.854\ldots, x_{\pm 1} = -4.248\ldots \pm \mathrm{i}\,6.383\ldots, x_{\pm 2} = -5.184\ldots \pm \mathrm{i}\,12.885\ldots \tag{2.13}$$

and so on. Let us note that the Laplace transform of $f_V^{\mathrm{II}}(v)$ with respect to v is given by

$$\widehat{f_V^{\mathrm{II}}}(x) = \frac{\mathrm{e}^x}{1 + \sqrt{\pi x}\,\mathrm{e}^x \,\mathrm{erf}\,\sqrt{x}}, \tag{2.14}$$

where the variable x is conjugate to v. This can be seen by taking the inverse Laplace transform of $\widehat{f_V^{\mathrm{II}}}(x)$ and noting that the residues of this function at the poles x_k are equal to $-2x_k\,\mathrm{e}^{x_k}$ (see (4.45) in section 4.6), which yields $f_V^{\mathrm{II}}(v)$ back. We can now interpret $f_V^{\mathrm{II}}(v)$ as the density of a random variable, because it is a positive function, normalised to unity since $\widehat{f_V^{\mathrm{II}}}(0) = 1$. The precise meaning of this density is given in section 3.3.

2.2. Proof of (2.10)

Let us now detail how to derive the continuum scaling limit (2.10) from (2.6), by an asymptotic analysis of the latter at large times. Since $f_n = u_{n-1} - u_n$ (see Appendix B), we have

$$1 - \tilde{f}_k(z) = u_k z^k + (1-z) \sum_{n=0}^{k-1} u_n z^n, \tag{2.15}$$

then setting $z = \mathrm{e}^{-s}$ and using (B.5), we obtain, when $k \to \infty$, $s \to 0$, with $x = ks$ fixed,

$$\begin{aligned} \tilde{v}_k(z) = \frac{1}{1 - \tilde{f}_k(z)} &\approx \frac{1}{\sqrt{s}} \frac{1}{(\pi k s)^{-1/2} \mathrm{e}^{-ks} + \operatorname{erf} \sqrt{ks}} \\ &\approx \sqrt{\pi k}\, \frac{\mathrm{e}^x}{1 + \sqrt{\pi x}\, \mathrm{e}^x \operatorname{erf} \sqrt{x}}. \end{aligned} \tag{2.16}$$

It is now simple to infer (2.10) from (2.16). In the continuum scaling limit (2.9), informally denoting the generating function $\tilde{v}_k(z)$ as a Laplace transform with respect to N (considered now as a real variable), yields

$$\tilde{v}_k(z) = \underset{N}{\mathcal{L}} \operatorname{Prob}(R_N \le k/N, S_{2N} = 0) = \underset{N}{\mathcal{L}} \operatorname{Prob}(V_N > N/k, S_{2N} = 0). \tag{2.17}$$

Rescaling N by k, the left side becomes the Laplace transform with respect to v, with x conjugate to v,

$$\underset{v}{\mathcal{L}} \operatorname{Prob}(V_N > v = N/k, S_{2N} = 0) \approx \sqrt{\frac{\pi}{k}} \widehat{f_V^{\mathrm{II}}}(x), \tag{2.18}$$

hence

$$\operatorname{Prob}(V_N > v, S_{2N} = 0) \approx \sqrt{\frac{\pi}{k}} f_V^{\mathrm{II}}(v). \tag{2.19}$$

Dividing both sides by $u_N \approx 1/\sqrt{\pi N}$ leads to (2.10).

2.3. Characterization of the density

The large v behaviour of the density $f_V^\star(v)$ can be read off from (2.10). At leading order

$$\overline{F}_V^\star(v) \approx 2|x_0|\pi\sqrt{v}\, \mathrm{e}^{-|x_0|v}, \tag{2.20}$$

from which $f_V^\star(v)$ ensues by derivation,

$$f_V^\star(v) \approx 2\pi\, x_0^2 \sqrt{v}\, \mathrm{e}^{-|x_0|v}. \tag{2.21}$$

As a consequence, $f_R^\star(r)$ has an essential singularity at the origin,

$$f_R^\star(r) \approx 2\pi\, x_0^2 \frac{\mathrm{e}^{-|x_0|/r}}{r^{5/2}}. \tag{2.22}$$

The density $f_V^\star(v)$ is a piece-wise continuous function, as is also the case of the densities $f_V^{\mathrm{I}}(v)$, $f_V^{\mathrm{II}}(v)$ and $f_V^{\mathrm{III}}(v)$ [3, 6] (f_V^{I} and f_V^{III} are defined later). The behaviour for $1 < v < 2$ is given in [1]:

$$f_V^\star(v) = \frac{1}{2\sqrt{v}}, \tag{2.23}$$

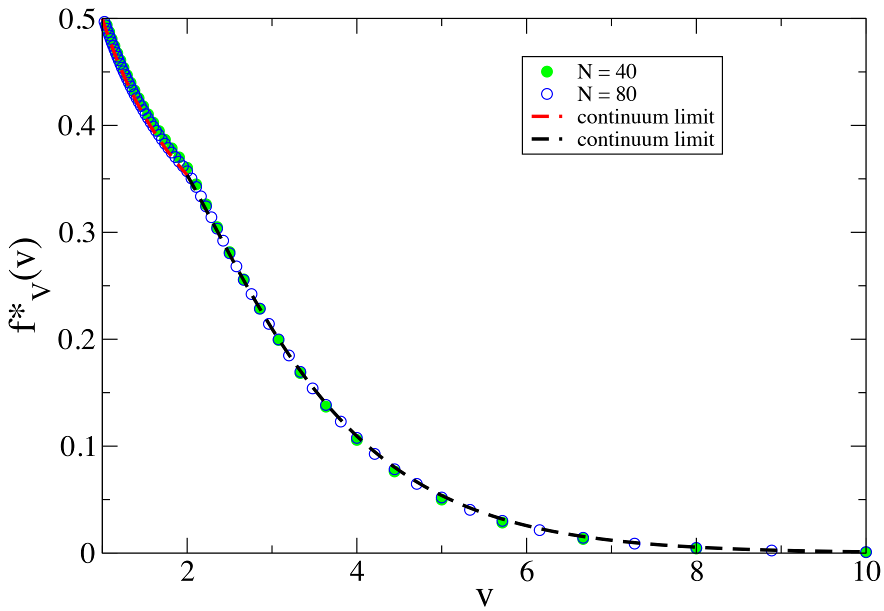

Figure 2. Probability density $f_V^\star(v)$ for the tied-down random walk obtained from (2.6) for $N = 40$ and $N = 80$. The black dashed curve ($v > 2$) corresponds to (2.21), the red dashed curve ($1 < v < 2$) to (2.23).

hence $f_R^\star(r) = r^{-3/2}/2$ for $1/2 < r < 1$. The reasoning is again due to Rosén. Consider

$$w_{N,k} = \mathrm{Prob}(I_N = 2k, S_{2N} = 0). \tag{2.24}$$

When $I_N = 2k > N$, then the longest interval is unique. This is case in figure 1, where $I_{15} = 18$. Decomposing a path into three contributions, we obtain

$$w_{N,k} = \sum_{n=0}^{N-k} u_n\, f_k\, u_{N-n-k} = f_k, \quad (2k > N), \tag{2.25}$$

the last equality resulting from $\sum_{j=0}^{m} u_j u_{m-j} = 1$. In figure 1 these three contributions correspond to $\tau_1 + \tau_2$, τ_3, and $\tau_4 + \tau_5$. So, using (B.5), we have

$$F_R^\star(r) = 1 - \lim_{N\to\infty} \sum_{k=Nr}^{N} \frac{w_{N,k}}{u_N} \tag{2.26}$$

$$= 1 - \int_r^1 \mathrm{d}y \frac{1}{2y^{3/2}} = 2 - \frac{1}{r^{1/2}}, \quad (1/2 < r < 1). \tag{2.27}$$

This result will be generalized in section 4.6. A method for the determination of $f_V^\star(v)$ in the successive intervals $(i, i+1)$ is given in [1]. Complementary information on this issue can be found in [4, 7, 8, 9].

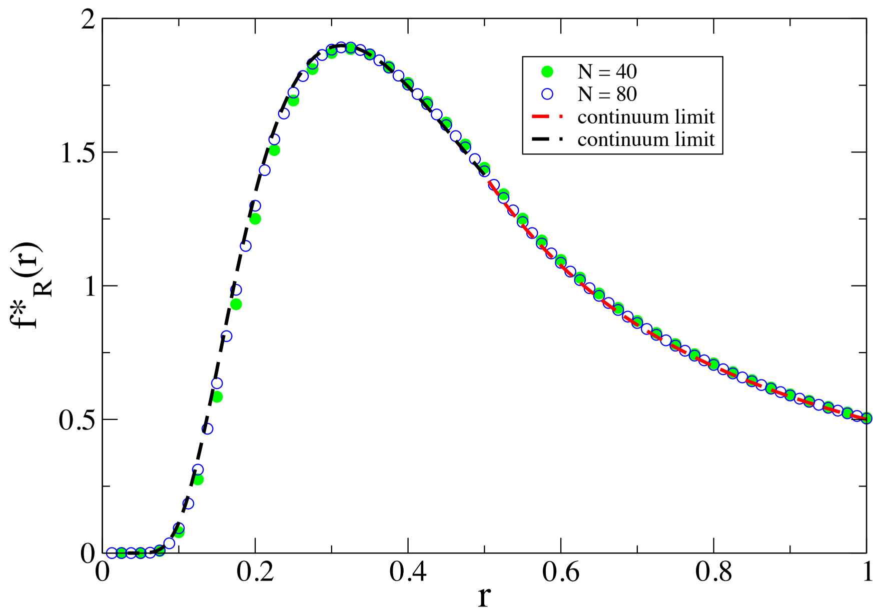

Figure 3. Probability density $f_R^\star(r)$ obtained from the data of figure 2.

Figure 2 depicts the density $f_V^\star(v)$ obtained by extracting $v_{N,k}$ from (2.6) by formal computation, with $0 \le k \le N$, for $N = 40$ and $N = 80$, then rescaling appropriately this sequence. Figure 3 depicts the density $f_R^\star(r)$ obtained from the same data. The scaling is already good for these rather small values of N. The dashed curves corresponds to the predictions (2.21), (2.22) and (2.23). It is striking to observe the quality of the two former ones for values of v or r beyond their a priori ranges of validity. The discontinuity of the derivative at $v = 2$ is clearly visible.

2.4. Average longest interval

Since

$$\mathrm{Prob}(I_N \le 2k, S_{2N} = 0) + \mathrm{Prob}(I_N > 2k, S_{2N} = 0) = u_N, \qquad (2.28)$$

the generating function of the sum adds to $1/\sqrt{1-z}$. In the continuum limit we thus find, using (2.16), that

$$\mathop{\mathcal{L}}_{N} \mathrm{Prob}(I_N > 2k, S_{2N} = 0) \approx \frac{1}{\sqrt{s}} \frac{1 - \sqrt{\pi x}\mathrm{e}^x \,\mathrm{erfc}\,\sqrt{x}}{1 + \sqrt{\pi x}\mathrm{e}^x \,\mathrm{erf}\,\sqrt{x}}. \qquad (2.29)$$

Defining

$$\widehat{f_V^{\mathrm{III}}}(x) = \frac{1 - \sqrt{\pi x}\,\mathrm{e}^x \,\mathrm{erfc}\,\sqrt{x}}{1 + \sqrt{\pi x}\,\mathrm{e}^x \,\mathrm{erf}\,\sqrt{x}} = 1 - \sqrt{\pi x}\,\widehat{f_V^{\mathrm{II}}}(x), \qquad (2.30)$$

whose meaning is given in section 3.3, we have

$$\mathop{\mathcal{L}}_{N} \langle I_N, S_{2N}=0\rangle = \mathop{\mathcal{L}}_{N} \int_0^\infty \mathrm{d}(2k)\ \mathrm{Prob}(I_N > 2k, S_{2N}=0)$$
$$= \frac{2}{s^{3/2}} \int_0^\infty \mathrm{d}x\, \widehat{f_V^{\mathrm{III}}}(x) = \frac{2}{s^{3/2}} 0.2417\ldots. \qquad (2.31)$$

By inversion of the Laplace transform and division by u_N, we obtain

$$\langle I_N | S_{2N}=0\rangle \approx 4N \times 0.2417\ldots. \qquad (2.32)$$

and finally

$$\langle R^\star\rangle = \lim_{N\to\infty} \langle R_N | S_{2N}=0\rangle = 0.4834\ldots. \qquad (2.33)$$

We shall comment and generalize this result later (cf. (4.40)).

2.5. The full joint probability for the tied-down random walk

An alternate method to recover the results above consists in considering the full joint probability of a configuration of the walk, in terms of the successive intervals $\tau_1, \tau_2, \ldots$ between zeros. Moreover this allows to investigate new quantities, such as the number of intervals up to time $2N$ or the probability of record breaking, i.e., the probability that the last interval be the longest. This formalism will also serve as a preparation for the sequel, where we consider the continuum renewal process generalizing the case of the random walk.

Let M_N be the number of intervals up to time $2N$. This random variable takes the values $m = 0, 1, 2, \ldots$. (For the tied-down walk, $m = 0$ necessarily implies $2N = 0$.) A configuration of the tied-down walk is specified by $\{\tau_1, \ldots, \tau_{M_N}, M_N\}$, whose realization is denoted by $\{2\ell_1, \ldots, 2\ell_{M_N}, m\}$. These definitions are illustrated in figure 1. The conditional probability for the walk to be tied-down is therefore

$$\mathrm{Prob}(\tau_1 = 2\ell_1, \ldots, \tau_{M_N} = 2\ell_{M_N}, M_N = m | S_{2N}=0) = \frac{f_{\ell_1}\ldots f_{\ell_m} \delta(\sum_{i=1}^m \ell_i, N)}{u_N}, \qquad (2.34)$$

where the denominator, obtained from the numerator by summing on the $\ell_i \ge 1$ and m, is precisely the probability u_N of return of the walk at time $2N$,

$$u_N = \sum_{m\ge 0} \sum_{\ell_1 \ldots \ell_m} f_{\ell_1}\ldots f_{\ell_m} \delta\Big(\sum_{i=1}^m \ell_i, N\Big). \qquad (2.35)$$

This can be checked by taking the generating function of the right side of (2.35)

$$\sum_{N\ge 0} z^N \sum_{m\ge 0} \sum_{\ell_1 \ldots \ell_m} f_{\ell_1}\ldots f_{\ell_m} \delta\Big(\sum_{i=1}^m \ell_i, N\Big) = \sum_{m\ge 0} \tilde{f}(z)^m \qquad (2.36)$$
$$= \frac{1}{1-\tilde{f}(z)} = \tilde{u}(z), \qquad (2.37)$$

which is indeed the generating function of the u_N. This can be alternatively be checked as follows. Let us denote the sum of the random number M_N of intervals by

$$t_{M_N} = \tau_1 + \cdots + \tau_{M_N}, \qquad (2.38)$$

and the corresponding sum when M_N is fixed equal to m by

$$t_m = \tau_1 + \cdots + \tau_m. \qquad (2.39)$$

Then, clearly,

$$\begin{aligned}\mathrm{Prob}(t_m = 2N) &= \mathrm{Prob}(M_N = m, S_{2N} = 0)\\ &= \sum_{\ell_1 \dots \ell_m} f_{\ell_1} \dots f_{\ell_m} \delta\Big(\sum_{i=1}^m \ell_i, N\Big). \end{aligned} \tag{2.40}$$

Hence, summing on m,

$$\sum_{m \ge 0} \mathrm{Prob}(t_m = 2N) = \mathrm{Prob}(t_{M_N} = 2N) \equiv u_N. \tag{2.41}$$

We can now easily express the probability distribution of the number of intervals (in the tied-down case) as

$$p_m^\star(N) = \mathrm{Prob}(M_N = m | S_{2N} = 0) = \frac{\mathrm{Prob}(t_m = 2N)}{u_N} \tag{2.42}$$

$$= \frac{\Big[\tilde{f}(z)^m\Big]_N}{u_N}, \tag{2.43}$$

where the notation $[\cdot]_N$ means the $N-$th coefficient of the series inside the brackets. In particular, the mean number of intervals is

$$\langle M_N | S_{2N} = 0 \rangle = \sum_{m \ge 0} m\, p_m^\star(N). \tag{2.44}$$

The generating function of the numerator of this expression reads

$$\sum_{m \ge 0} m \tilde{f}(z)^m = \frac{\tilde{f}(z)}{\big(1 - \tilde{f}(z)\big)^2}. \tag{2.45}$$

This yields

$$\langle M_N | S_{2N} = 0 \rangle = \frac{1}{u_N} - 1 = \frac{2^{2N}}{\binom{2N}{N}} - 1. \tag{2.46}$$

Hence, at long times $2N$,

$$\langle M_N | S_{2N} = 0 \rangle \approx \sqrt{\pi N}. \tag{2.47}$$

More generally, in the continuum limit where the series in the expressions above are dominated by z close to 1, the distribution $p_m^\star(N)$ has the scaling form

$$p_m^\star(N) \approx \frac{1}{2\sqrt{N}} y\, \mathrm{e}^{-y^2/4}, \qquad y = \frac{m}{\sqrt{N}}, \tag{2.48}$$

from which the asymptotic expressions of the higher moments ensue,

$$\langle (M_N)^p | S_{2N} = 0 \rangle = 2^{p-1} N^{p/2} p\, \Gamma\left(\frac{p}{2}\right). \tag{2.49}$$

2.6. The longest interval

Knowing (2.34) yields immediately the conditional probability (2.2)

$$\begin{aligned}\mathrm{Prob}(I_N \le 2k | S_{2N} = 0) &= \frac{v_{N,k}}{u_N}\\ &= \frac{1}{u_N} \sum_{m \ge 0} \sum_{\ell_1 = 1}^{k} \dots \sum_{\ell_m = 1}^{k} f_{\ell_1} \dots f_{\ell_m} \delta\Big(\sum_{i=1}^m \ell_i, N\Big). \end{aligned} \tag{2.50}$$

Taking the generating function of the numerator leads again to (2.6).

2.7. Probability of record breaking

When dealing with the successive extremes, or records, taken by a series of random variables, an important indicator of the statistics of records is the probability that the last random variable is the largest, named the probability of record breaking. For independent, identically distributed (i.i.d.) random variables this probability is equal to the inverse of the number of random variables in the series [10].

In the present case where the intervals are not independent, the determination of the probability of record breaking for the tied-down random walk, i.e.,

$$Q_N^\star = \mathrm{Prob}(I_N = \tau_{M_N} | S_{2N} = 0), \tag{2.51}$$

proceeds as follows (see [11] for similar reasonings). We have

$$Q_N^\star = \sum_{m \ge 1} \mathrm{Prob}(I_N = \tau_{M_N}, M_N = m | S_{2N} = 0) = \frac{Q_N}{u_N}, \tag{2.52}$$

where $Q_N = \sum_{m\ge 1} Q_{N,m}$, and

$$Q_{N,m} = \sum_{\ell_m \ge 1} \sum_{\ell_1=1}^{\ell_m} \cdots \sum_{\ell_{m-1}=1}^{\ell_m} f_{\ell_1} \dots f_{\ell_m} \delta\Big(\sum_{i=1}^m \ell_i, N\Big). \tag{2.53}$$

Taking the generating function of this quantity and summing on m we obtain

$$\tilde{Q}(z) = \sum_{N\ge 0} Q_N z^N = \sum_{\ell \ge 1} \frac{f_\ell z^\ell}{1 - \tilde{f}_\ell(z)}, \tag{2.54}$$

where $\tilde{f}_\ell(z)$ is defined in (2.5). This quantity appears in the appendix of [11], where it is found that, for $z \to 1$,

$$\tilde{Q}(z) \approx -\ln\sqrt{1-z} + c, \qquad c = \frac{1}{2}\Big(\gamma + \ln\frac{4}{\pi}\Big) \approx 0.409, \tag{2.55}$$

(γ is the Euler constant). By inversion and division by u_N, we thus finally obtain

$$Q_N^\star \approx \frac{\sqrt{\pi}}{2\sqrt{N}}. \tag{2.56}$$

Remark As recalled above, for i.i.d. random variables the probability of record breaking is equal to the inverse of the number of variables. The same holds if the random variables are exchangeable. In the present case, our intuition is that the intervals $\tau_1, \dots, \tau_{M_N}$ all play the same role. So we are led to compute the average of the inverse of the number of intervals M_N. Using (2.42), we have

$$\Big\langle \frac{1}{M_N} \Big\rangle = \sum_{m\ge 1} \frac{p_m^\star(N)}{m} = \frac{\big[-\ln\sqrt{1-z}\big]_N}{u_N} = \frac{1}{2N u_N}. \tag{2.57}$$

This expression is indeed asymptotically equal to (2.56). As we shall see in section 4.7, the same holds, not only asymptotically, but also at any time for the corresponding continuum renewal process.

3. A reminder on renewal processes

The present section and the next one generalize the tied-down random walk section 2 (which is actually a renewal process in discrete time) to a renewal process in continuous time, with an arbitrary distribution of intervals [12, 13]. We refer to table 1 for the correspondences between the notations for the discrete random walk and the continuum renewal process presented below.

Random walk	Renewal proc.
$2N$	t
$2k$	ℓ
I_N	$\tau_{\max}(t)$
u_N	$U(t)$
$v_{N,k}$	$F(t;\ell)$
f_n	$\rho(\tau)$
M_N	N_t
$\mathrm{Prob}(t_m = 2N)$	$f_{t_n}(t)$
$w_{N,k}$	$w(t;\ell)$

Table 1: Correspondences between the notations for the discrete random walk of section 2 and the continuum renewal process of sections 3 and 4.

3.1. Definitions and observables

We start afresh by reminding the definitions and notations used for renewal processes, following [2]. Events occur at the random epochs of time $t_1, t_2, \ldots$, from some time origin $t = 0$. These events are for instance the zero crossings of some stochastic process, (or zeros as for the simple random walk of section 2.1). We take the origin of time on a zero crossing. When the intervals of time between events, $\tau_1 = t_1, \tau_2 = t_2 - t_1, \ldots$, are independent and identically distributed random variables with common density $\rho(\tau)$, the process thus formed is a *renewal process*.

The probability $p_0(t)$ that no event occurred up to time t is simply given by the tail probability:

$$p_0(t) = \mathrm{Prob}(\tau_1 > t) = \int_t^\infty \mathrm{d}\tau\, \rho(\tau). \tag{3.1}$$

The density $\rho(\tau)$ can be either a narrow distribution with all moments finite, in which case the decay of $p_0(t)$, as $t \to \infty$, is faster than any power law, or a distribution characterized by a power-law fall-off with index $\theta > 0$

$$p_0(t) = \int_t^\infty \mathrm{d}\tau\, \rho(\tau) \approx \left(\frac{\tau_0}{t}\right)^\theta, \tag{3.2}$$

where τ_0 is a microscopic time scale. If $\theta < 1$ all moments of $\rho(\tau)$ are divergent, if $1 < \theta < 2$, the first moment $\langle\tau\rangle$ is finite but higher moments are divergent, and so on. In Laplace space, where s is conjugate to τ, for a narrow distribution we have

$$\mathop{\mathcal{L}}_{\tau}\, \rho(\tau) = \hat\rho(s) = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-s\tau} \rho(\tau) \mathop{=}_{s\to 0} 1 - \langle\tau\rangle\, s + \frac{1}{2}\left\langle\tau^2\right\rangle s^2 + \cdots \tag{3.3}$$

For a broad distribution, (3.2) yields

$$\hat\rho(s) \mathop{\approx}_{s\to 0} \begin{cases} 1 - a\, s^\theta & (\theta < 1) \\ 1 - \langle\tau\rangle\, s + a\, s^\theta & (1 < \theta < 2), \end{cases} \tag{3.4}$$

and so on, where

$$a = |\Gamma(1-\theta)|\tau_0^\theta. \tag{3.5}$$

From now on, unless otherwise stated, we shall only consider the case $0 < \theta < 1$.

The quantities naturally associated to a renewal process [12, 13, 2] are the following. The number of events which occurred between 0 and t, i.e., the largest

n such that $t_n \le t$, is a random variable denoted by N_t. The time of occurrence of the last event before t, that is of the N_t−th event, is therefore the sum of a random number of random variables‡

$$t_N = \tau_1 + \cdots + \tau_N. \tag{3.6}$$

The backward recurrence time A_t is defined as the length of time measured backwards from t to the last event before t, i.e.,

$$A_t = t - t_N. \tag{3.7}$$

It is therefore the age of the current, unfinished, interval at time t. Finally the forward recurrence time (or excess time) E_t is the time interval between t and the next event

$$E_t = t_{N+1} - t. \tag{3.8}$$

We have the simple relation $A_t + E_t = t_{N+1} - t_N = \tau_{N+1}$.

3.2. Joint probability densities

Consider the following sequences of intervals [3]

$$\begin{aligned}
&(\mathrm{I}): \quad \{\tau_1, \tau_2, \ldots, \tau_N, A_t\},\\
&(\mathrm{II}): \ \{\tau_1, \tau_2, \ldots, \tau_N, \tau_{N+1}\},\\
&(\mathrm{III}): \{\tau_1, \tau_2, \ldots, \tau_N\}.
\end{aligned} \tag{3.9}$$

To each of these sequences, supplemented by N_t, is associated a joint probability density [2, 3]. For the first sequence, this joint probability density is, with the notations of Appendix A,

$$f_{\vec{\tau},A_t,N_t}(t;\ell_1,\ldots,\ell_n,a,n) = \rho(\ell_1)\ldots\rho(\ell_n)\,p_0(a)\,\delta\Big(\sum_{i=1}^{n}\ell_i + a - t\Big). \tag{3.10}$$

Likewise, the joint probability density of $\tau_1, \ldots, \tau_{N+1}, N_t$ is

$$f_{\vec{\tau},\tau_{N+1},N_t}(t;\ell_1,\ldots,\ell_{n+1},n) = \rho(\ell_1)\ldots\rho(\ell_{n+1})\,I(t_n < t < t_n + \ell_{n+1}), \tag{3.11}$$

where $I(\cdot) = 1$ or 0 if the condition inside the parentheses is satisfied or not. Finally, for the third sequence, the joint probability density of $\tau_1, \ldots, \tau_N, N_t$ is

$$f_{\vec{\tau},N_t}(t;\ell_1,\ldots,\ell_n,n) = \rho(\ell_1)\ldots\rho(\ell_n)\int_0^\infty \mathrm{d}a\,p_0(a)\,\delta\left(\sum_{i=1}^{n}\ell_i + a - t\right), \tag{3.12}$$

which can alternatively be obtained from (3.10) or (3.11) by summing on a or ℓ_{n+1}, respectively. For short, we denote the joint probability densities (3.10)-(3.12) by

$$\begin{aligned}
f^{\mathrm{I}}(t;\ell_1,\ldots,\ell_n,a,n) &= f_{\vec{\tau},A_t,N_t}(t;\ell_1,\ldots,\ell_n,a,n),\\
f^{\mathrm{II}}(t;\ell_1,\ldots,\ell_{n+1},n) &= f_{\vec{\tau},\tau_{N+1},N_t}(t;\ell_1,\ldots,\ell_{n+1},n),\\
f^{\mathrm{III}}(t;\ell_1,\ldots,\ell_n,n) &= f_{\vec{\tau},N_t}(t;\ell_1,\ldots,\ell_n,n).
\end{aligned} \tag{3.13}$$

The explicit dependence in time t, which acts as a parameter in these densities, is enhanced by the notations above.

‡ From now on, when no ambiguity arises, we drop the time dependence of the random variable if the latter is itself in subscript.

3.3. Longest interval

To each of these sequences corresponds a longest interval, denoted by

$$
\begin{aligned}
&\tau^{\mathrm{I}}_{\max}(t) = \max(\tau_1, \tau_2, \dots, \tau_N, A_t),\\
&\tau^{\mathrm{II}}_{\max}(t) = \max(\tau_1, \tau_2, \dots, \tau_N, \tau_{N+1}),\\
&\tau^{\mathrm{III}}_{\max}(t) = \max(\tau_1, \tau_2, \dots, \tau_N).
\end{aligned}
\tag{3.14}
$$

It turns out that the ratios

$$
R^\alpha = \lim_{t\to\infty} \frac{\tau^\alpha_{\max}(t)}{t}, \quad V^\alpha = \frac{1}{R^\alpha}, \qquad (\alpha = \mathrm{I, II, III}) \tag{3.15}
$$

have limiting distributions, whose densities are denoted by $f^\alpha_R(r)$ and $f^\alpha_V(v)$. Explicit expressions for the Laplace transforms with respect to v of the latter, with x conjugate to v, are as follows:

$$
\widehat{f^{\mathrm{I}}_V}(x) = \frac{1}{1 + x^\theta \mathrm{e}^x \int_0^x \mathrm{d}u\, u^{-\theta} \mathrm{e}^{-u}} = \frac{1}{{}_1F_1(1, 1-\theta, x)}, \tag{3.16}
$$

$$
\widehat{f^{\mathrm{II}}_V}(x) = \mathrm{e}^x \widehat{f^{\mathrm{I}}_V}(x), \tag{3.17}
$$

$$
\widehat{f^{\mathrm{III}}_V}(x) = 1 - x^\theta \Gamma(1-\theta) \widehat{f^{\mathrm{II}}_V}(x), \tag{3.18}
$$

where ${}_1F_1(1, 1-\theta, x)$ is a confluent hypergeometric function, simply related to the incomplete gamma function

$$
\Gamma(\theta, x) = \int_x^\infty \mathrm{d}u\, u^{\theta-1} \mathrm{e}^{-u}, \tag{3.19}
$$

as follows,

$$
{}_1F_1(1, 1-\theta, x) = \mathrm{e}^x x^\theta \left[\Gamma(1-\theta) + \theta\Gamma(-\theta, x)\right]. \tag{3.20}
$$

The functions $f^{\mathrm{II}}_V(v)$ and $f^{\mathrm{III}}_V(v)$ encountered in (2.11) and (2.30), respectively, are precisely the densities of the random variables V^{II} and V^{III} defined for the second and third sequences (with $\theta = 1/2$).

The expression of the Laplace transform of the density (3.16) was originally found in [6], then derived by another method in [3], which also addresses the same question for the two other sequences II and III. Related studies can also be found in [11, 14, 15] in the context of record statistics of random walks and renewal processes.

4. The tied-down renewal process

The tied-down renewal process is defined by the condition $\{t_N = t\}$, or equivalently by the condition $\{A_t = 0\}$, which both express that the N_t−th event occurred at time t. This process generalizes the tied-down random walk of section 2, or its continuum limit, the Brownian bridge. The tied-down semi-stable process of order $0 < \theta < 1$ considered in [1] corresponds to the tied-down renewal process considered here when the tail index of the density $\rho(\tau)$ is $0 < \theta < 1$.

4.1. The tied-down conditional density

The tied-down conditional density, denoted for short by

$$
f^\star(t; \ell_1, \dots, \ell_n, n) \equiv f_{\vec{\tau}, N_t | t_N}(t; \ell_1, \dots, \ell_n, n | y = t), \tag{4.1}
$$

is a generalization of (2.34) (see Appendix C for more details). Its expression is

$$f^\star(t;\ell_1,\dots,\ell_n,n) = \frac{\rho(\ell_1)\dots\rho(\ell_n)\delta\left(\sum \ell_i - t\right)}{U(t)}, \tag{4.2}$$

where the denominator is obtained from the numerator by integration on the ℓ_i and summation on n,

$$\begin{aligned} U(t) &= \sum_{n\ge 0}\int_0^\infty \mathrm{d}\ell_1\dots\mathrm{d}\ell_n\,\rho(\ell_1)\dots\rho(\ell_n)\delta\Big(\sum_{i=1}^n \ell_i - t\Big) \\ &= \sum_{n\ge 0} f_{t_n}(t), \end{aligned} \tag{4.3}$$

denoting by $f_{t_n}(t)$ the density of the sum $t_n = \tau_1 + \cdots + \tau_n$, with n fixed (compare to (2.40) for the discrete case). The quantity (4.3), which is the continuum counterpart of the probability u_N of section 2.1 (cf. (2.35) and (2.41)), can be intuitively thought of as giving the fraction of 'tied-down' histories. It is the edge value of the probability density of t_N at its maximal value $t_N = t$ (see (C.7) and (C.12)). In Laplace space with respect to t, we have

$$\mathop{\mathcal{L}}_t f_{t_n}(t) = \hat\rho(s)^n, \tag{4.4}$$

so

$$\mathop{\mathcal{L}}_t U(t) = \sum_{n\ge 0}\hat\rho(s)^n = \frac{1}{1-\hat\rho(s)}, \tag{4.5}$$

which is the counterpart of (2.37). The right side behaves, when s is small, as $s^{-\theta}/a$. Thus, at long times, we finally obtain, using (3.5),

$$U(t) \approx \frac{\sin\pi\theta}{\pi}\frac{t^{\theta-1}}{\tau_0^\theta}. \tag{4.6}$$

Equations (4.2)-(4.6) are the cornerstone of the present study.

4.2. Number of renewals between 0 *and* t

Let us consider the conditional distribution of N_t, the number of renewals between 0 and t, for the tied-down renewal process,

$$p_n^\star(t) = \mathrm{Prob}(N_t = n|t_N = t) = \frac{f_{t_n}(t)}{U(t)}, \tag{4.7}$$

whose discrete counterpart is (2.42). We have

$$\langle N_t|t_N = t\rangle = \sum_{n>0} n p_n^\star(t) = \frac{\sum_{n>0} n f_{t_n}(t)}{U(t)}. \tag{4.8}$$

In Laplace space we have

$$\mathop{\mathcal{L}}_t \sum_{n>0} n f_{t_n}(t) = \frac{\hat\rho(s)}{(1-\hat\rho(s))^2} \approx \frac{1}{a^2 s^{2\theta}}. \tag{4.9}$$

Laplace inverting back and dividing by (4.6), we obtain, at large times

$$\langle N_t|t_N = t\rangle \approx A^\star(\theta)\left(\frac{t}{\tau_0}\right)^\theta, \quad A^\star(\theta) = \frac{\Gamma(\theta)}{\Gamma(1-\theta)\Gamma(2\theta)}. \tag{4.10}$$

By comparison, for the unconstrained renewal process [2],

$$\langle N_t \rangle \approx A(\theta) \left(\frac{t}{\tau_0} \right)^{\theta}, \quad A(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \tag{4.11}$$

Note that $A^{\star}(\theta) > A(\theta)$.

Likewise

$$\mathop{\mathcal{L}}_{t} \sum_{n\;0} n^2 f_{t_n}(t) = \frac{\hat{\rho}(s)(1+\hat{\rho}(s))}{(1-\hat{\rho}(s))^3} \approx \frac{2}{a^3 s^{3\theta}}. \tag{4.12}$$

By inversion and division by (4.6), we obtain $\langle N_t^2 | t_N = t \rangle \sim t^{2\theta}$. As for the unconstrained case [2], we can set

$$N_t = \left(\frac{t}{\tau_0} \right)^{\theta} Y_t, \tag{4.13}$$

where the random variable Y_t has a limiting distribution when $t \to \infty$. For instance, for $\theta = 1/2$, we obtain

$$f_Y(y) = \frac{\pi}{2} y\, \mathrm{e}^{-\pi y^2/4}, \qquad y = \frac{n}{\sqrt{t/\tau_0}}, \tag{4.14}$$

which is the counterpart of (2.48). More generally, we have

$$f_Y(y) = \frac{\pi}{\sin \pi\theta} \int \frac{\mathrm{d}z}{2\pi \mathrm{i}} \mathrm{e}^{z} \mathrm{e}^{-\Gamma(1-\theta) y z^{\theta}}. \tag{4.15}$$

4.3. Marginal statistics of a single interval

We want to determine the tied-down conditional average of one of the τ_i, say τ_1,

$$\langle \tau_1 | t_N = t \rangle = \sum_{n \ge 0} \int_0^{\infty} \mathrm{d}\ell_1 \ldots \mathrm{d}\ell_n\, \ell_1 f^{\star}(t; \ell_1, \ldots, \ell_n, n). \tag{4.16}$$

Laplace transforming the numerator of the right side yields

$$-\frac{\mathrm{d}\hat{\rho}(s)}{\mathrm{d}s} \frac{1}{1-\hat{\rho}(s)} \approx \frac{\theta}{s}. \tag{4.17}$$

By Laplace inverting and dividing by (4.6), we obtain

$$\langle \tau_1 | t_N = t \rangle \approx B^{\star}(\theta) \tau_0^{\theta} t^{1-\theta}, \quad B^{\star}(\theta) = \frac{\pi\theta}{\sin \pi\theta}, \tag{4.18}$$

which turns out to be equal to $t/\langle N_t \rangle$. By comparison, for the unconstrained renewal process [2],

$$\langle \tau_1 \rangle \approx B(\theta) \tau_0^{\theta} t^{1-\theta}, \quad B(\theta) = \frac{\theta}{1-\theta}. \tag{4.19}$$

We conclude that $\langle N_t | t_N = t \rangle \langle \tau_1 | t_N = t \rangle$ is proportional to t, as expected. Note that $\langle N_t \tau_1 \rangle = \langle t_N \rangle = t$, identically. Here again, the amplitude of the tied-down case $B^{\star}(\theta)$ is larger than the amplitude of the unconstrained case $B(\theta)$.

4.4. The longest interval

Let $\tau^\star_{\max}(t)$ be the longest interval of the sequence $\tau_1, \dots, \tau_N$ with the condition that their sum $t_N = t$. We want to compute the conditional distribution function

$$
\begin{aligned}
&F^\star(t;\ell) = \mathrm{Prob}(\tau^\star_{\max}(t) \le \ell | t_N = t) \\
&= \sum_{n\ge 0} \int_0^\ell \mathrm{d}\ell_1 \dots \int_0^\ell \mathrm{d}\ell_n f^\star(t;\ell_1 \dots, \ell_n, n, y = t) = \frac{F(t;\ell)}{U(t)},
\end{aligned}
\tag{4.20}
$$

where the numerator is

$$
F(t;\ell) = \sum_{n\ge 0} \int_0^\ell \mathrm{d}\ell_1\, \rho(\ell_1) \dots \int_0^\ell \mathrm{d}\ell_n\, \rho(\ell_n) \delta\Big(\sum_{i=0}^n \ell_i - t\Big). \tag{4.21}
$$

Equation (4.20) is the continuum counterpart of (2.50), with $F(t;\ell)$ playing the role of $v_{N,k}$. Laplace transforming (4.21) with respect to time, we get

$$
\mathop{\mathcal{L}}_{t} F(t;\ell) = \sum_{n\ge 0} \left(\int_0^\ell \mathrm{d}\ell_1\, \rho(\ell_1) \mathrm{e}^{-s\ell_1} \right)^n = \frac{1}{1 - \hat{\rho}(s;\ell)}, \tag{4.22}
$$

where

$$
\hat{\rho}(s;\ell) = \int_0^\ell \mathrm{d}\ell_1\, \rho(\ell_1) \mathrm{e}^{-s\ell_1}. \tag{4.23}
$$

The expression (4.22) is the continuum counterpart of $\tilde{v}_k(z)$ given in (2.6). It holds for any distribution of intervals $\rho(\tau)$. In the limit $\ell \to \infty$ the right side is equal to $1/(1 - \hat{\rho}(s))$, as it should (cf. (4.5)).

We now perform the asymptotic analysis of (4.22) along the lines of [3]. An integration by parts yields

$$
1 - \hat{\rho}(s;\ell) = p_0(\ell) \mathrm{e}^{-s\ell} + s\, \hat{p}_0(s;\ell), \tag{4.24}
$$

where $\hat{p}_0(s;\ell) = \int_0^\ell \mathrm{d}\tau\, p_0(\tau) \mathrm{e}^{-s\tau}$. Then, using the asymptotic estimates in the regime $s \to 0$, $\ell \to \infty$, with $s\ell$ fixed [3],

$$
\hat{p}_0(s;\ell) \approx \tau_0^\theta s^{\theta - 1} \int_0^{s\ell} \mathrm{d}u\, u^{-\theta} \mathrm{e}^{-u}, \tag{4.25}
$$

and

$$
1 - \hat{\rho}(s;\ell) \approx \tau_0^\theta s^\theta \left((s\ell)^{-\theta} \mathrm{e}^{-s\ell} + \int_0^{s\ell} \mathrm{d}u\, u^{-\theta} \mathrm{e}^{-u} \right), \tag{4.26}
$$

we obtain

$$
\mathop{\mathcal{L}}_{t} F(t;\ell) \approx \left(\frac{\ell}{\tau_0} \right)^\theta \frac{\mathrm{e}^{s\ell}}{1 + (s\ell)^\theta \mathrm{e}^{s\ell} \int_0^{s\ell} \mathrm{d}u\, u^{-\theta} \mathrm{e}^{-u}}. \tag{4.27}
$$

Laplace inverting with respect to s and dividing by $t^{\theta-1} \sin \pi\theta / (\pi \tau_0^\theta)$ (given in (4.6)), we finally obtain, using notations akin to (3.15),

$$
\begin{aligned}
F^\star_R(r) = \overline{F}^\star_V(v) &= \lim_{t\to\infty} F^\star(t;\ell), \\
&= \frac{\pi}{\sin \pi\theta} v^{1-\theta} f_V^{\mathrm{II}}(v),
\end{aligned}
\tag{4.28}
$$

where $f_V^{\mathrm{II}}(v)$ is the inverse Laplace transform of (3.17). Note that the microscopic scale τ_0 altogether disappeared in (4.28), which demonstrates the universality of the result with respect to the distribution $\rho(\tau)$. In particular (2.11), recovered for $\theta = 1/2$, is universal. Equation (4.28) can be alternatively written as

$$\mathop{\mathcal{L}}_{v} v^{\theta-1} \overline{F}_V^\star(v) = \frac{\pi}{\sin \pi\theta} \widehat{f_V^{\rm II}}(x), \tag{4.29}$$

$$\mathop{\mathcal{L}}_{v} v^{\theta-1} F_V^\star(v) = \frac{\Gamma(\theta)}{x^\theta} \widehat{f_V^{\rm III}}(x), \tag{4.30}$$

using (3.18). This expression, as well as its generalizations to the case of the second ($k=2$), third ($k=3$), ..., longest intervals, are given in [1] (cf. § 5, theorem 4),

$$\mathop{\mathcal{L}}_{v} v^{\theta-1} F_V^{(k)\star}(v) = \frac{\Gamma(\theta)}{x^\theta} \left(\widehat{f_V^{\rm III}}(x)\right)^k. \tag{4.31}$$

Note that $\Gamma(\theta)/x^\theta$ is equal to the Laplace transform of $v^{\theta-1}$. Ref. [1] also gives the generalization of the density (3.16) for the $k-$longest interval,

$$\widehat{f_V^{(k)\rm I}}(x) = \widehat{f_V^{\rm I}}(x) \left(\widehat{f_V^{\rm III}}(x)\right)^{k-1}. \tag{4.32}$$

Equations (4.31) and (4.32) are derived in Appendix D by elementary methods of order statistics theory. Likewise, one could show that

$$\widehat{f_V^{(k)\rm II}}(x) = \widehat{f_V^{\rm II}}(x) \left(\widehat{f_V^{\rm III}}(x)\right)^{k-1}, \tag{4.33}$$

and

$$\widehat{f_V^{(k)\rm III}}(x) = \left(\widehat{f_V^{\rm III}}(x)\right)^{k}, \tag{4.34}$$

which can be summarized as

$$\widehat{f_V^{(k)\alpha}}(x) = \widehat{f_V^{\alpha}}(x) \left(\widehat{f_V^{\rm III}}(x)\right)^{k-1}, \quad (\alpha = {\rm I}, {\rm II}, {\rm III}). \tag{4.35}$$

4.5. Average longest interval

The method follows that of section (2.4). The average longest interval is computed as

$$\langle \tau_{\max}^\star(t) \rangle = \int_0^\infty {\rm d}\ell\, \overline{F}^\star(t;\ell). \tag{4.36}$$

We have (see (4.20))

$$F(t;\ell) + \overline{F}(t;\ell) = U(t). \tag{4.37}$$

Laplace transforming this equation with respect to time, we get

$$\mathop{\mathcal{L}}_{t} \overline{F}(t;\ell) = \frac{1}{1-\hat{\rho}(s)} - \frac{1}{1-\hat{\rho}(s;\ell)} \tag{4.38}$$

$$\approx \frac{s^{-\theta}}{a} \left(1 - (s\ell)^\theta \Gamma(1-\theta) \widehat{f_V^{\rm II}}(s\ell)\right) = \frac{s^{-\theta}}{a} \widehat{f_V^{\rm III}}(s\ell). \tag{4.39}$$

After integration upon ℓ, inverse Laplace transform with respect to s, and division by $U(t)$ (given by (4.6)), we obtain

$$\langle R^\star \rangle = \lim_{t\to\infty} \left\langle \frac{\tau_{\max}^\star(t)}{t} \right\rangle = \frac{1}{\theta} \int_0^\infty {\rm d}x\, \widehat{f_V^{\rm III}}(x) \tag{4.40}$$

$$= \frac{1}{\theta} \lim_{t\to\infty} \left\langle \frac{\tau_{\max}^{\rm III}(t)}{t} \right\rangle = \frac{1}{\theta} \langle R^{\rm III} \rangle. \tag{4.41}$$

For $\theta = 1/2$ we recover (2.33).

Using the same method, we find, for the $k-$th longest interval,

$$\langle R^{(k)\star}\rangle = \lim_{t\to\infty}\left\langle \frac{\tau_{\max}^{(k)\star}(t)}{t}\right\rangle = \frac{1}{\theta}\int_0^\infty \mathrm{d}x\,\left(\widehat{f_V^{\rm III}}(x)\right)^k$$
$$= \frac{1}{\theta}\langle R^{(k)\rm III}\rangle. \qquad (4.42)$$

Since the sum of the intervals $\tau_{\max}^{(k)\star}(t)$ is, by definition of the process, equal to t, one should have

$$\sum_{k\ge 1}\langle R^{(k)\star}\rangle = 1. \qquad (4.43)$$

This result can indeed be proved by direct computation of the integral of the geometrical series in $\widehat{f_V^{\rm III}}(x)$ in the right side, which gives the value θ. One can also note that the $\langle R^{(k)\rm III}\rangle$ sums up to $\langle t_N/t\rangle$ and that $\langle t_N/t\rangle \to \theta$ [2]. (See also [16, 5].)

Remark The last equality in (4.42) can be obtained by the methods of Appendix D (see [16]). A list of values of $\langle R^{(k)\rm III}\rangle$ for $\theta = 1/2$ can be found in [16]. One could generalize the results of [16] to find

$$\langle R^{(k)\rm I}\rangle = \int_0^\infty \mathrm{d}x\,\widehat{f_V^{(k)\rm I}}(x), \quad \langle R^{(k)\rm II}\rangle = \int_0^\infty \mathrm{d}x\,\widehat{f_V^{(k)\rm II}}(x). \qquad (4.44)$$

4.6. Characterization of the densities $f_V^\star$ and $f_R^\star$

The denominator of $\widehat{f_V^{\rm II}}(x)$ in (3.16), $D(x) = 1 + x^\theta \mathrm{e}^x \int_0^x \mathrm{d}u\, u^{-\theta}\mathrm{e}^{-u}$, satisfies the following differential equation

$$xD'(x) = D(x)(x+\theta) - \theta. \qquad (4.45)$$

The residues at the poles x_k of $\widehat{f_V^{\rm II}}(x)$ are therefore equal to $-x_k \mathrm{e}^{x_k}/\theta$, so

$$F_R^\star(r) = \overline{F}_V^\star(v) = \frac{\pi}{\theta\sin\pi\theta} v^{1-\theta}\sum_{k=-\infty}^{\infty}(-x_k)\mathrm{e}^{x_k(1+v)}, \qquad (4.46)$$

which generalizes (2.10) [1]. Hence the asymptotic behaviour at large v (small r) of the corresponding densities is obtained as in section 2.3 (cf. (2.21), (2.22)), yielding

$$f_V^\star(v) \approx \frac{\pi x_0^2}{\theta\sin\pi\theta} v^{1-\theta}\,\mathrm{e}^{-|x_0|v}, \quad f_R^\star(r) \approx \frac{\pi\, x_0^2}{\theta\sin\pi\theta}\frac{\mathrm{e}^{-|x_0|/r}}{r^{3-\theta}}. \qquad (4.47)$$

The density $f_R^\star(r)$ for $1/2 < r < 1$ has a simple expression,

$$f_R^\star(r) = \frac{\theta\,\Gamma(\theta)^2\sin\pi\theta}{\pi\Gamma(2\theta)}\frac{(1-r)^{2\theta-1}}{r^{1+\theta}}, \qquad (4.48)$$

which, for $\theta = 1/2$, yields back $f_R^\star(r) = r^{-3/2}/2$. This expression is obtained by the following reasoning, adapted from that used in [1] for the tied-down random walk, and reproduced in section 2.3 (cf. (2.25)). Let $w(t;\ell)$ be the density

$$w(t;\ell) = \frac{\mathrm{d}F(t;\ell)}{\mathrm{d}\ell}, \qquad (4.49)$$

where $F(t;\ell)$ is defined in (4.21). If $\ell > t/2$, then the longest interval is unique. Decomposing an history into three contributions as in (2.25), we have

$$w(t;\ell) = \int_0^{t-\ell}\mathrm{d}\tau\, U(\tau)\rho(\ell)U(t-\ell-\tau). \qquad (4.50)$$

Noting that

$$\underset{T}{\mathcal{L}} \int_0^T \mathrm{d}\tau\, U(\tau)U(T-\tau) = \frac{1}{(1-\hat{\rho}(s))^2} \approx (as)^{-2\theta}, \tag{4.51}$$

we have, for $\ell > t/2$,

$$w(t;\ell) \approx \frac{1}{a^{2\theta}\Gamma(2\theta)} \rho(\ell)(t-\ell)^{2\theta-1}. \tag{4.52}$$

It follows that

$$F_R^\star(r) = 1 - \frac{C(\theta)}{t^{\theta-1}} \int_{rt}^t \mathrm{d}\ell\, \ell^{-1-\theta}(t-\ell)^{2\theta-1}, \quad (1/2 < r < 1), \tag{4.53}$$

where

$$C(\theta) = \frac{\theta\, \Gamma(\theta)^2 \sin \pi\theta}{\pi\Gamma(2\theta)}, \tag{4.54}$$

yielding (4.48), by derivation with respect to r. It is also possible to derive (4.48) from eq. (6.1) of [1]. As a consequence

$$f_V^\star(v) = \frac{\theta\, \Gamma(\theta)^2 \sin \pi\theta}{\pi\Gamma(2\theta)} v^{-\theta}(v-1)^{2\theta-1}. \tag{4.55}$$

4.7. Probability of record breaking $Q^\star(t)$

The probability that the last interval is the longest one is defined as

$$Q^\star(t) = \mathrm{Prob}(\tau_{\max}^\star(t) = \tau_N | t_N = t) = \mathrm{Prob}(\tau_N > \max(\tau_1, \dots, \tau_{N-1}) | t_N = t). \tag{4.56}$$

This probability is given by the sum (see [3] for similar reasonings)

$$Q^\star(t) = \sum_{n\ge1} Q_n^\star(t) = \sum_{n\ge1} \mathrm{Prob}\Big(\tau_N > \max(\tau_1,\dots,\tau_{N-1}), N_t = n | t_N = t\Big). \tag{4.57}$$

Explicitly,

$$Q_n^\star(t) = \int_0^\infty \mathrm{d}\ell_n \int_0^{\ell_n} \mathrm{d}\ell_1 \dots \int_0^{\ell_n} \mathrm{d}\ell_{n-1}\, f^\star(t;\ell_1,\dots,\ell_{n-1},\ell_n,n) \tag{4.58}$$

$$= \frac{Q_n(t)}{U(t)}, \tag{4.59}$$

where

$$Q_n(t) = \int_0^\infty \mathrm{d}\ell_n \int_0^{\ell_n} \mathrm{d}\ell_1 \dots \int_0^{\ell_n} \mathrm{d}\ell_{n-1}\, \rho(\ell_1)\dots\rho(\ell_n)\, \delta\Big(\sum_{i=0}^n \ell_i - t\Big). \tag{4.60}$$

In Laplace space, after summing on n, we have

$$\hat{Q}(s) = \int_0^\infty \mathrm{d}\ell\, \frac{\rho(\ell)\mathrm{e}^{-s\ell}}{1 - \int_0^\ell \mathrm{d}\tau\, \rho(\tau)\mathrm{e}^{-s\tau}} = \int_0^{\hat{\rho}(s)} \frac{\mathrm{d}\hat{\rho}(s;\ell)}{1-\hat{\rho}(s;\ell)}, \tag{4.61}$$

where $\hat{\rho}(s;\ell) = \int_0^\ell \mathrm{d}\tau\, \rho(\tau)\mathrm{e}^{-s\tau}$. Finally,

$$\hat{Q}(s) = -\ln(1-\hat{\rho}(s)). \tag{4.62}$$

The same result can be recovered by assuming that the N_t intervals $\tau_1, \dots, \tau_N$ should all play the same role, hence that the probability of record breaking is equal to the inverse number of these random variables, as for i.i.d. random variables. So, let us assume that

$$Q_n^\star(t) = \frac{p_n^\star(t)}{n}, \quad (n > 0), \tag{4.63}$$

where $p_n^\star(t) = \mathrm{Prob}(N_t = n | t_N = t)$ (see (4.7)). Thus

$$Q^\star(t) = \sum_{n \ge 1} Q_n^\star(t) = \left\langle N_t^{-1} | t_N = t \right\rangle . \tag{4.64}$$

In Laplace space, the numerator of this expression is

$$\begin{aligned}\hat{Q}(s) &= \sum_{n\ge 1} \frac{\hat{f}_{t_n}(s)}{n} = \sum_{n\ge 1} \frac{\hat{\rho}(s)^n}{n}, \\ &= -\ln(1 - \hat{\rho}(s)),\end{aligned} \tag{4.65}$$

which is (4.62) above. The last step consists in Laplace inverting with respect to s, then dividing by $U(t)$. We thus find, at large times,

$$Q^\star(t) \approx \frac{\pi\theta}{\sin \pi\theta} \left(\frac{\tau_0}{t}\right)^\theta \approx \frac{1}{\langle N_t \rangle}, \tag{4.66}$$

where the right side pertains to the unconstrained case (see (4.11)). There is no universality of the result with respect to the choice of distribution $\rho(\tau)$ since the microscopic scale τ_0 is still present. We also recall, for comparison, that $Q^{\mathrm{III}}(t) \sim \ln t / t^\theta$ [3].

4.8. Narrow distribution of intervals

The aim of this subsection is to determine the distribution of $\tau^\star_{\max}(t)$ and the probability of record breaking $Q^\star(t)$ for a narrow distribution of intervals, taking the exponential distribution of intervals, $\rho(\tau) = \mathrm{e}^{-\tau}$, as an example. We first note that, by inversion of (4.5), we have $U(t) = 1$ for $t > 0$.

The computation of $\langle \tau^\star_{\max}(t) \rangle$ relies on (4.36) and (4.38). We find

$$\int_0^\infty \mathrm{d}\ell \ \mathop{\mathcal{L}}_t \overline{F}(t;\ell) = \frac{1}{s} \ln\left(1 + \frac{1}{s}\right), \tag{4.67}$$

whose inverse is

$$\int_0^\infty \mathrm{d}\ell\, \overline{F}(t;\ell) = \mathrm{E}(t) \equiv \int_0^t \mathrm{d}u\, \frac{1 - \mathrm{e}^{-u}}{u}. \tag{4.68}$$

At large times, $\mathrm{E}(t) \approx \ln t + \gamma$, where γ is the Euler constant. we finally obtain

$$\langle \tau^\star_{\max}(t) \rangle = \mathrm{E}(t) \approx \ln t + \gamma. \tag{4.69}$$

We also have, for s small and ℓ large,

$$\mathop{\mathcal{L}}_t F(t;\ell) = \frac{1}{s + \mathrm{e}^{-\ell}}, \tag{4.70}$$

which by inversion yields

$$F(t;\ell) = F^\star(t;\ell) \approx \mathrm{e}^{-\mathrm{e}^{-(\ell - \ln t)}}. \tag{4.71}$$

So

$$\tau^\star_{\max}(t) \approx \ln t + Z^{\mathrm{G}}, \tag{4.72}$$

where Z^{G} follows the standard Gumbel distribution, with $\langle Z^{\mathrm{G}} \rangle = \gamma$. This behaviour coincides with that found for the three sequences (3.9) in [3]. We also find, from (4.65) that $Q^\star(t) \approx 1/t$. Since $\langle N_t | t_N = t \rangle \approx t$, as can be inferred from (4.9), $Q^\star(t)$ behaves qualitatively as if the N_t intervals were i.i.d. random variables. This is akin to what was found for the cases of $Q^{\mathrm{I}}(t)$ and $Q^{\mathrm{III}}(t)$, the probabilities of record breaking for the sequences I and III [3].

4.9. Broad distribution of intervals with $\theta > 1$

We first find, by inversion of (4.5), that

$$U(t) \approx \frac{1}{\langle \tau \rangle} + \frac{\tau_0^\theta}{(\theta - 1)\langle \tau \rangle^2} t^{1-\theta}. \tag{4.73}$$

We then compute the average number of renewals, using the first equality in (4.9). We obtain, after division by $1/\langle \tau \rangle$,

$$\langle N_t | t_N = t \rangle \approx \frac{t}{\langle \tau \rangle} + \frac{2\tau_0^\theta}{(\theta - 1)(2 - \theta)\langle \tau \rangle^2} t^{2-\theta}. \tag{4.74}$$

We restart from (4.22) in order to compute the distribution of $\tau^\star_{\max}(t)$. Following the asymptotic analysis made in [3], we find

$$\mathop{\mathcal{L}}_{t} F(t; \ell) \approx \frac{1}{\langle \tau \rangle} \frac{1}{s + (\ell/\tau_0)^{-\theta}/\langle \tau \rangle}, \tag{4.75}$$

hence

$$F(t; \ell) \approx \frac{1}{\langle \tau \rangle} \mathrm{e}^{-t/\langle \tau \rangle (\ell/\tau_0)^{-\theta}}. \tag{4.76}$$

Dividing this expression by the leading order $1/\langle \tau \rangle$ in (4.73), we have

$$F^\star(t; \ell) \approx \mathrm{e}^{-t/\langle \tau \rangle (\ell/\tau_0)^{-\theta}}. \tag{4.77}$$

Setting

$$\tau^\star_{\max}(t) = \tau_0 \left(\frac{t}{\langle \tau \rangle} \right)^{1/\theta} Z_t, \tag{4.78}$$

we have, as $t \to \infty$, $Z_t \to Z^F$, with limiting distribution

$$\mathrm{Prob}(Z^F < x) = \mathrm{e}^{-1/x^\theta} \tag{4.79}$$

which is the Fréchet law. Therefore

$$\langle \tau^\star_{\max}(t) \rangle \approx \tau_0 \left(\frac{t}{\langle \tau \rangle} \right)^{1/\theta} \underbrace{\langle Z^F \rangle}_{\Gamma(1-1/\theta)} . \tag{4.80}$$

This is exactly the result found for the three sequences (3.14) in ref. [3]. The tied-down condition does not change the asymptotic distribution of the longest interval if $\theta > 1$. Finally, from (4.65) we find

$$Q^\star(t) \approx \frac{\langle \tau \rangle}{t}, \tag{4.81}$$

which has therefore the same time dependence as $Q^{\mathrm{III}}(t)$ [3].

5. Discussion

The tied-down renewal process studied in the present paper is equivalent to the stable process considered in [1] if the tail exponent of the distribution of intervals is comprised between 0 and 1. The results of [1] concerning the statistics of the longest interval are thus recovered in a simple manner. This study is extended in several directions such as the statistics of the number of intervals or the probability of record breaking, both for the tied-down random walk, the Brownian bridge and the tied-down renewal process. We also discuss the cases of a narrow distribution of intervals or of a distribution with a tail exponent $\theta > 1$. A summary of some important results for the tied-down random walk and the tied-down renewal process (for $0 < \theta < 1$) is given in table 2.

tied-down random walk	tied-down renewal proc.
$\langle M_N \vert S_{2N} = 0\rangle \approx \sqrt{\pi N}$	$\langle N_t \vert t_N = t\rangle \approx \frac{\Gamma(\theta)}{\Gamma(1-\theta)\Gamma(2\theta)} \left(\frac{t}{\tau_0}\right)^\theta$
$\langle R^{(k)\star}\rangle = 2\langle R^{(k)\mathrm{III}}\rangle$	$\langle R^{(k)\star}\rangle = \frac{1}{\theta}\langle R^{(k)\mathrm{III}}\rangle$
$Q_N^\star \approx \frac{\sqrt{\pi}}{2\sqrt{N}}$	$Q^\star(t) \approx \frac{\pi\theta}{\sin\pi\theta}\left(\frac{\tau_0}{t}\right)^\theta$

Table 2: Some important results for the tied-down random walk of section 2 (starting and ending at the origin) and the continuum renewal process of section 4 (with $0 < \theta < 1$). The results in the left column correspond, respectively, to the mean number of interval (2.47), a generalisation of (2.33) for the asymptotic mean ratio of the longest interval to the total length of time, and the probability of record breaking (2.56). Those in the right column correspond to the equivalent quantities (4.10), (4.42), and (4.66), in the continuum formalism.

Some related works, that we now review, can be put in perspective with the present study.

(i) As mentioned in the course of this study, there are close connections between the tied-down renewal process, including the Brownian bridge, and the cases (I, II and III) considered in [3, 11]. In particular, the connection with case III is clearly seen in table 2.

(ii) In the recent past, the distribution of the longest interval for the tied-down random walk and the Brownian bridge of section 2 was investigated in [4]. The results of this analysis can be usefully completed by the studies made in [1, 7, 9] and in the present work.

(iii) Recently, a study of the longest domain in a specific one-dimensional system of Ising spins has been given in [5]. In this model, introduced in [18, 19], the probability density associated to the Boltzmann weight of a spin configuration can be expressed in terms of the lengths of domains. It turns out that, at criticality, this density can be seen as the discrete version (2.34) of the tied-down conditional density (4.2), for a particular choice of distribution of intervals f_ℓ. Thus, the expression given in [5] for the distribution of the longest spin domain at criticality (with $0 < \theta < 1$) coincides with the distribution of the longest interval (4.30) for the tied-down renewal process, as a consequence of the universality of this result with respect to the choice of distribution of intervals $\rho(\tau)$, when $0 < \theta < 1$, as demonstrated in the present work.

In a companion paper [20] we will complete the study done here by addressing the statistics of other quantities, such as the occupation time or the two-time correlation function.

Appendix A. Notations

The distribution function of the random variable X is denoted by

$$F_X(x) = \mathrm{Prob}(X \le x). \tag{A.1}$$

If X is a continuous random variable, it has a density

$$f_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x}. \tag{A.2}$$

For several random variables we have

$$F_{X_1,X_2,\ldots}(x_1, x_2, \ldots) = \mathrm{Prob}(X_1 \le x_1, X_2 \le x_2, \ldots), \tag{A.3}$$

with associated density $f_{X_1,X_2,\ldots}(x_1, x_2, \ldots)$. When permitted by the context, we will omit the variables in subscript.

Let X and Y two random variables with joint density $f_{X,Y}(x, y)$ and marginal densities $f_X(x)$ and $f_Y(y)$. For discrete random variables the conditional distribution function of X given $Y = y$ is simply

$$\mathrm{Prob}(X \le x|Y = y) = F_{X|Y}(x|y) = \frac{\mathrm{Prob}(X \le x, Y = y)}{\mathrm{Prob}(Y = y)}. \tag{A.4}$$

For continuous random variables, the conditional distribution function of X given $Y = y$ is defined as follows [17],

$$\mathrm{Prob}(X \le x|Y = y) = F_{X|Y}(x|y) = \int_0^x \mathrm{d}u\, \frac{f_{X,Y}(u, y)}{f_Y(y)}. \tag{A.5}$$

Therefore the conditional density reads

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{f_{X,Y}(x, y)}{\int \mathrm{d}x\, f_{X,Y}(x, y)}. \tag{A.6}$$

Appendix B. First return probability f_n for the simple random walk

Let

$$u_n = \mathrm{Prob}(S_{2n} = 0) = (-1)^n \binom{-\frac{1}{2}}{n} = \frac{1}{2^{2n}} \binom{2n}{n}, \tag{B.1}$$

and

$$f_n = \mathrm{Prob}(\text{first return to zero occurs at time } 2n) = (-1)^{n-1} \binom{\frac{1}{2}}{n}. \tag{B.2}$$

Thus, $u_0 = 1, u_1 = \frac{1}{2}, u_2 = \frac{3}{8} \ldots ; f_1 = \frac{1}{2}, f_2 = \frac{1}{8}, f_3 = \frac{1}{16} \ldots$. These probabilities obey $f_n = u_{n-1} - u_n$, and their generating functions are

$$\tilde{u}(z) = \sum_{n\ge0} u_n z^n = \frac{1}{\sqrt{1 - z}}, \tag{B.3}$$

$$\tilde{f}(z) = \sum_{n\ge0} f_n z^n = 1 - \sqrt{1 - z}. \tag{B.4}$$

At large n,

$$u_n \approx \frac{1}{\sqrt{\pi n}}, \qquad f_n \approx \frac{1}{2\sqrt{\pi} n^{3/2}}. \tag{B.5}$$

Appendix C. The tied-down conditional density

Let us add some details on the definition of the tied-down conditional density (4.2) given in the bulk of the paper. Consider the conditional probability

$$\mathrm{Prob}(\vec{\tau} \le \vec{\ell}, N_t = n | t_N = y), \tag{C.1}$$

where $\vec{\ell} = \{\ell_1, \dots, \ell_n\}$ is a realization of the sequence of intervals

$$\vec{\tau} = \{\tau_1, \dots, \tau_N\}. \tag{C.2}$$

The associated conditional density is a generalization of (A.6), with $X = \{\vec{\tau}, N_t\}$ and $Y = t_N$,

$$f_{\vec{\tau},N_t|t_N}(\ell_1, \dots, \ell_n, n|y) = \frac{f_{\vec{\tau},N_t,t_N}(t; \ell_1, \dots, \ell_n, n, y)}{f_{t_N}(y)}. \tag{C.3}$$

The numerator is explicitly obtained by multiplying $f^{\mathrm{III}}(\cdot)$ (given by (3.12)) by $\delta(\sum \ell_i - y)$, i.e.,

$$\begin{aligned} &f_{\vec{\tau},N_t,t_N}(t; \ell_1, \dots, \ell_n, n, y) \\ &= \rho(\ell_1) \dots \rho(\ell_n) \int_0^\infty \mathrm{d}a \, p_0(a) \, \delta\Big(\sum_{i=1}^n \ell_i + a - t\Big) \delta\Big(\sum_{i=1}^n \ell_i - y\Big). \end{aligned} \tag{C.4}$$

The denominator, $f_{t_N}(t; y)$, obtained from the numerator by integration on $\ell_1, \dots, \ell_n$ and summation on n, is the probability density of the random variable t_N. The double Laplace transforms with respect to t and y of the numerator and of the denominator are respectively given in (C.9) and (C.10).

The tied-down conditional density (C.6) is defined as (C.3), however with the condition that $t_N = t$. Setting $y = t$ in (C.4) amounts to suppressing the first delta function in the right side of the equation. The remaining integral upon a is equal to 1, so

$$f_{\vec{\tau},N_t,t_N}(t; \ell_1, \dots, \ell_n, n, y = t) = \rho(\ell_1) \dots \rho(\ell_n) \delta\Big(\sum_{i=1}^n \ell_i - t\Big). \tag{C.5}$$

The same result can also be obtained by setting $a = 0$ in (3.10). Thus, using a shorter notation for the tied-down conditional density, we have

$$\begin{aligned} f^\star(t; \ell_1, \dots, \ell_n, n) &= f_{\vec{\tau},N_t|t_N}(t; \ell_1, \dots, \ell_n, n|y = t) \\ &= \frac{\rho(\ell_1) \dots \rho(\ell_n) \delta\left(\sum \ell_i - t\right)}{f_{t_N}(t; y = t)}, \end{aligned} \tag{C.6}$$

where the denominator reads§

$$\begin{aligned} f_{t_N}(t; y = t) &= \sum_{n \ge 0} \int_0^\infty \mathrm{d}\ell_1 \dots \mathrm{d}\ell_n \, \rho(\ell_1) \dots \rho(\ell_n) \delta\Big(\sum_{i=1}^n \ell_i - t\Big) \\ &= \sum_{n \ge 0} f_{t_n}(t), \end{aligned} \tag{C.7}$$

denoting by $f_{t_n}(t)$ the density of the sum $t_n = \tau_1 + \dots + \tau_n$, with n fixed. In the bulk of the paper we use the shorter notation

$$U(t) \equiv f_{t_N}(t; y = t) \tag{C.8}$$

for the edge value of the probability density of t_N at its maximal value $y = t$.

§ $n = 0$ corresponds to $\delta(t)$ in (C.5), and therefore to 1 in Laplace space.

Remark Laplace transforming (C.4) with respect to t and y (with s conjugate to t and u conjugate to y), yields

$$\begin{aligned}&\mathop{\mathcal{L}}_{t,y} f_{\vec\tau,N_t,t_N}(t;\ell_1,\dots,\ell_n,n,y)\\ &=\rho(\ell_1)\mathrm{e}^{-(s+u)\ell_1}\dots\rho(\ell_n)\mathrm{e}^{-(s+u)\ell_1}\,\frac{1-\hat\rho(s)}{s}.\end{aligned}\tag{C.9}$$

Then summing upon the ℓ_i and n yields the double Laplace transform of $f_{t_N}(t;y)$ [2]

$$\mathop{\mathcal{L}}_{t,y} f_{t_N}(t;y)=\mathop{\mathcal{L}}_{t}\langle\mathrm{e}^{-ut_N}\rangle=\frac{1}{1-\hat\rho(s+u)}\frac{1-\hat\rho(s)}{s}.\tag{C.10}$$

In order to get the edge value of this density at $y=t$, we invert (C.10),

$$f_{t_N}(t;y)=\int\frac{\mathrm{d}u}{2i\pi}\mathrm{e}^{uy}\int\frac{\mathrm{d}s}{2i\pi}\mathrm{e}^{st}\frac{1}{1-\hat\rho(s+u)}\frac{1-\hat\rho(s)}{s},\tag{C.11}$$

we then set $y=t$ and $w=s+u$, yielding, with the shorter notation (C.8),

$$U(t)=\int\frac{\mathrm{d}w}{2i\pi}\mathrm{e}^{wt}\frac{1}{1-\hat\rho(w)}\int\frac{\mathrm{d}s}{2i\pi}\frac{1-\hat\rho(s)}{s}.\tag{C.12}$$

The second integral is equal to 1, since it represents $p_0(t)$ for $t=0$. We thus recover (4.5).

Appendix D. Second, third, . . . , longest intervals

For independent, identically distributed (i.i.d.) random variables $X_1,\dots,X_n$, the distribution function of the $k-$th largest random variable $X^{(k)}$ can be obtained by noting that the event $\{X^{(k)}\le\ell\}$ means that at most $k-1$ variables X_i are larger than ℓ, so

$$F^{(k)}(\ell)=\mathrm{Prob}(X^{(k)}\le\ell)=\sum_{j=0}^{k-1}\mathrm{Prob}(j\ \mathrm{r.v.}\ X_i>\ell)\tag{D.1}$$

$$=\sum_{j=0}^{k-1}\binom{n}{j}\overline{F}(\ell)^jF(\ell)^{n-j},\tag{D.2}$$

where $F(\ell)=\mathrm{Prob}(X\le\ell)$, $\overline{F}(\ell)=\mathrm{Prob}(X>\ell)$.

For the intervals $\tau_1,\dots,\tau_N$, the conditional distribution function

$$F^{(k)\star}(t;\ell)=\mathrm{Prob}(\tau_{\max}^{(k)\star}\le\ell|t_N=t)\tag{D.3}$$

still obeys (D.1). We have likewise, using (C.6),

$$\mathrm{Prob}(j\ \mathrm{r.v.}\ \tau_i>\ell)=\frac{1}{U(t)}\tag{D.4}$$

$$\sum_{n\ge0}\binom{n}{j}\underbrace{\int_\ell^\infty\mathrm{d}\ell_1\,\rho(\ell_1)\cdots}\ \underbrace{\int_0^\ell\mathrm{d}\ell_1\,\rho(\ell_1)\cdots}\delta\Big(\sum_{i=1}^n\ell_i-t\Big),\tag{D.5}$$

where the first group of integrals is done j times, and the second group $n-j$ times. Summing on j and Laplace transforming with respect to time, we obtain for the numerator of $F^{(k)\star}(t;\ell)$, denoted by

$$F^{(k)}(t;\ell)=\mathrm{Prob}(\tau_{\max}^{(k)\star}\le\ell,t_N=t),\tag{D.6}$$

the expression

$$\mathop{\mathcal{L}}_{t} F^{(k)}(t;\ell) = \sum_{j=0}^{k-1} \sum_{n\ge 0} \binom{n}{j} [\hat\rho(s) - \hat\rho(s;\ell)]^j \, [\hat\rho(s;\ell)]^{n-j} \tag{D.7}$$

$$= \sum_{j=0}^{k-1} \frac{[\hat\rho(s) - \hat\rho(s;\ell)]^j}{[1-\hat\rho(s;\ell)]^{j+1}} = \frac{1}{1-\hat\rho(s)} \left(1 - \left[\frac{\hat\rho(s) - \hat\rho(s;\ell)}{1-\hat\rho(s;\ell)}\right]\right)^k . \tag{D.8}$$

In the scaling limit of large times, i.e., $s \to 0$, using the asymptotic estimate (4.26), this expression becomes

$$\mathop{\mathcal{L}}_{t} F^{(k)}(t;\ell) = \frac{1 - \left(\widehat{f_V^{\mathrm{III}}}(s\ell)\right)^k}{a s^\theta}. \tag{D.9}$$

By Laplace inversion with respect to s, and division by (4.6), we obtain Wendel's result (4.31). A similar computation is done in [5] for the case of a specific choice of discrete distribution of intervals.

We can finally derive (4.32) by the same methods. We start from the distribution $f^{\mathrm{I}}(\cdot)$, given by (3.10), for the $N_t + 1$ intervals $\tau_1, \dots, \tau_N, A_t$. In order to evaluate the probability of having j of these random variables larger than ℓ, we have to separate the cases where A_t belongs to the group of random variables smaller than ℓ or to the group of random variables larger than ℓ. Hence

$$\mathrm{Prob}(j \text{ r.v. } (\tau_i \text{ and } A_t) > \ell) =$$

$$\sum_{n\ge 0} \binom{n}{j-1} \underbrace{\int_\ell^\infty \mathrm{d}\ell_1\, \rho(\ell_1) \cdots} \underbrace{\int_0^\ell \mathrm{d}\ell_1\, \rho(\ell_1) \cdots} \int_\ell^\infty \mathrm{d}a\, p_0(a)\, \delta\Big(\sum_{i=1}^n \ell_i + a - t\Big) +$$

$$\sum_{n\ge 0} \binom{n}{j} \underbrace{\int_\ell^\infty \mathrm{d}\ell_1\, \rho(\ell_1) \cdots} \underbrace{\int_0^\ell \mathrm{d}\ell_1\, \rho(\ell_1) \cdots} \int_0^\ell \mathrm{d}a\, p_0(a)\, \delta\Big(\sum_{i=1}^n \ell_i + a - t\Big). \tag{D.10}$$

In the first line of the right side, the first group of integrals is done $j-1$ times, and the second group $n-j+1$ times, with $j \ge 1$, while in the second line, the first group of integrals is done j times, and the second group $n-j$ times, with $j \ge 0$. The rest of the computation follows as above, using the asymptotic estimates (4.25) and (4.26).

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