

Abelian Z-theory: NLSM amplitudes and alpha'-corrections from the open string

John Joseph M. Carrasco, Carlos R. Mafra, Oliver Schlotterer

▶ **To cite this version:**

John Joseph M. Carrasco, Carlos R. Mafra, Oliver Schlotterer. Abelian Z-theory: NLSM amplitudes and alpha'-corrections from the open string. 2017. cea-01426644

HAL Id: cea-01426644
https://cea.hal.science/cea-01426644

Preprint submitted on 4 Jan 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:1608.02569v1 [hep-th] 8 Aug 2016

Abelian Z-theory: NLSM amplitudes and α'-corrections from the open string

John Joseph M. Carrasco[a,c], Carlos R. Mafra[b,c] and Oliver Schlotterer[d]

[a] *Institut de Physique Théorique, CEA–Saclay, F–91191 Gif-sur-Yvette cedex, France*

[b] *STAG Research Centre and Mathematical Sciences, University of Southampton, UK*

[c] *Institute for Advanced Study, School of Natural Sciences, Einstein Drive, Princeton, NJ 08540, USA*

[d] *Max–Planck–Institut für Gravitationsphysik, Albert–Einstein–Institut, Am Mühlenberg 1, 14476 Potsdam, Germany*

In this paper we derive the tree-level S-matrix of the effective theory of Goldstone bosons known as the non-linear sigma model (NLSM) from string theory. This novel connection relies on a recent realization of tree-level open-superstring S-matrix predictions as a *double copy* of super-Yang–Mills theory with Z-theory — the collection of putative scalar effective field theories encoding all the α'-dependence of the open superstring. Here we identify the color-ordered amplitudes of the NLSM as the low-energy limit of abelian Z-theory. This realization also provides natural higher-derivative corrections to the NLSM amplitudes arising from higher powers of α' in the abelian Z-theory amplitudes, and through double copy also to Born–Infeld and Volkov–Akulov theories. The Kleiss–Kuijf and Bern–Carrasco–Johansson relations obeyed by Z-theory amplitudes thereby apply to all α'-corrections of the NLSM. As such we naturally obtain a cubic-graph parameterization for the abelian Z-theory predictions whose kinematic numerators obey the duality between color and kinematics to all orders in α'.

August 2016

[a] email: john-joseph.carrasco@cea.fr

[b] email: c.r.mafra@soton.ac.uk

[d] email: olivers@aei.mpg.de

Contents

1 Introduction 2

2 Review 4

2.1. Double-copy construction 4

2.2. Z-theory amplitudes 7

2.3. The field-theory limit 9

2.4. Abelian limit 9

2.5. α'-expansion 10

3 NLSM amplitudes from string theory 11

4 Higher-derivative corrections to the NLSM 12

4.1. Four points 12

4.2. Six points 13

4.3. All order-systematics 14

4.4. Simplifications in the odd zeta sector 18

5 Color-kinematic satisfying numerators 18

6 Conclusions and outlook 23

A The six-point $\alpha'^3\zeta_3$ correction to the NLSM 24

1. Introduction

It is well known that string theory provides a powerful and unified framework to study the sea of field theories that arise in the limit when the size of the strings approaches zero; some of the most celebrated examples being the maximally supersymmetric super-Yang–Mills and supergravity theories [1].

Since the gluon and graviton belong to the massless excitations of the string, their scattering amplitudes naturally emerge from low-energy limits of the string-theory S-matrix. By the same token, one might suspect that scattering amplitudes of field theories absent in the (naive) string spectrum may be difficult to study within string theory. As we will see, such an expectation is surpassed by a long-hidden double-copy structure, secretly and deftly encoded in open-string theory—a structure which applies universally to a broad set of point-like quantum field theories [2]. Perturbative predictions in double-copy quantum field theories [3,4,5] can be completely fixed by knowing the predictions of two possibly distinct input theories[1].

In recent work by Broedel, Stieberger, and one of the current authors [2] it was demonstrated that open-superstring amplitudes [7,8] can be understood as a double copy of color-stripped Yang–Mills amplitudes and certain Z-functions which behave like scalar partial amplitudes. These Z-functions are iterated integrals over the boundary of a disk worldsheet and naturally incorporate two notions of ordering. One ordering, $Q = (q_1, q_2, \ldots, q_n)$, refers to the integrand, and one to the integration domain $P = (p_1, p_2, \ldots, p_n)$, (see section 2.2)

$$Z_P(q_1, q_2, \ldots, q_n) \equiv {\alpha'}^{n-3} \int_{D(P)} \frac{\mathrm{d}z_1 \, \mathrm{d}z_2 \, \cdots \, \mathrm{d}z_n}{\mathrm{vol}(SL(2,\mathbb{R}))} \frac{\prod_{i<j}^{n} |z_{ij}|^{\alpha' s_{ij}}}{z_{q_1 q_2} z_{q_2 q_3} \ldots z_{q_{n-1} q_n} z_{q_n q_1}} \,. \tag{1.1}$$

It was shown in ref. [2] that these doubly-ordered functions obey Kleiss–Kujif (KK) [9] and Bern–Carrasco–Johansson (BCJ) [4] field-theory amplitude relations along its integrand ordering Q, and the string-theory monodromy relations [10,11] along the integration domain P.

Given the field-theory amplitude relations satisfied by the Z-functions along their integrand ordering, it is natural to take them seriously as color-stripped amplitudes in some set of color-kinematics satisfying scalar effective field theories. These scalar amplitudes in

[1] For example, scattering in the $\mathcal{N} = 5$ supergravity theory is completely determined by a double copy consisting of color-stripped amplitudes of $\mathcal{N} = 4$ and $\mathcal{N} = 1$ super Yang–Mills theories. Note that double copy holds at the integrand level [6] for multiloop amplitudes.

turn retain the fingerprints of string-theory relevance through their second ordering governed by string monodromy relations. We will refer to this set of monodromy related field theories collectively as *Z-theory*. In the time-honored tradition of periodically reflecting upon the venerable query, "What is string theory?", we find ourselves struck by the ubiquity of double-copy constructions – not only in unifying open- and closed-string predictions a la Kawai, Lewellen and Tye (KLT) [3], but also in constraining the effective-field-theory (EFT) modifications to super-Yang–Mills resulting in open superstring tree-level predictions. We suspect a strategic path forward may arise from the more modest question driving this current manuscript, "What is Z-theory?".

A critical first clue was given in [2], where it was demonstrated that the $\alpha' \to 0$ limit of the Z-functions lands on the inverse of the field-theory KLT matrix. This same limit was later recognized to correspond to the (double-partial) tree-level amplitudes of a scalar bi-adjoint theory [12]. Therefore, even though the string spectrum does not include bi-adjoint scalars, string tree-level amplitudes, through Z-amplitudes, contain all their tree-level predictions.

In this paper we will identify a novel correspondence: the tree-level S-matrix of Goldstone bosons described by the nonlinear sigma model (NLSM) can also be obtained from open-string theory. The color-ordered amplitudes of the NLSM emerge from abelianizing Z-theory amplitudes along their second string-monodromy related ordering, and taking the surviving low-energy limit. Specifically we find that NLSM amplitudes are given by[2]

$$A_{\rm NLSM}(1,2,\ldots,n) = \lim_{\alpha'\to 0} (\alpha')^{2-n} \int_{\mathbb{R}^n} \frac{{\rm d}z_1\ {\rm d}z_2\ \cdots\ {\rm d}z_n}{{\rm vol}(SL(2,\mathbb{R}))} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{z_{12}z_{23}\ldots z_{n-1,n}z_{n,1}}\ , \tag{1.2}$$

where the color-ordering of the NLSM legs is reflected by the integrand $(z_{12}z_{23}\ldots z_{n,1})^{-1}$. At lowest multiplicities, for example, (1.2) yields

$$A_{\rm NLSM}(1,2,3,4) = \pi^2(s_{12}+s_{23})\,, \tag{1.3}$$

$$A_{\rm NLSM}(1,2,\ldots,6) = \pi^4\Big[s_{12} - \frac{1}{2}\frac{(s_{12}+s_{23})(s_{45}+s_{56})}{s_{123}} + {\rm cyclic}(1,2,3,4,5,6)\Big]\,. \tag{1.4}$$

As will be explained below, the string integrals in (1.2) appear in the n-point amplitude of massless open-string states obtained in [7,8] upon specialization to *abelian* gauge bosons.

[2] In order to avoid cluttering of factors of two, we have rescaled α' such that the standard open-string conventions are recovered by setting $\alpha' \to 2\alpha'$ in the equations of this work.

In contrast to the doubly-partial amplitudes of bi-adjoint scalars, the NLSM amplitudes arise from nonzero orders $(\alpha')^{n-2}$ singled out by the leading low-energy contribution to n-point disk integrals in absence of an ordering in the integration domain. In addition, from the subleading terms in the α'-expansion of (1.2), one naturally derives stringy higher-derivative corrections to the NLSM amplitudes. This summarizes our primary result for which we report on explicit calculations through nine points. We additionally provide a general approach to generating associated local color-kinematic satisfying numerators from these amplitudes. Along the way we provide a nice recursive form of the KLT matrix $S[\cdot|\cdot]_1$, inspired by ref. [13], and propose a strikingly simple form for all-multiplicity $n = 2k$ NLSM color-kinematic satisfying master numerators

$$N_{1|\rho(23\ldots 2k-1)|2k} = (-1)^k S[\rho(23\ldots 2k{-}1)|\rho(23\ldots 2k{-}1)]_1 \ . \tag{1.5}$$

This manuscript is organized as follows: After a review of disk integrals in section 2, we state and prove our main result (1.2) in section 3. Examples and the systematics of higher-derivative corrections to the NLSM are elaborated in section 4, and the construction of explicit BCJ numerators for the α'-corrected NLSM can be found in section 5.

2. Review

2.1. Double-copy construction

Due to the seminal work [3] of KLT, it has long been recognized that the tree-level predictions of open strings entirely encode[3] the predictions of tree-level closed strings. The amplitudes admit a representation in the form of the sum over products of color-stripped gauge (open-string) amplitudes. This made a particular impact in its low-energy limit in the study of field-theory gravitational scattering amplitudes in the 90's where various closed-form representations were identified [16,17].

[3] Given the all-multiplicity tree-level relations between gravity and gauge theory, one might rightfully ask if classical general-relativity solutions are encoded in classical gauge-theory solutions. Answering this question is an active area of investigation see e.g. refs. [14] for explicit solution relationships, and refs. [15] for considerations of classical symmetries and duality-groups, as well as references therein.

Admittedly, many properties of the KLT double-copy construction were mysterious (including the emergence of the necessary permutation symmetry of the gravity amplitudes), and the original formulation of sums over shuffle-ordered products of permutations could be dizzying, especially at higher multiplicities. Nevertheless, this approach, as it would factor over the state-sum of unitarity cuts, proved critical for gathering information about the spectacular UV behavior of the maximally supersymmetric supergravity theory through four-loops [18]. In the process of such exploration, the field of scattering amplitudes acquired a new set of insights relevant to the double-copy story when Bern, Johansson, and one of the current authors (BCJ) observed [4] that there was a very direct path to gravity-theory predictions, where the double-copy construction could be made manifest graph by graph. This has particular value at the multiloop integrand level [6], where integrand labeling ambiguities can create obstacles for realizing generalized-gauge-invariant double-copy relationships, outside of certain kinematic limits like unitarity cuts.

The BCJ double-copy approach relies critically on the realization [4] that gauge-theory predictions (and their supersymmetric partners) admit a color-kinematic duality satisfying representation (color-dual representations are ones where graph by graph color-weights and kinematic weights obey the same generic algebraic properties). The existence of such color-dual representations resulted in the discovery of new relations between color-ordered amplitudes known now as the BCJ relations. With such a dual representation, color-factors could be consistently replaced[4] by kinematic weights, recycling a small set of kinematic predictions to describe a wide variety of theories [6]. Additionally, color-dual kinematic weights could be solved for in terms of color-ordered amplitudes, thus allowing for the generation of generalized KLT relations.

While a Lagrangian understanding of the organizing principle is only available in the four-dimensional self-dual case [20], many theories, including the NLSM [21], in a variety of spacetime dimensions, admit the duality between color and kinematics, and associated double-copy construction [22]. This new perspective on field-theory predictions has proven critical in developing aspects of our understanding of non-planar scattering amplitudes over the last decade, both formally as well as through practical reach in computation. Jacobi relations drastically constrain the independent information relevant to a given scattering calculation. For instance, the closely related double-copy constructions of multiloop gravity

[4] Established to some finite multiplicity at tree-level in [4], but later proven via BCFW and the KLT relations in [19].

amplitudes [6] has allowed many explicit calculations that can probe the possible onset of ultraviolet divergences in supergravity theories [23].

String theory continues to provide key insights probing the color-kinematics duality and its associated representation of the double-copy construction: from the powerful proof [10,11] of the $(n-3)!$-basis of Yang–Mills tree amplitudes as the low-energy limit of the related string monodromy relations[5], to the elegant construction of explicit local tree-level numerators [25,26], to the construction of string-inspired BCJ numerators [27] at loop level. The fact that the BCJ-duality also applies to the NLSM [21], can now be appreciated either as a *consequence* of the BCJ relations satisfied [2] by Z-theory as in (1.2), or as a *requirement* for the NLSM to be able to participate in Z-theory's construction of the open superstring. Following the recent result of Du and Fu [13] who present an elegant closed-form construction of local color-kinematics satisfying numerators in the NLSM, we will discuss its applicability to all orders in α'.

It is worth mentioning a related [28] approach to constructing double-copy representations known as the Cachazo–He–Yuan (CHY) formalism [5,29,12], which generalizes the four-dimensional connected prescription of Roiban, Spradlin, Volovich, and Witten [30,31] to general dimensions. Similar to string theory, scattering amplitudes in the CHY framework are derived from punctured Riemann surfaces[6]. Exploiting a CHY description of Yang–Mills theory and the NLSM model, ref. [36] offered the first double-copy realization of self-dual Born–Infeld [37] scattering amplitudes. The idea that there can exist a duality between electric and magnetic field densities is as old as gauge theory. Satisfied by sourceless Maxwell electrodynamics, this natural duality has inspired analysis and generalizations that have been key to understanding aspects of supersymmetry, symmetry breaking, and string theory, starting with perhaps most famously the Born–Infeld non-linear generalization of electromagnetics [38]. The emergence of duality invariance in the form of Born–Infeld scattering due to a double-copy interplay between YM and the low-energy limit of abelian Z-theory is remarkable. In concordance with the structure of open-string amplitudes given as a double copy between Yang–Mills constituents and Z-theory disk-integrals [2], the double-copy representation of Born–Infeld amplitudes as its

[5] See [24] for a recent higher-loop generalization.

[6] The CHY integrands for gluon and graviton scattering have direct antecedents in the heterotic string and the type-II superstring, respectively [32,33]. Also see ref. [34] for a careful discussion of subtle differences between CHY (tree-level) integrands for Einstein–Yang–Mills [35] and correlation functions of the heterotic string.

surviving abelian low-energy limit serves as a key check that our observation (1.2) holds to all multiplicities, beyond the explicit verification at $n \leq 9$ points we report on here.

2.2. Z-theory amplitudes

Tree-level scattering amplitudes of open-string states are determined by iterated integrals on the boundary of a disk worldsheet. Massless n-point amplitudes of the open superstring [7] and conjecturally those of the open bosonic string [39] possess cyclic integrands of the following form:

$$Z_P(q_1, q_2, \ldots, q_n) \equiv \alpha'^{n-3} \int\limits_{D(P)} \frac{\mathrm{d}z_1 \, \mathrm{d}z_2 \, \cdots \, \mathrm{d}z_n}{\mathrm{vol}(SL(2,\mathbb{R}))} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{z_{q_1 q_2} z_{q_2 q_3} \ldots z_{q_{n-1} q_n} z_{q_n q_1}} \, . \tag{2.1}$$

The universal and permutation-symmetric Koba-Nielsen factor $\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}$ built from differences $z_{ij} \equiv z_i - z_j$ is accompanied by a cyclic product of propagators $z_{q_i q_{i+1}}^{-1}$ indicated by the labels $(q_1, q_2, \ldots, q_n)$ on the left hand side. The additional subscript $P \equiv p_1 p_2 \ldots p_n$ encodes the ordering for the iterated integrals,

$$D(P) \equiv \{(z_1, z_2, \ldots, z_n) \in \mathbb{R}^n \mid -\infty < z_{p_1} < z_{p_2} < \ldots < z_{p_n} < \infty\} \, , \tag{2.2}$$

and thereby the accompanying color trace over gauge-group generators $t_{p_1} t_{p_2} \ldots t_{p_n}$. The inverse volume of the conformal Killing group of the disk instructs one to drop any three variables of integration z_i, z_j, z_k and to compensate with a Jacobian $z_{ij} z_{ik} z_{jk}$, e.g.

$$\int\limits_{D(12\ldots n)} \frac{\mathrm{d}z_1 \, \mathrm{d}z_2 \, \cdots \, \mathrm{d}z_n}{\mathrm{vol}(SL(2,\mathbb{R}))} = z_{1,n-1} z_{1,n} z_{n-1,n} \int\limits_{z_1}^{z_{n-1}} \mathrm{d}z_{n-2} \int\limits_{z_1}^{z_{n-2}} \mathrm{d}z_{n-3} \ldots \int\limits_{z_1}^{z_4} \mathrm{d}z_3 \int\limits_{z_1}^{z_3} \mathrm{d}z_2 \, . \tag{2.3}$$

The unintegrated variables can then be fixed to any real values such as $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$. Finally, the Mandelstam variables are defined in terms of lightlike momenta k_i:

$$s_{ij} \equiv k_i \cdot k_j, \quad s_{i_1 i_2 \ldots i_p} \equiv \frac{1}{2}(k_{i_1} + k_{i_2} + \cdots + k_{i_p})^2 \, . \tag{2.4}$$

Their appearance in the open-superstring amplitudes leads us to view the integrals (2.1) as defining the tree-level S-matrix of Z-theory, the collection of putative scalar effective field theories that incorporate all the α'-corrections on a disk worldsheet.

2.2.1. Symmetries

For a fixed choice of the integration domain $D(P)$, the integrals (2.1) associated with different permutations of $q_1, q_2, \ldots, q_n$ satisfy the same relations as color-ordered YM amplitudes. Apart from the obvious cyclic symmetry and reflection parity,

$$Z_P(q_1, q_2, q_3 \ldots, q_n) = Z_P(q_2, q_3, \ldots, q_n, q_1) \tag{2.5}$$

$$Z_P(q_1, q_2, \ldots, q_n) = (-1)^n Z_P(q_n, \ldots, q_2, q_1)\,, \tag{2.6}$$

partial fraction rearrangements of the integrand and integration-by-parts relations can be written as [2],

$$0 = Z_P(1, A, n, B) - (-1)^{|B|} \sum_{\sigma \in A \sqcup\!\sqcup \tilde{B}} Z_P(1, \sigma, n)\,, \quad \forall A, B \tag{2.7}$$

$$0 = \sum_{j=2}^{n-1} (k_{q_1} \cdot k_{q_2 q_3 \ldots q_j}) Z_P(q_2, q_3, \ldots, q_j, q_1, q_{j+1}, \ldots, q_n)\ , \tag{2.8}$$

where $A = a_1 a_2 \ldots a_{|A|}$ and $B = b_1 b_2 \ldots b_{|B|}$ represent arbitrary sets of particle labels, and $\tilde{B}$ denotes the transpose of the set B. Furthermore, the shuffle product is defined by [40]

$$\emptyset \sqcup\!\sqcup A = A \sqcup\!\sqcup \emptyset = A, \qquad A \sqcup\!\sqcup B \equiv a_1(a_2 \ldots a_{|A|} \sqcup\!\sqcup B) + b_1(b_2 \ldots b_{|B|} \sqcup\!\sqcup A)\,. \tag{2.9}$$

Note that (2.7) and (2.8) take exactly the same form as the KK relations [9] and the BCJ relations [4] among $A_{\rm YM}(q_1, q_2, \ldots, q_n)$ (see also [41]), which are well-known to yield an $(n{-}3)!$-element basis. Analogous relations among disk integrals (2.1) with the same integrands $Q = (q_1, q_2, \ldots, q_n)$ but different orders $P = p_1 p_2 \ldots p_n$ include cyclicity and reflection

$$Z_{p_1 p_2 \ldots p_n}(Q) = Z_{p_2 p_3 \ldots p_n p_1}(Q) = (-1)^n Z_{p_n \ldots p_2 p_1}(Q)\ , \tag{2.10}$$

and additional relations follow from monodromy properties of the worldsheet [10,11]

$$0 = \sum_{j=2}^{n-1} \exp\big[i\pi\alpha'(k_{p_1} \cdot k_{p_2 p_3 \ldots p_j})\big] Z_{p_2 p_3 \ldots p_j p_1 p_{j+1} \ldots p_n}(Q)\ , \tag{2.11}$$

which also yield an $(n{-}3)!$ basis of integration domains.

These symmetry properties underpin our viewpoint on (2.1) as the doubly-partial amplitudes of Z-theory which by (2.7) and (2.8) satisfy the color-kinematics duality in the integrand orderings to all orders in α'. The additional α'-dependence in the relations (2.11) among the integration domain orderings, on the other hand, imprint the monodromy properties of the disk worldsheets on the S-matrix of Z-theory.

2.3. The field-theory limit

In the field-theory limit $\alpha' \to 0$, the disk integrals (2.1) yield kinematic poles that correspond to the propagators of cubic diagrams [42,43]. As a convenient tool to describe the pole structure, we recall the theory of a bi-adjoint scalar $\phi \equiv \phi_{a|b} t_a \otimes \tilde{t}_b$ with a cubic interaction

$$\mathcal{L}_{\rm bi-adjoint} = \frac{1}{2}\partial_m \phi_{a|b} \partial^m \phi_{a|b} + \frac{1}{3} f_{acg} \tilde{f}_{bdh} \phi_{a|b} \phi_{c|d} \phi_{g|h} \ . \tag{2.12}$$

Doubly-partial amplitudes $m[P|Q]$ are defined to track the traces of gauge-group generators t_a and $\tilde{t}_b$ in the tree amplitudes of the above scalar theory [12],

$$M_{\phi^3} = \sum_{\sigma,\rho \in S_{n-1}} {\rm Tr}(t_1 t_{\sigma(2)} \dots t_{\sigma(n)}) {\rm Tr}(\tilde{t}_1 \tilde{t}_{\rho(2)} \dots \tilde{t}_{\rho(n)})\, m[1,\sigma(2,\dots,n)|1,\rho(2,\dots,n)] \,. \tag{2.13}$$

Following the all-multiplicity techniques of [8], the field-theory limits of disk integrals have been written in terms of double-partial amplitudes as [12],

$$\lim_{\alpha' \to 0} Z_P(Q) = m[P|Q] \,, \tag{2.14}$$

identifying the bi-adjoint scalar theory (2.12) as the low-energy limit of Z-theory, see [26] for an efficient Berends–Giele implementation of (2.14).

2.4. Abelian limit

Recall that the color-dressed n-point tree amplitude of the open superstring is given by

$$M_{\rm gluon}(\alpha') = \sum_{\sigma \in S_{n-1}} {\rm Tr}\big[\, t_{a_1}\, t_{a_{\sigma(2)}} \cdots t_{a_{\sigma(n)}} \,\big]\, A_{\rm gluon}(1,\sigma(2,\dots,n-1,n);\alpha') \ , \tag{2.15}$$

where the color-stripped amplitudes determined in [7] were later identified in [2] to exhibit a KLT-like structure

$$\begin{aligned} A_{\rm gluon}(1,\sigma(2,3,\dots,n);\alpha') = \sum_{\rho,\tau \in S_{n-3}} & Z_{1\sigma(2,3,\dots,n)}(1,\rho(2,3,\dots,n-2),n,n-1) \\ & \times S[\rho(23\dots n-2)|\tau(23\dots n-2)]_1 A_{\rm YM}(1,\tau(2,3,\dots,n-2),n-1,n) \,. \end{aligned} \tag{2.16}$$

The symmetric matrix $S[\rho|\tau]_1$ in (2.16) encodes the field-theory limit ($\alpha' \to 0$) of KLT relations [3] to all multiplicities [17] and admits the following recursive definition[7],

$$S[A,j|B,j,C]_i = (k_{iB}\cdot k_j) S[A|B,C]_i, \qquad S[\emptyset|\emptyset]_i \equiv 1 \,, \tag{2.17}$$

[7] The field theory KLT matrix was originally defined in non-symmetric form in [17], later rewritten in [44,41] with the symmetric form used in [2]. Inspired by equation (3.8) of [13] we arrived at the novel recursive definition (2.17), which generalizes to all orders in α' in an obvious manner [45].

where A, B and C are arbitrary multiparticle labels such that $|A| = |B| + |C|$ and the multiparticle momentum is defined by $k_{iB} \equiv k_i + k_{b_1} + \cdots + k_{b_{|B|}}$. For example $S[2,3,4|2,4,3]_1 = (k_{12}{\cdot}k_4)S[2,3|2,3]_1 = (k_{12}{\cdot}k_4)(k_{12}{\cdot}k_3)S[2|2]_1 = (k_{12}{\cdot}k_4)(k_{12}{\cdot}k_3)(k_1{\cdot}k_2)$. The doubly-partial amplitudes (2.13) of bi-adjoint scalars furnish the inverse of this matrix [12].

Note that the color-ordering σ of the string amplitude in (2.16) enters globally as the integration domain of the $Z_{1\sigma(2,3,\ldots,n)}(\ldots)$ and does not interfere with the permutation sums over ρ and τ in (2.16). Accordingly, its specialization to abelian gauge bosons (henceforth referred to as photons) is obtained by setting all the color traces to unity and yields,

$$M_{\rm photon}(\alpha') = \sum_{\rho,\tau\in S_{n-3}} Z_\times(1,\rho(2,3,\ldots,n-2),n,n-1) \times S[\rho(23\ldots n-2)|\tau(23\ldots n-2)]_1 A_{\rm YM}(1,\tau(2,3,\ldots,n-2),n-1,n)\,, \tag{2.18}$$

where

$$Z_\times(q_1,q_2,\ldots,q_n) \equiv \sum_{\sigma\in S_{n-1}} Z_{1\sigma(2,3,\ldots,n)}(q_1,q_2,\ldots,q_n) \tag{2.19}$$

defines the *abelian disk integrals* or the partial amplitude of abelian Z-theory whose α'-expansion will be discussed below.

2.5. α'-expansion

The α'-expansion of the disk integrals (2.1) gives rise to multiple zeta values (MZVs),

$$\zeta_{n_1,n_2,\ldots,n_r} \equiv \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1}k_2^{-n_2}\ldots k_r^{-n_r}\ , \qquad n_r\geq 2\ , \tag{2.20}$$

which are characterized by their weight $w = n_1+n_2+\ldots+n_r$ and depth r. More precisely, the order $(\alpha')^w$ of disk integrals is accompanied by products of MZVs with total weight w (where the weight is understood to be additive in products of MZVs); a property known as *uniform transcendentality*. This has been discussed in the literature of both mathematics [46] and physics [8,47,48] and can for instance be proven by the recursive construction[8] of disk integrals using the Drinfeld associator [50].

[8] At multiplicities five, six and seven, explicit results for the leading orders are available for download on [49], along with the building blocks for eight and nine points.

The combination of all integration orders to obtain abelian disk integrals projects out a variety of MZVs from the α'-expansion of (2.19). As elaborated in section 4.3, these cancellations include the field-theory limit (2.14) and the coefficients of odd Riemann zeta values ζ_{2k+1} without accompanying factors of ζ_{2n}. Moreover, abelian disk integrals of odd multiplicity vanish at all orders in α' by the reflection property (2.10),

$$Z_\times(q_1, q_2, \ldots, q_{2k+1}) = 0 \ . \tag{2.21}$$

It turns out that the leading low-energy contribution to abelian disk integrals of even multiplicity n arises from the order α'^{n-2} and stems solely from the even Riemann zeta values such as (B_{2k} are the Bernoulli numbers)

$$\zeta_2 = \frac{\pi^2}{6}\,, \qquad \zeta_4 = \frac{\pi^4}{90}\,, \qquad \zeta_6 = \frac{\pi^6}{945}\,, \qquad \ldots \qquad \zeta_{2k} = (-1)^{k-1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!}\,. \tag{2.22}$$

The impact of these selection principles on the α'-expansion of abelian Z-theory in connection with NLSM amplitudes will be explored in section 4. In light of the ubiquitous appearance of MZVs in both abelian and non-abelian Z-theory, one might be tempted to derive the capital letter in the theory's name from "zeta".

3. NLSM amplitudes from string theory

Although the superstring spectrum does not include any bi-adjoint scalar[9], the doubly-partial amplitudes (2.13) emerge naturally from the low-energy limit of the Z-theory amplitudes (2.1) contributing to the open string. In this work, we show that the NLSM tree-level amplitudes can be obtained from the abelian disk integrals (2.19).

To see this, note that the Born–Infeld action emerges as the leading low-energy contribution to photon amplitudes in string theory [51]. Therefore, the expression for $M_{\text{photon}}(\alpha')$ on the right-hand side of (2.18) must reduce to the Born–Infeld amplitude whose KLT-like double-copy structure has recently been identified by Cachazo, He and Yuan [36],

$$\begin{aligned} M_{\text{BI}} = \sum_{\rho,\tau\in S_{n-3}} & A_{\text{NLSM}}(1,\rho(2,3,\ldots,n-2),n,n-1) \\ & \times S[\rho(23\ldots n-2)|\tau(23\ldots n-2)]_1 A_{\text{YM}}(1,\tau(2,3,\ldots,n-2),n-1,n)\ . \end{aligned} \tag{3.1}$$

[9] Its tentative closed-string vertex operator would include two decoupled systems of Kac-Moody currents $J^a(z)\tilde{J}^b(\overline{z})$ whose multitrace contributions would cause tachyon propagation.

Comparing (2.18) with (3.1) and assuming linear independence of the YM partial amplitudes in the BCJ basis leads to the conclusion that the abelian Z-amplitudes (2.19) reduce to color-ordered NLSM tree amplitudes at low energies,

$$A_{\rm NLSM}(1,2,\ldots,n) = \lim_{\alpha'\to 0}(\alpha')^{2-n} Z_\times(1,2,\ldots,n)\,. \tag{3.2}$$

This has been explicitly verified up to $n=9$, using the expansion method of [50] to probe the α'^6-order at the highest non-trivial multiplicity $n=8$. As an immediate consistency condition for the validity of (3.2), note that the KK and BCJ relations satisfied by the NLSM amplitudes [21] correspond to the following identities of the abelian integrals,

$$\begin{aligned} 0 &= Z_\times(1,A,n-1,B) - (-1)^{|B|}\sum_{\sigma\in A\sqcup\!\sqcup\tilde{B}} Z_\times(1,\sigma,n-1)\,, \quad \forall A,B \\ 0 &= \sum_{j=2}^{n-1}(k_{q_1}\cdot k_{q_2q_3\ldots q_j}) Z_\times(q_2,q_3,\ldots,q_j,q_1,q_{j+1},\ldots,q_n)\ , \end{aligned} \tag{3.3}$$

which are a consequence of (2.7) and (2.8).

Besides reproducing NLSM amplitudes, higher α'-orders of abelian disk integrals (2.19) yield natural higher-mass dimension extensions of the NLSM which will all satisfy KK and BCJ relations (3.3). More precisely, the symmetry properties (3.3) hold separately at each order in α', and in fact for the coefficients of any MZV which is conjecturally linearly independent over $\mathbb{Q}$. Hence, abelian disk integrals can be viewed as a factory for effective theories with any number of derivatives, and each such theory obeys the duality between color and kinematics. The discussion of these α'-corrections to the NLSM will be the main focus of section 4.

4. Higher-derivative corrections to the NLSM

4.1. Four points

At four points, monodromy relations (2.11) [10,11] allow to compactly express the abelian disk integral (2.19) in terms of any $Z_P(q_1,q_2,q_3,q_4)$, e.g.

$$Z_\times(1,2,4,3) = 2\Big(1+\frac{\sin(\alpha'\pi s_{23})}{\sin(\alpha'\pi s_{13})}+\frac{\sin(\alpha'\pi s_{12})}{\sin(\alpha'\pi s_{13})}\Big)Z_{1234}(1,2,4,3) \tag{4.1}$$

It is straightforward to see using the form of the Veneziano amplitude

$$s_{12}Z_{1234}(1,2,4,3) = \frac{\Gamma(1+\alpha' s_{12})\Gamma(1+\alpha' s_{23})}{\Gamma(1+\alpha'(s_{12}+s_{23}))} \tag{4.2}$$

together with the identities

$$\sin(\pi x) = \frac{\pi}{\Gamma(1-x)\Gamma(x)}, \quad \ln(\Gamma(1+x)) = -\gamma x + \sum_{k=2}^{\infty} \frac{\zeta_k}{k}(-x)^k \tag{4.3}$$

that the four-point abelian integral (4.1) can be written as

$$Z_\times(1,2,4,3) = \frac{2\big[\sin(\pi\alpha' s_{12}) + \text{cyc}(1,2,3)\big]}{\pi\alpha' s_{12}s_{13}} \exp\Big(\sum_{k=2}^{\infty} \frac{\zeta_k}{k}(-\alpha')^k \big[s_{12}^k + s_{23}^k + s_{13}^k\big]\Big) . \tag{4.4}$$

The abelian integral (4.4) not only reproduces the standard four-point NLSM amplitude $A_{\text{NLSM}}(1,2,3,4) = -\pi^2 s_{13}$ at its lowest α' order (note the swap of legs $3 \leftrightarrow 4$), but also implies an infinite series of higher-derivative corrections,

$$\begin{aligned} Z_\times(1,2,3,4) = -\alpha'^2\pi^2 s_{13} \times \Big(1 &+ \frac{1}{2}\zeta_2\sigma_2 + \zeta_3\sigma_3 + \frac{3}{10}\zeta_2^2\sigma_2^2 + (\zeta_5 + \frac{1}{2}\zeta_3\zeta_2)\sigma_2\sigma_3 \\ &+ \frac{1}{2}\zeta_3^2\sigma_3^2 + \frac{\zeta_2^3}{280}(31\sigma_3^2 + 51\sigma_2^3) + (\zeta_7 + \frac{1}{2}\zeta_5\zeta_2 + \frac{3}{10}\zeta_3\zeta_2^2)\sigma_2^2\sigma_3 \\ &+ (\zeta_3\zeta_5 + \frac{1}{4}\zeta_2\zeta_3^2)\sigma_2\sigma_3^2 + \frac{\zeta_2^4\sigma_2}{1400}(67\sigma_3^2 + 31\sigma_2^3) + \dots\Big) , \end{aligned} \tag{4.5}$$

where we defined $\sigma_2 \equiv \frac{1}{2}\alpha'^2(s_{12}^2 + s_{13}^2 + s_{23}^2)$ and $\sigma_3 \equiv -\alpha'^3 s_{12}s_{23}s_{13}$. Note that the terms inside parenthesis in (4.5) are invariant under permutations, thereby manifesting the BCJ and KK relations (2.7) obeyed by $Z_\times(1,2,3,4)$.

4.2. Six points

The α'-expansion of six-point disk integrals (2.1) was pioneered in [52,53] and later on aligned into systematic all-multiplicity methods using polylogarithms [2] or the Drinfeld associator [50] (see also [54]). When summing over the 5! integration domains to obtain an abelian six-point disk integral (2.19), the leading α'-orders associated with s_{ij}^{-3}, $\alpha'^2\zeta_2 s_{ij}^{-1}$ and $\alpha'^3\zeta_3$ turn out to cancel, see section 4.3 for further details. The first non-vanishing order $\sim \alpha'^4\zeta_4$ coincides with the six-point NLSM amplitude (1.4),

$$\begin{aligned} Z_\times(1,2,3,4,5,6) &= \alpha'^4\pi^4\Big\{s_{12} - \frac{(s_{12}+s_{23})(s_{45}+s_{56})}{2s_{123}} + \text{cyc}(1,2,3,4,5,6)\Big\} + \mathcal{O}(\alpha'^6) \\ &= \alpha'^4 A_{\text{NLSM}}(1,2,3,4,5,6) + \mathcal{O}(\alpha'^6) , \end{aligned} \tag{4.6}$$

in agreement with the general claim (3.2). Beyond the order $\alpha'^4\zeta_4$ of the NLSM, an infinite tower of corrections occurs in the expansion of $Z_\times(1,2,\ldots,6)$, starting with $\alpha'^6\zeta_6$, $\alpha'^7\zeta_4\zeta_3$, $\alpha'^8\zeta_8$, $\alpha'^9\zeta_6\zeta_3$ and $\alpha'^9\zeta_4\zeta_5$. The lowest-order corrections are given by

$$\begin{aligned} Z_\times(1,2,3,4,5,6)\,\big|_{\alpha'^6} = \frac{\pi^6}{12}\Big[&-\frac{(s_{12}+s_{23})(s_{12}^2+s_{12}s_{23}+s_{23}^2)(s_{45}+s_{56})}{s_{123}} + 4s_{12}s_{23}s_{234} \\ &+ 4s_{12}s_{23}s_{345} - 4s_{12}s_{23}s_{34} + 2s_{12}s_{23}s_{56} + 2s_{12}s_{23}s_{45} + 2s_{12}s_{34}s_{123} + 2s_{12}s_{34}s_{234} \\ &+ s_{12}s_{34}s_{345} + s_{12}^3 + 2s_{12}^2s_{45} + 2s_{12}^2s_{234} - 2s_{12}s_{234}^2 - 4s_{12}s_{123}s_{234} - 2s_{23}s_{123}s_{234} \\ &- 4s_{34}s_{123}s_{234} - \frac{1}{2}s_{12}s_{45}s_{123} - \frac{1}{2}s_{12}s_{45}s_{345} + s_{123}^2s_{234} + s_{123}s_{234}^2 + \frac{1}{3}s_{12}s_{34}s_{56} \\ &+ \frac{4}{3}s_{123}s_{234}s_{345} + \mathrm{cyc}(1,2,3,4,5,6)\Big]\,, \end{aligned} \tag{4.7}$$

and the expression for the terms of order $\alpha'^7\zeta_4\zeta_3$ is given in (A.1).

4.3. All order-systematics

In order to discuss the all-multiplicity systematics of the α'-expansion of abelian disk integrals, we recall the patterns of MZVs in open-string amplitudes identified in [48]. A particularly convenient basis for that purpose is furnished by the $(n-3)!\times(n-3)!$ integrals[10] appearing in the n-gluon amplitude (2.16) of the superstring [7,2]

$$F_\Sigma{}^\rho \equiv \sum_{\tau\in S_{n-3}} S[\rho(23\ldots n{-}2)|\tau(23\ldots n{-}2)]_1 Z_{1,\Sigma(23\ldots n-2),n-1,n}(1,\tau(2,3,\ldots,n{-}2),n,n{-}1)\,. \tag{4.8}$$

These integrals form a square matrix indexed by integration domains Σ and integrands ρ, and the multiplication with the KLT matrix $S[\cdot|\cdot]_1$ defined in (2.17) ensures that all the entries are analytic in α', i.e. that there are no poles in any $s_{i_1\ldots i_p}$. The pattern of MZVs in the power-series expansion is based on matrix multiplications [48]:

$$\begin{aligned} F = &(1+\zeta_2P_2+\zeta_2^2P_4+\zeta_2^3P_6+\zeta_2^4P_8+\ldots) \\ &\times\Big(1+\zeta_3M_3+\zeta_5M_5+\frac{1}{2}\zeta_3^2M_3^2+\zeta_7M_7+\zeta_3\zeta_5M_5M_3+\frac{1}{5}\zeta_{3,5}[M_5,M_3]+\ldots\Big)\,. \end{aligned} \tag{4.9}$$

[10] Note that in a frame where $(z_1,z_{n-1},z_n)=(0,1,\infty)$, the integrals in (4.8) take the form [7]

$$\begin{aligned} F_\Sigma{}^\rho = \int_{0\le z_{\Sigma(2)}\le z_{\Sigma(3)}\le\cdots\le z_{\Sigma(n-2)}\le 1} &\mathrm{d}z_2\,\mathrm{d}z_3\,\ldots\,\mathrm{d}z_{n-2}\prod_{i<j}^{n-1}|z_{ij}|^{\alpha' s_{ij}}\,\frac{s_{1\rho(2)}}{z_{1\rho(2)}}\left(\frac{s_{1\rho(3)}}{z_{1\rho(3)}}+\frac{s_{\rho(2)\rho(3)}}{z_{\rho(2)\rho(3)}}\right) \\ &\times\left(\frac{s_{1\rho(4)}}{z_{1\rho(4)}}+\frac{s_{\rho(2)\rho(4)}}{z_{\rho(2)\rho(4)}}+\frac{s_{\rho(3)\rho(4)}}{z_{\rho(3)\rho(4)}}\right)\ldots\left(\frac{s_{1\rho(n-2)}}{z_{1\rho(n-2)}}+\frac{s_{\rho(2)\rho(n-2)}}{z_{\rho(2)\rho(n-2)}}+\ldots+\frac{s_{\rho(n-3)\rho(n-2)}}{z_{\rho(n-3)\rho(n-2)}}\right)\,. \end{aligned}$$

Both P_w and M_w denote $(n-3)!\times(n-3)!$ matrices whose entries are degree-w polynomials in $\alpha' s_{ij}$ with rational coefficients. The explicit form of these entries can be determined from polylogarithm manipulations [2] or the Drinfeld associator [50], and examples at multiplicity $n \leq 7$ are available for download from [49]. As a first non-trivial statement of (4.9), for instance, the coefficient of $\zeta_2\zeta_3$ is given by the matrix product $P_2 M_3$ combining the constituents at the ζ_2- and ζ_3-orders of F.

4.3.1. Selection rule for the zero'th order in ζ_2

The monodromy relations (2.11) among different color-orderings of $A_{\rm gluon}(\ldots)$ can be viewed as deformation of the BCJ relations by even powers of $\alpha'\pi s_{ij}$, i.e. by ζ_{2k} according to (2.22). These α'-corrections only interfere with the left-multiplicative factors of $\zeta_{2k}P_{2k}$ in the first line of (4.9). Therefore the entire second line of (4.9) – in fact any product of matrices M_{2k+1} – preserves the BCJ and KK relations[11] for $(M_{2k_1+1}\ldots M_{2k_n+1})A_{\rm YM}$.

Accordingly, the disk integrals' coefficient of ζ_{2k+1}, $\zeta_3\zeta_5$, $\zeta_{3,5}$ as well as suitable generalizations at higher weight and depth [57,48][12] satisfy the BCJ and KK relations. Once we collectively denote any MZV in the second line of (4.9) by $\zeta_M \in \{\zeta_{2k+1}, \zeta_3\zeta_5, \zeta_{3,5}, \ldots\}$, this can be written as

$$\sum_{j=2}^{n-1}(k_{p_1}\cdot k_{p_2p_3\ldots p_j})Z_{p_2p_3\ldots p_jp_1p_{j+1}\ldots p_n}(q_1,q_2,\ldots,q_n)\,\big|_{\zeta_M} = 0\ . \tag{4.10}$$

Hence, these MZVs drop out from abelian disk integrals,

$$Z_\times(q_1,q_2,\ldots,q_n)\,\big|_{\zeta_M} = 0\ . \tag{4.11}$$

4.3.2. Selection rule for the first order in ζ_2

Similarly, the α'-deformed BCJ relations of $P_{2k}A_{\rm YM}$ encoded in the monodromy relations directly carry over to the matrix products $P_{2k}M_{2\ell_1+1}\ldots M_{2\ell_n+1}A_{\rm YM}$ with $\ell_j \in \mathbb{N}$. This

[11] This argument firstly appeared in the discussion of BCJ relations among amplitudes from higher-mass dimension operators [55], and a similar statement in the context of the heterotic string can be found in [56].

[12] The choice of MZVs at a given weight to represent disk integrals is ambiguous, and we will follow the conventions of [57,48] to take $\{\zeta_8,\ \zeta_3\zeta_5,\ \zeta_2\zeta_3^2,\ \zeta_{3,5}\}$ as the conjectural $\mathbb{Q}$-basis of weight-eight MZVs. Different choices lead to redefinitions of the matrices P_w, M_w, e.g. P_8 is shifted by a rational multiple of $[M_3, M_5]$ when trading $\zeta_{3,5}$ for another basis MZV of depth ≥ 2.

has a direct implication for abelian disk integrals: Whenever the coefficient of ζ_2^k vanishes by the monodromy relations to that order, the same vanishing statement applies to all products of ζ_2^k with any term $\sim \zeta_M$ of the second line in (4.9),

$$Z_\times(q_1,q_2,\ldots,q_n)\,\big|_{\zeta_2^k}=0 \quad \Rightarrow \quad Z_\times(q_1,q_2,\ldots,q_n)\,\big|_{\zeta_2^k\,\zeta_M}=0\,. \tag{4.12}$$

The "KK-like" relations among the ζ_2-orders of gluon amplitudes [53,58,59] for instance are known to annihilate permutation sums at multiplicities $n\geq 5$ and therefore

$$Z_\times(q_1,q_2,\ldots,q_n)\,\big|_{\zeta_2\,\zeta_M}=0, \qquad \forall\ n\geq 5\,. \tag{4.13}$$

4.3.3. Selection rule for higher orders in ζ_2

Although the symmetry patterns associated with the $\zeta_4,\zeta_6,\ldots$-orders of gluon amplitudes have not yet been studied, there is an indirect argument to extend the selection rule (4.13) to higher orders: Since the low-energy limit of the photon amplitude (2.18) is known to stem from the Born–Infeld action [51], the n-point amplitude cannot have contributions of orders below α'^{n-2}. In particular, this implies

$$Z_\times(q_1,q_2,\ldots,q_n)\,\big|_{\zeta_2^k}=0, \qquad \forall\ k<\frac{n}{2}-1 \tag{4.14}$$

and leads to an infinity of additional vanishing statements by (4.12),

$$Z_\times(q_1,q_2,\ldots,q_n)\,\big|_{\zeta_2^k\,\zeta_M}=0\ , \qquad \forall\ k<\frac{n}{2}-1\ , \tag{4.15}$$

with ζ_M again referring to any MZV in the second line of (4.9).

Examples of the above selection rules on the low-energy regime of abelian disk integrals are summarized in the subsequent table:

n	ζ_2	ζ_3	ζ_4	ζ_5	$\zeta_2\zeta_3$	ζ_6	ζ_3^2	ζ_7	$\zeta_2\zeta_5$	$\zeta_4\zeta_3$	ζ_8	$\zeta_3\zeta_5$	$\zeta_{3,5}$	$\zeta_2\zeta_3^2$
4	✓	×	✓	×	✓	✓	×	×	✓	✓	✓	×	×	✓
6	×	×	✓	×	×	✓	×	×	×	✓	✓	×	×	×
8	×	×	×	×	×	✓	×	×	×	×	✓	×	×	×
10	×	×	×	×	×	×	×	×	×	×	✓	×	×	×

Table 1. Overview of the MZVs of weight $w\leq 8$ present in abelian disk integrals at multiplicities $n=4,6,8,10$. In each of the fields marked by ×, the selection rules (4.14) and (4.15) forbid the appearance of the respective MZV.

4.3.4. Comments on explicit checks

The above selection rules call for explicit verification at low multiplicities and orders in α'. In principle, the expansion of $Z_\times(\ldots)$ can be extracted from the known P_w and M_w matrices in the expression (4.9) for F by means of the monodromy relations (2.11). However, their explicit solution becomes cumbersome at higher multiplicity, so we instead apply the following procedure:

1. compute the first line of ${F_\Sigma}^\rho$ with $\Sigma = 2, 3, \ldots, n-2$ the canonical ordering
2. promote the resulting expression for

$$A_{\text{gluon}}(1, 2, \ldots, n) = \sum_{\rho \in S_{n-3}} {F_{23\ldots n-2}}^\rho A_{\text{YM}}(1, \rho(2, 3, \ldots, n-2), n-1, n) \tag{4.16}$$

   to a function of the labels $2, 3, \ldots, n \to i_2, i_3, \ldots, i_n$
3. assemble the photon amplitude by summing over all $(n-1)!$ permutations thereof,

$$M_{\text{photon}} = \sum_{\Sigma \in S_{n-1}} A_{\text{gluon}}(1, \Sigma(2, 3, \ldots, n)) \tag{4.17}$$

4. expand the $(n-1)!$ amplitudes $A_{\text{YM}}(1, j_2, j_3, \ldots, j_n)$ resulting from (4.17) in an $(n-3)!$-dimensional BCJ basis $\{A_{\text{YM}}(1, \tau(2, \ldots, n-2), n-1, n), \tau \in S_{n-3}\}$ to rewrite

$$M_{\text{photon}} \equiv \sum_{\tau \in S_{n-3}} {F_\times}^\tau A_{\text{YM}}(1, \tau(2, 3, \ldots, n-2), n-1, n)\,, \tag{4.18}$$

   which defines $(n-3)!$ abelian integrals ${F_\times}^\tau$
5. compare (4.18) with the corresponding KLT-like expression (2.18) to obtain the abelian disk integrals (note the swap of legs $n-1 \leftrightarrow n$)

$$Z_\times(1, \sigma(2, \ldots, n-2), n, n-1) = \sum_{\tau \in S_{n-3}} {F_\times}^\tau S^{-1}[\tau(2, \ldots, n-2)|\sigma(2, \ldots, n-2)]_1 \tag{4.19}$$

   by multiplication with the inverse of the KLT matrix (2.17)

$$S^{-1}[\tau(2, \ldots, n-2)|\sigma(2, \ldots, n-2)]_1 = \phi_{1,\tau(2,\ldots,n-2)|1,\sigma(2,\ldots,n-2)}\,, \tag{4.20}$$

   whose entries are determined by Berends–Giele double currents $\phi_{A|B}$ [26].

These steps have been followed to generate the orders ${\alpha'}^{n \leq 7}$ of $Z_\times(1, 2, \ldots, 6)$ and ${\alpha'}^{n \leq 6}$ of $Z_\times(1, 2, \ldots, 8)$, and the results are compatible with the above selection rules. However, the need to perform a sum over $(n-1)!$ permutations of (large) expressions of disk integral expansions at order ${\alpha'}^{n-2}$ coupled with a BCJ basis reduction turns the procedure inefficient at high multiplicities. These complications will be tremendously reduced upon introduction of a Berends–Giele description of Z-theory's disk integrals in future work.

4.4. Simplifications in the odd zeta sector

It turns out that the laborious procedure to determine the α'-expansion of $Z_\times(\ldots)$ can be bypassed for all the M_w matrices. Once we have determined the contributions of the type $\zeta_{2k}P_{2k}$ from the first line of (4.9),

$$Z_\times^{\text{even}}(q_1,q_2,\ldots,q_n) \equiv Z_\times(q_1,q_2,\ldots,q_n)\,\big|_{\zeta_M\to 0}\ , \tag{4.21}$$

then the coefficient of ζ_{2k+1} or any other MZV in the second line of (4.9) can be inferred by matrix multiplication

$$\begin{aligned} Z_\times(1,\tau(2,3,\ldots,n-2),n-1,n) = \sum_{\sigma\in S_{n-3}} \Big(1+\zeta_3 M_3+\zeta_5 M_5+\frac{1}{2}\zeta_3^2 M_3^2 \\ +\zeta_7 M_7+\zeta_3\zeta_5 M_3M_5-\frac{1}{5}\zeta_{3,5}[M_5,M_3]+\ldots\Big)_\tau{}^\sigma\, Z_\times^{\text{even}}(1,\sigma(2,\ldots,n-2),n-1,n)\ . \end{aligned} \tag{4.22}$$

Note, however, that the multiplication order of M_w matrices is reversed in (4.22) as compared to (4.9). As before, matrix multiplication with any sequence of M_{2k+1} propagates the BCJ and KK relations of $Z_\times^{\text{even}}$ to the full integral $Z_\times$.

One might wonder if the structure in (4.22) can be refined and if the appearance of any ζ_{2k} in $Z_\times$ can be captured by combining a BCJ basis of $A_{\text{NLSM}}(1,\sigma(2,\ldots,n{-}2),n{-}1,n)$ with polynomials in Mandelstam variables. When insisting on local coefficients for the basis of NLSM amplitudes, this scenario can be ruled out from a simple six-point example: In an ansatz of the form

$$Z_\times(1,\tau(2,3,4),5,6)\,\big|_{\zeta_6} = \alpha'^4\sum_{\sigma\in S_3}(M_2)_\tau{}^\rho A_{\text{NLSM}}(1,\sigma(2,3,4),5,6)\ , \tag{4.23}$$

with the left-hand side given by (4.7), the entries of the 6×6 matrix M_2 cannot be chosen as degree-two polynomials in $\alpha' s_{ij}$. Hence, there is no local degree-two counterpart M_2 of the M_{2k+1} matrices at six-points which preserves the BCJ and KK relations.

5. Color-kinematic satisfying numerators

As we will review, the fact that color-stripped NLSM amplitudes satisfy the BCJ relations ensures [60,45] that they admit a color-kinematic satisfying representation at tree-level by virtue of the existence of the KLT decomposition. The fact that this holds to all multiplicity

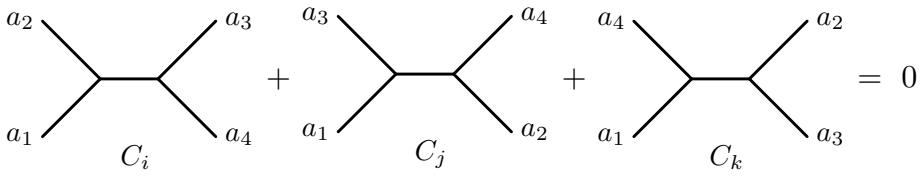


Fig. 1 The Jacobi identity $f^{a_1a_2b}f^{ba_3a_4} + \text{cyc}(a_1, a_2, a_3)$ implies the vanishing of the color factors C_i, C_j and C_k associated to triplets of cubic graphs. In the above diagrams, the legs $a_1, \ldots, a_4$ may represent arbitrary cubic tree-level subdiagrams. The duality between color and kinematics states that their corresponding kinematic numerators N_i built from polynomials of Mandelstam invariant for the cases of interest can be chosen such that $N_i + N_j + N_k = 0$ whenever $C_i + C_j + C_k = 0$ [4].

suggests that the integrands of these theories, effective though they are, should also admit color-kinematic satisfying numerators. This intriguing possibility motivates exploring what various closed forms for color-kinematic satisfying tree-level numerators can be found.

The BCJ relations (3.3) among abelian disk integrals hold separately at each order in α', more precisely for the coefficients of all the MZVs which are conjecturally linearly independent over $\mathbb{Q}$. Following the original derivation of BCJ relations for YM amplitudes from the duality between color and kinematics [4], one should expect each MZV coefficient of $Z_\times(\ldots)$ to admit a cubic-graph organization, where the s_{ij}-dependent numerators satisfy kinematic Jacobi relations. The latter apply to any triplet of cubic diagrams whose color factors under a generic gauge group (obtained from dressing each vertex by a structure constant f^{abc} of a non-abelian gauge group) sum to zero by the Jacobi identity $f^{a_1a_2b}f^{ba_3a_4} + \text{cyc}(a_1, a_2, a_3)$, see fig. 1.

While the original outline for finding tree-level Jacobi-satisfying numerators relied on manually inverting the propagator matrix and exploiting the residual gauge freedom to establish locality, it was not long before the community realized that the KLT matrix, or momentum kernel, does indeed represent an inversion of the propagator matrix relevant to finding Jacobi-satisfying numerators [60,45]. The prescription is to define the masters as the half-ladder diagrams with external legs k_1, and k_n as fixed farthest rungs, allowing all permutations of legs $\{2, \ldots, n-1\}$, as in fig. 2. All such master numerators for all permutations without label $n-1$ as the second to last argument are set to vanish, with the remaining $(n-3)!$ masters set to be

$$N_{1|\,\tau(23\ldots n-2),n-1\,|n} \equiv \sum_{\rho\in S_{n-3}} A(1,\rho(2,3\ldots,n{-}2),n,n{-}1)S[\rho(23\ldots n{-}2)\,|\,\tau(23\ldots n{-}2)]_1\,. \tag{5.1}$$

All numerators follow via Jacobi from these master numerators. These numerators are manifestly non-local (although of course all physical observables have the appropriate poles).

All poles belonging to the vanishing masters have been absorbed by the non-vanishing masters. As pointed out in [4], as long as the color-stripped amplitudes obey kinematic Jacobi relations on residues, one can find a generalized gauge transformation (cf. ref. [61]) consistent with Jacobi pushing these poles into the appropriate master numerators.

One should expect that if such a local representation is always possible for a theory then there should be a closed form for local masters[13]. Indeed, the authors of [13] present such a closed-form construction for the NLSM, making the key-insight that the symmetric $(n-2)! \times (n-2)!$ form of the momentum kernel has the necessary freedom to allow for locality, while recognizing the need for an off-shell regulation. The naive on-shell attempt fails as,

$$\sum_{\rho \in S_{n-2}} S[\sigma(23\ldots n-1) \,|\, \rho(23\ldots n-1)]_1 A(1, \rho(2,3,\ldots,n-1), n) = 0\,. \tag{5.2}$$

Indeed, this on-shell failure was realized in the first symmetric $(n-2)!\times(n-2)!$ construction of a momentum kernel [44]. The authors of ref. [44] proposed a regulation of such a $S[\cdot|\cdot]_1$ in the practice of building gravitational amplitudes symmetrically from a KK basis. They did so by regulating the product of the symmetric sum with $1/s_{12\ldots n-1}$ to cancel an overall $s_{12\ldots n-1}$ and then taking the appropriate $s_{12\ldots n-1} \to 0$ limit. The authors of [13] invoke such a regulation in building local NLSM master numerators proposing

$$\begin{aligned} N_{1|\sigma(23\ldots n-1)|n} \equiv \lim_{s_{12\ldots n-1}\to 0} s^{-1}_{12\ldots n-1} \sum_{\rho\in S_{n-2}} & S[\sigma(23\ldots n-1)|\rho(23\ldots n-1)]_1 \\ & \times A_{\rm NLSM}(1,\rho(2,3,\ldots,n-1),n)\,. \end{aligned} \tag{5.3}$$

Locality does indeed arise when the scattering amplitudes are expressed in an appropriate basis of Mandelstam variables as we now describe. In a similar fashion as in the Berends–Giele [62] description of NLSM amplitudes [63], one can extend NLSM amplitudes to an off-shell momentum $k_n^2 \neq 0$ by using an overcomplete set of Mandelstam variables s_{ij} with $1 \leq i < j \leq n-1$. Accordingly, the sum over ρ in (5.3) gives rise to an overall factor of $s_{12\ldots n-1} = \sum_{i<j}^{n-1} s_{ij}$ which cancels the propagator $s^{-1}_{12\ldots n-1}$. This, in turn, yields a well-defined expression upon the elimination $s_{1,n-1} \to -\sum_{i<j}^{n-2} s_{ij} - \sum_{i=2}^{n-2} s_{i,n-1}$ which implements the on-shell limit $s_{12\ldots n-1} \to 0$.

[13] Locality of numerators for Jacobi-descendant graphs when expressed in terms of local numerators from masters follows from the color-stripped amplitudes satisfying Jacobi on all poles [4].

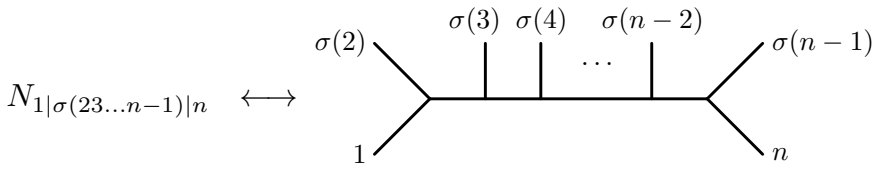


Fig. 2 Master diagrams with respect to Jacobi relations which are associated with the master numerators $N_{1|\sigma(23\ldots n-1)|n}$ defined by (5.3).

The four- and six-point numerators (5.3) corresponding to the amplitudes (1.3) and (1.4) read

$$N_{1|23|4} = \pi^2 s_{12}(s_{13}+s_{23}) \tag{5.4}$$

$$N_{1|2345|6} = -\pi^4 s_{12}(s_{13}+s_{23})(s_{14}+s_{24}+s_{34})(s_{15}+s_{25}+s_{35}+s_{45}) \ . \tag{5.5}$$

The numerators of any descendant cubic diagram are then simply defined by a sequence of kinematic Jacobi identities as well as antisymmetry under flips of cubic vertices. Remarkably, the four- and six-point numerators in (5.4) and (5.5) coincide with the diagonal entries of the KLT matrix (2.17). On these grounds, we propose the following all-multiplicity formula for NLSM master numerators (5.3),

$$N_{1|\rho(23\ldots 2k-1)|2k} = (-1)^k S[\rho(23\ldots 2k{-}1)|\rho(23\ldots 2k{-}1)]_1 \ . \tag{5.6}$$

We have verified their validity through multiplicity $2k = 8$. While indeed surprising, one should note that ref. [13] arrived at a master numerator representation involving sums over permutations of KLT matrix elements.

The form of the local numerators depend on the choice of implementing momentum conservation in $A_{\rm NLSM}(\ldots)$. For instance, the on-shell equivalent expression $-\pi^2 s_{13}$ for $A_{\rm NLSM}(1,2,3,4)$ instead of $\pi^2(s_{12}+s_{23})$ yields $-\pi^2 s_{12}s_{13}$ for the numerator $N_{1|23|4}$ instead of $\pi^2 s_{12}(s_{13}+s_{23})$. The six-point numerator (5.5) is obtained from the NLSM amplitude (1.4) after converting the Mandelstam invariants into the nine-element basis $\{s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{25}, s_{34}, s_{35}, s_{45}\}$. The n-point generalization of this basis choice applicable to (5.6) reads $\{s_{ij} \mid 1 \leq i < j \leq n{-}1 \ \& \ (i,j) \neq (1,n{-}1)\}$.

At generic multiplicity, the connection between color-ordered NLSM amplitudes and master numerators (5.3) is captured by doubly-partial amplitudes in (2.13) and (2.14),

$$A_{\rm NLSM}(\Sigma(1,2,\ldots,n-1),n) = \sum_{\rho\in S_{n-2}} m[\Sigma(12\ldots n-1)n|1\rho(23\ldots n-1)n]N_{1|\rho(23\ldots n-1)|n} \ , \tag{5.7}$$

which leads to the following expressions at four points,

$$\begin{aligned} A_{\rm NLSM}(1,2,3,4) = A_{\rm NLSM}(3,2,1,4) &= \frac{N_{1|23|4}}{s_{12}} + \frac{N_{1|23|4} - N_{1|32|4}}{s_{23}} \\ A_{\rm NLSM}(1,3,2,4) = A_{\rm NLSM}(2,3,1,4) &= \frac{N_{1|32|4}}{s_{13}} + \frac{N_{1|32|4} - N_{1|23|4}}{s_{23}} \\ A_{\rm NLSM}(2,1,3,4) = A_{\rm NLSM}(3,1,2,4) &= -\frac{N_{1|23|4}}{s_{12}} - \frac{N_{1|32|4}}{s_{13}} \ . \end{aligned} \tag{5.8}$$

One can verify from (5.4) that the four-point amplitude (1.3) is correctly reproduced.

As pointed out in [64] one can always symmetrize Jacobi-satisfying numerators to arrive at a crossing symmetric function for the generically dressed half-ladder topology in a manner that preserves linear relations (like Jacobi). One can note that fully crossing-symmetric local numerators of [13] were arrived at by evaluating the Berends–Giele currents in the pion parameterization scheme. This suggests an interesting connection between generalized gauge transformations at the amplitude level and field redefinitions in the context of Lagrangians.

The above prescription to convert amplitudes subject to BCJ relations into local and Jacobi-satisfying kinematic numerators is straightforwardly applied to the full-fledged abelian disk integrals (2.19). As in (5.3), we define $(n-2)!$ master numerators associated with the half-ladder diagrams in fig. 2

$$\begin{aligned} N^{\times}_{1|\sigma(23\ldots n-1)|n}(\alpha') \equiv (\alpha')^{2-n} \lim_{s_{12\ldots n-1}\to 0} s^{-1}_{12\ldots n-1} \\ \times \sum_{\rho\in S_{n-2}} S[\sigma(23\ldots n-1)|\rho(23\ldots n-1)]_1 Z_{\times}(1,\rho(2,3,\ldots,n-1),n)\,, \end{aligned} \tag{5.9}$$

and corresponding α'-corrected NLSM amplitudes:

$$(\alpha')^{2-n} Z_{\times}(\Sigma(1,\ldots,n-1),n) = \sum_{\rho\in S_{n-2}} m[\Sigma(1\ldots n-1)n|1\rho(2\ldots n-1)n] N^{\times}_{1|\rho(23\ldots n-1)|n}(\alpha'). \tag{5.10}$$

For example, the α'-corrections to the abelian four-point disk integrals (4.5) generalize the master numerator (5.4) to (recalling that $\sigma_2 \equiv \frac{\alpha'^2}{2}(s_{12}^2+s_{13}^2+s_{23}^2)$ and $\sigma_3 \equiv -\alpha'^3 s_{12}s_{23}s_{13}$)

$$N^{\times}_{1|23|4}(\alpha') = \pi^2 s_{12}(s_{13}+s_{23}) \times \Big(1 + \frac{1}{2}\zeta_2\sigma_2 + \zeta_3\sigma_3 + \frac{3}{10}\zeta_2^2\sigma_2^2 + \mathcal{O}(\alpha'^5)\Big)\ . \tag{5.11}$$

The analogous six-point corrections to (5.5) at the order of ζ_6 and $\zeta_4\zeta_3$ are attached as ancillary files to the arXiv submission of this work; they yield (4.7) and (A.1) according to (5.10).

Of course, the construction of α'-dependent numerators can be truncated to each desired order in α' and refined to any MZV which is conjecturally linearly independent over $\mathbb{Q}$. For example, the $\alpha'^2\zeta_2$ and $\alpha'^3\zeta_3$ orders of $N^{\times}_{1|\sigma(23\ldots n-1)|n}(\alpha')$ generate BCJ numerators in an effective theory where the NLSM interactions are supplemented by higher-derivative corrections with an extra $\alpha'^2\zeta_2\partial^4$ and $\alpha'^3\zeta_3\partial^6$, respectively.

6. Conclusions and outlook

In this paper, we interpret the disk integrals in open-string tree-level amplitudes as the S-matrix of a collection of putative scalar effective theories we refer to as Z-theory. Our key result (1.2) establishes that the low-energy limit of abelian Z-theory amplitudes yields the n-point amplitudes of the NLSM at order α'^{n-2}, while the next orders define new higher-derivative corrections which admit color-dual representations. Using this setup, we obtain α'-corrections to the local BCJ-satisfying numerators recently identified by Du and Fu [13], and indeed, a novel all-multiplicity expression (1.5) for the master numerators of the NLSM.

Given the high orders of α' involved in extracting the NLSM amplitudes and corrections from the Z-amplitudes, the straightforward organization of the string-theory calculations presented in this paper is not optimized to probe high multiplicities. Here, we chose to instead emphasize the relationship between open-string predictions, abelian Z-amplitudes, and explicit α'-corrections. In a future work, efficient calculations will be addressed by a Berends–Giele recursion for the α'-expansion of non-abelian disk integrals using an extension of the method described in [26].

Towards identifying patterns within Z-theory, we find ourselves encouraged to investigate the relevant higher-derivative corrections to the Lagrangian description of the NLSM which reproduces the higher α'-corrections of the amplitudes discussed in this work. Preliminary considerations suggest that these are not the only higher-derivative corrections consistent with color-kinematics. Accordingly, additional guiding principles may need to be invoked to arrive at the selection rules and the patterns of MZVs realized by worldsheets of disk topology. It has not escaped our notice that apprehending such guiding principles could indeed prove a fruitful line of inquiry.

In addition to providing higher derivative corrections to the NLSM we have through double copy, *en passant*, generated predictions for a set of higher-derivative corrections to Born–Infeld, and its supersymmetric partners including Volkov–Akulov from the fermionic

sector. It would be interesting to contrast with higher-derivative corrected Born–Infeld-type theories existing in the literature, cf. the set of self-dual theories constructed in ref. [65].

Acknowledgements: We are grateful to Nima Arkani-Hamed, Zvi Bern, James Drummond, Song He, Yu-tin Huang, Henrik Johansson, Renata Kallosh, and Radu Roiban for combinations of inspiring discussions and related collaboration. We would like to especially thank Radu Roiban for valuable comments on an initial draft. The authors would like to thank IAS at Princeton where this work was initiated as well as the Higgs Center, Nordita and the Isaac Newton Institute, and relevant program organizers, for providing stimulating atmosphere, support, and hospitality through the "QCD meets Gravity", "Aspects of Amplitudes", and "Gravity, Twistors, and Amplitudes" programs, respectively. JJMC is supported by the European Research Council under ERC-STG-639729, *Strategic Predictions for Quantum Field Theories*. CRM is supported by a University Research Fellowship from the Royal Society, and gratefully acknowledges support from NSF grant number PHY 1314311 and the Paul Dirac Fund and EPSRC grant no EP/K032208/1.

Appendix A. The six-point $\alpha'^3\zeta_3$ correction to the NLSM

In this appendix, we display the subleading α'-correction to the NLSM model as obtained from the abelian six-point disk integral at the order α'^7:

$$
\begin{aligned}
Z_\times(1,2,3,4,5,6)\,\big|_{\alpha'^7} = \pi^4\zeta_3\Big\{ &-\frac{s_{12}s_{23}(s_{12}+s_{23})^2(s_{45}+s_{56})}{s_{123}} - s_{12}^2 s_{23}s_{34} - 3s_{12}s_{23}^2 s_{34} \\
&- s_{12}s_{23}s_{34}^2 + s_{12}^3 s_{45} + 2s_{12}^2 s_{23}s_{45} + 2s_{12}s_{23}^2 s_{56} - s_{12}s_{23}s_{34}s_{56} + s_{12}^2 s_{34}s_{123} \\
&+ 2s_{12}s_{23}s_{34}s_{123} + s_{12}^2 s_{23}s_{345} + s_{12}s_{34}^2 s_{123} + 3s_{23}s_{34}^2 s_{123} + s_{34}^3 s_{123} + s_{12}s_{23}s_{45}s_{123} \\
&- s_{12}s_{23}s_{56}s_{234} + 3s_{12}^2 s_{23}s_{234} + s_{12}s_{23}^2 s_{234} - s_{12}^2 s_{234}^2 + s_{12}s_{23}s_{234}^2 + s_{12}s_{23}s_{56}s_{123} \\
&+ s_{12}^2 s_{34}s_{234} + 2s_{12}s_{23}s_{34}s_{234} + s_{12}s_{34}^2 s_{234} + 4s_{12}s_{23}s_{45}s_{234} + 4s_{12}s_{34}s_{45}s_{234} \\
&- s_{12}^2 s_{123}s_{234} - 2s_{12}s_{23}s_{123}s_{234} - s_{12}s_{23}s_{45}s_{345} - s_{12}^2 s_{123}s_{345} - 2s_{12}s_{23}s_{123}s_{345} \\
&- s_{12}s_{123}^2 s_{234} + s_{12}s_{123}s_{234}^2 - s_{12}s_{34}s_{123}s_{345} + s_{12}s_{34}s_{56}s_{345} - 2s_{12}s_{34}s_{123}s_{234} \\
&- s_{12}s_{34}s_{234}s_{345} - s_{12}s_{234}s_{345}^2 - 4s_{12}s_{23}s_{234}s_{345} + s_{12}s_{23}s_{345}^2 - s_{12}^2 s_{234}s_{345} \\
&+ s_{12}s_{234}^2 s_{345} - s_{12}s_{234}^3 - s_{12}^2 s_{34}s_{56} - 2s_{12}s_{23}s_{45}s_{56} - 2s_{12}s_{45}s_{234}^2 - \frac{1}{2}s_{12}s_{45}s_{123}^2 \\
&- \frac{1}{2}s_{12}s_{45}s_{345}^2 + s_{123}^2 s_{234}s_{345} + \frac{1}{2}s_{123}^3 s_{234} + \frac{1}{2}s_{123}^3 s_{345} + \mathrm{cyc}(1,2,\ldots,6)\Big\}\,. \qquad \text{(A.1)}
\end{aligned}
$$

References

[1] M. B. Green, J. H. Schwarz and L. Brink, "N=4 Yang-Mills and N=8 Supergravity as Limits of String Theories," Nucl. Phys. B **198**, 474 (1982).

[2] J. Broedel, O. Schlotterer and S. Stieberger, "Polylogarithms, Multiple Zeta Values and Superstring Amplitudes," Fortsch. Phys. **61**, 812 (2013). [arXiv:1304.7267 [hep-th]].

[3] H. Kawai, D.C. Lewellen and S.H.H. Tye, "A Relation Between Tree Amplitudes of Closed and Open Strings," Nucl. Phys. B **269**, 1 (1986).

[4] Z. Bern, J.J.M. Carrasco and H. Johansson, "New Relations for Gauge-Theory Amplitudes," Phys. Rev. D **78**, 085011 (2008). [arXiv:0805.3993 [hep-ph]].

[5] F. Cachazo, S. He and E.Y. Yuan, "Scattering equations and Kawai-Lewellen-Tye orthogonality," Phys. Rev. D **90**, no. 6, 065001 (2014). [arXiv:1306.6575 [hep-th]].

[6] Z. Bern, J. J. M. Carrasco and H. Johansson, "Perturbative Quantum Gravity as a Double Copy of Gauge Theory," Phys. Rev. Lett. **105**, 061602 (2010). [arXiv:1004.0476 [hep-th]].

[7] C.R. Mafra, O. Schlotterer and S. Stieberger, "Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation," Nucl. Phys. B **873**, 419 (2013). [arXiv:1106.2645 [hep-th]].

[8] C.R. Mafra, O. Schlotterer and S. Stieberger, "Complete N-Point Superstring Disk Amplitude II. Amplitude and Hypergeometric Function Structure," Nucl. Phys. B **873**, 461 (2013). [arXiv:1106.2646 [hep-th]].

[9] R. Kleiss and H. Kuijf, "Multi - Gluon Cross-sections and Five Jet Production at Hadron Colliders," Nucl. Phys. B **312**, 616 (1989).

[10] N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, "Minimal Basis for Gauge Theory Amplitudes," Phys. Rev. Lett. **103**, 161602 (2009). [arXiv:0907.1425 [hep-th]].

[11] S. Stieberger, "Open & Closed vs. Pure Open String Disk Amplitudes," [arXiv:0907.2211 [hep-th]].

[12] F. Cachazo, S. He and E.Y. Yuan, "Scattering of Massless Particles: Scalars, Gluons and Gravitons," JHEP **1407**, 033 (2014). [arXiv:1309.0885 [hep-th]].

[13] Y. J. Du and C. H. Fu, "Explicit BCJ numerators of nonlinear sigma model," [arXiv:1606.05846 [hep-th]].

[14] R. Monteiro, D. O'Connell and C. D. White, "Black holes and the double copy," JHEP **1412**, 056 (2014). [arXiv:1410.0239 [hep-th]]. ;
A. Luna, R. Monteiro, D. O'Connell and C. D. White, "The classical double copy for Taub-NUT spacetime," Phys. Lett. B **750**, 272 (2015). [arXiv:1507.01869 [hep-th]]. ;
R. Monteiro, D. O'Connell and C. D. White, "Gravity as a double copy of gauge theory: from amplitudes to black holes," Int. J. Mod. Phys. D **24**, no. 09, 1542008 (2015). ;
A. Luna, R. Monteiro, I. Nicholson, D. O'Connell and C. D. White, "The double copy: Bremsstrahlung and accelerating black holes," JHEP **1606**, 023 (2016). [arXiv:1603.05737 [hep-th]].

[15] L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy, "Magic Square from Yang-Mills Squared," Phys. Rev. Lett. **112**, no. 13, 131601 (2014). [arXiv:1301.4176 [hep-th]]. ;
A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy, "Super Yang-Mills, division algebras and triality," JHEP **1408**, 080 (2014). [arXiv:1309.0546 [hep-th]]. ;
A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy, "A magic pyramid of supergravities," JHEP **1404**, 178 (2014). [arXiv:1312.6523 [hep-th]]. ;
A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy, "Yang-Mills origin of gravitational symmetries," Phys. Rev. Lett. **113**, no. 23, 231606 (2014). [arXiv:1408.4434 [hep-th]]. ;
A. Anastasiou, L. Borsten, M. J. Hughes and S. Nagy, "Global symmetries of Yang-Mills squared in various dimensions," JHEP **1601**, 148 (2016). [arXiv:1502.05359 [hep-th]]. ;
L. Borsten and M. J. Duff, "Gravity as the square of Yang-Mills," Phys. Scripta **90**, 108012 (2015). [arXiv:1602.08267 [hep-th]].

[16] F. A. Berends, W. T. Giele and H. Kuijf, "On relations between multi - gluon and multi-graviton scattering," Phys. Lett. B **211**, 91 (1988).

[17] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, "Multileg one loop gravity amplitudes from gauge theory," Nucl. Phys. B **546**, 423 (1999). [hep-th/9811140].

[18] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, "On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences," Nucl. Phys. B **530**, 401 (1998). [hep-th/9802162]. ;
Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, "Three-Loop Superfiniteness of N=8 Supergravity," Phys. Rev. Lett. **98**, 161303 (2007). [hep-th/0702112]. ;
Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "Manifest Ultraviolet Behavior for the Three-Loop Four-Point Amplitude of N=8 Supergravity," Phys. Rev. D **78**, 105019 (2008). [arXiv:0808.4112 [hep-th]]. ;
Z. Bern, J. J. Carrasco, D. Forde, H. Ita and H. Johansson, "Unexpected Cancellations in Gravity Theories," Phys. Rev. D **77**, 025010 (2008). [arXiv:0707.1035 [hep-th]]. ;
Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "The Ultraviolet Behavior of N=8 Supergravity at Four Loops," Phys. Rev. Lett. **103**, 081301 (2009). [arXiv:0905.2326 [hep-th]].

[19] Z. Bern, T. Dennen, Y. t. Huang and M. Kiermaier, "Gravity as the Square of Gauge Theory," Phys. Rev. D **82**, 065003 (2010). [arXiv:1004.0693 [hep-th]].

[20] R. Monteiro and D. O'Connell, "The Kinematic Algebra From the Self-Dual Sector," JHEP **1107**, 007 (2011). [arXiv:1105.2565 [hep-th]].

[21] G. Chen and Y.J. Du, "Amplitude Relations in Non-linear Sigma Model," JHEP **1401**, 061 (2014). [arXiv:1311.1133 [hep-th]].

[22] T. Bargheer, S. He and T. McLoughlin, "New Relations for 3-Dimensional Supersymmetric Scattering Amplitudes," Phys. Rev. Lett. **108**, 231601 (2012). [arXiv:1203.0562 [hep-th]]. ;

J. J. M. Carrasco, M. Chiodaroli, M. Gnaydin and R. Roiban, "One-loop four-point amplitudes in pure and matter-coupled N $\leq$ 4 supergravity," JHEP **1303**, 056 (2013). [arXiv:1212.1146 [hep-th]]. ;

Y. t. Huang and H. Johansson, "Equivalent D=3 Supergravity Amplitudes from Double Copies of Three-Algebra and Two-Algebra Gauge Theories," Phys. Rev. Lett. **110**, 171601 (2013). [arXiv:1210.2255 [hep-th]]. ;

Y. t. Huang, H. Johansson and S. Lee, "On Three-Algebra and Bi-Fundamental Matter Amplitudes and Integrability of Supergravity," JHEP **1311**, 050 (2013). [arXiv:1307.2222 [hep-th]]. ;

M. Chiodaroli, Q. Jin and R. Roiban, "Color/kinematics duality for general abelian orbifolds of N=4 super Yang-Mills theory," JHEP **1401**, 152 (2014). [arXiv:1311.3600 [hep-th]]. ;

H. Johansson and A. Ochirov, "Pure Gravities via Color-Kinematics Duality for Fundamental Matter," JHEP **1511**, 046 (2015). [arXiv:1407.4772 [hep-th]]. ;

M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, "Scattering amplitudes in $\mathcal{N} = 2$ Maxwell-Einstein and Yang-Mills/Einstein supergravity," JHEP **1501**, 081 (2015). [arXiv:1408.0764 [hep-th]]. ;

H. Johansson and A. Ochirov, "Color-Kinematics Duality for QCD Amplitudes," JHEP **1601**, 170 (2016). [arXiv:1507.00332 [hep-ph]]. ;

M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, "Spontaneously Broken Yang-Mills-Einstein Supergravities as Double Copies," [arXiv:1511.01740 [hep-th]]. ;

M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, "Complete construction of magical, symmetric and homogeneous N=2 supergravities as double copies of gauge theories," Phys. Rev. Lett. **117**, no. 1, 011603 (2016). [arXiv:1512.09130 [hep-th]]. ;

M. Chiodaroli, "Simplifying amplitudes in Maxwell-Einstein and Yang-Mills-Einstein supergravities," [arXiv:1607.04129 [hep-th]].

[23] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes," Phys. Rev. D **85**, 105014 (2012). [arXiv:1201.5366 [hep-th]]. ;

Z. Bern, S. Davies, T. Dennen and Y. t. Huang, "Absence of Three-Loop Four-Point Divergences in N=4 Supergravity," Phys. Rev. Lett. **108**, 201301 (2012). [arXiv:1202.3423 [hep-th]]. ;

Z. Bern, S. Davies, T. Dennen, A. V. Smirnov and V. A. Smirnov, "Ultraviolet Properties of N=4 Supergravity at Four Loops," Phys. Rev. Lett. **111**, no. 23, 231302 (2013). [arXiv:1309.2498 [hep-th]]. ;

Z. Bern, S. Davies and T. Dennen, "Enhanced ultraviolet cancellations in $\mathcal{N} = 5$ supergravity at four loops," Phys. Rev. D **90**, no. 10, 105011 (2014). [arXiv:1409.3089 [hep-th]].

[24] P. Tourkine and P. Vanhove, "Higher-loop amplitude monodromy relations in string and gauge theory," [arXiv:1608.01665 [hep-th]].

[25] C.R. Mafra, O. Schlotterer and S. Stieberger, "Explicit BCJ Numerators from Pure Spinors," JHEP **1107**, 092 (2011). [arXiv:1104.5224 [hep-th]]. ;
C. R. Mafra and O. Schlotterer, "Berends-Giele recursions and the BCJ duality in superspace and components," JHEP **1603**, 097 (2016). [arXiv:1510.08846 [hep-th]].

[26] C.R. Mafra, "Berends-Giele recursion for double-color-ordered amplitudes," JHEP **1607**, 080 (2016). [arXiv:1603.09731 [hep-th]].

[27] C.R. Mafra and O. Schlotterer, "Towards one-loop SYM amplitudes from the pure spinor BRST cohomology," Fortsch. Phys. **63**, no. 2, 105 (2015). [arXiv:1410.0668 [hep-th]]. ;
C.R. Mafra and O. Schlotterer, "Two-loop five-point amplitudes of super Yang-Mills and supergravity in pure spinor superspace," JHEP **1510**, 124 (2015). [arXiv:1505.02746 [hep-th]]. ;
S. He, R. Monteiro and O. Schlotterer, "String-inspired BCJ numerators for one-loop MHV amplitudes," JHEP **1601**, 171 (2016). [arXiv:1507.06288 [hep-th]].

[28] S. Litsey and J. Stankowicz, "Kinematic numerators and a double-copy formula for N=4 super-Yang-Mills residues," Phys. Rev. D **90**, no. 2, 025013 (2014). [arXiv:1309.7681 [hep-th]]. ;
R. Monteiro and D. O'Connell, "The Kinematic Algebras from the Scattering Equations," JHEP **1403**, 110 (2014). [arXiv:1311.1151 [hep-th]]. ;
N. E. J. Bjerrum-Bohr, J. L. Bourjaily, P. H. Damgaard and B. Feng, "Manifesting Color-Kinematics Duality in the Scattering Equation Formalism," [arXiv:1608.00006 [hep-th]].

[29] F. Cachazo, S. He and E. Y. Yuan, "Scattering of Massless Particles in Arbitrary Dimensions," Phys. Rev. Lett. **113**, no. 17, 171601 (2014). [arXiv:1307.2199 [hep-th]].

[30] E. Witten, "Perturbative gauge theory as a string theory in twistor space," Commun. Math. Phys. **252**, 189 (2004). [hep-th/0312171].

[31] R. Roiban, M. Spradlin and A. Volovich, "On the tree level S matrix of Yang-Mills theory," Phys. Rev. D **70**, 026009 (2004). [hep-th/0403190].

[32] H. Gomez and E. Y. Yuan, "N-point tree-level scattering amplitude in the new Berkovits' string," JHEP **1404**, 046 (2014). [arXiv:1312.5485 [hep-th]].

[33] N. Berkovits, "Infinite Tension Limit of the Pure Spinor Superstring," JHEP **1403**, 017 (2014). [arXiv:1311.4156 [hep-th]].

[34] O. Schlotterer, "Amplitude relations in heterotic string theory and Einstein-Yang-Mills," [arXiv:1608.00130 [hep-th]].

[35] F. Cachazo, S. He and E. Y. Yuan, "Einstein-Yang-Mills Scattering Amplitudes From Scattering Equations," JHEP **1501**, 121 (2015). [arXiv:1409.8256 [hep-th]].

[36] F. Cachazo, S. He and E. Y. Yuan, "Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM," JHEP **1507**, 149 (2015). [arXiv:1412.3479 [hep-th]].

[37] M. Born and L. Infeld, "Foundations of the new field theory," Proc. Roy. Soc. Lond. A **144**, 425 (1934).

[38] E. Schrödinger, "Contributions to Born's New Theory of the Electromagnetic Field," Proc. Roy. Soc. Lond. A **150**, 465 (1935). ;
P.A.M. Dirac, "An Extensible model of the electron," Proc. Roy. Soc. Lond. A **268**, 57 (1962). ;
D. V. Volkov and V. P. Akulov, "Is the Neutrino a Goldstone Particle?," Phys. Lett. B **46**, 109 (1973). ;
S. Deser and C. Teitelboim, "Duality Transformations of Abelian and Nonabelian Gauge Fields," Phys. Rev. D **13**, 1592 (1976). ;
M. K. Gaillard and B. Zumino, "Duality Rotations for Interacting Fields," Nucl. Phys. B **193**, 221 (1981). ;
S. Cecotti and S. Ferrara, "Supersymmetric Born-infeld Lagrangians," Phys. Lett. B **187**, 335 (1987). ;
G. W. Gibbons and D. A. Rasheed, "Electric - magnetic duality rotations in nonlinear electrodynamics," Nucl. Phys. B **454**, 185 (1995). [hep-th/9506035]. ;
J. Bagger and A. Galperin, "A New Goldstone multiplet for partially broken supersymmetry," Phys. Rev. D **55**, 1091 (1997). [hep-th/9608177]. ;
M. K. Gaillard and B. Zumino, "Nonlinear electromagnetic selfduality and Legendre transformations," In *Cambridge 1997, Duality and supersymmetric theories* 33-48. [hep-th/9712103]. ;
S. V. Ketov, "A Manifestly N=2 supersymmetric Born-Infeld action," Mod. Phys. Lett. A **14**, 501 (1999). [hep-th/9809121]. ;
S. V. Ketov, "Many faces of Born-Infeld theory," [hep-th/0108189]. ;
M. Rocek and A. A. Tseytlin, "Partial breaking of global D = 4 supersymmetry, constrained superfields, and three-brane actions," Phys. Rev. D **59**, 106001 (1999). [hep-th/9811232]. ;
A. A. Tseytlin, "Born-Infeld action, supersymmetry and string theory," In *Shifman, M.A. (ed.): The many faces of the superworld* 417-452. [hep-th/9908105]. ;
S. M. Kuzenko and S. Theisen, "Supersymmetric duality rotations," JHEP **0003**, 034 (2000). [hep-th/0001068]. ;
S. M. Kuzenko and S. Theisen, "Nonlinear selfduality and supersymmetry," Fortsch. Phys. **49**, 273 (2001). [hep-th/0007231]. ;
S. Bellucci, E. Ivanov and S. Krivonos, "Towards the complete N=2 superfield Born-Infeld action with partially broken N=4 supersymmetry," Phys. Rev. D **64**, 025014 (2001). [hep-th/0101195]. ;
S. Bellucci, E. Ivanov and S. Krivonos, "N=2 and N=4 supersymmetric Born-Infeld theories from nonlinear realizations," Phys. Lett. B **502**, 279 (2001). [hep-th/0012236]. ;
E. A. Ivanov and B. M. Zupnik, "New representation for Lagrangians of selfdual nonlinear electrodynamics," [hep-th/0202203]. ;
E. A. Ivanov and B. M. Zupnik, "New approach to nonlinear electrodynamics: Dualities as

symmetries of interaction," Phys. Atom. Nucl. **67**, 2188 (2004), [Yad. Fiz. **67**, 2212 (2004)]. [hep-th/0303192].

[39] Y. t. Huang, O. Schlotterer and C. Wen, "Universality in string interactions," [arXiv:1602.01674 [hep-th]].

[40] C. Reutenauer, "Free Lie Algebras", London Mathematical Society Monographs, 1993.

[41] N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, "Proof of Gravity and Yang-Mills Amplitude Relations," JHEP **1009**, 067 (2010). [arXiv:1007.3111 [hep-th]].

[42] J. Scherk, "Zero-slope limit of the dual resonance model," Nucl. Phys. B **31**, 222 (1971).

[43] P. Frampton, "Dual Resonance Models," Frontiers in Physics, Benjamin 1974.

[44] N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, "Gravity and Yang-Mills Amplitude Relations," Phys. Rev. D **82**, 107702 (2010). [arXiv:1005.4367 [hep-th]].

[45] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, "The Momentum Kernel of Gauge and Gravity Theories," JHEP **1101**, 001 (2011). [arXiv:1010.3933 [hep-th]].

[46] T. Terasoma, "Selberg integrals and multiple zeta values," Compositio Mathematica **133**, 1 (2002). ;
K. Aomoto, "Special values of hyperlogarithms and linear difference schemes," Illinois J. Math. **34** (2), 191 (1990). ;
F. Brown, "Multiple zeta values and periods of moduli spaces $\mathcal{M}_{0,n}$," Ann. Sci. Ec. Norm. Super. **42** (4), 371 (2009), [math/0606419].

[47] S. Stieberger, "Constraints on Tree-Level Higher Order Gravitational Couplings in Superstring Theory," Phys. Rev. Lett. **106**, 111601 (2011). [arXiv:0910.0180 [hep-th]].

[48] O. Schlotterer and S. Stieberger, "Motivic Multiple Zeta Values and Superstring Amplitudes," J. Phys. A **46**, 475401 (2013). [arXiv:1205.1516 [hep-th]].

[49] J. Broedel, O. Schlotterer and S. Stieberger, `http://mzv.mpp.mpg.de`

[50] J. Broedel, O. Schlotterer, S. Stieberger and T. Terasoma, "All order α'-expansion of superstring trees from the Drinfeld associator," Phys. Rev. D **89**, no. 6, 066014 (2014). [arXiv:1304.7304 [hep-th]].

[51] R. R. Metsaev, M. Rakhmanov and A. A. Tseytlin, "The Born-Infeld Action as the Effective Action in the Open Superstring Theory," Phys. Lett. B **193**, 207 (1987).

[52] D. Oprisa and S. Stieberger, "Six gluon open superstring disk amplitude, multiple hypergeometric series and Euler-Zagier sums," [hep-th/0509042].

[53] S. Stieberger and T. R. Taylor, "Multi-Gluon Scattering in Open Superstring Theory," Phys. Rev. D **74**, 126007 (2006). [hep-th/0609175].

[54] J. M. Drummond and E. Ragoucy, "Superstring amplitudes and the associator," JHEP **1308**, 135 (2013). [arXiv:1301.0794 [hep-th]].

[55] J. Broedel and L. J. Dixon, "Color-kinematics duality and double-copy construction for amplitudes from higher-dimension operators," JHEP **1210**, 091 (2012). [arXiv:1208.0876 [hep-th]].

[56] S. Stieberger and T. R. Taylor, "Closed String Amplitudes as Single-Valued Open String Amplitudes," Nucl. Phys. B **881**, 269 (2014). [arXiv:1401.1218 [hep-th]].

[57] J. Blumlein, D. J. Broadhurst and J. A. M. Vermaseren, "The Multiple Zeta Value Data Mine," Comput. Phys. Commun. **181**, 582 (2010). [arXiv:0907.2557 [math-ph]].

[58] N. E. J. Bjerrum-Bohr, P. H. Damgaard, H. Johansson and T. Sondergaard, "Monodromy–like Relations for Finite Loop Amplitudes," JHEP **1105**, 039 (2011). [arXiv:1103.6190 [hep-th]].

[59] C.R. Mafra and O. Schlotterer, "The Structure of n-Point One-Loop Open Superstring Amplitudes," JHEP **1408**, 099 (2014). [arXiv:1203.6215 [hep-th]].

[60] M. Kiermaier, "Gravity as the Square of Gauge Theory"[14], Amplitudes 2010, Queen Mary, University of London.

[61] S. Lee, C. R. Mafra and O. Schlotterer, "Non-linear gauge transformations in $D = 10$ SYM theory and the BCJ duality," JHEP **1603**, 090 (2016). [arXiv:1510.08843 [hep-th]].

[62] F.A. Berends and W.T. Giele, "Recursive Calculations for Processes with n Gluons," Nucl. Phys. B **306**, 759 (1988).

[63] K. Kampf, J. Novotny and J. Trnka, "Recursion relations for tree-level amplitudes in the $SU(N)$ nonlinear sigma model," Phys. Rev. D **87**, no. 8, 081701 (2013). [arXiv:1212.5224 [hep-th]]. ;
K. Kampf, J. Novotny and J. Trnka, "Tree-level Amplitudes in the Nonlinear Sigma Model," JHEP **1305**, 032 (2013). [arXiv:1304.3048 [hep-th]].

[64] C. H. Fu, Y. J. Du and B. Feng, "Note on symmetric BCJ numerator," JHEP **1408**, 098 (2014). [arXiv:1403.6262 [hep-th]]. ;
S. G. Naculich, "Scattering equations and virtuous kinematic numerators and dual-trace functions," JHEP **1407**, 143 (2014). [arXiv:1404.7141 [hep-th]].

[65] W. Chemissany, R. Kallosh and T. Ortin, "Born-Infeld with Higher Derivatives," Phys. Rev. D **85**, 046002 (2012). [arXiv:1112.0332 [hep-th]].

[14] `http://www.strings.ph.qmul.ac.uk/~theory/Amplitudes2010/Talks/MK2010.pdf`