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IPhT-T17/013

arXiv:1702.01156v2 [hep-th] 6 Mar 2017

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Abstract: We study heterotic backgrounds with non-trivial H-flux and non-vanishing expectation values of fermionic bilinears, often referred to as gaugino condensates. The gaugini appear in the low energy action via the gauge-invariant three-form bilinear $\Sigma_{MNP} = \mathrm{tr}\,\overline{\chi}\Gamma_{MNP}\chi$. For Calabi-Yau compactifications to four dimensions, the gaugino condensate corresponds to an internal three-form Σ_{mnp} that must be a singlet of the holonomy group. This condition does not hold anymore when an internal H-flux is turned on and $\mathcal{O}(\alpha')$ effects are included. In this paper we study flux compactifications to three and four-dimensions on G-structure manifolds. We derive the generic conditions for supersymmetric solutions. We use integrability conditions and Lichnerowicz type arguments to derive a set of constraints whose solution, together with supersymmetry, is sufficient for finding backgrounds with gaugino condensate.

Keywords: Heterotic Supergravity, Supersymmetry, Gaugino Condensates

Contents

1 Introduction

The study of fermion, notably gaugino, condensates in heterotic string was primarily motivated by attempts to break supersymmetry while preserving a zero cosmological constant at tree level [1–3], based on earlier work in supergravity [4–6]. More recently fermionic condensates have also been considered in the context of supersymmetric compactifications, often in association with a non-trivial H-flux (see e.g. [7–17]).

In the supersymmetry transformations and the low energy action only the gaugino bilinear with three gamma matrices

$$\Sigma_{MNP} = \mathrm{tr}\,\overline{\chi}\Gamma_{MNP}\chi\,, \tag{1.1}$$

appears [18]. At a formal level, in order to preserve supersymmetry at order α' (and eventually satisfy the bosonic equations of motion), an interplay between the three-form Σ, the NS three-from flux H and the geometric data of the internal space is required.

Poincaré invariance requires the vev's of the individual fermions to vanish, which does not need to be the case for vev's of fermion bilinears such as Σ. As far as Poincaré invariance goes, one can roughly impose on Σ the same type of conditions as for fluxes. This has very restrictive implications for compactifications to four dimensions, where only the components of the gaugino bilinear in the internal six-dimensional space are allowed. Moreover, if the internal manifold has $SU(3)$ holonomy, i.e. a Calabi-Yau space, the gaugino condensate must be a singlet of the holonomy group. This is due to the fact that the component of χ that is massless in four dimensions is an $SU(3)$ singlet [2]. This forces the internal three-form to be proportional to the holomorphic top-form:

$$\langle\Sigma\,\rangle = \Lambda\Omega + c.c.\,, \tag{1.2}$$

where $\Lambda \sim \langle \mathrm{tr}\,\overline{\chi}^{(4)}(1+\gamma_5)\chi^{(4)}\rangle$ can be viewed as the "four-dimensional condensate". When compactifying to three dimensions, similar arguments can be made when the internal manifold is a compact space of G_2 holonomy. Now the condensate is forced to be proportional to the associative three-form φ. However, an external "spacetime filling" component of Σ is allowed by Poincaré invariance and has to be turned on [16].

In this paper we are interested in heterotic string backgrounds that involve non-trivial gaugino condensates and a non-trivial H-flux. Turning on the H-flux necessitates the inclusion of $\mathcal{O}(\alpha')$ effects and the consideration of internal spaces that are not Ricci-flat, and hence are not of special holonomy. As it is usual in flux compactifications, one is interested in manifolds that support nowhere vanishing (not necessarily covariantly constant) spinors, or equivalently a so-called G-structure. We will consider compactifications to four-dimensions on $SU(3)$ structure manifolds and to three-dimensional G_2 structure manifolds. Considering non compact internal seven-manifolds that are foliations over suitably chosen six-dimensional compact spaces allows to describe also four-dimensional domain-wall backgrounds. Importantly for our purposes, when studying the lower-dimensional effective theories on such manifolds, one is no longer obliged to restrict to a strictly massless lower-dimensional spectrum. As a consequence the gaugino condensate is no longer constrained to be a singlet of the structure group. Clearly we still have to require that, upon inclusion of $\langle\Sigma\rangle$ and $\mathcal{O}(\alpha')$ effects, the ten-dimensional equations of motion and Bianchi identities are all satisfied.

Our strategy will be to start from ten-dimensional equations, and include nontrivial condensate $\langle\Sigma\rangle$.[1] This may appear to be as bad as having fermion vevs, since the ten-dimensional Poincaré invariance is broken. However we shall then restrict ourselves to backgrounds with three- and four-dimensional Poincaré invariance. Fermion vevs will still not be allowed but some components of Σ are compatible with the symmetry. We remark that there are bi-spinor contributions to Einstein and dilaton equations of motion, of the form $\sim \alpha' \langle \chi^\dagger \gamma_{(M}\nabla_{N)}\chi\rangle$ and $\sim \alpha' \langle \chi^\dagger \not{\nabla}\chi\rangle$ respectively. We shall assume that only flux-like objects, i.e. Σ, can have nontrivial expectation values, and hence from now on we can ignore the gaugino kinetic terms.

As already mentioned, for backgrounds of the the form $M_{10} = M_3 \times X_7$ Poincaré invariance is compatible with non-zero external components for both Σ and the H-flux.[2] Note that we only need to require that the internal manifold is spin, as this automatically leads to existence of nowhere vanishing spinors and hence G_2 structures.[3] In spite of the formal similarity with the associative three-form φ, the condensate Σ does not define an alternative G_2 structure, as it may vanish pointwise. As we shall see in section 4, the conditions imposed by supersymmetry imply the H equation of motion and are somewhat under-constrained. For $M_{10} = M_4 \times X_6$ only internal components of H and Σ are allowed. Even in this case the conditions imposed by supersymmetry are under-constrained.

Without condensate, by a generalisation of Lichnerowicz formula, the bosonic action can be written as [19]

$$\begin{aligned} S_b = BPS^2 =& 4\int_{M_{10}} e^{-2\phi}[(\not{D}^0\epsilon)^\dagger \not{D}^0\epsilon - (D^0_M\epsilon)^\dagger D^{0\,M}\epsilon \\ &+ \frac{\alpha'}{96}\Big(\mathrm{tr}\,\epsilon^\dagger \not{F}\not{F}\epsilon - \mathrm{tr}\,\epsilon^\dagger \not{R}^- \not{R}^-\epsilon\Big)] + \mathcal{O}(\alpha'^2)\,, \end{aligned} \tag{1.3}$$

where we took the ten-dimensional supersymmetry parameter ϵ to have norm one ($\epsilon^\dagger\epsilon = 1$), and D^0 and D^0_M are differential operators appearing in the (modified) dilatino and gravitino variations (2.3) and (2.2a). The superscript 0 denotes the absence of condensate Σ. Note that the equality (1.3) implies integration by parts

[1] When this does not lead to confusion, we shall drop the brackets $\langle\ \rangle$ when talking about vev's and just use Σ for the components of the condensate. When talking about Σ, terms as "gaugino condensate" and "gaugino bilinear" will be used interchangeably.

[2] Note that since Σ is a bilinear of ten-dimensional spinors, both its internal and external components will be products of internal and external spinor bilinears.

[3] In fact, every spin seven-manifold has an S^3 of G_2 structures, or equivalently an $SU(2)$ structure. This allows to decompose the ten-dimensional spinors into twisted products of external and internal spinors, and eventually may lead to solutions with higher amount of preserved supersymmetry, than allowed by the direct product spinorial ansatz (or preservation of supersymmetry in cases where the direct product ansatz of internal and external spinors cannot preserve any). It may be of interest to study these cases further, however they stay outside the scope of our paper. As far as the geometric structures go, we shall make use of a single G_2 structure on the internal seven-manifold and hence employ the direct product anstaz.

and vanishing boundary conditions for the fields. Moreover (1.3) reproduces the heterotic action only after imposing Bianchi identity for H

$$\mathrm{d}H - \frac{\alpha'}{4}\left(\mathrm{tr}(R^-)^2 - \mathrm{tr}F^2\right) = 0\,. \tag{1.4}$$

From (1.3) it follows that the action vanishes for ten-dimensional supersymmetric solutions. A variation of (1.3) further implies that supersymmetric solutions also solve the equations of motion, provided the Bianchi identity is satisfied.[4]

While there are reasons to believe that generic supersymmetric theories should satisfy this kinds of generalised Lichnerowicz theorem, this clearly applies only to the bosonic action (indeed D^0 and D^0_M appear in the fermionic variations, and there are no analogues for the bosonic ones). So the condensate violates the BPS^2 property, and solving supersymmetry conditions (and the Bianchi identity) no longer guarantees solving equations of motion. Yet we shall show that, under the assumption that one can integrate by parts the Lichnerowicz formula can be generalised to cast the action in the form[5]

$$S_b = BPS^2 + \Delta S(\Sigma)\,. \tag{1.5}$$

This simplifies the analysis of the equations of motion significantly. Indeed the equations of motion for the dilaton, B-field and metric derived from (1.5) have the supersymmetry constrains already taken into account. On a more formal level, this exercise may also turn out to be useful in understanding the limits of application of the generalised Lichnerowicz theorem to supersymmetric theories.

As mentioned above, the derivation of (1.3) involves integration by parts and hence assumes the vanishing of the fields at infinity. This will not be true for AdS backgrounds or for domain walls. For these cases we derive the integrability conditions for the supersymmetry variations corrected by the fermion bilinears. The result are equations of the form

$$BPS^2 = E.O.M + f(\Sigma) \tag{1.6}$$

where $f(\Sigma)$ is a function of the gaugino condensate. The l.h.s of the above equation vanishes due to supersymmetry and the equations of motion are implied if Σ satisfies the constraints $f(\Sigma) = 0$. One can show that when the internal space is compact or when the boundary conditions at infinity allow for integration by parts the two types of analysis agree.

The integrability conditions can also be reformulated in a way that allows to discuss non-supersymmetric solutions and involves a curios phenomenon labeled as

[4]Note that supersymmetry (see the variation of the covariant derivative of the gravitino) implies that the curvature of the torsionful connection R^- satisfies the Hermitean Yang-Mills equation to the appropriate order in α'.

[5] We formally treat the condensate as three-form in ten dimensions Σ. Of course, we should be aware that Σ is breaking the ten-dimensional Poincaré invariance.

"fake supersymmetry". The crucial observation is that the equations of motion with non-trivial condensate have the same form as with zero condensate but with the replacement

$$H \to \overline{H} = H + \frac{\alpha'}{2}\Sigma\,. \tag{1.7}$$

One can define "fake supersymmetry" equations, $BPS(\overline{H})$, that have the same form as the BPS equations with zero condensate but with H replaced by $\overline{H}$. Squaring them we find

$$BPS^2(\overline{H}) = E.O.M\,. \tag{1.8}$$

Then solutions of the equations of motion can be found by solving the fake BPS conditions. The solutions will not be supersymmetric. Since the fake BPS conditions are linear equations, this formulation can be a useful tool to study supersymmetry breaking by a gaugino condensate, as was the original idea of [1–3].

The structure of the paper is as follows. In section 2 we give the supersymmetry conditions of heterotic theory with non-zero gaugino condensate and we recall Lichnerowicz formula. Then we discuss the integrability conditions and the generalisation of Lichnerowicz argument. We end this section with a discussion of the fake supersymmetry conditions. Sections 3 and 4 are devoted to the analysis of the general supersymmetry conditions for compactifications to four and three dimensions on manifolds of $SU(3)$ and G_2 structure, respectively. The latter also include domain walls in four dimensions. An explicit example of domain wall solution is presented in section 5. Some background and technical material can be found in the appendices.

2 Heterotic Supergravity with non trivial gaugino bilinears

The field content of $E_8 \times E_8$ heterotic supergravity consists of the metric, the NS two-form B, the dilaton ϕ, the gauge-field A, plus the gravitino ψ_M, the dilatino λ and the gaugino χ. The bosonic action and equations of motion, as well as the supersymmetry variations, are given in appendix A.

In this paper we are interested in solutions of the theory with non-trivial fermionic bilinears. Specifically, we consider what happens if we allow for gaugino condensation. This means that some components of the three-form

$$\Sigma_{MNP} = \mathrm{tr}\,\overline{\chi}\Gamma_{MNP}\chi \tag{2.1}$$

can take non-zero values.

The equations of motion and supersymmetry conditions with non-zero fermionic bilinears have been derived in [18]. Keeping only the gaugino bilinears they read[6]

$$\delta\psi_M = D_M\epsilon = \nabla_M\epsilon + \frac{1}{4}H_M\epsilon + \frac{\alpha'}{16}\not{\Sigma}\Gamma_M\epsilon + \mathcal{O}(\alpha'^2)\,, \tag{2.2a}$$

$$\delta\lambda = -\frac{\sqrt{2}}{4}\mathcal{P}\epsilon = -\frac{\sqrt{2}}{4}\left[\not{\partial}\phi + \frac{1}{2}\not{H} - \frac{\alpha'}{8}\not{\Sigma}\right]\epsilon + \mathcal{O}(\alpha'^2)\,, \tag{2.2b}$$

$$\delta\chi = -\frac{1}{2\sqrt{2}}\not{F}\epsilon + \mathcal{O}(\alpha'^2)\,, \tag{2.2c}$$

where ϵ and χ are Majorana-Weyl spinors of positive chirality, $H_M = \frac{1}{2}H_{MNP}\Gamma^{NP}$ and $\not{A}_p = \frac{1}{p!}A_{M_1\ldots M_p}\Gamma^{M_1\ldots M_p}$. We shall be using connections with torsion defined as $\nabla^{\pm}_M = \nabla_M \pm \frac{1}{4}H_M$.

It is also convenient to introduce the modified dilatino variation

$$\delta\rho = \Gamma^M(\delta\psi_M) - \delta\lambda = \not{D}\epsilon = \left(\not{\nabla} - \not{\partial}\phi + \frac{1}{4}\not{H} - \frac{\alpha'}{8}\not{\Sigma}\right)\epsilon + \mathcal{O}(\alpha'^2)\,. \tag{2.3}$$

With non-trivial gaugino bilinears the bosonic action (A.1) and the bosonic equations of motion (A.5a)-(A.5d) have the same form with the replacement [7, 20]

$$H \to \overline{H} = H + \frac{\alpha'}{2}\Sigma\,. \tag{2.4}$$

Notice that, on the contrary, the presence of a gaugino condensate does not affect the Bianchi identity at $\mathcal{O}(\alpha')$

$$\mathrm{d}H - \frac{\alpha'}{4}\left(\mathrm{tr}(R^-)^2 - \mathrm{tr}F^2\right) = 0\,. \tag{2.5}$$

We will mostly be interested in supersymmetric solutions and, as usual, we will determine them by solving the supersymmetry conditions (2.2a)-(2.2c). For zero-condensate a solution of the supersymmetry constraints that satisfies the Bianchi identities for H is also a solution of the whole set of equations of motion [21]. In presence of a condensate we do not expect this to be true, and we would like to derive the set of extra constraints required for a solution of the supersymmetry variations to be also a solution of the equations of motion. We will address the question by two different methods. One approach consists in extending the standard integrability arguments of [21, 22]. The other is based on a generalisation of the Lichnerowicz theorem as in [19]. The idea is to rewrite the ten-dimensional action as the sum of a term that is the square of the supersymmetry variations and some extra terms involving Σ. Even if this approach is less general than the integrability conditions, due to the use of integration by parts, it could provide an interesting way of computing solutions.

[6]Note that, in order to have a well-defined theory at first order in α', the gaugino supersymmetry variation need only be specified to zero-th order. This is due to the fact that the gauge fields of heterotic supergravity are already an $\mathcal{O}(\alpha')$ effect.

2.1 Integrability of the supersymmetry equations

In this section we provide a general analysis of the relation between supersymmetry and equations of motion in terms of the integrability conditions of the supersymmetry variations. Strictly speaking, gaugino condensates only make sense in the lower dimensional effective actions obtained from compactification. However, we perform the analysis of integrability in the full ten-dimensional theory, treating the condensate as a formal object. The aim is to derive the most general set of constraints, which can then be applied to specific compactifications.

In absence of a condensate, it is possible to build combinations of squares of the supersymmetry variations that reproduce the equations of motion [21, 22]

$$\Gamma^M D^0_{[N} D^0_{M]}\epsilon - \frac{1}{2} D^0_N(\mathcal{P}^0 \epsilon) + \frac{1}{2}\mathcal{P}^0 D^0_N \epsilon = -\frac{1}{4}\mathcal{E}^0_{NP}\Gamma^P \epsilon + \frac{1}{8}\mathcal{B}^0_{NP}\Gamma^P\epsilon + \frac{1}{2}\iota_N \mathrm{d}H^0 \epsilon\,,$$
$$\not{D}^0 \not{D}^0 \epsilon - (D^{0\,M} - 2\partial^M\phi) D^0_M \epsilon = -\frac{1}{8}\mathcal{D}^0\epsilon + \frac{1}{4}\mathrm{d}H^0\epsilon\,, \tag{2.6}$$

where $\mathcal{E}^0_{NP}$, $\mathcal{B}^0_{NP}$ and $\mathcal{D}^0$ denote the Einstein, B-field and dilaton equations of motion (A.5a) - (A.5c). $\mathrm{d}H$ is the Bianchi identity.[7] Since the left-hand side of both equations in (2.6) vanishes on supersymmetric solutions, the equations of motion are also satisfied, provided the Bianchi identity holds.

When the condensate is included, the first equation in (2.6) becomes

$$\Gamma^M D_{[N} D_{M]}\epsilon - \frac{1}{2} D_N(\mathcal{P}\epsilon) + \frac{1}{2}\mathcal{P} D_N\epsilon = -\frac{1}{4}\mathcal{E}_{NP}\Gamma^P\epsilon + \frac{1}{8}\mathcal{B}_{NP}\Gamma^P\epsilon$$
$$+ \frac{1}{2}\iota_N \mathrm{d}H\epsilon - \frac{\alpha'}{32} A_N(\Sigma)\epsilon\,, \tag{2.7}$$

where now $\mathcal{E}_{NM}$, $\mathcal{B}_{NM}$, $\mathcal{D}$ are the Einstein, B-field and dilaton equations of motion with non-zero condensate, and the extra term $A_N(\Sigma)$ is given by

$$\begin{aligned} A_N(\Sigma)\epsilon &= A_{NP}\Gamma^P\epsilon + A_{NPQR}\Gamma^{PQR}\epsilon + A_{NPQRST}\Gamma^{PQRST}\epsilon \\ &= \big[e^{2\phi}\nabla^M(e^{-2\phi}\Sigma_{MNP}) + \frac{1}{2}\Sigma_{NRS}H^{RS}{}_P + H\Sigma\delta_{NP}\big]\Gamma^P\epsilon \\ &\quad + \frac{1}{2}\big[e^{2\phi}\nabla^M(e^{-2\phi}\Sigma_{MPQ})\delta_{NR} + \frac{1}{3}e^{-2\phi}\nabla_N(e^{2\phi}\Sigma_{PQR}) \\ &\qquad + \nabla_R\Sigma_{NPQ} - H_{PST}\Sigma_Q{}^{ST}\delta_{NR}\big]\Gamma^{PQR}\epsilon \\ &\quad - \frac{1}{6}\big[\nabla_S\Sigma_{PQR}\delta_{NT} + \frac{1}{2}H_{NPQ}\Sigma_{RST}\big]\Gamma^{PQRST}\epsilon\,. \end{aligned} \tag{2.8}$$

Note that the tensors $\{A_{NM}, A_{NPQR}, A_{NPQRST}\}$ do not have any symmetry property. The analogue of the second equation in (2.6) contains extra terms in Σ that cancel

[7] Note that, for simplicity, we have ignored the contribution of the gauge fields in (2.6). As the gauge sector does not see the condensate at this order in α', we are free to ignore it in this computation.

non-tensorial terms

$$\not{D}\not{D}\epsilon - (D^{0\,M} - 2\partial^M\phi - \frac{\alpha'}{16}\not{\Sigma}\Gamma^M - \frac{\alpha'}{4}\Sigma^M)D_M\epsilon = -\frac{1}{8}\mathcal{D}\epsilon + \frac{1}{4}\mathrm{d}H\epsilon - \frac{\alpha'}{32}B(\Sigma)\epsilon\,, \tag{2.9}$$

where the extra contribution from Σ is

$$\begin{aligned} B(\Sigma)\epsilon &= B\epsilon + B_{NP}\Gamma^{NP}\epsilon + B_{MNPQ}\Gamma^{MNPQ}\epsilon \\ &= 6H\lrcorner\Sigma\epsilon + 3\big[e^{2\phi}\nabla^M(e^{-2\phi}\Sigma_{MNP}) + H_{NQR}\Sigma_P{}^{QR}\big]\Gamma^{NP}\epsilon \\ &\quad + \frac{1}{3}\big[e^{-2\phi}\nabla_M\left(e^{2\phi}\Sigma_{NPQ}\right) - \frac{3}{2}\,H_{MNS}\Sigma_{PQ}{}^S\big]\Gamma^{MNPQ}\epsilon\,. \end{aligned} \tag{2.10}$$

The left-hand sides of (2.7) and (2.9) still vanish because of the supersymmetry variations, but now the analysis of the right-hand sides is more involved. For zero condensate, after imposing the Bianchi identity, the only terms left are $\mathcal{E}^0_{NM}$ and $\mathcal{B}^0_{NM}$ multiplying one gamma matrix, and they must vanish separately because of their symmetry properties. In presence of condensate, the extra terms $A(\Sigma)$ and $B(\Sigma)$ in (2.7) and (2.9) contain several terms involving different numbers of gamma matrices, which, in ten dimensions, are not independent and hence cannot be set to zero separately.

To determine the set of independent constraints implied by the equations above we need to project them on a basis of spinors in ten dimensions. We follow closely the discussion in [23]. The supersymmetry parameter ϵ defines a vector

$$K^M = \frac{1}{32}\bar{\epsilon}\Gamma^M\epsilon \tag{2.11}$$

that is null and annihilates the spinor ϵ

$$\not{K}\epsilon = K^M\Gamma_M\epsilon = 0\,. \tag{2.12}$$

Since K is null, there are eight vectors orthogonal to it. We can use K to define a natural frame in $\mathbb{R}^{1,9}$. This is given by $\hat{e}_- = K$, $\hat{e}_\alpha$, with $\alpha = 1,\ldots,8$, spanning the eight directions orthogonal to K, and another vector $e_+ = \tilde{K}$ that is not orthogonal to K

$$\hat{e}_-\cdot\hat{e}_+ = \frac{1}{2}\,, \qquad \hat{e}_\pm\cdot\hat{e}_\pm = 0\,, \qquad \hat{e}_\pm\cdot\hat{e}_\alpha = 0\,. \tag{2.13}$$

The ten-dimensional gamma matrices can be taken to be real[8] and decomposed as

$$\Gamma^\pm = \gamma^\pm_{(2)}\otimes\mathbb{I} \qquad \Gamma^\alpha = \tilde{\gamma}_{(2)}\otimes\hat{\gamma}^\alpha\,, \tag{2.14}$$

where $\hat{\gamma}^\alpha$ are eight-dimensional gamma matrices and

$$\gamma^+_{(2)} = 2\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\,,\quad \gamma^-_{(2)} = 2\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\,. \tag{2.15}$$

[8]This means that $\Gamma^T_M = \Gamma^0\Gamma_M\Gamma^0$.

The two-dimensional chirality operator is $\tilde{\gamma}_{(2)} = \frac{1}{4}(\gamma^-_{(2)}\gamma^+_{(2)} - \gamma^+_{(2)}\gamma^-_{(2)}) = \sigma_3$ and $\hat{\gamma}_{(8)} = \prod_{\alpha=1}^{8} \hat{\gamma}^\alpha$ is the eight-dimensional one. The ten-dimensional chirality is then $\Gamma_{(10)} = \Gamma^0 \ldots \Gamma^9 = \tilde{\gamma}_{(2)} \otimes \hat{\gamma}_{(8)}$. In this basis, (2.12) becomes

$$\Gamma^+ \epsilon = \Gamma_- \epsilon = 0\,, \tag{2.16}$$

which implies that the supersymmetry parameter decomposes as

$$\epsilon = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \eta = |\downarrow> \otimes\, \eta\,, \tag{2.17}$$

where η is an eight-dimensional Majorana-Weil spinor of negative chirality, defining a $Spin(7)$ structure in eight-dimensions (see appendix B for the definition of the relevant G-structures). A basis for the 16-dimensional Majorana-Weyl spinors is given by

$$\epsilon\,, \qquad \omega_{\mathbf{7}}^{\alpha\beta}\Gamma_{\alpha\beta}\epsilon = |\downarrow> \otimes \Pi_{\mathbf{7}\ \gamma\delta}^{\alpha\beta}\hat{\gamma}^{\gamma\delta}\eta\,, \qquad \Gamma^{\alpha -}\epsilon = 2|\uparrow> \otimes \hat{\gamma}^\alpha \eta\,, \tag{2.18}$$

where $\Pi_{\mathbf{7}\ \gamma\delta}^{\alpha\beta}$ is the projector onto the representation $\mathbf{7}$ of $Spin(7)$ and is given in Appendix B. The first two terms above span the space of the negative chirality spinors and the last term that of positive chirality ones.

The spinor ϵ defines a $Spin(7) \ltimes \mathbb{R}^8$ structure. As usual the structure is equivalently given in terms of forms that are bilinears in ϵ. In this case these are

$$K \qquad\qquad \Psi = K \wedge \Phi_4 \tag{2.19}$$

where Φ_4 is the four-form associated to the $Spin(7)$ structure in the eight-dimensional space spanned by the vectors orthogonal to K.

In order to find the set of independent integrability conditions, we have to decompose (2.7) and (2.9) on the ten-dimensional basis (2.18). We always assume that the Bianchi identity is satisfied and that the left-hand sides vanish because of supersymmetry. We consider first the dilatino equation, (2.9). Clearly only the term of $B(\Sigma)$ proportional to the singlet can contribute to the dilaton equation of motion. Then, plugging (2.18) into (2.9), we obtain that for the dilaton equation of motion to be satisfied it is sufficient to require that

$$B + 4\,B_{+-} + B_{\alpha\beta\gamma\delta}\Phi^{\alpha\beta\gamma\delta} = 0\,, \tag{2.20}$$

where the tensors B, B_{+-} and $B_{\alpha\beta\gamma\delta}$ are defined in (2.10).

Next let us consider the Einstein and B-field equations. Decomposing (2.7) in the spinorial basis (2.18), we find that the components $\mathcal{E}_{N-}$ and $\mathcal{B}_{N-}$ vanish if

$$A_{N-} + 5\,A_{N[\alpha\beta\gamma\kappa-]}\Phi^{\alpha\beta\gamma\kappa} = 0\,, \tag{2.21}$$

and, similarly, the components $\mathcal{E}_{N\alpha}$ and $\mathcal{B}_{N\alpha}$ are implied if

$$A_{N\alpha} + 12\, A_{N[\alpha+-]} - \; A_{N\beta\gamma\delta}\Phi^{\beta\gamma\delta}{}_{\alpha} + 5\, A_{N[\beta\gamma\delta\epsilon\alpha]}\Phi^{\beta\gamma\delta\epsilon} - 40\, A_{N[+-\beta\gamma\delta]}\Phi^{\beta\gamma\delta}{}_{\alpha} = 0\,. \tag{2.22}$$

Notice that, since $\Gamma^{+}\epsilon = 0$, the $++$ component of the Einstein equation is always projected out in (2.7) [24]. Imposing these constraints on Σ, we find again that supersymmetry and the Bianchi identity for H imply all other equations of motion with the exclusion of

$$\mathcal{E}_{++} = 0\,. \tag{2.23}$$

Note that the conditions (2.20) and (2.21) can be written in a base independent way using the structure K, $\tilde{K}$ and Ψ

$$B + 4\,\tilde{K}^{M}K^{N}B_{MN} + 2B_{NPQR}\tilde{K}_{M}\Phi^{MNPQR} = 0 \tag{2.24a}$$

$$A_{NP}K^{P} + A_{NPQRTS}\Psi^{PQRTS} = 0\,. \tag{2.24b}$$

2.2 Supersymmetry and equations of motion

In this section we describe an alternative approach to study the relation between supersymmetry variations and equations of motion that generalisese Lichnerowicz theorem.

Let us consider first the case when the condensate is zero. The starting point is the Bismut-Lichnerowicz identity [19]. Provided, the Bianchi identity is satisfied, the Bismut-Lichnerowicz identity allows to write the bosonic Lagrangian (A.1) as

$$\frac{1}{4}\mathcal{L}_{b}\epsilon + \mathcal{O}(\alpha'^2) = D^{0}_{M}D^{0\,M}\epsilon - \not{D}^{0}\not{D}^{0}\epsilon + \frac{\alpha'}{16}\left(\mathrm{tr}\,\not{F}\not{F}\epsilon - \mathrm{tr}\,\not{R}^{-}\not{R}^{-}\epsilon\right) - 2\nabla^{M}\phi D^{0}_{M}\epsilon\,, \tag{2.25}$$

where D^{0}_{M} and $\not{D}^{0}$ are the gravitino and modified dilatino equations, (2.2a) and (2.3), with zero condensate. R^{-} is the curvature two-form derived from the torsionful connection ∇^{-} (note that the gravitino variation involves ∇^{+}). Multiplying (2.25) by $e^{-2\phi}\epsilon^{\dagger}$ and integrating it, gives the action

$$\begin{aligned} S_{b} = -4\int_{M_{10}}\sqrt{-g}e^{-2\phi}\Big[&\epsilon^{\dagger}\not{D}^{0}\not{D}^{0}\epsilon - \epsilon^{\dagger}D^{0}_{M}D^{0\,M}\epsilon \\ &+ 2\nabla^{M}\phi\,\epsilon^{\dagger}D^{0}_{M}\epsilon - \frac{\alpha'}{16}\left(\mathrm{tr}\,\epsilon^{\dagger}\not{F}\not{F}\epsilon - \mathrm{tr}\,\epsilon^{\dagger}\not{R}^{-}\not{R}^{-}\epsilon\right)\Big] + \mathcal{O}(\alpha'^2)\,, \end{aligned} \tag{2.26}$$

where we assumed that $\epsilon^{\dagger}\epsilon = 1$.

We would like to write the action as a BPS-squared expression, since its variations will give the equations of motion in terms of the supersymmetry variations. If the theory were Euclidean, we could integrate (2.26) by parts, and end up with such an action. Unfortunately, when the metric has Lorentzian signature it is not possible, in general, to reconstruct the supersymmetry operators D^0_M and $\not{D}^0$ after the integration by parts. Since the problematic terms always involve components of the fields with one leg along the time direction, one can restrict to solutions where none of the fields has components of this type. This means

$$H_{0MN} = \Sigma_{0MN} = F_{0M} = 0\,, \tag{2.27}$$

and, for the metric[9],

$$\mathrm{d}s^2_{10} = -e^{2A}\mathrm{d}t^2 + g_{mp}\mathrm{d}x^m\mathrm{d}x^p\,, \tag{2.28}$$

where $A = A(x^m)$, $g = g(x^m)$ and $\{x^m, x^n, ..\}$ denote spatial coordinates. Note also that, in order to perform the integration by parts, we assume that the fields vanish at infinity where the metric is flat.[10] Under these assumptions the action can be integrated by parts to give

$$\begin{aligned}\int_{M_{10}} e^{-2\phi}\mathcal{L}_b =& 4\int_{M_{10}} e^{-2\phi}[(\not{D}^0\epsilon)^\dagger\not{D}^0\epsilon - (D^0_M\epsilon)^\dagger D^{0\,M}\epsilon \\ &+ \frac{\alpha'}{16}\Big(\mathrm{tr}\,\epsilon^\dagger\not{F}\not{F}\epsilon - \mathrm{tr}\,\epsilon^\dagger\not{R}^-\not{R}^-\epsilon\Big)] + \mathcal{O}(\alpha'^2)\,. \end{aligned}\tag{2.29}$$

The variations of (2.29) with respect to the metric, dilaton and B-field give the corresponding equations of motion written in terms of the supersymmetry conditions. Thus solutions of the supersymmetry constraints also automatically solve the equations of motion.[11]

For non-trivial condensate the same analysis gives, up to $\mathcal{O}(\alpha'^2)$ terms,

$$\begin{aligned}S_b = BPS^2 &+ \frac{\alpha'}{8}\int_{M_{10}}\sqrt{-g}e^{-2\phi}\Big[(\not{D}\epsilon)^\dagger\not{\Sigma}\epsilon + (\not{\Sigma}\epsilon)^\dagger\not{D}\epsilon \\ &+ \frac{1}{2}\left((D_M\epsilon)^\dagger\not{\Sigma}\Gamma^M\epsilon + (\not{\Sigma}\Gamma^M\epsilon)^\dagger D_M\epsilon\right) - H\lrcorner\Sigma\Big]\,,\end{aligned}\tag{2.30a}$$

where $H\lrcorner\Sigma = \frac{1}{3!}H^{MNP}\Sigma_{MNP}$, and BPS^2 denotes the part of the action that can be written as the square of the supersymmetry variations

$$BPS^2 = \int_{M_{10}}\sqrt{-g}e^{-2\phi}\Big[(\not{D}\epsilon)^\dagger\not{D}\epsilon - (D_M\epsilon)^\dagger D^M\epsilon + \frac{\alpha'}{16}\Big(\mathrm{tr}\,\epsilon^\dagger\not{F}\not{F}\epsilon - \mathrm{tr}\,\epsilon^\dagger\not{R}^-\not{R}^-\epsilon\Big)\Big]\,. \tag{2.30b}$$

Note that because of the extra terms involving Σ on the right-hand side of (2.30a), the action can no longer be written as a BPS-squared expression. The equations of

[9] Note that given (2.27) and (2.28) for H and the metric, R^- has no temporal legs.

[10] Without this assumption there might be unwanted boundary terms. The assumption holds for Minkowski compactifications but excludes certain types of solutions, like domain walls.

[11] The variation of the second line in (2.29) also vanishes. Its variation is proportional to

$$\not{R}^-_{MN}\,\epsilon = R^-_{PQMN}\Gamma^{PQ}\epsilon = R^+_{MNPQ}\Gamma^{PQ}\epsilon + \mathcal{O}(\alpha') = \mathcal{O}(\alpha')\,,$$

which vanishes on supersymmetric solutions to the appropriate order in α'.

motion for g_{MN}, ϕ and B_{MN} obtained by varying (2.30a) will contain a term coming from its BPS part, which vanishes for supersymmetric configurations, and terms in Σ, which will provide additional constraints to be imposed on the supersymmetric solution. These extra terms have a relatively simple structure and in particular do not contain the curvatures F and R^-. More concretely, the extra terms in the dilaton and B-field equations of motion are respectively

$$H_{MNP}\Sigma^{MNP} + e^{2\phi}\nabla^M\left(e^{-2\phi}\epsilon^\dagger\Gamma_{MNPQ}\Sigma^{NPQ}\epsilon\right) = 0 + \mathcal{O}(\alpha')\,, \tag{2.31a}$$

$$e^{2\phi}\nabla^M\left(e^{-2\phi}\,\epsilon^\dagger\{\Gamma_{MPQ},\not{\Sigma}\}\epsilon\right) = 0 + \mathcal{O}(\alpha')\,. \tag{2.31b}$$

Both equations are only linear in derivatives. Note the resemblance between (2.31b) and the usual flux equation of motion (A.5c). The metric equation of motion evaluated on a supersymmetric solutions gives the equation[12]

$$e^{2\phi}\nabla_N\left[e^{-2\phi}\left(\epsilon^\dagger\Gamma^{N(P}\not{\Sigma}\Gamma^{M)}\epsilon - 2g^{MP}\epsilon^\dagger\Gamma^N\not{\Sigma}\epsilon + 2g^{N(M}\epsilon^\dagger\Gamma^{P)}\not{\Sigma}\epsilon\right)\right] = \nabla^{(M}\phi\,\epsilon^\dagger\Gamma^{P)}\not{\Sigma}\epsilon$$
$$+\,2\,g^{MP}H\lrcorner\Sigma -\, H_{RS}{}^{(M}\Sigma^{P)RS} - 3\,H_{NR}{}^{(M}\epsilon^\dagger\Gamma^{P)N}\not{\Sigma}\Gamma^R\epsilon + \mathcal{O}(\alpha')\,, \tag{2.32}$$

The equations above can can be further simplified, using the fact that

$$H = \mathrm{d}\phi = 0 + \mathcal{O}(\alpha')\,. \tag{2.33}$$

This follows from imposing that the fields must vanish at infinity. Indeed, substracting $1/4$ of the trace of (A.5a) from (A.5b) we find

$$\frac{1}{2}\nabla^2\phi - (\nabla\phi)^2 + \frac{1}{4}H^2 = 0 + \mathcal{O}(\alpha')\,, \tag{2.34a}$$

or, alternatively,

$$e^{2\phi}\nabla^M\left(e^{-2\phi}\nabla_M\phi\right) + \frac{1}{2}H^2 = 0 + \mathcal{O}(\alpha')\,. \tag{2.34b}$$

Multiplying by $e^{-2\phi}$ and integrating over spacetime then gives

$$\int_{M_{10}} e^{-2\phi}H^2 = 0 + \mathcal{O}(\alpha')\,. \tag{2.35}$$

As the integrand is positive, it has to vanish point-wise, which implies (2.33). Integrating (2.34a), we also find that $\mathrm{d}\phi = 0 + \mathcal{O}(\alpha')$. Using this result, the equations (2.31a) -(2.32) become (modulo $\mathcal{O}(\alpha')$ terms)

$$\tilde{\Psi}^{MNPQ}\,\nabla_{[M}\Sigma_{NPQ]} = 0\,, \tag{2.36a}$$

$$\nabla^P\Sigma_{PMN} + \tilde{\Psi}_{PQR[M}\nabla^P\Sigma^{QR}{}_{N]} - \frac{1}{2}\tilde{\Psi}_{MNPQ}\nabla^R\Sigma_R{}^{PQ} = 0\,, \tag{2.36b}$$

$$\tilde{\Psi}_{MPQ(R}\nabla^M\Sigma_{N)}{}^{PQ} - \frac{1}{6}\tilde{\Psi}^{TPQ}_{(R}\nabla_{N)}\Sigma_{TPQ} = 0\,, \tag{2.36c}$$

[12]We also need to vary the vielbeine in the gamma matrices as $\delta\Gamma^M = \delta e_A{}^M\Gamma^A$ and in the gaugino bilinear $\Sigma_{MNP} = \overline{\chi}\Gamma_{MNP}\chi = \overline{\chi}\Gamma_{ABC}\chi e_M{}^A e_N{}^B e_P{}^C$.

where, for simplicity of notation, we have defined $\tilde{\Psi}_{MNPQ} = \epsilon^\dagger\Gamma_{MNPQ}\epsilon$.

As in the previous section, to extract the set of independent constraints from (2.36a)-(2.36c) we need to decompose them on the spinorial basis (2.18). It is straightforward to see that the only non-zero component of $\tilde{\Psi}$ is purely in the eight directions orthogonal to the vector K and reduces to the $Spin(7)$ invariant form

$$\tilde{\Psi}_{\pm MNP} = 0\,, \quad \tilde{\Psi}_{\alpha\beta\gamma\delta} = \Phi_{\alpha\beta\gamma\delta}\,. \tag{2.37}$$

The condition that there are no components in the time direction, (2.27), implies

$$\Sigma_{+\alpha\beta} = \Sigma_{-\alpha\beta}\,, \qquad \Sigma_{+-\alpha} = 0\,. \tag{2.38}$$

It is immediate to see that (2.36a) reduces to

$$\Phi\lrcorner \mathrm{d}\Sigma = 0\,. \tag{2.39}$$

Combining the $+\alpha$ components of (2.36b) and (2.36c), one finds

$$\nabla^\beta\Sigma_{\beta\alpha+} + \frac{1}{2}\Phi_{\beta\gamma\delta\alpha}\nabla^\beta\Sigma^{\gamma\delta}{}_+ = 0\,, \tag{2.40a}$$

$$\Phi_{\gamma\delta\epsilon\alpha}\nabla^\gamma\Sigma^{\delta\epsilon}{}_+ + \frac{1}{6}\Phi_\alpha{}^{\gamma\delta\epsilon}\nabla_+\Sigma_{\gamma\delta\epsilon} = 0\,. \tag{2.40b}$$

Then the α, β components of (2.36b) and (2.36c) reduce to

$$\left(2\nabla^+\Sigma_{+\gamma\delta} + \nabla^\epsilon\Sigma_{\epsilon\gamma\delta}\right)\left(\delta^{\gamma\delta}_{\alpha\beta} - \frac{1}{2}\Phi_{\alpha\beta}{}^{\gamma\delta}\right) + \Phi_{\delta\epsilon\sigma[\alpha}\nabla^\delta\Sigma_{\beta]}{}^{\epsilon\sigma} = 0 \tag{2.41a}$$

$$\Phi_{\gamma\delta\epsilon(\alpha}\nabla^\epsilon\Sigma_{\beta)}{}^{\gamma\delta} - \frac{1}{6}\Phi_{\gamma\delta\epsilon(\alpha}\nabla_{\beta)}\Sigma^{\gamma\delta\epsilon} = 0\,. \tag{2.41b}$$

We would like to compare these conditions with those obtained from integrability in section 2.1. Recall that in our analysis we neglect terms of order α'^2 in the equations of motion and that (2.20) – (2.22) are already of order α'. Since for static solutions $H = \mathrm{d}\phi = \mathcal{O}(\alpha')$, we can discard all terms involving products of H and Σ in (2.20) – (2.22). Using also (2.38) it is easy to see that (2.20) reduces to

$$\nabla_\alpha\Sigma_{\beta\gamma\sigma}\Phi^{\alpha\beta\gamma\sigma} = 0\,, \tag{2.42}$$

which agrees with (2.39). For (2.21) one can show that only the $N = \alpha$ component gives a non trivial condition

$$\nabla^\beta\Sigma_{\beta\alpha-} - \frac{2}{3}\Phi_\alpha{}^{\beta\gamma\delta}\nabla_{[-}\Sigma_{\beta\gamma\delta]} = 0\,. \tag{2.43}$$

From (2.22) we get two equations. One for the component $N = +$

$$3\nabla^\beta\Sigma_{\beta\alpha+} + \frac{1}{2}\nabla_+\Sigma_{\beta\gamma\delta}\Phi^{\beta\gamma\delta}{}_\alpha - \frac{5}{2}\nabla_\beta\Sigma_{\gamma\delta+}\Phi^{\beta\gamma\delta}{}_\alpha = 0\,, \tag{2.44}$$

and another for the component $N = \beta$, whose symmetric and anti-symmetric parts are

$$\left(2\nabla^+\Sigma_{+\gamma\epsilon} + \nabla^\sigma\Sigma_{\sigma\gamma\epsilon}\right)\left(\delta^{\gamma\epsilon}_{\alpha\beta} - \frac{1}{2}\Phi_{\alpha\beta}{}^{\gamma\epsilon}\right) + \frac{1}{3}\Phi^{\gamma\delta\epsilon}{}_{[\alpha}\nabla_{\beta]}\Sigma_{\gamma\delta\epsilon} = 0 \tag{2.45a}$$

$$\Phi_{(\alpha}{}^{\epsilon\gamma\delta}\nabla_{|\epsilon|}\Sigma_{\beta)\gamma\delta} = 0\,. \tag{2.45b}$$

Using (2.43) to get rid of the first term in (2.44), we find precisely (2.40b). This can then be inserted back into (2.43) to obtain (2.40a). To compare the remaining equations, we need to take into account that, in order to derive the action (2.30a), we assumed that the ten-dimensional spinor ϵ has constant norm. This implies that

$$\Phi_{\alpha\beta\gamma\delta}\Sigma^{\beta\gamma\delta} = 0 \tag{2.46}$$

on the supersymmetric locus, as it can be shown from

$$0 = \partial_M\left(\epsilon^\dagger\epsilon\right) = \left(\nabla^{H^+}_M\epsilon\right)^\dagger\epsilon + \epsilon^\dagger\nabla^{H^+}_M\epsilon = \frac{\alpha'}{48}\Psi_{MNPQ}\Sigma^{NPQ}\,, \tag{2.47}$$

where we have used (2.2a) and $H^+ = H + \frac{\alpha'}{4}\Sigma$. Using (2.46), we see that (2.41b) and (2.45b) coincide, while (2.45a) reduces to

$$\left(2\nabla^+\Sigma_{+\gamma\epsilon} + \nabla^\sigma\Sigma_{\sigma\gamma\epsilon}\right)\left(\delta^{\gamma\epsilon}_{\alpha\beta} - \frac{1}{2}\Phi_{\alpha\beta}{}^{\gamma\epsilon}\right) = 0\,. \tag{2.48}$$

This is to be compared with (2.41a). It can be checked that the projection of the two-form $\Phi_{\delta\epsilon\sigma[\alpha}\nabla^\delta\Sigma_{\beta]}{}^{\epsilon\sigma}$ into **21** vanishes, and hence **7** is its only surviving $Spin(7)$ representation. It can further be shown that this is the same as the projection to **7** of $\Phi^{\gamma\delta\epsilon}{}_{[\alpha}\nabla_{\beta]}\Sigma_{\gamma\delta\epsilon}$.[13] The latter vanishes by virtue of (2.46) and $\nabla_M\epsilon \sim \mathcal{O}(\alpha')$.

To summarise, for static solutions vanishing at infinity, both set of constraints (2.41a) - (2.41b) and (2.45a) - (2.45b) reduce to

$$\Pi_7\left(2\nabla^+\Sigma_{+\gamma\epsilon} + \nabla^\sigma\Sigma_{\sigma\gamma\epsilon}\right) \equiv \left(2\nabla^+\Sigma_{+\gamma\epsilon} + \nabla^\sigma\Sigma_{\sigma\gamma\epsilon}\right)\left(\delta^{\gamma\epsilon}_{\alpha\beta} - \frac{1}{2}\Phi_{\alpha\beta}{}^{\gamma\epsilon}\right) = 0 \tag{2.49a}$$

$$\Phi_{(\alpha}{}^{\epsilon\gamma\delta}\nabla_{|\epsilon|}\Sigma_{\beta)\gamma\delta} = 0\,. \tag{2.49b}$$

Note that only $(\mathrm{d}\Sigma)_{\mathbf{27}}$ may still be non-vanishing.

Regardless of the above considerations, one may be tempted to impose by hand the stronger condition

$$\mathrm{d}\Sigma = 0\,, \tag{2.50}$$

[13] To show this, note that $\nabla\Sigma$ - an object with 1+3 indices corresponds to $\mathbf{8}\times(\mathbf{35}+\mathbf{21}) \to \mathbf{7}+\mathbf{21}+\mathbf{35}+\mathbf{35}+\mathbf{21}+\mathbf{35}+\mathbf{105}+\mathbf{189}$. But then one contracts it with Φ leaving only two free indices. In other words every such contraction is a projection into $\mathbf{35}+\mathbf{21}+\mathbf{7}$ (there could also be a trace part, i.e. **1**, but it is not there in the original decomposition, and as mentioned is vanishing by a separate condition (2.39) and (2.43)). Note that 2 different projections appear $\Phi_{\gamma\delta\epsilon\alpha}\nabla^\epsilon\Sigma_\beta{}^{\gamma\delta}$ and $\Phi_{\gamma\delta\epsilon\alpha}\nabla_\beta\Sigma^{\gamma\delta\epsilon} = 0$ at the supersymmetric locus. But there is only a single **7** so it must be the same in both projections.

which can be interpreted as a "Bianchi identity" for the condensate three-form. Indeed, we recall that when the condensate is turned on the bosonic action is obtained from the usual one by the replacement

$$H \to \overline{H} = H + \frac{\alpha'}{2}\Sigma\,. \tag{2.51}$$

With (2.50) imposed, H and $\overline{H}$ satisfy the same Bianchi identity. As we shall see, this condition allows to construct solutions (possibly non-supersymmetric) with nontrivial condensate form a solution with $\Sigma = 0$.

2.3 Solutions with Fake Supersymmetry

A similar approach to the integrability of section 2.1 provides a nice method to find solutions of the equations of motion that are not supersymmetric. Solutions of this type are in the spirit of [1–3], where it is shown that gaugino condensation provides a natural mechanism for supersymmetry breaking.

As already mentioned several times, the heterotic equations of motion with non trivial condensate are formally the same as for zero condensate, (A.5a) - (A.5d), but with the replacement

$$H \to \overline{H} = H + \frac{\alpha'}{2}\Sigma\,. \tag{2.52}$$

Let us define "fake" supersymmetry variations that have the same form as those for zero condensate (2.2a)-(2.3) with H replace by $\overline{H}$

$$\delta\psi_M = \overline{D}_M\epsilon = \nabla_M\epsilon + \frac{1}{4}\overline{H}_M\epsilon + \mathcal{O}(\alpha'^2)\,, \tag{2.53a}$$

$$\delta\lambda = -\frac{\sqrt{2}}{4}\overline{\mathcal{P}}\epsilon = -\frac{\sqrt{2}}{4}\left[\partial\!\!\!/\phi + \frac{1}{2}\overline{H\!\!\!/}\right]\epsilon + \mathcal{O}(\alpha'^2)\,, \tag{2.53b}$$

$$\delta\rho = \overline{D\!\!\!/}\epsilon = \left(\nabla\!\!\!\!/ - \partial\!\!\!/\phi + \frac{1}{4}\overline{H\!\!\!/}\right)\epsilon + \mathcal{O}(\alpha'^2)\,. \tag{2.53c}$$

Then if we square the equations above as in section 2.1 we find

$$\Gamma^M\overline{D}_{[N}\overline{D}_{M]}\epsilon - \frac{1}{2}\left[\overline{D}_N, \overline{\mathcal{P}}\right]\epsilon = -\frac{1}{4}\mathcal{E}_{NP}\Gamma^P\epsilon + \frac{1}{8}\mathcal{B}_{NP}\Gamma^P\epsilon + \frac{1}{2}\iota_N\mathrm{d}\overline{H}\epsilon\,, \tag{2.54a}$$

$$\overline{D\!\!\!/}\,\overline{D\!\!\!/}\epsilon - (\overline{D}^M - 2\partial^M\phi)\overline{D}_M\epsilon = -\frac{1}{8}\mathcal{D}\epsilon + \frac{1}{4}\mathrm{d}\overline{H}\epsilon\,, \tag{2.54b}$$

where $\mathcal{E}_{NP}$, $\mathcal{B}_{NP}$ and $\mathcal{D}$ are the Einstein, B-field and dilaton equations of motion with non-zero condensate. Notice that the term involving the Bianchi identity contains a correction is Σ

$$\mathrm{d}\overline{H} = \mathrm{d}H + \frac{\alpha'}{2}\mathrm{d}\Sigma\,. \tag{2.55}$$

From (2.54a) we see that a background satisfying

$$\overline{D}_M\epsilon = 0 \qquad \overline{D\!\!\!/}\epsilon = 0 \qquad \overline{\mathcal{P}}\epsilon = 0 \tag{2.56}$$

plus the Bianchi identity $\mathrm{d}H = 0$ and the closure of Σ

$$\mathrm{d}\Sigma = 0 \tag{2.57}$$

is also a solution of the equations of motion. The advantage of this approach is that, to find non-supersymmetric solutions, it is sufficient to solve susy-like first order equations, which are generally simpler than the full equations of motion. Generically the solutions will not be supersymmetric, hence the name fake supersymmetry.

One can also ask what extra conditions must be imposed for the solution to be supersymmetric. Comparing (2.53a) with (2.2a) we see that the former reduces to the latter if

$$\Sigma_{NPQ}\Gamma^{NPQ}{}_{M}\epsilon = 3\,\Sigma_{MNP}\Gamma^{NP}\epsilon\,. \tag{2.58}$$

This condition also implies that $\not{\Sigma}$ vanishes and hence (2.53b) and (2.53c) also reduce to the supersymmetric ones. The class of supersymmetric backgrounds obtained this way is more reduced than what we would obtain by solving the supersymmetry equations (2.2a) - (2.3) plus the integrability conditions, but have the advantage that one only needs to solve first order equations. We will return to some explicit solutions of this type in section 5.

3 Compactifications to four dimensions

We consider first the supersymmetry conditions for compactifications to four-dimensional maximally symmetric spaces, Minkowski or Anti de Sitter. The metric is of the form

$$\mathrm{d}s^2_{10} = e^{2\Delta}\mathrm{d}s^2_4 + \mathrm{d}s^2_6\,, \tag{3.1}$$

where the warp factor only depends on the internal coordinates and the internal manifold is assumed to have $SU(3)$ structure.

We denote four-dimensional indices by greek letters, $\mu, \nu, \ldots = 0, \ldots, 3$ and internal indices by latin ones, $m, n, \ldots = 1, \ldots, 6$. The ten-dimensional gamma matrices decompose as

$$\begin{aligned}\Gamma^\mu &= e^{\Delta}\gamma^\mu \otimes \mathbf{1}\\ \Gamma^m &= \gamma_{(4)} \otimes \gamma_m\,,\end{aligned} \tag{3.2}$$

where γ^μ are four-dimensional gamma matrices of the unwarped metric and $\gamma_{(4)}$ is the chiral gamma in four dimensions. The six-dimensional matrices γ^m are hermitian. The supersymmetry parameter decomposes as

$$\epsilon = e^{A}\zeta_+ \otimes \eta_+ + e^{A}\zeta_- \otimes \eta_-\,, \tag{3.3}$$

where $\zeta_\pm$ and $\eta_\pm$ are Weyl spinors of positive and negative chirality in four and six dimensions, respectively (see appendix B.2 for more conventions).

The ten-dimensional gravitino splits as

$$\chi_{(10)} = \chi_+^{(4)} \otimes \chi_+ + \chi_-^{(4)} \otimes \chi_- \,, \tag{3.4}$$

where $\chi_\pm^{(4)}$ and $\chi_\pm$ are four- and six-dimensional Weyl spinors. In Calabi-Yau compactifications, the internal spinor χ is taken to be a singlet of the $SU(3)$ holonomy group in order to have a massless four-dimensional gaugino. In this paper we are interested in the larger class of $SU(3)$ structure compactifications and we a-priori take χ_+ to be a generic six-dimensional Weyl spinor.

Poincaré invariance in four dimensions forces the three-form flux H and the three-form condensate to have only non-trivial components in the internal space. We are still taking a ten-dimensional expectation value, $\langle \Sigma_{MNP} \rangle$. We should also remember that the ten-dimensional gauginos are in the adjoint of $E_8 \times E_8$ or $SO(32)$. Denoting the external four-dimensional gauge group as G, and the internal group as H (G is the stabilizer of H in the ten-dimensional group), we may decompose the ten-dimensional product representation as $\mathbf{496} \otimes \mathbf{496} \to \sum_i (R(G)_i, R(H)_i)$. Of course the details very much depend on the choice of G and H, but in general the ten-dimensional trace over fermion bilinears will break into a sum of many terms:

$$\begin{aligned}
\langle \Sigma \rangle_{mnp} &= \langle \mathrm{tr} \overline{\chi}_{(10)} \Gamma_{mnp} \chi_{(10)} \rangle \\
&= \sum_i \langle \mathrm{tr}_{R(G)_i} \overline{\chi}_+^{(4)} \chi_-^{(4)} \cdot \mathrm{tr}_{R(H)_i} \chi_+^\dagger \gamma_{mnp} \chi_- \rangle - \sum_i \langle \mathrm{tr}_{R(G)_i} \overline{\chi}_-^{(4)} \chi_+^{(4)} \cdot \mathrm{tr}_{R(H)_i} \chi_-^\dagger \gamma_{mnp} \chi_+ \rangle \\
&= -2 \sum_i \mathrm{Re} \langle \Lambda_i \Sigma^i_{mnp} \rangle \,.
\end{aligned} \tag{3.5}$$

Here we have defined internal three-forms Σ^i_{mnp} as

$$\Sigma^i_{mnp} = \mathrm{tr}_{R(H)_i} \chi_-^\dagger \gamma_{mnp} \chi_+ \,, \tag{3.6}$$

and a four-dimensional condensate *vector* Λ_i as

$$\Lambda_i = \mathrm{tr}_{R(G)_i} \overline{\chi}_-^{(4)} \chi_+^{(4)} \,. \tag{3.7}$$

From now on we shall suppress all the traces and the G and H representation indices. To simplify the notation, in the rest of this section we will set $\langle \Sigma \rangle = \hat{\Sigma} = -2\mathrm{Re}(\Lambda \Sigma)$.

As shown in appendix D, the ten-dimensional supersymmetry equations can be written as a set of conditions on the forms Ω and J defining the $SU(3)$ structure. The set of independent equations are

$$H \lrcorner \Omega = 2i\overline{\mu} e^{-\Delta} \qquad i\frac{\alpha'}{8} \hat{\Sigma} \lrcorner \Omega = i\overline{\mu} e^{-\Delta} \,, \tag{3.8a}$$

and the differential conditions

$$\mathrm{d}J = (2\mathrm{d}\phi - 4\mathrm{d}\Delta) \wedge J + *\overline{H} + 3\,e^{-\Delta}\mathrm{Re}\,(\mu\,\Omega) \tag{3.8b}$$

$$\mathrm{d}\Omega = (2\mathrm{d}\phi - 3\mathrm{d}\Delta) \wedge \Omega - ie^{-\Delta}\,\overline{\mu}\,J \wedge J \tag{3.8c}$$

$$J \wedge \mathrm{d}J = (\mathrm{d}\phi - \mathrm{d}\Delta) \wedge J \wedge J \tag{3.8d}$$

$$\frac{\alpha'}{4} * \hat{\Sigma} \wedge J = \mathrm{d}\Delta \wedge J \wedge J \tag{3.8e}$$

where we define $\overline{H} = H + \frac{\alpha'}{2}\hat{\Sigma}$. A similar set of supersymmetry conditions was also derived in [12]. Note that the flux equation of motion is automatically satisfied

$$\mathrm{d}\left(e^{-2\phi+4\Delta} * \overline{H}\right) = 0\,, \tag{3.9}$$

while the integrability conditions of section 2.1 have to be imposed to ensure that also the other equations of motion are satisfied. This generically gives non-trivial conditions on the three-form $\hat{\Sigma}$.

It is interesting to see whether the supersymmetry equations are enough to completely determine the solution. To this extent, we decompose H and $\hat{\Sigma}$ in $SU(3)$ representations

$$H = \frac{3}{2}\mathrm{Im}(H_{\mathbf{1}}\overline{\Omega}) + H_{(3)} \wedge J + H_{\mathbf{6}}\,, \tag{3.10a}$$

$$\hat{\Sigma} = \frac{3}{2}\mathrm{Im}(\hat{\Sigma}_{\mathbf{1}}\overline{\Omega}) + \hat{\Sigma}_{(3)} \wedge J + \hat{\Sigma}_{\mathbf{6}}\,, \tag{3.10b}$$

where $H_{(3)} = H_{\mathbf{3}} + H_{\overline{\mathbf{3}}}$ and (i is a complex index)

$$H_{\mathbf{1}} = \frac{1}{36}\Omega^{mnp}H_{mnp} \tag{3.11a}$$

$$H_{(3)\,i} = \frac{1}{4}H_{imn}J^{mn} \tag{3.11b}$$

$$H_{\mathbf{6}\,mnp} = H_{mnp} - \frac{3}{2}\mathrm{Im}(H_{\mathbf{1}}\overline{\Omega})_{mnp} - 3H_{(3)m}J_{mp]}\,, \tag{3.11c}$$

and similarly for $\hat{\Sigma}$. Comparing the supersymmetry conditions (3.8b) - (3.8e) with the torsion equations

$$\mathrm{d}J = \frac{3}{2}\mathrm{Im}(W_1\overline{\Omega}) + W_4 \wedge J + W_3 \tag{3.12a}$$

$$\mathrm{d}\Omega = W_1\,J \wedge J + W_2 \wedge J + \overline{W}_5 \wedge \Omega\,, \tag{3.12b}$$

we can see how many of the $SU(3)$ irreducible representations are fixed by supersymmetry. We find the following table

$SU(3)$ repr.	parameters	susy eq.
$\mathbf{1}$	$W_{\mathbf{1}}$ $e^{-\Delta}\mu$ $\overline{H}_{\mathbf{1}}$ $\hat{\Sigma}_{\mathbf{1}}$	3
$\mathbf{3}+\overline{\mathbf{3}}$	W_4 W_5 $\mathrm{d}\phi$ $\mathrm{d}\Delta$ $\overline{H}_{\mathbf{3}}$, $\overline{H}_{\overline{\mathbf{3}}}$ $\hat{\Sigma}_{\mathbf{3}}$, $\hat{\Sigma}_{\overline{\mathbf{3}}}$	4
$\mathbf{6}$	W_3 $\overline{H}_{\mathbf{6}}$ $\hat{\Sigma}_{\mathbf{6}}$	1
$\mathbf{8}$	W_2	1

From (3.12b) and (3.8c) we immediately see that the **8** representation in the torsion is set to zero

$$W_2 = 0\,. \tag{3.13}$$

It is also easy to check that all the singlet representations are fixed

$$W_1 = -i\overline{\mu}e^{-\Delta} \tag{3.14a}$$

$$\overline{H}_{\mathbf{1}} = 3\,\overline{\mu}\,e^{-\Delta} \tag{3.14b}$$

$$\alpha'\,\mathrm{Re}(\Lambda\Sigma)_{\mathbf{1}} = \frac{1}{3}\overline{\mu}\,e^{-\Delta} \tag{3.14c}$$

On the other hand, we are left with one undetermined quantity in the $\mathbf{3}+\overline{\mathbf{3}}$ representation

$$W_4 = 2\mathrm{d}\phi - 4\mathrm{d}\Delta \tag{3.15a}$$

$$W_5 = 2\mathrm{d}\phi - 3\mathrm{d}\Delta \tag{3.15b}$$

$$\overline{H}_{\mathbf{3}} = i(\mathrm{d}\phi - 3\,\mathrm{d}\Delta)^{(1,0)} \tag{3.15c}$$

$$\alpha'\mathrm{Re}(\Lambda\Sigma)_{\mathbf{3}} = 2i(\mathrm{d}\Delta)^{(1,0)} \tag{3.15d}$$

and also for the **6**, where $\mathrm{Re}(\Lambda\Sigma)_{\mathbf{6}}$ is left undetermined

$$W_3 = \mathrm{d}J_{\mathbf{6}} = (*\overline{H})_{\mathbf{6}}\,. \tag{3.16}$$

To have a solution of the equations of motion, in addition to the above set of supersymmetry conditions, we must also satisfy the conditions derived from the integrability and the Bianchi identity (2.5). On the four-dimensional external space there is no H-flux and the Bianchi identity is trivially zero. One might worry that whenever there is a non vanishing cosmological constant, the right-hand side of the Bianchi identity can give a non-trivial contribution in the four-dimensional spacetime, coming from the curvature of AdS. However such a piece is by (3.8a) of $\mathcal{O}(\alpha')$ and the corresponding contribution to the Bianchi identity is $\mathcal{O}(\alpha'^2)$. For the same reason (see (3.15d)) there is no contribution from the warp factor at this order. Then we are left with the purely six-dimensional equation

$$\mathrm{d}H = \frac{\alpha'}{4}\left(\mathrm{tr}\,F\wedge F - \mathrm{tr}\,R^-\wedge R^-\right)\,. \tag{3.17}$$

Up to now we did not make any assumption on the nature of the internal six-dimensional space. When the manifold is compact, which is the most relevant case for string phenomenology, one can show that "fake supersymmetry" solutions of the type of section 2.3 are the only allowed ones. Notice first that (3.8a) implies that at zeroth order in α' there are no AdS_4 solutions.[14] Then a well-known no-go theorem

[14]Compact solutions with condensates and non trivial AdS_4 parameter have been found before in the literature, see e.g. [12–15]. Note however that such solutions inevitably require $\mathcal{O}(1/\alpha')$ effects or similar to make sense.

[21] states that at zeroth order in α' warping, dilaton and H-flux vanish and the internal geometry must be Calabi-Yau. We denote the CY geometry by (X_0, Ω_0, J_0) where

$$\mathrm{d}\Omega_0 = 0 \,, \quad \mathrm{d}J_0 = 0 \,, \tag{3.18}$$

and we express the full solution as an expansion in α'

$$\Omega = \Omega_0 + \alpha'\Omega_1 + .. \tag{3.19a}$$

$$J = J_0 + \alpha' J_1 + .. \,, \tag{3.19b}$$

and similarly for the other fields, where H-flux, dilaton, warping and condensate have no $(\alpha')^0$ terms. Consider then (3.8c), which, at first order, reads

$$\mathrm{d}\Omega_1 = (2\mathrm{d}\phi_1 - 3\mathrm{d}\Delta_1) \wedge \Omega_0 - i\overline{\mu}_1\, J_0 \wedge J_0 \,, \tag{3.20}$$

If $\mu_1 \neq 0$, this equation implies that $J_0 \wedge J_0$ is an exact four-form, which is not possible on a compact Calabi-Yau three-fold of finite volume. We conclude that $\mu_1 = 0$ for supersymmetric solutions, and we can drop the cosmological constant to the order we are working at. A similar result was also derived in [17].

Next we consider the integrability condition obtained from (2.10), which on the first order geometry reduces to

$$\nabla^m \partial_m \Delta_1 = 0 \,, \tag{3.21}$$

where we have used the BPS equation (3.8e). It follows that Δ_1 is constant on a compact geometry. The BPS conditions for such geometries can hence, without loss of generality, be taken to be

$$J \wedge \mathrm{d}J = \mathrm{d}\phi \wedge J \wedge J \tag{3.22a}$$

$$\mathrm{d}J = 2\,\mathrm{d}\phi \wedge J + * \left(H + \frac{\alpha'}{2}\hat{\Sigma} \right) \tag{3.22b}$$

$$\mathrm{d}\Omega = 2\,\mathrm{d}\phi \wedge \Omega \,, \tag{3.22c}$$

where $\hat{\Sigma}$ is some primitive $(2,1)+(1,2)$ form. Primitivity follows from $\mathrm{d}\Delta = 0$ and (3.8e). Using the integrability conditions (2.8) and (2.10) one can then show that $\hat{\Sigma}$ must satisfy

$$\nabla_{[m}\hat{\Sigma}_{npq]}\gamma^{npq}\eta_+ = 0 \,, \tag{3.23}$$

which in turn implies $\mathrm{d}\Sigma = 0$. Hence we recover the fake supersymmetry conditions of section (2.3).

4 Backgrounds with three-dimensional Poincaré invariance

In this section we consider compactifications to three dimensions. The ten-dimensional spacetime is a warped product of a maximally symmetric three-dimensional space M_3 and a seven-dimensional manifold X_7. The metric is

$$\mathrm{d}s^2 = e^{2\Delta}\mathrm{d}s_3^2 + \mathrm{d}s_7^2\,, \tag{4.1}$$

where the warp factor Δ only depends on the coordinates on X_7. The three-dimensional space can be either Minkowski or Anti de Sitter, while X_7 is not necessarily compact and admits a G_2 structure. Taking a non compact X_7 allows to describe also four-dimensional domain-wall solutions and standard compactifications to four dimensions. As we will see, generic seven-dimensional compactifications allow for more components of fluxes and, more importantly for this paper, condensate configurations.

With the ansatz (4.1), the ten-dimensional gamma matrices decompose as

$$\begin{aligned}\Gamma_\mu &= e^{\Delta}\gamma_\mu \otimes \sigma_3 \otimes \mathbb{I}_{(8)}\\ \Gamma_m &= \mathbb{I}_{(2)} \otimes \sigma_1 \otimes \gamma_m\,,\end{aligned} \tag{4.2}$$

where we denote the three-dimensional and seven-dimensional indices by $\{\mu, \nu, ..\}$ and $\{m, n, ..\}$, respectively. γ_μ are chosen to be real, while γ_m are purely imaginary. The ten-dimensional chiral operator is $\Gamma_{(10)} = -\mathbb{I}_{(2)} \otimes \sigma_2 \otimes \mathbb{I}_{(8)}$.

The ten-dimensional spinors split accordingly. The supersymmetry parameter ϵ decomposes as

$$\epsilon = e^{A}\zeta \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix} \otimes \eta\,, \tag{4.3}$$

where ζ is a three-dimensional spinor and η a seven-dimensional spinor of unitary norm. The spinor η is globally defined and defines a G_2 structure on X_7.

Similarly, the gaugino takes the following form

$$\chi_{(10)} = \frac{1}{\sqrt{2}}\xi \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix} \otimes \chi\,, \tag{4.4}$$

where again ξ and χ are a three- and seven-dimensional spinors. The internal spinor χ may be expressed in terms of the G_2 spinor η as follows

$$\chi^i = c^i\eta + c^i_m\gamma_m\eta\,, \tag{4.5}$$

where i is the internal gauge index, and c^i and c^i_m are generic real functions of the internal coordinates. The norm of χ is $C = \chi^\dagger\chi = \mathrm{tr}(c^2 + c_m c_m)$.[15]

[15] As in section 3, all the traces and three- and seven-dimensional representations are suppressed. C has to be understood as a vector. The same applies to Λ the internal three-form Σ to be defined shortly.

The split (4.4) allows for two non-vanishing components of the three-form Σ_{MNP} that are compatible with the three-dimensional Poincaré symmetry, one with all indices in the three-dimensional space and one with all indices in the internal space. Recall that Σ_{MNP} is a gaugino bilinear, and as in the previous section $\langle\Sigma_{MNP}\rangle$ has to be interpreted as ten-dimensional vev. Using (4.2) and (B.22) one can show that in flat indices

$$\langle\Sigma_{ABC}\rangle = \begin{cases} \langle\overline{\chi}_{(10)}\Gamma_{\alpha\beta\gamma}\chi_{(10)}\rangle = \langle C\,\Lambda\rangle\,\epsilon_{\alpha\beta\gamma} & \alpha,\beta,\gamma = 1,2,3 \\ \langle\overline{\chi}_{(10)}\Gamma_{abc}\chi_{(10)}\rangle = -i\langle\Lambda\Sigma_{abc}\rangle\,, & \end{cases} \tag{4.6}$$

where we have defined $\Lambda = \mathrm{tr}\overline{\xi}\xi$ and $\Sigma_{abc} = \chi^\dagger\gamma_{abc}\chi$. The fact that the gaugino χ is Majorana ensures that Σ_{abc} is imaginary.

Poincaré invariance also implies that the only non-trivial components of the H-flux are

$$H_{MNP} = \begin{cases} H_{\mu\nu\rho} = \tilde{H}\epsilon_{\mu\nu\rho}\,, \\ H_{mnp}\,. \end{cases} \tag{4.7}$$

The ansatz (4.1), (4.7) and (4.6) for the metric, H-flux and condensate can be used, together with the splitting (4.2) and (4.3), to reduce the ten-dimensional supersymmetry variations to a set of equations on the forms φ and ψ defining the G_2-structure on the internal manifold.[16] These consist of two algrebraic relations

$$H\lrcorner\varphi = e^{-3\Delta}(2\mu\, e^{2\Delta} - \frac{\alpha'}{2}\Lambda C)\,, \tag{4.8a}$$

$$i\frac{\alpha'}{4}\Lambda\Sigma\lrcorner\varphi = e^{-3\Delta}\,(\tilde{H} + \frac{\alpha'}{4}C\Lambda - 2\mu e^{2\Delta})\,, \tag{4.8b}$$

together with the differential conditions

$$\mathrm{d}\varphi = (2\mathrm{d}\phi - 3\mathrm{d}\Delta)\wedge\varphi - (*H - i\frac{\alpha'}{2}*(\Lambda\Sigma)) + 2\mu\, e^{-\Delta}\,\psi \tag{4.9a}$$

$$\mathrm{d}\psi = (2\mathrm{d}\phi - 2\mathrm{d}\Delta)\wedge\psi\,, \tag{4.9b}$$

$$\mathrm{d}\Delta = i\frac{\alpha'}{8}\Lambda\,\Sigma\lrcorner\psi\,. \tag{4.9c}$$

For consistency of the theory, we must also satisfy the Bianchi identity. The three-dimensional external part is trivial. Since the warping by (4.9c) gives an $\mathcal{O}(\alpha'^2)$ contribution to the right-hand side, the internal Bianchi identity is

$$\mathrm{d}H = \frac{\alpha'}{4}\left(\mathrm{tr}\,F\wedge F - \mathrm{tr}\,R^-\wedge R^-\right)\,. \tag{4.10}$$

Decomposing H (and similarly Σ) in G_2 representations

$$H_{mnp} = \frac{1}{7}H_{\mathbf{1}}\phi_{mnp} - \frac{1}{4}H_{\mathbf{7}}^q\psi_{qmnp} + \frac{3}{2}H_{\mathbf{27}\,q[m}\phi_{np]}{}^q\,, \tag{4.11a}$$

$$(*H)_{mnpq} = \frac{1}{7}H_{\mathbf{1}}\psi_{mnpq} + H_{\mathbf{7}[m}\phi_{npq]} - 2H_{\mathbf{27}\,e[r}\psi^r{}_{npq]}\,, \tag{4.11b}$$

[16] Definitions and conventions for the forms φ and ψ are given in appendix B. In appendix C we give the derivation of the equations below.

with (and similarly for Σ)

$$H_{\mathbf{1}} = \frac{1}{6}\phi^{mnp}H_{mnp} \qquad H_{\mathbf{7}\,m} = \frac{1}{6}H^{npq}\psi_{npqm} \qquad H_{\mathbf{27}\,mn} = \frac{1}{2}H_{pq(m}\phi_{n)}{}^{pq} - \frac{3}{7}H_{\mathbf{1}}\delta_{mn}\,,$$

and comparing with the torsion of the G_2 structure

$$\mathrm{d}\varphi = 3\,W_{\mathbf{7}}\wedge\varphi + W_{\mathbf{1}}\psi + *W_{\mathbf{27}} \tag{4.12a}$$
$$\mathrm{d}\psi = 4\,W_{\mathbf{7}}\wedge\psi + *W_{\mathbf{14}}\,, \tag{4.12b}$$

it is easy to show that supersymmetry is not enough to completely determine the solution. This can already be seen by simple counting of the full set degrees of freedom in the solution and the number of constraints imposed by supersymmetry, as summarised in the table below

G_2 repr.	parameters	susy eq.
1	$W_{\mathbf{1}}\quad e^{-\Delta}\mu\quad e^{-3\Delta}\tilde{H}\quad e^{-3\Delta}C\Lambda\quad H_{\mathbf{1}}\quad \Lambda\Sigma_{\mathbf{1}}$	3
7	$W_{\mathbf{7}}\quad \mathrm{d}\phi\quad \mathrm{d}\Delta\quad H_{\mathbf{7}}\quad \Lambda\Sigma_{\mathbf{7}}$	3
14	$W_{\mathbf{14}}$	1
27	$W_{\mathbf{27}}\quad H_{\mathbf{27}}\quad \Lambda\Sigma_{\mathbf{27}}$	1

More explicitly, combining (4.8a) – (4.9a) and (4.12a) we can fix three of the singlets

$$W_{\mathbf{1}} = \frac{8}{7}\mu e^{-\Delta} + \frac{1}{7}e^{-3\Delta}(2\tilde{H} + \alpha'\Lambda C)\,, \tag{4.13a}$$
$$H_{\mathbf{1}} = e^{-3\Delta}(2\mu\, e^{2\Delta} - \frac{\alpha'}{2}\Lambda C)\,, \tag{4.13b}$$
$$\Lambda\Sigma_{\mathbf{1}} = -\frac{4i}{\alpha'}e^{-3\Delta}\,(\tilde{H} + \frac{\alpha'}{4}C\Lambda - 2\mu e^{2\Delta})\,. \tag{4.13c}$$

Similarly, (4.9a), (4.9c) with (4.12a) determine some of the forms in the **7**

$$W_{\mathbf{7}} = \frac{1}{4}H_{\mathbf{7}} \qquad \Lambda\Sigma_{\mathbf{7}} = -\frac{8i}{\alpha'}\mathrm{d}\Delta \tag{4.14}$$

and in the **27**

$$W_{\mathbf{27}} \sim H_{\mathbf{27}} - i\frac{\alpha'}{2}\Lambda\Sigma_{\mathbf{27}}\,. \tag{4.15}$$

Finally notice that, even with a generic condensate, it is not possible to generate the representation **14** of the torsion classes. G_2-structures of this type are often referred to as *integrable*. They have strong similarities with even-dimensional complex manifolds [25] and might be useful for a better understanding of heterotic G_2-compactifications and their moduli (see e.g. [26, 27] for work in this direction).

As in the previous section, in order to solve the equations of motion a solution of the supersymmetry variation has to satisfy the integrability constraints (2.9) – (2.10), imposing further constraints on Σ.

4.1 Four-dimensional domain walls

The ansatz (4.1) and (4.3) also describes (3+1)-dimensional domain walls. For these solutions the seven-dimensional metric decomposes as

$$\mathrm{d}s_7^2 = e^{2A(x)}\mathrm{d}y^2 + \mathrm{d}s_6^2 \,, \tag{4.16}$$

where y is the normal coordinate to the domain wall and x^m are coordinates on the six-dimensional internal manifold M_6. In this paper we assume it to have $SU(3)$ structure. With this ansatz, the G_2-structure forms can be written as

$$\varphi = e^A \mathrm{d}y \wedge J + \Omega_- \tag{4.17a}$$

$$\psi = e^A \mathrm{d}y \wedge \Omega_+ - \frac{1}{2} J \wedge J \,, \tag{4.17b}$$

where $\Omega = \Omega_+ + i\Omega_-$ and J are the holomorphic three-form and the almost hermitian two-form defining the $SU(3)$ structure on M_6. The three-form flux H and condensate three-form Σ decompose as

$$H = \mathrm{d}y \wedge H_2 + H_3 \tag{4.18a}$$

$$\Sigma = \mathrm{d}y \wedge \Sigma_2 + \Sigma_3 \,, \tag{4.18b}$$

where H_2, Σ_2 and H_3 Σ_3 are two and three-forms on M_6.

Plugging the ansatz above in the system (4.8a)-(4.9c), we find

$$-\frac{1}{2}e^{-A}H_2 \wedge J \wedge J + \Omega_+ \wedge H_3 = *(2\mu e^{-\Delta} - \frac{\alpha'}{2}C\Lambda e^{-3\Delta}) \tag{4.19a}$$

$$-i\frac{\alpha'}{4}(\frac{1}{2}e^{-A}\Lambda\Sigma_2 \wedge J \wedge J - \Omega_+ \wedge \Lambda\Sigma_3) = *(e^{-3\Delta}\tilde{H} + \frac{\alpha'}{4}e^{-3\Delta}C\Lambda - 2\mu\, e^{-\Delta}) \,, \tag{4.19b}$$

where $*$ denotes the six-dimensional Hodge star. The differential conditions become

$$\begin{aligned} \mathrm{d}J &= -\mathrm{d}A \wedge J + e^{-A}\Omega_-' - (2\phi' - 3\Delta')e^{-A}\Omega_- + (2\mathrm{d}\phi - 3\mathrm{d}\Delta) \wedge J \\ &\quad + *\,(H_3 - i\frac{\alpha'}{2}\Lambda\Sigma_3) - 2\mu\, e^{-\Delta}\Omega_+ \end{aligned} \tag{4.20a}$$

$$\mathrm{d}\Omega_- = (2\mathrm{d}\phi - 3\mathrm{d}\Delta) \wedge \Omega_- - *(H_2 - i\frac{\alpha'}{2}\Lambda\Sigma_2)e^{-A} - \mu\, e^{-\Delta} J \wedge J \tag{4.20b}$$

$$\mathrm{d}\Omega_+ = -e^{-A} J \wedge J' + e^{-A}(\phi' - \Delta')J \wedge J + (2\mathrm{d}\phi - 2\mathrm{d}\Delta - \mathrm{d}A) \wedge \Omega_+ \tag{4.20c}$$

$$J \wedge \mathrm{d}J = (\mathrm{d}\phi - \mathrm{d}\Delta) \wedge J \wedge J \tag{4.20d}$$

$$*\Delta' = i\frac{\alpha'}{8}e^A\Lambda\Sigma_3 \wedge \Omega_- \tag{4.20e}$$

$$*\mathrm{d}\Delta = i\frac{\alpha'}{8}(e^{-A}\Lambda\Sigma_2 \wedge \Omega_- - \Lambda\Sigma_3 \wedge J) \,, \tag{4.20f}$$

where the prime denotes derivatives with respect to the y direction. This system of equations should also be supplemented with the extra integrability constraints needed to solve the equations of motion, properly reduced to the domain wall situation. With the ansatz (4.18a) for H, the Bianchi identity splits into the two equations

$$\mathrm{d}\overline{H}_3 = \frac{\alpha'}{4}\left(\mathrm{tr}\, F_2 \wedge F_2 - \mathrm{tr}\, R_2^- \wedge R_2^-\right) \tag{4.21a}$$

$$\overline{H}_3' = \mathrm{d}\overline{H}_2 + \frac{\alpha'}{2}\left(\mathrm{tr}\, F_1 \wedge F_2 - \mathrm{tr}\, R_1^- \wedge R_2^-\right) , \tag{4.21b}$$

where $\overline{H} = H - i\frac{\alpha'}{2}\Lambda\Sigma$ and we have defined

$$F = F_2 + \mathrm{d}y \wedge F_1 \tag{4.22a}$$

$$R^- = R_2^- + \mathrm{d}y \wedge R_1^- . \tag{4.22b}$$

5 Symplectic half-flat domain walls

Explicit solutions are not the focus of this paper, but we would like to reexamine an already known solution, which provides an illustration of the fake supersymmetry approach of section 2.3. The idea is to look for geometries such that are solutions of the equations (2.56) and (2.57).

We look for a domain wall solution as in section 4.1. The ten-dimensional metric is

$$\mathrm{d}s^2 = \mathrm{d}s_3^2 + e^{2A}\mathrm{d}y^2 + \mathrm{d}s_6^2 , \tag{5.1}$$

where we set to zero the warp factor Δ.[17] The fake supersymmetry equations are

$$-\frac{1}{2}e^{-A}\overline{H}_2 \wedge J \wedge J + \Omega_+ \wedge \overline{H}_3 = 2 * \mu , \quad \tilde{\overline{H}} = 2\mu , \tag{5.2}$$

and the differential conditions

$$J \wedge \mathrm{d}J = \mathrm{d}\phi \wedge J \wedge J \tag{5.3a}$$

$$\mathrm{d}(e^A\Omega_+) = -J \wedge J' + \phi' J \wedge J + 2\mathrm{d}\phi \wedge e^A\Omega_+ \tag{5.3b}$$

$$\mathrm{d}J = -\mathrm{d}A \wedge J + e^{-A}\Omega_-' - 2\phi' e^{-A}\Omega_- + 2\mathrm{d}\phi \wedge J + *\overline{H}_3 - 2\mu\Omega_+ \tag{5.3c}$$

$$\mathrm{d}\Omega_- = 2\mathrm{d}\phi \wedge \Omega_- - *\overline{H}_2 e^{-A} - \mu J \wedge J , \tag{5.3d}$$

where d now denotes the exterior derivative in six dimensions and the prime denotes the derivative along y. The Bianchi identity imposes the further conditions (4.21a) and (4.21b)

$$\mathrm{d}\overline{H}_3 = \frac{\alpha'}{4}\left(\mathrm{tr}\, F_2 \wedge F_2 - \mathrm{tr}\, R_2^- \wedge R_2^-\right) \tag{5.4a}$$

$$\overline{H}_3' = \mathrm{d}\overline{H}_2 + \frac{\alpha'}{2}\left(\mathrm{tr}\, F_1 \wedge F_2 - \mathrm{tr}\, R_1^- \wedge R_2^-\right) . \tag{5.4b}$$

[17] This follows from assuming that the equations are formally the same as with no condensate.

For zero condensate, this system has been analyzed to a great degree in the literature before, see e.g. [12, 15, 28–33]. Here we will see how it can also admit solutions with non trivial condensate.

Consider first the system (5.2)-(5.3d) together with the Bianchi identities (5.4a) and (5.4b). A relatively simple solution is obtained assuming that the internal six-dimensional geometry is an half-flat manifold

$$\mathrm{d}J = 0 \tag{5.5a}$$
$$\mathrm{d}\Omega_+ = 0 \tag{5.5b}$$
$$\mathrm{d}\Omega_- = W_2 \wedge J\,, \tag{5.5c}$$

where W_2 is purely imaginary and taking the flux to be closed, i.e. we solve the Bianchi identity exactly.[18] The Bianchi identity in this case reduces to

$$\mathrm{d}\overline{H}_3 = 0\,, \quad H'_{(3)} = \mathrm{d}\overline{H}_2\,, \quad \mathrm{d}\mu = \mu' = 0\,. \tag{5.6}$$

In our example the internal six-dimensional manifold is a particular solvmanifold defined in [34]. Let G be the Lie group of matrices of the form

$$\begin{pmatrix} e^{\lambda z} & 0 & 0 & x \\ 0 & e^{-\lambda z} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}\,, \tag{5.7}$$

where $\{x, y, z\}$ are real numbers and $\lambda = \log \frac{3+\sqrt{5}}{2}$. Then G is a connected solvable Lie group admitting a cocompact lattice Γ, so that the quotient $N = G/\Gamma$ is a three-dimensional compact parallelizable solvmanifold [35]. The six-dimensional solvmanifold is the product $M_0 = N \times N$. M_0 admits a coframe that satisfies

$$\begin{aligned} \mathrm{d}\alpha_1 &= -\lambda\, \alpha_1 \wedge \alpha_3 \qquad & \mathrm{d}\alpha_4 &= -\lambda\, \alpha_4 \wedge \alpha_6 \\ \mathrm{d}\alpha_2 &= \lambda\, \alpha_2 \wedge \alpha_3 & \mathrm{d}\alpha_5 &= \lambda\, \alpha_5 \wedge \alpha_6 \\ \mathrm{d}\alpha_3 &= 0 & \mathrm{d}\alpha_6 &= 0 \end{aligned} \tag{5.8}$$

where $\{\alpha_1, \alpha_2, \alpha_3\}$ is a coframe on N. It is easy to check that

$$J_0 = \alpha_1 \wedge \alpha_2 + \alpha_4 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6 \tag{5.9a}$$
$$\Omega_0 = \frac{(1-i)}{\sqrt{2}}\,(\alpha_1 + i\,\alpha_2) \wedge (\alpha_4 + i\,\alpha_5) \wedge (\alpha_3 + i\,\alpha_6) \tag{5.9b}$$

define an half-flat structure. Moreover

$$\Omega_{0+} = \frac{1}{\sqrt{2}\lambda}\mathrm{d}P_0 \tag{5.10}$$

[18] This can be achieved by embedding the gauge-connection in the ∇^- connection.

with $P_0 = \alpha_{14} + \alpha_{15} - \alpha_{24} + \alpha_{25}$[19] is a primitive two-form

$$P_0 \wedge J_0 \wedge J_0 = 0\,. \tag{5.11}$$

In our solution we take the six-dimensional metric to be

$$\mathrm{d}s_6^2 = e^{\phi(y)}\mathrm{d}s_0^2 \tag{5.12}$$

where $\mathrm{d}s_0^2$ is the metric on the solvmanifold M_0 and, correspondingly,

$$J = e^{\phi} J_0 \qquad \Omega = e^{\frac{3\phi}{2}}\Omega_0\,. \tag{5.13}$$

Then it is easy to see that (5.3a) is solved if the dilaton only depends on the domain wall direction $\phi = \phi(y)$ so that

$$\mathrm{d}\phi = 0\,. \tag{5.14}$$

while (5.3b) implies that the warp factor must be constant (for simplicity we take $A = 0$). (5.3c) and (5.3d) become

$$\overline{H}_3 = -\frac{1}{2}e^{\frac{3\phi}{2}}(\phi'\Omega_{0+} - 4\mu\Omega_{0-}) \tag{5.15a}$$

$$*\overline{H}_2 = -e^{\frac{3\phi}{2}}\mathrm{d}\Omega_{0-}\,. \tag{5.15b}$$

Then the Bianchi identity $\mathrm{d}\overline{H}_3 = 0$ in (5.6) together with (5.5a) imply that the three-dimensional space is Minkowksi, $\mu = 0$. Using the explicit form of Ω_0 we find that the fluxes are

$$\overline{H}_2 = e^{\frac{\phi}{2}}\sqrt{2}\lambda\, P_0 \tag{5.16a}$$

$$\overline{H}_3 = -\frac{1}{2}\phi' e^{\frac{3\phi}{2}}\Omega_{0+} = -\frac{1}{3}\left(e^{\frac{3}{2}\phi}\right)'\Omega_{0+} \tag{5.16b}$$

$$\tilde{H} = -\frac{\alpha'}{2}C\Lambda \tag{5.16c}$$

We still have to impose the Bianchi identity $\overline{H}_3' = \mathrm{d}\overline{H}_2$, which reduces to a differential equation for the dilaton

$$\left(e^{\phi}\right)'' + \frac{1}{2}\left(\phi'\right)^2 e^{\phi} + 4\lambda^2 = 0\,. \tag{5.17}$$

From (5.17) we see that e^{ϕ} always has a negative second order derivative. There are two possibilities, either e^{ϕ} is bounded on an interval, or it tends to some linear function from below. The latter option is not possible however, as from (5.17) this would imply that

$$\frac{1}{2}\left(\phi'\right)^2 e^{\phi} + 4\lambda^2 \to 0\,.$$

[19] We use the notation $\alpha_{mn} = \alpha_m \wedge \alpha_n$.

By rescaling the y-coordinate, $\mathrm{d}y = e^{\frac{1}{2}\phi}\mathrm{d}t$, we find that the equation reduces to

$$\partial_t^2 e^{\phi} + 4\lambda^2 e^{\phi} = 0 \,, \tag{5.18}$$

with solution

$$e^{\phi} = \left| \cos\left(2\lambda(t-t_0)\right) \right| \,, \tag{5.19}$$

where we have set an overall constant to one, so that $\phi_0 = \phi(t=t_0) = 0$. Similar equations were discussed in [26]. The solution (5.19) is periodic, and discontinuous at points where the cosine vanishes. The discontinuity leads to the following profile for $\mathrm{d}\overline{H}$, where now $\mathrm{d} = \mathrm{d}t\,\partial_t + \mathrm{d}_6$

$$\mathrm{d}\overline{H} = -4\lambda^2 \sum_n \delta\left[2\lambda(t-t_0) - \frac{\pi}{2}(1+2n)\right] \mathrm{d}t \wedge \Omega_{+0} \,, \tag{5.20}$$

which suggest that there are sources localised along the t-direction at intervals of $\Delta t = \pi/2\lambda$, while they wrap a trivial internal three-cycle, Poincaré dual to the exact form Ω_{+0}.

Let us also comment on the scalar curvature close to the source. Write the seven-dimensional metric as

$$\mathrm{d}s_7^2 = \mathrm{d}y^2 + e^{\phi(y)} g_{0\,mn}\mathrm{d}x^m\mathrm{d}x^n = e^{\phi(t)}\left(\mathrm{d}t^2 + g_{0\,mn}\mathrm{d}x^m\mathrm{d}x^n\right) \,. \tag{5.21}$$

Consider the metric in the near-source region, i.e.

$$t - t_0 = \frac{\pi}{4\lambda} + \Delta t \,, \quad \Delta t > 0 \,, \tag{5.22}$$

for small enough Δt. In this region the warp factor behaves like

$$e^{\phi} = \sin\left(2\lambda\Delta t\right) = 2\lambda\Delta t + \mathcal{O}(\Delta t^2) \,. \tag{5.23}$$

With this, we find the metric close to the source

$$\mathrm{d}s_7^2 = \mathrm{d}y^2 + (3\lambda y)^{2/3}\,\mathrm{d}s_0^2 \,. \tag{5.24}$$

Computing the Ricci scalar, we find

$$\mathcal{R} = \frac{1}{\sin\left(2\lambda\Delta t\right)}\left(\mathcal{R}_0 + 24\lambda^2 - 6\cot(2\lambda\Delta t)\right) \,, \tag{5.25}$$

where $\mathcal{R}_0$ is the Ricci scalar of g_0. This has the following pole expansion

$$\mathcal{R} = -\frac{3}{4\lambda(\Delta t)^3} + \frac{\mathcal{R}_0 + 27\lambda^2}{2\lambda\Delta t} + \mathcal{O}(\Delta t) \,. \tag{5.26}$$

Any half-flat symplectic solution of this form will have this behaviour. In particular, note that at some point close enough to the source the vacuum energy becomes negative.

We conclude with two comments: The solution of the system above is generically not supersymmetric. The additional constraint (2.58), which leads to a supersymmetric solution, in this case reduces to

$$-\frac{1}{2}e^{-A}\Sigma_2 \wedge J \wedge J + \Omega_+ \wedge \Sigma_3 = -i * C\Lambda \tag{5.27a}$$

$$\Sigma_3 \wedge \Omega_- = 0 \tag{5.27b}$$

$$e^{-A}\Sigma_2 \wedge \Omega_- = \Sigma_3 \wedge J\,. \tag{5.27c}$$

One may see that the "canonical" form of a seven-dimensional condensate $\Sigma \propto \varphi$ will not work here. For instance, if we consider condensates of the form

$$\Sigma = a\,\Omega_{0+} + b\,\Omega_{0-} + c\,\mathrm{d}y \wedge P_0 + d\,\mathrm{d}y \wedge J_0\,, \tag{5.28}$$

where a, b, c and d are y-dependent functions. From (5.27b) we see that we need $a = 0$, while the closure condition on Σ gives

$$b\,\mathrm{d}\Omega_{0-} + b'\,\mathrm{d}y \wedge \Omega_{0-} - c\,\mathrm{d}y \wedge \mathrm{d}P_0 = 0\,, \tag{5.29}$$

from which we get $b = c = 0$. Hence, the only type of supersymmetry preserving solutions of this type are given by

$$\Sigma_3 = 0\,, \quad \Sigma_2 = d\,J_0\,. \tag{5.30}$$

The value of $C\Lambda$ is the determined through (5.27a).

Finally, let us return to the sources in (5.20). They clearly are of codimension four, and one may be tempted to interpret them as NS5-branes localised along $\mathrm{d}t$. It would be interesting to be able to confirm this and to study the stability properties of the soluton.

Acknowledgments

We would like to thank N. Halmagyi, C. Strickland-Constable, S. Theisen and A. Tomasiello for helpful discussions. RM thanks the Korea Institute for Advanced Study and the Simons Center for Geometry and Physics, MP thanks Galileo Galilei Institute, and EES thanks the Theoretical Physics Department at CERN for hospitality during the course of the work. This work of RM was supported in part by the Agence Nationale de la Recherche under the grant 12-BS05-003-01. The work of EES, made within the Labex Ilp (reference Anr-10- Labx-63), was supported by French state funds managed by the Agence nationale de la recherche, as part of the programme Investissements davenir under the reference Anr-11-Idex-0004-02.

A Heterotic Supergravity

The bosonic sector of heterotic supergravity consists of the metric g, the NS two-form B, the dilaton ϕ and the $E_8 \times E_8$ gauge-field A. The corresponding fermionic fields are the graviton ψ_M, the dilatino λ and the gaugino χ. The action for the bosonic sector is

$$S_0 = \int_{M_{10}} e^{-2\phi}\Big[\mathcal{R} + 4\nabla_M\phi\nabla^M\phi - \frac{1}{12}H_{MNP}H^{MNP} - \frac{\alpha'}{8}\Big(\mathrm{tr}\, F_{MN}F^{MN} - \mathrm{tr}\, R^-_{MN}R^{-MN}\Big)\Big]\,, \tag{A.1}$$

where the field strength H is defined as

$$H = \mathrm{d}B + \frac{\alpha'}{4}\Big(\omega_{CS}(A) - \omega_{CS}(\nabla^-)\Big)\,, \tag{A.2}$$

and satisfies the Bianchi identity

$$\mathrm{d}H = \frac{\alpha'}{4}\left(\mathrm{tr}\, F\wedge F - \mathrm{tr}\, R^-\wedge R^-\right)\,. \tag{A.3}$$

The derivatives $\nabla^\pm$ have connection symbols given by

$$\Gamma^{\pm}{}_{KL}{}^{M} = \Gamma^{\mathrm{LC}}{}_{KL}{}^{M} \pm \frac{1}{2}\hat{H}_{KL}{}^{M}\,. \tag{A.4}$$

The ten-dimensional bosonic equations of motion are

$$\mathcal{R}_{MN} + 2\nabla_M\nabla_N\phi - \frac{1}{2}H_M H_N - \frac{\alpha'}{4}\Big(\mathrm{tr}\, F_M\lrcorner F_N - \mathrm{tr}\, R^-{}_M\lrcorner R^-_N\Big) = 0 + \mathcal{O}(\alpha'^2) \tag{A.5a}$$

$$\frac{1}{4}\mathcal{R} + \nabla^2\phi - (\nabla\phi)^2 - \frac{1}{8}H^2 - \frac{\alpha'}{16}\Big(\mathrm{tr}\, F^2 - \mathrm{tr}\, R^{-2}\Big) = 0 + \mathcal{O}(\alpha'^2) \tag{A.5b}$$

$$e^{2\phi}\nabla^M\Big(e^{-2\phi}H_{MNP}\Big) = 0 + \mathcal{O}(\alpha'^2) \tag{A.5c}$$

$$e^{2\phi}\mathrm{d}_A(e^{-2\phi} * F) - F\wedge *H = 0 \, + \mathcal{O}(\alpha')\,, \tag{A.5d}$$

where the symbol $\lrcorner$ denotes the contractions. In particular, given a p-form A_p and a q-form B, we defined

$$A_M\lrcorner A_N = \frac{1}{(p-1)!}A_{MM_1..M_{p-1}}A^{MM_1..M_{p-1}} \tag{A.6a}$$

$$(A\lrcorner B)_{M_{p+1}\ldots M_q} = \frac{1}{p!}A^{N_1\ldots N_p}B_{N_1\ldots N_pM_{p+1}\ldots M_q} \tag{A.6b}$$

Since we are interested in solution respecting Poincaré invariance, we will always take the vacuum expectation value (vev) of individual fermions to vanish

$$\langle\psi_M\rangle = \langle\chi\rangle = \langle\lambda\rangle = 0\,, \tag{A.7}$$

so that the only relevant equations of motion are the bosonic ones.

The heterotic supersymmetry transformations with no fermionic bilinears are

$$\delta\psi_M = \nabla_M \epsilon + \frac{1}{8} H_{MNP} \Gamma^{NP} \epsilon + \mathcal{O}(\alpha'^2) \tag{A.8a}$$

$$\delta\lambda = -\frac{\sqrt{2}}{4} \left[\partial\!\!\!/ \phi + \frac{1}{2} H\!\!\!/ \right] + \mathcal{O}(\alpha'^2) \tag{A.8b}$$

$$\delta\chi = -\frac{1}{2\sqrt{2}} F\!\!\!/ \epsilon + \mathcal{O}(\alpha'^2) \,, \tag{A.8c}$$

where $A\!\!\!/$ denote the Clifford product

$$A\!\!\!/ = \frac{1}{p!} A_{M_1..M_p} \Gamma^{M_1..M_p} \,, \tag{A.9}$$

The connection Γ^- in (A.4) is the bosonic part of an $SO(9,1)$ Yang-Mills gauge supermultiplet $(\Gamma^{-}_{KL}{}^{M}, \psi_{PQ})$, whose fermionic part is the supercovariant curvature

$$\psi_{MN} = \nabla^+_M \psi_N - \nabla^+_N \psi_M \,. \tag{A.10}$$

The fields $(\Gamma^{-}_{KL}{}^{M}, \psi_{PQ})$ are constructed so as to transform as a $SO(9,1)$ Yang-Mills supermultiplet, modulo terms of $\mathcal{O}(\alpha')$. Since these fields only appear in the theory at $\mathcal{O}(\alpha')$, the theory is supersymmetric modulo terms $\mathcal{O}(\alpha'^2)$. Note that this holds even when including the higher order fermionic terms in the action [20].

B Conventions and G-structures in various dimensions

We Under the compactification

$$M_{10} = M_{10-d} \times X_d \,, \tag{B.1}$$

we use the following index conventions for our coordinates

$$\begin{aligned} \text{10-dimensional} &= I, J, K, .. \\ d\text{-dimensional} &= m, n, p, .. \\ (10\text{-}d)\text{-dimensional} &= \mu, \nu, \rho, .. \end{aligned}$$

For frame indices we will use $A, B, ...$ in ten dimensions, $\alpha, \beta, ...$ and $a, b, ...$ for $10-d$ and d-dimensional quantities, respectively. We also take a moment to collect some other conventions used. a p-form α is defined as

$$\alpha = \frac{1}{p!} \alpha_{m_1..m_p} \mathrm{d}x^{m_1..m_p} \,, \tag{B.2}$$

with its Hodge dual given by

$$* \, \alpha = \frac{1}{(d-p)!\, p!} \epsilon_{n_1..n_{d-p} m_1..m_p} \alpha^{m_1..m_p} \mathrm{d}x^{n_1..n_{d-p}} \,. \tag{B.3}$$

We denote the contraction of a p-form α by a q-form β where $(p > q)$ as

$$\beta \lrcorner \alpha = \frac{1}{(p-q)!\, q!} \beta^{m_1..m_q} \alpha_{m_1..m_p} \mathrm{d}x^{m_{q+1}..m_p} \, . \tag{B.4}$$

Note that

$$* \, (\alpha \wedge \beta) = \alpha \lrcorner * \beta \, . \tag{B.5}$$

With this, in the case of compact geometries the adjoint of the exterior derivative and Laplacian read

$$\mathrm{d}^\dagger = *\mathrm{d}* \, , \quad \Delta = \mathrm{d} * \mathrm{d} * + * \mathrm{d} * \mathrm{d} \, . \tag{B.6}$$

B.1 G-structures in different dimensions

In this appendix we summarise the basic features of the G-structures we need in this paper. These are G_2 structures in seven dimensions and Spin(7) structures in eight dimensions.

B.2 $SU(3)$ structure in six dimensions

In compactification to four dimensions we consider the ten-dimensional metric takes the form

$$\mathrm{d}s^2_{10} = e^{2\Delta(y)} \mathrm{d}s^2_4 + \mathrm{d}s^2_6 \tag{B.7}$$

where $\mathrm{d}s^2_6$ is the metric on a compact six-dimensional manifold X_6 and y denotes its coordinates. For the ten-dimensional gamma-matrices we take the following decomposition

$$\begin{aligned} \Gamma^\mu &= e^\Delta \gamma^\mu \otimes \mathbf{1} \\ \Gamma^m &= \gamma_{(4)} \otimes \gamma_m \, , \end{aligned} \tag{B.8}$$

where $\mu = 0, \ldots, 3$ and $m = 1, \ldots, 6$ and $\gamma_{(4)} = i\gamma^{0123}$ is the four-dimensional chiral gamma. The six-dimensional gamma-matrices γ_m are hermitian and purely imaginary and we define the six-dimensional chiral gamma as

$$\gamma_{(6)} = -i\gamma^{123456} \, . \tag{B.9}$$

Then ten-dimensional chiral gamma is

$$\Gamma_{(10)} = \prod_{A=0}^{9} \Gamma_A = \gamma_{(4)} \otimes \gamma_{(6)} \, . \tag{B.10}$$

We decompose the ten-dimensional Majorana-Weil spinor ϵ as

$$\epsilon = e^A \zeta_+ \otimes \eta_+ + e^A \zeta_- \otimes \eta_- \, , \tag{B.11}$$

where $\zeta_\pm$ are four-dimensional Weil spinors of positive and negative chirality, while $\eta_\pm$ are six-dimensional Weil spinors of opposite chirality[20]

$$\gamma_4 \zeta_\pm = \pm \zeta_\pm \, , \qquad \gamma \eta_\pm = \pm \eta_\pm \, . \tag{B.12}$$

We assume that the spinor η_+ is globally defined on X_6 and hence defines an $SU(3)$-structure. This is alternatively specified by the two globally-defined forms

$$\Omega_{mnp} = i\eta_-^\dagger \gamma_{mnp} \eta_+ \tag{B.13a}$$
$$J_{mn} = -i\eta_+^\dagger \gamma_{mn} \eta_+ = i\eta_-^\dagger \gamma^{mn} \eta_- \, . \tag{B.13b}$$

These satisfy the usual $SU(3)$ structure relations[21]

$$* \, 1 = \frac{i}{8} \Omega \wedge \overline{\Omega} = -\frac{1}{6} J \wedge J \wedge J \, , \quad \Omega \wedge J = 0 \, . \tag{B.14}$$

We we also need the bilinears

$$\eta_\pm^\dagger \gamma_{mnpq} \eta_\pm = -3 \, J_{[mn} J_{pq]} \tag{B.15a}$$
$$\eta_\pm^\dagger \gamma_{mnprst} \eta_+ = i\eta_\pm^\dagger \epsilon_{mnprst} \gamma_{(6)} \eta_+ \, . \tag{B.15b}$$

We also have the following duality properties

$$* \Omega = i\Omega \, , \quad * \Omega_+ = -\Omega_- \, , \quad * \Omega_- = \Omega_+ \tag{B.16a}$$
$$* J = -\frac{1}{2} J \wedge J \, . \tag{B.16b}$$

The exterior derivatives of the J and Ω can be decomposed into irreducible representation of $SU(3)$

$$\mathrm{d}J = \frac{3}{2} \mathrm{Im}(W_1 \overline{\Omega}) + W_4 \wedge J + W_3 \tag{B.17a}$$
$$\mathrm{d}\Omega = W_1 \, J \wedge J + W_2 \wedge J + \overline{W}_5 \wedge \Omega \, . \tag{B.17b}$$

where W_1, W_2, W_3, W_4 and W_5 are the $SU(3)$ torsion classes. Recall that W_1 is a complex singlet zero-form, W_2 is a complex primitive two-form in the $\mathbf{8} \oplus \overline{\mathbf{8}}$, W_3 is a real primitive $(2,1)$ plus $(1,2)$ form in $\mathbf{6} \oplus \overline{\mathbf{6}}$, W_4 is a real one-form in $\mathbf{3} \oplus \overline{\mathbf{3}}$, and W_5 is a complex one-form in $\mathbf{3} \oplus \overline{\mathbf{3}}$.

[20]We choose the gamma matrices in such a way that $(\eta_+)^* = \eta_-$.

[21]Sometimes conventions are used where the volume form is $*1 = \frac{1}{6} J^3$. This can be understood as a redefinition $J \to -J$.

B.3 G_2 structure in seven dimensions

In this section we recall our conventions for compactifications to three dimensions

$$M_{10} = M_3 \times X_7 \,, \tag{B.18}$$

where M_3 is maximally symmetric three-dimensional spacetime, and X is a seven-dimensional possibly non-compact internal space. The space (B.18) is equipped with the metric

$$\mathrm{d}s^2 = e^{2\Delta(y)}\mathrm{d}s_3^2 + \mathrm{d}s_7^2 \,. \tag{B.19}$$

With this ansatz, the ten-dimensional gamma matrices decompose as[22]

$$\begin{aligned}\Gamma_\mu &= e^{\Delta}\gamma_\mu \otimes \sigma_3 \otimes 1 \\ \Gamma_m &= 1 \otimes \sigma_1 \otimes \gamma_m \,,\end{aligned} \tag{B.21}$$

where $\mu = 0, 1, 2$ and $m = 1, \ldots, 7$. The three-dimensional gamma-matrices γ_μ are real and the seven-dimensional matrices γ_m are chosen purely imaginary. The matrices γ_μ and γ_m satisfy

$$\gamma_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}\,\mathbb{I}_{(2)} \qquad \gamma_{1\ldots 7} = i\mathbb{I}_{(8)} \tag{B.22}$$

with $\epsilon_{012} = 1$ and $\epsilon^{012} = -1$. Then ten-dimensional chirality operator is $\Gamma_{(10)} = -\mathbb{I} \otimes \sigma_2 \otimes \mathbb{I}_{(8)}$.

The supersymmetry parameter ϵ has positive chirality in 10 dimensions and splits as

$$\epsilon = e^{A(y)}\zeta \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix} \otimes \eta \,, \tag{B.23}$$

where η is a a globally defined spinor of unitary norm defining a G_2-structure on X_7.

The G_2-structure can be also defined in terms of bilinears of η, namely a three-form φ and dual four-form $\psi = *\varphi$ which are G_2 singlets. If we take the spinor η to have norm one φ and ψ are given by

$$\varphi_{mnp} = -i\eta^\dagger\gamma_{mnp}\eta \,, \quad \psi_{mnpq} = \eta^\dagger\gamma_{mnpq}\eta \,. \tag{B.24}$$

The exterior derivatives of the G_2 invariant forms can be decomposed into irreducible representation of G_2

$$\mathrm{d}\varphi = 3\,W_{\mathbf{7}} \wedge \varphi + W_{\mathbf{1}}\psi + *W_{\mathbf{27}} \tag{B.25a}$$

$$\mathrm{d}\psi = 4\,W_{\mathbf{7}} \wedge \psi + *W_{\mathbf{14}} \,. \tag{B.25b}$$

where $\{W_{\mathbf{1}}, W_{\mathbf{7}}, W_{\mathbf{14}}, W_{\mathbf{27}}\}$ are the four G_2 torsion classes and the boldface numbers denote the various G_2 representations. In particular $W_{\mathbf{1}}$ is a real scalar, $W_{\mathbf{7}}$ a real vector, $W_{\mathbf{14}}$ and $W_{\mathbf{27}}$ is a symmetric, traceless tensor.

[22] Our conventions for the Pauli-matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{B.20}$$

B.4 Spin(7) structure in eight dimensions

In eight-dimension a globally defined spinor η gives rise to a $Spin(7)$ structure, whose fundamental four-form is given by

$$\Phi_{\alpha\beta\gamma\rho} = \eta^\dagger \gamma_{\alpha\beta\gamma\rho}\eta\,. \tag{B.26}$$

The 16-dimensional spinors split into positive and negative chirality ones. If we choose η to be of negative chirality, a basis for the space of positive chirality spinors is given by $\gamma_\alpha\eta$, while for the negative chirality ones we have

$$\eta\,, \qquad \Pi_{\mathbf{7}}{}^{\alpha\beta}{}_{\gamma\delta}\hat{\gamma}^{\gamma\delta}\eta\,. \tag{B.27}$$

$\Pi_{\mathbf{7}}{}^{\alpha\beta}{}_{\gamma\delta}$ is the projector onto the representation $\mathbf{7}$ and is given by

$$\Pi_{\mathbf{7}\alpha\beta\gamma\delta} = \frac{1}{8}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma} - \Phi_{\alpha\beta\gamma\delta})\,. \tag{B.28}$$

Using the fundamental form one can decompose the action of all other gamma matrices in terms of the spinorial basis [36]

$$\hat{\gamma}_{\alpha\beta}\eta = -\frac{1}{6}\Phi_{\alpha\beta\gamma\rho}\hat{\gamma}^{\gamma\rho}\eta \tag{B.29a}$$

$$\hat{\gamma}_{\alpha\beta\gamma}\eta = \Phi_{\rho\alpha\beta\gamma}\hat{\gamma}^{\rho}\eta \tag{B.29b}$$

$$\hat{\gamma}_{\alpha\beta\gamma\rho}\eta = \Phi_{\alpha\beta\gamma\rho}\eta + \Phi_{[\alpha\beta\gamma}{}^{\kappa}\hat{\gamma}_{\rho]\kappa}\eta \tag{B.29c}$$

$$\hat{\gamma}_{\alpha\beta\gamma\rho\kappa}\eta = 5\,\Phi_{[\alpha\beta\gamma\rho}\hat{\gamma}_{\kappa]}\eta\,. \tag{B.29d}$$

We will need to decompose Spin(7) into G_2. The eight-dimensional gamma matrices $\hat{\gamma}_\alpha$ are given in terms of the seven-dimensional ones by

$$\hat{\gamma}^i = \sigma_2 \otimes \gamma^i \qquad\qquad \hat{\gamma}^8 = -i\sigma_1 \otimes \gamma^8 = \sigma_1 \otimes 1\,, \tag{B.30}$$

with $i = 1, \dots, 7$ and where we defined $\gamma^8 = -\gamma^1 \dots \gamma^7$. Moreover, the fundamental four-form decomposes as

$$\Phi = \varphi \wedge \mathrm{d}x^8 + \psi\,, \tag{B.31}$$

where φ is the fundamental three-form specifying the G_2-structure, and x^8 is the eight-dimensional special direction.

C Derivation of the supersymmetry conditions for G_2 structures in seven dimensions

In this section we discuss how to derive the set of equations (4.8a)-(4.9c). For simplicity of notation we define the combination

$$H^\pm = H \pm \frac{\alpha'}{4}\langle\Sigma\rangle \tag{C.1}$$

whose non-trivial components are

$$H^{\pm}_{mnp} = H_{mnp} \mp \frac{i}{4}\alpha' \Lambda \Sigma_{mnp} \tag{C.2a}$$

$$\tilde{H}^{\pm}_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}\tilde{H}^{\pm} = \epsilon_{\mu\nu\rho}(\tilde{H} \pm \frac{\alpha'}{4}\Lambda C)\,. \tag{C.2b}$$

Let us consider first the dilatino equation (2.2b). Splitting into three- and seven-dimensional indices

$$\Gamma^m \partial_m \phi \epsilon + \frac{1}{2} H^-_{mnp}\Gamma^{mnp}\epsilon + \frac{1}{12} H^-_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\epsilon = 0\,, \tag{C.3}$$

using the decompositions (B.21), (B.23) and (B.22), we can reduce it to an equation on the internal space only

$$\left(\partial\!\!\!/\phi + \frac{1}{2} H\!\!\!/^- - \frac{i}{2} e^{-3\Delta}\tilde{H}^-\right)\eta = 0\,. \tag{C.4}$$

Consider now the gravitino variation (2.2a). Choosing the frame

$$e^A_M = \left\{e^\Delta e_\mu{}^\alpha(x)\,, e_m{}^a(y)\right\} \tag{C.5}$$

for the metric (4.1), it is straightforward to see that the spin-connection[23]

$$\begin{aligned}\omega_M{}^{AB} = {}& \frac{1}{2} e^{N\,A}\left(\partial_M e^B_N - \partial_N e^B_M\right) - \frac{1}{2} e^{N\,B}\left(\partial_M e^A_N - \partial_N e^A_M\right) \\ & - \frac{1}{2} e^{P\,A} e^{Q\,B}\left(\partial_P e_{Q\,C} - \partial_Q e_{P\,C}\right) e^C_M\,,\end{aligned} \tag{C.7}$$

has non-zero components

$$\omega_\mu{}^{\alpha\beta}\,, \quad \omega_\mu{}^{\alpha b} = e^\Delta e_\mu{}^\alpha e^{nb}\partial_n\Delta\,, \quad \omega_m{}^{ab}\,. \tag{C.8}$$

Then, using (4.6), the $M = \mu$ component of the gravitino variation reduces to

$$\delta\psi_\mu = \nabla_\mu\epsilon + \frac{1}{2}\partial_n\Delta\Gamma_\mu\Gamma_n\epsilon + \frac{1}{8} H^+_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon - i\frac{\alpha'}{96}\Lambda\Sigma_{mnp}\Gamma^{mnp}\Gamma_\mu\epsilon = 0\,, \tag{C.9}$$

where we used the properties of gamma matrices to reconstruct the tensor $H^+_{\mu\nu\rho}$ given in (C.2a). The three-dimensional covariant derivative acts as

$$\nabla^{(3)}_\mu \zeta = \frac{\mu}{2}\gamma_\mu\zeta\,. \tag{C.10}$$

[23] We recall that the ten-dimensional Levi-Civita equation for a generic spinor ϵ is given by

$$\nabla_M\epsilon = \partial_M\epsilon + \frac{1}{4}\omega_M{}^{AB}\Gamma_{AB}\,\epsilon\,, \tag{C.6}$$

where $\{A, B, ..\}$ are ten-dimensional frame-indices.

where $-|\mu|^2$ the Anti-de-Sitter scalar curvature. Plugging (C.10) (B.21), (B.23) in (C.9), we find again a purely internal equation

$$\left(\partial\!\!\!/\Delta + i\mu\, e^{-\Delta} - \frac{i}{2} e^{-3\Delta} \tilde{H}^{+} + i\frac{\alpha'}{8}\Lambda \Sigma\!\!\!/\right)\eta = 0\,. \tag{C.11}$$

Similarly the $M = m$ component of the gravitino equation can be written as

$$\delta\psi_m = \nabla_m\epsilon + \frac{1}{8}H^{+}_{mnp}\Gamma^{np}\epsilon - i\frac{\alpha'}{96}\Lambda\Sigma_{npq}\Gamma^{npq}{}_m\epsilon + \frac{\alpha'}{96}\langle\Sigma_{\mu\nu\rho}\rangle\Gamma^{\mu\nu\rho}\Gamma_m\epsilon = 0\,, \tag{C.12}$$

and using (B.21) and (B.23) together with (4.6) we find

$$\nabla_m\eta + \partial_m A + \frac{1}{8}H^{+}_{mnp}\gamma^{np}\eta - i\frac{\alpha'}{96}\Lambda\Sigma_{npq}\gamma^{npq}{}_m\eta + i\frac{\alpha'}{16}C\Lambda\gamma_m\eta = 0\,. \tag{C.13}$$

Notice that combining (C.13) with the condition that η has unit norm we find

$$0 = \nabla_m(\eta^\dagger\eta) = -2\partial_m A + i\frac{\alpha'}{48}\Lambda\,\Sigma_{npq}\psi^{npq}{}_m\,, \tag{C.14}$$

where ψ is the G_2 structure four-form (B.24), and we used the fact that Σ_{mnp} is purely imaginary while Λ is real.

It is convenient to rewrite the equations (C.4), (C.11) and (C.13) as a set of conditions on the forms φ and ψ defining the G_2 structure. The procedure is standard: multiplying each equation by $\eta^\dagger\gamma^m$, $\eta^\dagger\gamma^{mn}$ up to $\eta^\dagger\gamma^{mnpq}$ gives a set of equations for φ and ψ. For the dilatino equation (C.4) we obtain the following set of equations

$$e^{-3\Delta}\tilde{H}^{-} = \frac{1}{3!}H^{-}_{mnp}\varphi^{mnp} \tag{C.15a}$$

$$\partial_m\phi = \frac{1}{12}H^{-}_{npq}\psi^{npq}{}_m \tag{C.15b}$$

$$\partial_{[q}\phi\,\psi_{rstu]} = H^{-}_{m[qr}\,\psi_{stu]}{}^m \tag{C.15c}$$

$$4\partial_{[m}\phi\,\varphi_{npr]} = \frac{1}{2}\left(*H^{-}\right)_{mnpr} - 3\,H^{-}_{w[mn}\,\varphi_{pr]}{}^w - \frac{1}{2}e^{-3\Delta}\tilde{H}^{-}\psi_{mnpr}\,. \tag{C.15d}$$

The last two equations are not independent, they can be obtained from the first two using the properties of the G_2 structure forms. We give them since we will need them to simply expressions later on. More importantly, it should be stressed that that these equations contain only information about the **1** and **7** representations of the G_2-structure.

The external gravitino (C.11) gives

$$\left(-2\mu\, e^{-\Delta} + e^{-3\Delta}\tilde{H}^{+}\right) = i\frac{\alpha'}{24}\Lambda\Sigma_{mnp}\varphi^{mnp} \tag{C.16a}$$

$$\partial_m\Delta = i\frac{\alpha'}{48}\Lambda\Sigma_{npq}\psi^{npq}{}_{m} \tag{C.16b}$$

$$\partial_{[m}\Delta\,\psi_{npqr]} = i\frac{\alpha'}{4}\Lambda\Sigma_{s[mn}\,\psi_{pqr]}{}^{s} \tag{C.16c}$$

$$\begin{aligned} 4\partial_{[m}\Delta\,\varphi_{npq]} = & -i\frac{3}{4}\alpha'\Lambda\,\Sigma_{r[mn}\,\varphi_{pq]}{}^{r} + i\frac{\alpha'}{8}\Lambda\,(*\Sigma)_{mnpq} \\ & -\frac{1}{2}\left(e^{-3\Delta}\tilde{H}^{+} - 2\mu\, e^{-\Delta}\right)\psi_{mnpq}\,. \end{aligned} \tag{C.16d}$$

As before, the last two equations are redundant, but we include them since we will use them to simplify later expressions.

From (C.16b) it follows immediately that, in order to allow for a non-zero condensate, the warp factor need not be constant $\partial_m\Delta \neq 0$. Moreover, combining (C.16b) and (C.14) we find

$$\mathrm{d}A = \frac{1}{2}\mathrm{d}\Delta\,. \tag{C.17}$$

Using equation (C.13), together with the dilatino equations (C.15a)-(C.15d), and the external gravitino (C.16a)-(C.16d) we find that

$$\mathrm{d}\varphi = (2\mathrm{d}\phi - 3\mathrm{d}\Delta)\wedge\varphi - (*H - i\frac{\alpha'}{2}*(\Lambda\Sigma)) + (2\mu\, e^{-\Delta})\,\psi \tag{C.18a}$$

$$\mathrm{d}\psi = (2\mathrm{d}\phi - 2\mathrm{d}\Delta)\wedge\psi\,. \tag{C.18b}$$

D Derivation of the supersymmetry conditions for $SU(3)$-structures in six dimensions

In this section we derive the set of equations (3.8a)-(3.8c). We use the conventions appendix B.2 for gamma matrices, spinors and SU(3) structure. Splitting into four and six-dimensional indices the supersymmetry variations (2.2a)-(2.2c) reduce to

$$\nabla_\mu\epsilon + \frac{1}{2}\Gamma_\mu\partial_m\Delta\Gamma^m\epsilon + \frac{\alpha'}{96}\langle\Sigma\rangle_{mnp}\Gamma^{mnp}\Gamma_\mu\epsilon = 0 \tag{D.1a}$$

$$\nabla_m\epsilon + \frac{1}{8}H_{mnp}\Gamma^{np}\epsilon + \frac{\alpha'}{96}\langle\Sigma\rangle_{rst}\Gamma^{rst}\Gamma_m\epsilon = 0 \tag{D.1b}$$

$$\left(\Gamma^m\partial_m\phi + \frac{1}{12}H_{mnp}\Gamma^{mnp} - \frac{\alpha'}{48}\langle\Sigma\rangle_{mnp}\Gamma^{mnp}\right)\epsilon = 0\,. \tag{D.1c}$$

The only non-trivial components of the three-form flux H and the gaugino three-form Σ are purely internal and we define

$$H^{\pm}_{mnp} = H_{mnp} \pm \frac{\alpha'}{4}\langle\Sigma_{mnp}\rangle\,. \tag{D.2}$$

Consider first the external gravitino (D.1a). The ten-dimensional spin connection decomposes as in (C.8) and the four-dimensional covariant derivative is

$$\nabla^{(4)}_\mu \zeta_+ = \frac{\mu}{2}\gamma_\mu \zeta_- \,, \tag{D.3}$$

where μ is a complex parameter related and the cosmological constant is equal to $-|\mu|^2$. ζ_+ is the four-dimensional susy parameter in (B.11). Using the splittings (B.8) the equation reduces to

$$\mu\, e^{-\Delta}\, \eta_+ - \partial\!\!\!/ \Delta \eta_- + \frac{\alpha'}{8}\langle \Sigma\!\!\!/ \rangle \eta_- = 0\,. \tag{D.4}$$

Multiplying (D.4) by $\eta_+^\dagger$ we find

$$\mu e^{-\Delta} = i\frac{\alpha'}{8}\langle \Sigma \rangle \lrcorner \overline{\Omega}\,, \tag{D.5}$$

while multiplying by $\eta_-^\dagger \gamma_r$ and taking the real part gives

$$\partial_m \Delta = \frac{\alpha'}{16}\langle \Sigma \rangle^{rst} J_{[rs} J_{tm]}\,, \tag{D.6}$$

which can also be written as

$$\frac{\alpha'}{4} * \langle \Sigma \rangle \wedge J = \mathrm{d}\Delta \wedge J \wedge J\,. \tag{D.7}$$

This exhausts the set of independent equations that can be derived from (D.1a). It is however useful to consider the equation obtained multiplying (D.4) by $\eta_-^\dagger \gamma_{mnp}$ (and taking the real real part) and by $\eta_+^\dagger \gamma_{mnpq}$

$$\frac{3\alpha'}{8}\langle \Sigma \rangle^t{}_{[mn} J_{p]t} = \frac{\alpha'}{8}\left(*\langle \Sigma \rangle\right)_{mnp} + e^{-\Delta}\,\mathrm{Re}\left(\mu\, \Omega_{mnp}\right) - 3\partial_{[m}\Delta\, J_{np]}\,, \tag{D.8a}$$

$$-\frac{3\alpha'}{8}\langle \Sigma \rangle^s{}_{[mn} \Omega_{pq]s} = -2\partial_{[m}\Delta\, \Omega_{npq]} - i\frac{3}{2}e^{-\Delta}\overline{\mu}\, J_{[mn} J_{pq]}\,. \tag{D.8b}$$

With the splitting (B.8) and (B.11) the dilatino equation (D.1c) becomes

$$\left(\partial\!\!\!/ \phi + \frac{1}{2} H\!\!\!/^{-}\right)\eta_+ = 0\,. \tag{D.9}$$

As for the external gravitino, we can derive form equations by multiplying by the basis spinors $\eta_-^\dagger$, $\eta_+^\dagger \gamma_m$

$$H^- \lrcorner \Omega = 0\,, \tag{D.10a}$$

$$\partial_m \phi = -\frac{1}{4} H^{-rst} J_{[rs} J_{tm]}\,. \tag{D.10b}$$

The second equation can also be written as

$$*H^{-}\wedge J = -J\wedge J\wedge \mathrm{d}\phi\,. \tag{D.11}$$

Moreover multiplying (D.9) by $\eta_+^\dagger\gamma_{mnp}$ and $\eta_-^\dagger\gamma_{mnpq}$ we find the two additional equations

$$H^{-t}{}_{[mn}J_{p]t} = 2\partial_{[m}\phi\, J_{np]} + \frac{1}{3}\left(*H^{-}\right)_{mnp}\,, \tag{D.12a}$$

$$\frac{3}{2}H^{-s}{}_{[mn}\Omega_{pq]s} = -2\partial_{[m}\phi\,\Omega_{npq]}\,. \tag{D.12b}$$

We now turn to the internal gravitino equation, which reduces to

$$\nabla_m\eta_+ + \partial_m A\eta_+ + \frac{1}{8}H^{+}_{mnp}\gamma^{np}\eta_+ + \frac{\alpha'}{96}\langle\Sigma\rangle_{rst}\gamma^{rst}{}_m\eta_+ = 0\,. \tag{D.13}$$

As in the seven dimensional case, we insist that the internal spinors are normalised unit norm, from which it follows that

$$0 = \nabla_m\left(\eta_+^\dagger\eta_+\right) = -2\partial_m A + \frac{\alpha'}{16}\langle\Sigma\rangle_{rst}J^{rs}J^{t}{}_m\,. \tag{D.14}$$

Using this and (D.6), we find

$$\mathrm{d}\Delta = 2\mathrm{d}A\,. \tag{D.15}$$

We can use (D.13) and the definitions (B.13b) and (B.13a) to compute the exterior derivatives of J and Ω. For the J we find

$$\nabla_{[p}J_{mn]} = -\partial_{[p}\Delta\, J_{mn]} + H^{+t}{}_{[mn}J_{p]t} - \frac{\alpha'}{8}\langle\Sigma\rangle^{t}_{[mn}J_{p]t} + \frac{\alpha'}{8}(*\langle\Sigma\rangle)_{mnp}\,. \tag{D.16}$$

Using (D.8a) and (D.12a), (D.16) reduces to

$$\mathrm{d}J = (2\mathrm{d}\phi - 4\mathrm{d}\Delta)\wedge J + *\overline{H} + 3\,e^{-\Delta}\mathrm{Re}\left(\mu\,\Omega\right)\,, \tag{D.17}$$

where $\overline{H} = H + \frac{\alpha'}{2}\langle\Sigma\rangle$

We next compute the exterior derivative of Ω to find

$$\nabla_{[m}\Omega_{npq]} = -\partial_{[m}\Delta\,\Omega_{npq]} - \frac{3}{2}H^{-s}{}_{[mn}\Omega_{pq]s} - \frac{3\alpha'}{8}\langle\Sigma\rangle^{s}{}_{[mn}\Omega_{pq]s}\,, \tag{D.18}$$

which using (D.8b) and (D.12b) becomes

$$\mathrm{d}\Omega = (2\mathrm{d}\phi - 3\mathrm{d}\Delta)\wedge\Omega - ie^{-\Delta}\,\overline{\mu}\,J\wedge J\,. \tag{D.19}$$

Note that equations (D.10a) can be derived by contracting (D.17) with Ω and using (D.5) and (D.19). Furthermore, by wedging (D.17) with J and using (D.7) and (D.11) we find

$$J\wedge\mathrm{d}J = (\mathrm{d}\phi - \mathrm{d}\Delta)\wedge J\wedge J\,. \tag{D.20}$$

To summarise, the set of indepentent four dimensional BPS equations are

$$\mu e^{-\Delta} = i\frac{\alpha'}{8}\langle\Sigma\rangle \lrcorner \overline{\Omega}\,, \tag{D.21}$$

with the differential conditions

$$\frac{\alpha'}{4} * \langle\Sigma\rangle \wedge J = \mathrm{d}\Delta \wedge J \wedge J \tag{D.22a}$$

$$J \wedge \mathrm{d}J = (\mathrm{d}\phi - \mathrm{d}\Delta) \wedge J \wedge J \tag{D.22b}$$

$$\mathrm{d}J = (2\mathrm{d}\phi - 4\mathrm{d}\Delta) \wedge J + *\overline{H} + 3\,e^{-\Delta}\mathrm{Re}\,(\mu\,\Omega) \tag{D.22c}$$

$$\mathrm{d}\Omega = (2\mathrm{d}\phi - 3\mathrm{d}\Delta) \wedge \Omega - ie^{-\Delta}\,\overline{\mu}\,J \wedge J\,. \tag{D.22d}$$

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