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► **To cite this version:**

Antonio Amariti, Davide Forcella, Claudius Klare, Domenico Orlando, Susanne Reffert. The braneology of 3D dualities. *Journal of Physics A: Mathematical and Theoretical*, IOP Publishing, 2015, 48 (26), 10.1088/1751-8113/48/26/265401 . cea-01336464

**HAL Id: cea-01336464**

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Submitted on 23 Jun 2016

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# The braneology of 3D dualities

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## Abstract

In this paper we study the reduction of four-dimensional Seiberg duality to three dimensions from a brane perspective.

We reproduce the non-perturbative dynamics of the three-dimensional field theory via a T-duality at finite radius and the action of Euclidean D-strings. In this way we also overcome certain issues regarding the brane description of Aharony duality. Moreover we apply our strategy to more general dualities, such as toric duality for M2-branes and dualities with adjoint matter fields.

# 1 Introduction

Gaining a better understanding of the IR properties of strongly coupled field theories is an ambitious, yet non-trivial goal. One approach which has been successfully applied many times over the last decades is duality. On the two sides of a duality, different degrees of freedom are used to describe the same physical phenomenon. One of the main strengths of this idea is that strongly coupled theories can have a weakly coupled counterpart which is accessible via perturbative analysis. It has however been difficult to find examples of such dualities. An exception are supersymmetric theories, where a plethora of dualities has been formulated.

In the minimal case of four supercharges in 3+1 space-time dimensions, Seiberg duality [1] has played a major role in the investigation of field theory dynamics. Many checks and applications have since followed. This duality, originally formulated for  $SU(N_c)$  SQCD with  $N_f$  fundamentals and anti-fundamentals, is a quite general property of 3+1-dimensional supersymmetric gauge theories. It has been extended to cases with real gauge groups and more complicated matter content (see for example [2–4]). Also cases with product groups have been studied, and it was realized that this duality plays an important role in the study of RG flows in the AdS/CFT correspondence [5]. Another extension of this duality is to 2+1 dimensions. Some early attempts were made in the 90s [6–12], but the 2+1-dimensional version of Seiberg duality has become more popular only in recent years.

There are two main reasons for this renewed interest. On the one hand, localization has allowed the exact computation of the partition function of theories that preserve some supercharges in curved spaces. On the other hand, the discovery of the Aharony–Bergman–Jafferis–Maldacena (ABJM) model [13] has extended the AdS/CFT duality to three dimensions. The simplest case of  $AdS_4 \times S^7/Z_k$  has been found to be dual to a Chern–Simons (CS) quiver gauge theory with  $\mathcal{N} = 6$  supersymmetry. After the ABJM model, the correspondence has been studied also for cases with less supersymmetry. Many theories have been conjectured to describe the IR dynamics of M2-branes probing a Calabi–Yau cone  $X = C(Y)$  over the seven-dimensional Sasaki–Einstein manifold  $Y$  [14–16]. They preserve generally  $\mathcal{N} = 2$  supersymmetry, the same number of supercharges as in  $\mathcal{N} = 1$  in 3+1 dimensions.

Motivated by these observations, many extensions of the duality in the 2+1-dimensional  $\mathcal{N} = 2$  case have been proposed and analyzed [17–24]. Examples have been found and it was observed that instead of a single duality like in four dimensions, there are many possibilities. These are often close to the four-dimensional parent duality but differ from each other in some aspects. There are essentially two reasons for this proliferation of dualities in 2+1 dimensions:

- the first is that in three dimensions it is possible to write a topological CS action for the gauge field, which affects the IR dynamics quite strongly;
- secondly, the moduli space has a complicated structure due to the presence of a real scalar field in the vector multiplet, coming from the reduction of the last component of the 3+1-dimensional gauge field.

This scalar moreover allows the existence of real (non-holomorphic) mass terms in the action. The absence of anomalies in 2+1 dimensions allows also the existence of

chiral<sup>1</sup> real masses and cs terms which are very closely connected. By integrating out a certain amount of matter with non-vanishing real mass, one can generate a cs action. This mechanism is crucial in the study of three-dimensional dualities.

It is natural to wonder whether the various three-dimensional dualities are somehow related and whether they can be classified in a unified way. At first sight there are analogies between these dualities and the four-dimensional case. Many of these relations are more evident from a stringy perspective. An interesting starting point to classify and possibly find new dualities in three-dimensional  $\mathcal{N} = 2$  gauge theories consists in compactifying the four-dimensional dualities on a circle at finite radius [25]. By shrinking this circle one expects to obtain a duality in three dimensions. This idea has been deeply investigated in [26], where a non-canonical dimensional reduction was performed. By naive dimensional reduction of two dual phases one obtains a new duality that is valid only below a too small energy scale. This is because the holomorphic scale associated to the gauge coupling decreases with the radius of the circle. There is a different way to consider this limit, by interpreting the theory on the finite circle as an effective three-dimensional theory. When the low-energy spectrum of the duality on the circle is considered, the two phases remain dual thanks to the non-perturbative effects generated by the circle itself. This idea has been applied in [26] to SQCD with flavor in the fundamental representation of unitary symplectic groups and in [27] for the orthogonal case<sup>2</sup>. New dualities have been found in this way, and it has been shown that from these one can also flow to previously known examples (see *e.g.* [11] and [18]). The procedure is more general and can be applied to other configurations. The problem is that the non-perturbative structure of the theories at finite radius strongly depends on the details of the gauge and matter content. The counting of the zero modes and the presence of a compact moduli space in the theory on the circle can strongly affect the non-perturbative dynamics, so one has to study the different cases separately. For example, the case with adjoint matter and unitary groups has been discussed in [29], while the analysis of the s-confining case was started in [30].

In this paper we discuss the mechanism to generate three-dimensional dualities from four dimensions based on the brane description of the associated field theories. We also make progress towards the geometric description of the duality of Aharony [11]. The string realization is a  $uv$  completion of the gauge theories. One might wonder if the resulting extra massive modes spoil a duality proof based on the brane construction. This is not the case: the matching of points in moduli space (*i.e.* the equivalence of gauge theories) is protected by supersymmetry; only the form of the Kähler potential depends on the details of the string  $uv$  completion.

We start by considering the four-dimensional representation of Seiberg duality as seen from a brane setup. This can be represented as a type IIA system, with  $N_c$  D4-branes stretched between two non-parallel NS5-branes and  $N_f$  D6-branes. This system reproduces the low-energy spectrum of  $\mathcal{N} = 1$   $SU(N_c)$  SQCD with  $N_f$  fundamentals and anti-fundamentals in four dimensions. Seiberg duality is obtained by exchanging the two NS-branes. The reduction of this duality in terms

<sup>1</sup> Note that even if the notion of chirality is absent in three dimensions, by “chiral” we refer to the chirality the four-dimensional parent theory.

<sup>2</sup> See also [28] for an earlier attempt.

of branes can be sketched as follows. First one compactifies one space-like direction of the spacetime, for example  $x_3$ . Then a  $T$ -duality is performed along this compact direction. This duality transforms the  $Dp$ -branes considered above into  $D(p-1)$ -branes, leaving the NS-branes unchanged. If the T-dual radius is large enough we can consider the theory as effectively three-dimensional. The non-perturbative dynamics, coming from the presence of the circle, is captured at the brane level by D1-strings stretched between the D3 and the NS-branes. This analysis reproduces the field theory results found in [26] in terms of brane dynamics. In field theory, by adding some mass deformation to this duality, one obtains Aharony duality. Here we reproduce this RG flow from the brane perspective, where many aspects of this flow have a simple physical interpretation.

The brane analysis allows us to study, in a general framework, theories with a more complicated matter content (*e.g.* tensor matter) and gauge groups (orthogonal and symplectic). Also in these cases Seiberg duality corresponds to the exchange of the NS-branes. The reduction of these theories to three dimensions follows from the steps discussed above. Indeed we (re)-obtain the reduction discussed in [29] for the duality with adjoint matter by using the brane representation of this theory. We also propose the reduction of the duality for quiver gauge theories and discuss the relation of the three-dimensional dualities obtained by dimensional reduction and the dualities obtained in [20] for M2-branes on CY fourfolds. We conclude by proposing the extension of our procedure to the case with tensor matter and real gauge groups, higher supersymmetry and lower dimensionality.

## 2 Mini review of known results

### 2.1 Field theory reduction of 4D dualities to 3D

In this section we review the relevant aspects of three-dimensional field theories and of the dimensional reduction of the four-dimensional Seiberg duality for  $U(N_c)$  SQCD with  $N_f$  flavors to three dimensions.

**Some aspects of  $\mathcal{N} = 2$  three-dimensional field theories.** Three-dimensional theories with four supercharges have an additional (real) scalar  $\sigma$  in the vector multiplet with respect to the four-dimensional case. Classically, this scalar implies the existence of a Coulomb branch parametrized by the vacuum expectation value (VEV)  $\langle \sigma \rangle$ , which generically breaks the rank  $r$  gauge group  $G$  to  $U(1)^r$ . Another feature of three-dimensional gauge theory is that a  $U(1)$  gauge field  $A_\mu$  can be dualized to a scalar  $\phi = d\star F$ , where  $F$  is the  $U(1)$  field strength. On the Coulomb branch we have  $r$  such dual photons, one from each  $U(1)$  factor. We can dualize the  $i$ th vector multiplet to a chiral one with lowest component  $Y_i \equiv e^{i\phi_i + \sigma_i / e_3^2}$ , where  $e_3$  is the gauge coupling.

Quantum corrections can lift some of the directions on the Coulomb branch. For  $U(N_c)$  SQCD with  $N_f > N_c$  only two directions  $Y \equiv Y_1$  and  $\tilde{Y} \equiv Y_{N_c}$  remain unlifted [10]. The Coulomb branch coordinates  $Y, \tilde{Y}$  have a UV interpretation as monopole operators in the field theory [10], which are excitations of magnetic flux  $(\pm 1, 0, \dots, 0)$  in the Cartan subgroup  $U(1)^r$ . Note that  $Y, \tilde{Y}$  are charged under the topological symmetry  $U(1)_J$ , that shifts the dual photon.

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_R$	$U(1)_I$
$Q$	$N_f$	1	$\Delta$	0
$\tilde{Q}$	1	$\overline{N_f}$	$\Delta$	0
$q$	$\overline{N_f}$	1	$1 - \Delta$	0
$\tilde{q}$	1	$N_f$	$1 - \Delta$	0
$M$	$N_f$	$\overline{N_f}$	$2\Delta$	0
$Y$	1	1	$N_f(1 - \Delta) - N_c + 1$	1
$\tilde{Y}$	1	1	$N_f(1 - \Delta) - N_c + 1$	-1
$y$	1	1	$-N_f(1 - \Delta) + N_c + 1$	1
$\tilde{y}$	1	1	$-N_f(1 - \Delta) + N_c + 1$	-1

Table 1: Global symmetries of the three-dimensional fields in the reduction of Seiberg duality. Note that the monopoles are charged under  $U(1)_R$  because of quantum corrections [32–34].

**Duality at finite radius.** Let us consider a  $U(N_c)$  gauge theory with  $N_f$  fundamentals and anti-fundamentals  $Q$  and  $\tilde{Q}$  without superpotential as the four-dimensional electric theory. For  $N_f > N_c + 1$  the theory admits a dual description<sup>3</sup> (the so-called Seiberg-dual) in terms of a magnetic theory with gauge group  $U(N_f - N_c)$ ,  $N_f$  fundamentals and anti-fundamentals  $q$  and  $\tilde{q}$ , a meson  $M = Q\tilde{Q}$  and a superpotential  $W = Mq\tilde{q}$ .

If we put both the electric and the magnetic theory on  $\mathbb{R}^3 \times S^1$  with finite circle radius  $R_3$ , a three-dimensional description is obtained by keeping the scales  $\Lambda$ ,  $\tilde{\Lambda}$  and the radius  $R_3$  fixed and by looking at energies  $E \ll \Lambda, \tilde{\Lambda}, 1/R_3$ . In this limit the dynamics is effectively three-dimensional and, as discussed in [26], the 4D duality reduces to a new duality in 3D. Crucially, we can still see the effect of the finite circle radius through the non-perturbative superpotentials

$$W_\eta = \eta Y\tilde{Y}, \quad W_{\eta'} = \eta' y\tilde{y}, \quad (2.1)$$

where  $y, \tilde{y}$  are the monopoles of the magnetic theory and  $\eta = e^{-8\pi/(R_3 e_3^2)} = \Lambda^{2b}$  (recall that  $2\pi R_3 e_3^2 = e_4^2$ ). The power  $b$  is the coefficient of the one-loop Novikov–Shifman–Vainshtein–Zakharov [31] (NSVZ)  $\beta$ -function; analogous definitions hold for the magnetic  $\eta'$ .

The superpotentials (2.1) completely lift the Coulomb branch. Note also that (2.1) break the axial symmetry  $U(1)_A$  which in four dimensions is broken by anomalies. The global symmetries coincide with the ones of the four-dimensional parent theory. The charges of the 3D fields are given in Table 1.

**Aharony duality.** If we want to recover a more *classic* three-dimensional duality, we have to integrate out some matter fields. If we consider for example  $N_f + 2$  flavors and integrate out two pairs of fundamentals and anti-fundamentals, one with positive large mass and one with opposite large mass, we obtain a  $U(N_c)$  gauge theory with  $N_f$  flavors. In the magnetic theory the dual quarks and the mesons acquire large masses fixed by their charges with respect to the global symmetries.

<sup>3</sup> The confining case  $N_f = N_c + 1$  was discussed recently in [30].

	0	1	2	3	4	5	6	7	8	9
NS	×	×	×	×	×	×				
NS'	×	×	×	×					×	×
D4	×	×	×	×			×			

Table 2: Brane configuration for  $\mathcal{N} = 1, d = 4$  SYM. The D4-branes are suspended between the two NS5s.

To preserve the duality one has to higgs the gauge symmetry by assigning a large vev to some components of the scalar in the vector multiplet [26]. The vev breaks the gauge symmetry as  $U(N_f + 2 - N_c) \rightarrow U(N_f - N_c) \times U(1)^2$ . Each  $U(1)$  sector has a fundamental and an anti-fundamental field and a singlet. The three gauge sectors are coupled via Affleck–Harvey–Witten [35] (AHW) superpotentials. The superpotential of this dual theory is given by

$$W_m = Mq\tilde{q} + M_1q_1\tilde{q}_1 + M_2q_2\tilde{q}_2 + y\tilde{y}_1 + \tilde{y}y_2 + \eta'y_1\tilde{y}_2. \quad (2.2)$$

The  $U(1)$  sectors can be dualized to sectors containing only singlets [6, 9, 10]. Observe that in absence of the singlets  $M_i$  each mirror sector coincides with an XYZ model [10], which is a Wess–Zumino model with three chiral fields  $X, Y$  and  $Z$  and superpotential  $W = XYZ$ . Here the dual mesons  $q_i\tilde{q}_i$  become massive because of the superpotential. Moreover, the term  $y_1\tilde{y}_2$  is a mass term in the mirror theory, because  $y_i, \tilde{y}_i$  are singlets. In the limit  $E \ll \Lambda, \tilde{\Lambda}, 1/R_3$  we can integrate out this mass term. Finally the superpotential in Eq. (2.2) becomes

$$W_m = Mq\tilde{q} + yY + \tilde{y}\tilde{Y}, \quad (2.3)$$

where the identifications  $Y = \tilde{y}_1$  and  $\tilde{Y} = y_2$  follow from the quantum charges of the singlets. The superpotential (2.3) reproduces the one expected from the duality of Aharony.

## 2.2 Non-perturbative superpotentials from the brane picture

In this section we review some aspects of the generation of the non-perturbative superpotential for SQCD at finite radius from the brane perspective. The discussion will be relevant in the next section when studying the reduction of four-dimensional Seiberg duality to three dimensions.

**Super Yang–Mills.** Consider  $\mathcal{N} = 1$  super Yang–Mills (SYM). The theory is described by an NS5–brane, an NS5'–brane and  $N_c$  D4–branes extended as shown in Table 2. The D4s are suspended between the NS5 and the NS5'. The four-dimensional gauge coupling is  $e_4^2 = g_4^2/\ell_6 = (2\pi)^2\sqrt{\alpha'}/\ell_6$  where  $\ell_6$  is the distance between the NS5–branes and  $g_4^2 = (2\pi)^2\sqrt{\alpha'}$  is the D–brane coupling constant.

We consider compact  $x_3 \sim x_3 + 2\pi R_3$  and perform a T–duality along that direction. In the resulting type IIB frame the D4s have turned into D3–branes while the NS and NS'–branes remain unchanged. This setup describes  $U(N_c)$  SYM on  $\mathbb{R}^3 \times S^1$ . The theory has the entire Coulomb branch lifted and  $N_c$  isolated vacua [36].

The isolated vacua correspond to stable supersymmetric configurations of the brane system. As we will review below, in absence of D5-branes there is a repulsive force between the D3s. A stable brane configuration corresponds to distributing the D3-branes along  $x_3$  at equal distances. Here all moduli are lifted as the D3-branes cannot move freely due to the repulsive force.

The repulsive force is a non-perturbative quantum effect. From the 3D field theory point of view, a non-perturbative superpotential induced by three-dimensional instantons is generated. From the 4D perspective these instantons are monopole configurations. In the brane picture these monopoles are represented by Euclidean D1-strings stretched between each pair of D3-branes and the NS and NS'-branes, as depicted in figure 1 as shaded area along  $x_6$  and  $x_3$  respectively. The contribution of the monopoles can be computed following [9, 37] as  $e^{-S}$ , where  $S$  is the D1 world-sheet action. This action has two pieces, the Nambu-Goto action and a contribution from the boundary of the D1s. The Nambu-Goto term is proportional to the area of the D1 and involves the scalar  $\sigma$  that parametrizes the position of the D3. The boundary term is proportional to the dual photon  $\phi$ . By combining everything together one obtains the monopole contribution as

$$W = \sum_{i=1}^{N_c-1} e^{\Sigma_i - \Sigma_{i+1}}, \quad (2.4)$$

where  $\Sigma_i = \sigma_i/e_3^2 + i\phi_i$  and  $e_3^2 = (\sqrt{\alpha'}/R_3)(g_3^2/\ell_6) = 2\pi\sqrt{\alpha'}/(R_3\ell_6)$  is the three-dimensional gauge coupling<sup>4</sup>. Observe that the result is expressed in terms of the Coulomb branch coordinates, in terms of operators one can associate  $e^{\Sigma_i}$  to the monopole operators.

As is easily seen from the brane picture, for  $x^3$  being compact and finite there is another contribution from D1-branes stretching from the  $N_c$ th to the 1st D3-branes, which sit at positions  $\sigma_{N_c}$  and  $\sigma_1 + 2\pi R_3$  respectively. By following the calculation of [38, 39] one finds

$$W_\eta = \eta e^{\Sigma_{N_c} - \Sigma_1}, \quad (2.5)$$

where  $\eta$  incorporates the radius dependence from the position of the 1st D3-brane. This is the  $\eta$ -superpotential (2.1) which proves important in the reduction of 4D to 3D dualities.

**Fundamental matter.** The picture becomes more interesting when we introduce matter fields. Here we consider only the case  $N_f > N_c$  as we are interested in the dimensional reduction of Seiberg duality. As mentioned above two directions of the moduli space remain unlifted, which can be seen from the brane picture as follows. In the type IIA frame fundamental matter is associated to  $N_f$  D6-branes extended along 0123789 and sitting on the NS'-brane. In the T-dual frame they become D5-branes. Strings between the stack of D3-branes and the D5s correspond to  $N_f$  massless fundamentals  $Q$  and anti-fundamentals  $\tilde{Q}$ .

When D5-branes sitting at  $x_3 = 0$  intersect the worldsheet of the D1-strings, they contribute two additional zero modes to the D1-instanton and the superpotential in Eq. (2.4) is not generated. In this sense the D5-branes screen the repulsive force

<sup>4</sup> Here  $g_3^2 = 2\pi$ , and  $\sqrt{\alpha'}/R_3$  is the contribution of the type IIB dilaton so that  $e_4^2/e_3^2 = 2\pi R_3$ .



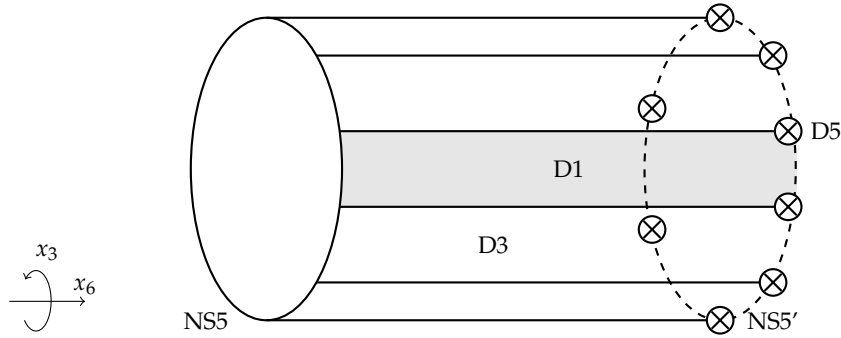


Figure 1: D1-branes (in grey) stretched between D3-branes with compact  $x_3$ . The D5-branes are represented by  $\otimes$  symbols, the NS5 by a continuous line and the NS5' by a dashed line.

	0	1	2	3	4	5	6	7	8	9
NS	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$				
NS'	$\times$	$\times$	$\times$	$\times$					$\times$	$\times$
D4	$\times$	$\times$	$\times$	$\times$			$\times$			
D6	$\times$	$\times$	$\times$	$\times$				$\times$	$\times$	$\times$

Table 3: Brane configuration for  $\mathcal{N} = 1, d = 4$  SQCD with  $N_f$  flavors. The  $N_c$  D4-branes are extended between the NS5 and the  $N_f$  D6-branes sit on the NS5'.

between the D3-branes [40, 41]. One finds that the screening happens for the 1st and the  $N_c$ th D3-brane, which are still free to move without being subjected to any force. There is however the interaction given by superpotential Eq. (2.5), which lifts one modulus.

For  $x_3$  being non-compact, there is no superpotential (2.5) and we remain with a two-dimensional moduli space corresponding to the forceless motion of the 1st and the  $N_c$ th D3-brane.

### 3 The braneology of the reduction

In this section we study the reduction of four-dimensional Seiberg duality for SQCD to three dimensions from the perspective of brane dynamics. Note that we consider the four-dimensional gauge symmetry to be  $U(N_c)$  rather than  $SU(N_c)$ . In field theory this enhancement is obtained by gauging the baryonic symmetry.

We start by considering a stack of  $N_c$  D4-branes, one NS5-brane, one NS5'-brane and  $N_f$  D6-branes. In this type IIA description the branes are extended as shown in Table 3. The D4-branes are suspended between the NS5-branes. The distance between the NS5-branes is proportional to the inverse gauge coupling of the four-dimensional theory. There are two possible configurations: in the first (electric) the NS5 is on the left and the NS5' on the right with  $N_c$  D4-brane in between. Moving the NS5 to the right we obtain the second configuration (magnetic) with  $(N_f - N_c)$  suspended D4-branes. The next step consists of compactifying both dual phases along  $x_3$  and studying the two dual theories at finite radius.

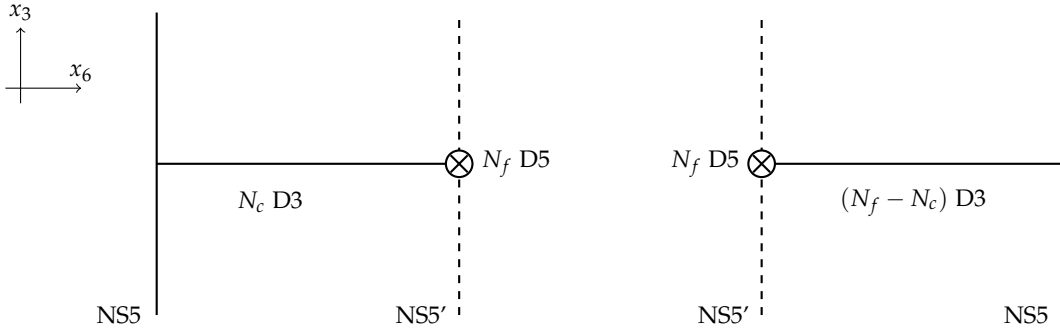


Figure 2: Brane description of three-dimensional Seiberg duality. From the brane description one can see that the electric superpotential is  $W_e = 0$  and the magnetic one is  $W_m = Mq\tilde{q}$ .

	0	1	2	3	4	5	6	7	8	9
NS	×	×	×	×	×	×				
NS'	×	×	×	×					×	×
D3	×	×	×				×			
D5	×	×	×					×	×	×

Table 4: Brane configuration for  $\mathcal{N} = 2, d = 3$  SQCD. The  $N_c$  D3-branes are extended between the NS5 and the  $N_f$  D5-branes sit on the NS5'.

### 3.1 Duality at finite radius

If we consider  $x_3$  to be compact we can perform a T-duality and obtain an effective three-dimensional  $\mathcal{N} = 2$  theory. In this case the D4 and the D6-branes become D3s and D5s respectively, while the NS and NS'-branes are left unchanged. In Figure 2 the electric and magnetic Seiberg-dual theories are shown, where the horizontal direction is  $x_6$  and the vertical one is  $x_3$ . In the type IIB description the branes are extended as shown in Table 4. This is a  $U(N_c)$  gauge theory with  $N_f$  fundamentals and anti-fundamentals as discussed in Section 2.2. The global symmetry group is  $SU(N_f)_L \times SU(N_f)_R \times U(1)_R \times U(1)_J$ . The axial symmetry  $U(1)_A$  under which  $Q$  and  $\tilde{Q}$  have the same charge is broken by the superpotential in Eq. (2.5).

Let us see how the global symmetries are realized in the brane picture.

- The brane system is invariant under rotations in the (4, 5) and in the (8, 9)-plane. The corresponding symmetry  $U(1)_{45} \times U(1)_{89}$  is part of the Lorentz group in 9+1 dimensions and rotates the supercharges. It is hence an  $R$ -symmetry<sup>5</sup>. However, the axial  $U(1)$  subgroup leaves the supercharges invariant while it rotates  $Q$  and  $\tilde{Q}$  in the same way, appearing as axial symmetry  $U(1)_A$  in the field theory. While admissible in 3D, on  $\mathbb{R}^3 \times S^1$  it is broken as will be discussed in the next paragraph.
- The non-Abelian flavor symmetry comes from the stack of  $N_f$  D5-branes. The chiral nature comes from the fact that the branes can be broken at the intersection with the

<sup>5</sup> Recall that in four dimensions, where we have a *quantum* description of the field theory through M-theory, one can see that only one particular combination of  $U(1)_{45}$  and  $U(1)_{89}$  is a symmetry in the lift to 11d (due to bending of the branes). This singles out the anomaly-free  $R$ -symmetry. Here we find both classical symmetries potentially realized in the field theory.

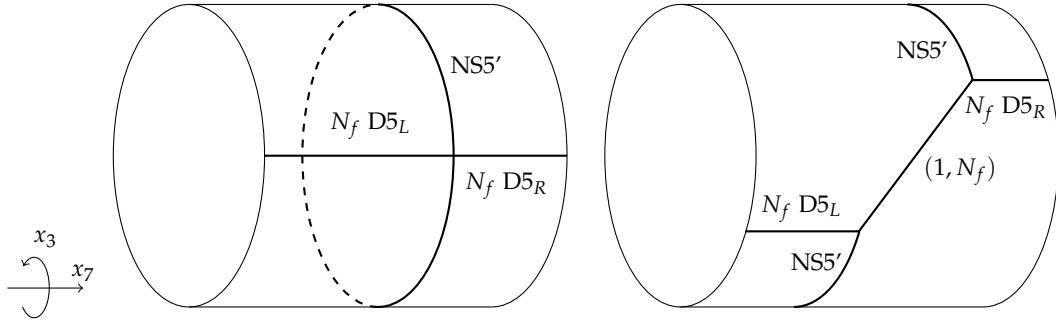


Figure 3: Geometric realization of the breaking of  $U(1)_A$  for compact  $x_3$ . If the stacks of D5s are moved apart to give a real mass, the NS5'–brane cannot be closed in the direction  $x_3$  without a displacement in  $x_7$ .

NS5', leading to two semi-infinite stacks,  $D5_L$  and  $D5_R$ , one extended in  $x_7 > 0$  and the other in  $x_7 < 0$ . The freedom to move the branes in each stack independently along  $x_3$  signals the presence of two independent  $SU(N_f)$  rotations [42]. The flavor group is  $U(N_f) \times U(N_f)$  but it turns out that one combination of the two  $U(1)$  factors appears as baryonic and one as axial symmetry. Let us make this more precise. The  $U(1) \subset U(N_f)$  real mass, *i.e.* the vev of the scalar in the corresponding  $U(1)$  vector multiplet, corresponds to collectively shifting the stack of semi-infinite D5s. The *axial*  $U(1)$  subgroup in  $U(N_f) \times U(N_f)$  corresponds to moving the two stacks in opposite directions; the *diagonal* subgroup corresponds to shifting both stacks (hence the original D5–brane) together. The latter is equivalent to a shift of the stack of the D3–branes, which is a  $U(1) \subset U(N_c)$  gauge transformation corresponding to the gauged baryonic symmetry. The former affects the fundamentals and the anti-fundamentals as does the axial subgroup of  $U(1)_{45} \times U(1)_{89}$  and is identified with turning on a real mass for  $U(1)_A$ .

On a circle (when  $x_3$  is compact) the  $U(1)_A$  is broken. In the brane picture this breaking can be visualized as follows. When moving the stack of  $D5_L$ s in  $x_3 < 0$  and the stack of  $D5_R$ s in  $x_3 > 0$ , charge conservation requires the generation of a  $(1, N_f)$  fivebrane along the directions  $x_3$  and  $x_7$  [43, 44] (see Figure 3). The NS–brane now cannot close anymore on the circle without breaking supersymmetry. This obstruction precludes the realization of the  $U(1)_A$  symmetry in the brane setup. This explains how the compactness of  $x_3$  geometrically reproduces the role of the  $\eta$ -superpotential in breaking the  $U(1)_A$  symmetry<sup>6</sup>. One might wonder what happens to the  $U(1)_B$  and why it is still a good symmetry. This symmetry is realized by the motion of the entire stack of D5–branes with respect to the stack of D3–branes along  $x_3$ . In this case the D5–branes can slide together on the NS–brane along  $x_3$ , without generating any  $(1, N_f)$  fivebrane, and the symmetry is realized in the brane picture. Observe that in the three-dimensional picture this symmetry is gauged, it is associated to the motion of the D3–branes rather than the motion of the D5–branes.

- The last symmetry is the topological  $U(1)_J$  shifting the dual photon. In the type III description the dual photon is not visible and a geometric interpretation would

<sup>6</sup> This mechanism is in spirit similar to the breaking of the axial symmetry in four-dimensional gauge theories. In fact, if we T–dualize and lift our configuration to M–theory, the system of D5s and NS' is lifted to a single M5–brane wrapped on a Riemann surface (*brane bending*).

require the lift of the configuration to M–theory<sup>7</sup> where the dual photon appears as a shift in  $x_{10}$ . This is related to the fact that this symmetry is not manifest in the Lagrangian but is an effect coming from the Bianchi identity. In the type IIB description we still have control of the real mass parameter associated to this symmetry, the Fayet–Iliopoulos (FI) term. In the brane picture it corresponds to the displacement of the NS and NS′–branes along  $x_7$ .

The same reduction can be performed in the magnetic picture. The discussion follows the one above and in the dual frame one obtains the same result when discussing the dual  $U(N_f - N_c)$  gauge group. Also in this case there are  $N_f$  dual fundamentals and anti-fundamentals and a meson  $M$ . The superpotential  $M q \tilde{q}$  is combined with the  $\eta'$  piece. This is the geometrical version of the reduction of the four-dimensional Seiberg duality for SQCD in terms of T–duality and branes. At finite radius, this theory can be treated as an effective three-dimensional theory if the radius of the T–dual circle is large enough, and in this sense it represents a new three-dimensional duality.

### 3.2 Aharony duality

One can flow to the duality of Aharony by turning on real mass terms for some of the quarks [26]. Here we reproduce this flow from the type IIB brane perspective where the real masses are generated by moving D5–branes in the  $x_3$  direction.

Consider the case with  $N_f + 2$  flavors. The flow is generated by breaking the  $SU(N_f + 2)^2$  flavor symmetry down to  $SU(N_f)^2 \times U(1)_A$ . We study the theory at an energy scale  $E < 1/\tilde{R}_3$  (where  $\tilde{R}_3 = \alpha'/R_3$  is the T–dual radius) so that  $x_3$  is effectively non-compact. In this large mass limit, the  $\eta$ –superpotential disappears and the axial  $U(1)_A$  is restored. The flavor symmetry is broken in the brane description by moving one D5–brane in the  $x_3 > 0$  direction and one D5 in the  $x_3 < 0$  direction (the stack of  $N_c$  D3–branes sits at  $x_3 = 0$ ). The configuration that we obtain is shown on the left-hand side of Figure 4. We end up with a  $U(N_c)$  gauge theory with  $N_f$  fundamentals and anti-fundamentals and vanishing superpotential. This is the electric theory studied in Aharony.

The dual picture is shown on the right-hand side of Figure 4. In this case, each D5–brane drags one D3 along the  $x_3$  direction and there are three gauge sectors,  $U(N_f - N_c) \times U(1)^2$  as expected from field theory. The chiral multiplets connecting the two sectors acquire a large mass as long as the D5–branes separate along  $x_3$ . However, in this case there are one fundamental and one anti-fundamental massless flavors and a meson with a superpotential interaction in each  $U(1)$  sector. The three sectors interact on the Coulomb branch. Indeed, if we consider the Coulomb branch in the brane picture, the D3s separate along  $x_3$  at equal distance. The  $U(1)$  sectors are crucial for understanding the structure of the moduli space of the dual phase. In the large mass limit, when the two D5s are far away in the  $x_3$  direction, two D3s of the stack of  $N_f - N_c$  D3–branes are free to move, and this parametrizes the Coulomb branch. Differently from the electric case, these D3s cannot be pushed

<sup>7</sup> Lifting our type IIB picture to M–theory leads to a two-dimensional theory. While there are analogies with the three-dimensional systems at hand, we cannot use it to make precise statements. This is essential in our case as there are important physical effects that are related to global boundary conditions (*i.e.* the fact that the  $x_3$  direction is periodic).

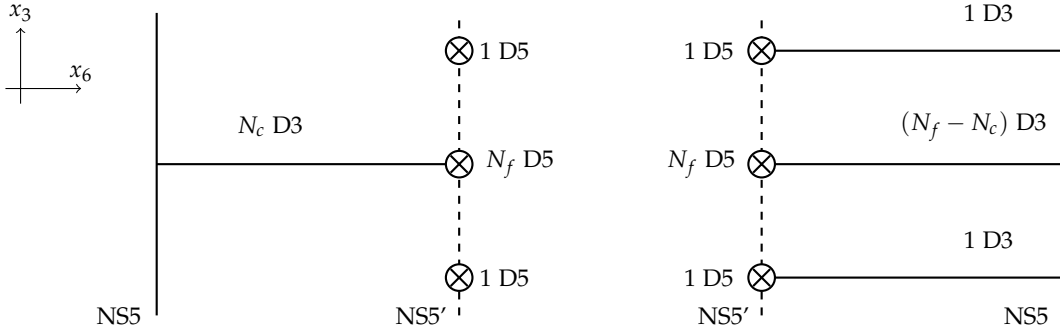


Figure 4: Electric and magnetic brane system for SQCD on  $\mathbb{R}^3 \times S^1$

to  $\pm\infty$ , because there are two extra D3-branes, the  $U(1)$  sectors. This constraint is reflected in an interaction between the  $U(N_f - N_c)$  sector and the  $U(1)$  sectors, this interaction is an AHW superpotential, coming from the broken  $U(N_f - N_c + 2)$  theory. At the level of the brane system it is due to the D1-strings between the NS-branes, the two D3s that are pushed far away in the  $x_3$  direction by the D5s, and the two D3s that parametrize the Coulomb branch of the  $U(N_f - N_c)$  sector.

As discussed in Section 2.2 the D1-branes create the non-perturbative AHW superpotential

$$W_{\text{AHW}} = e^{\Sigma^{(1)} - \Sigma_{\tilde{N}_c}} + e^{\Sigma_1 - \Sigma^{(2)}} + \eta' e^{\Sigma^{(2)} - \Sigma^{(1)}} = \tilde{y}_1 y + \tilde{y} y_2 + \eta' \tilde{y}_2 y_1. \quad (3.1)$$

Here  $\Sigma^1$  ( $\Sigma^2$ ) corresponds to the Coulomb branch coordinate of the 1st (2nd)  $U(1)$ -sector. Similarly,  $\Sigma_1$  and  $\Sigma_{\tilde{N}_c}$  correspond to the Coulomb branch coordinates that remain unlifted by (2.4), which we do not repeat here.

By going to large  $R_3$ , we can think of the dual version of each  $U(1)$  sector at  $x_3 > 0$  and  $x_3 < 0$ . Recall that a  $U(1)$  gauge theory with one pair of fundamental and anti fundamental is dual to the XYZ model. This duality is realized in the brane system by exchanging the NS and the D5-branes. Our  $U(1)$  sectors are similar to the gauge theory dual of the XYZ models, yet there is a slight modification due to the additional superpotential interaction  $M^{(i)} q^{(i)} \tilde{q}^{(i)}$  in each sector  $i = 1, 2$ . By duality, the  $i$ th sector corresponds to a deformation of the XYZ model, where the singlets  $X, Y$  and  $Z$  are identified with the meson  $N^{(i)} = q^{(i)} \tilde{q}^{(i)}$  and the Coulomb branch coordinates  $y_i, \tilde{y}_i$ . The resulting superpotential is given by

$$W_{\text{m}}^{(i)} = M^{(i)} N^{(i)} + N^{(i)} y_i \tilde{y}_i. \quad (3.2)$$

Upon integrating out the massive fields, the superpotential  $W_{\text{m}}^{(1)} + W_{\text{m}}^{(2)} + W_{\text{AHW}}$  becomes the interaction  $yY + \tilde{y}\tilde{Y}$  expected for the Aharony duality. This is done by interpreting  $\tilde{y}_1$  and  $y_2$  as the singlets  $Y$  and  $\tilde{Y}$  corresponding to the monopole operators of the electric phase.

We conclude this section with a general remark. For four-dimensional Seiberg duality, the brane construction is well understood. Using dimensional reduction, we have been able to describe Aharony duality in terms of branes, since the four-dimensional theories naturally provide a uv completion. This idea can be general-

ized and applied to other uv pairs flowing to Aharony duality. One can for example use the RG flow from Giveon–Kutasov [18] (GK) duality to Aharony duality recently studied in [45–48].

## 4 Extensions

In this section we introduce some extensions of the procedure explained above. The dimensional reduction of four-dimensional dualities to three dimensions can be generalized to more general gauge theories [26]. Many details of this construction are model-dependent: gauge group and matter content of a theory determine the IR dynamics which in turn affect the structure of the reduced theory. The brane construction provides a unified realization of the reduction of four-dimensional dualities and here we apply it to some dualities with a type IIA description.

We first use the brane picture to flow from the duality of Aharony to GK duality. This flow generates CS levels and is of importance for the analysis of other dualities in three dimensions.

Next we will discuss three-dimensional dualities for quiver gauge theories. By generating CS levels we obtain quivers describing the IR dynamics of M2–branes. This is of interest as it can link the dynamics of M2–branes to AdS<sub>5</sub>/CFT<sub>4</sub>.

Another generalization includes theories with a richer matter content. Here we explain how to reduce the dualities of [4] for theories with adjoint matter using the brane description. Our analysis reproduces the field theory results of [29].

### 4.1 Giveon–Kutasov duality

In this subsection we study the RG flow of four-dimensional Seiberg duality to GK duality for three-dimensional  $U(N)_k$  SQCD, where  $k$  is the Chern–Simons level. In field theory this has been studied in [49]. One can start from the Aharony electric phase with  $N_f + k$  flavors and use the global symmetry to assign a large mass to  $k$  fundamentals and anti-fundamentals, with the same (e.g. positive) sign. At large mass it generates the level  $k$  in the electric theory. In the dual theory after integrating out the massive fields (also the monopoles acquire a mass term) one is left with  $U(N_f - N_c + |k|)_{-k}$  SQCD with superpotential  $W_m = M q \tilde{q}$ . In the brane picture we first consider  $N_f + k$  D5–branes instead of  $N_f$ . Then we separate  $k$  D5s on the NS'–brane, and obtain two semi-infinite stacks. We move half of the stack to  $x_3 > 0$  and the other half to  $x_3 < 0$ . This process creates a  $(1, k)$  five-brane. In the limit when the  $(1, k)$  fivebrane becomes infinite a CS term is generated [43, 44]. At the level of field theory this reproduces the electric side of the GK duality. In the dual phase the situation is more complicated. The number of D3s created by this process is still  $N_f + k + 2$ , the gauge theory is now  $U(1)^2 \times U(N_f - N_c + k)$ . The motion of the D5s along  $x_3$  generates a non-trivial FI term for the  $U(1)$  sectors. This term is proportional to the axial real mass (in the flow  $U(1)_J$  and  $U(1)_A$  do mix indeed). This implies that the monopoles are massive as expected and they disappear. At the end one is left with the dual GK configuration.

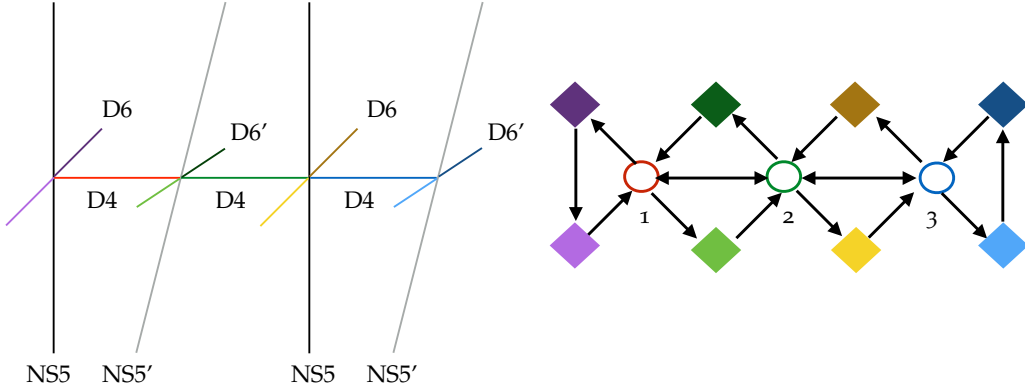


Figure 5: Brane description and linear quiver representing the product of four-dimensional  $SU(N)$  gauge group with flavor. We use the colors to identify the gauge and flavor symmetries associated to the D3 and D5-branes respectively.

## 4.2 Product groups

Consider a class of four-dimensional gauge theories that consist of products of  $G U(N)$  gauge groups with bifundamental and fundamental matter fields. This construction was first discussed in four dimensions in [50] for the case of the conifold. These systems have many possible Seiberg-dual phases in four dimensions. Here we discuss how this duality reduces to three dimensions along the lines explained above. We engineer this construction in a brane system and eventually relate this duality to the toric duality for M2-branes discussed in [20].

We start by considering a system of D4-branes suspended between  $G + 1$  NS and NS'-branes. On each NS (NS') brane we put a D6 (D6') brane. The NS-branes are extended along  $0123(45, 89)_\alpha$ , where  $(x, y)_\alpha$  denotes a rotation in the directions  $x, y$ . The NS' branes are extended along  $0123(45, 89)_{\alpha'}$ , where the prime denotes the susy-preserving complementary angle. Furthermore, D6 branes are along  $0123(45, 89)_{\alpha}7$  and D6' along  $0123(45, 89)_{\alpha'}7$ . In the language of quiver gauge theory we have a product of  $SU(N_i)$  gauge groups, where each  $N_i$  is the number of D4-branes between two consecutive fivebranes. There are bifundamental fields  $Q_{ij}$  connecting two consecutive gauge nodes. In our notation,  $Q_{ij}$  is in the fundamental representation of  $SU(N_i)$  and in the anti-fundamental representation of  $SU(N_j)$ . There are also fundamental matter fields, associated to the D6 and the D6'-branes: each gauge factor  $U(N_i)$  has a pair of fundamental and anti-fundamental  $(q_i, \tilde{q}_i)$  coming from the D6 (or D6') on the right and a pair  $(p_i, \tilde{p}_i)$  from the one on the left. The number of D6s and D6's has to be chosen consistently with the s-rule [37, 44]. In general there are  $SU(N_f)$  and  $U(1)$  flavor groups,  $G U(1)$  baryonic symmetry groups and the  $U(1)$  R-symmetry groups, but some of the  $U(1)$  flavor symmetries are anomalous. The brane system and the quiver for this case are shown in Figure 5.

There are two types of superpotential interactions. One involves only the bifun-

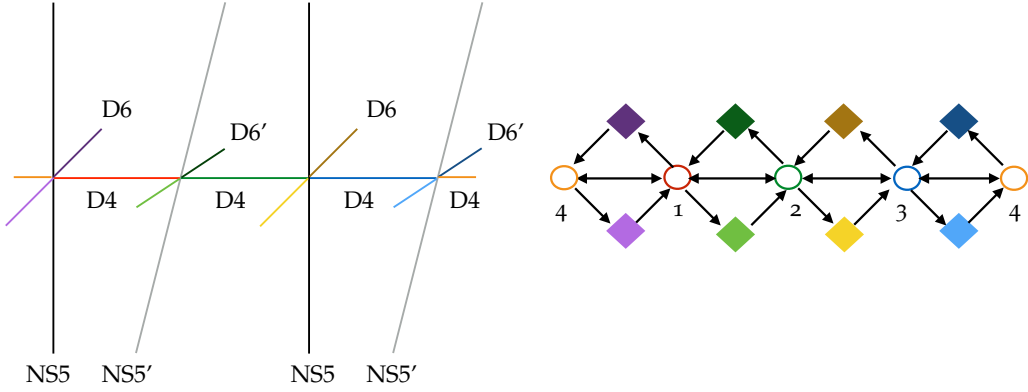


Figure 6: Brane description and circular quiver representing the product of four-dimensional  $SU(N)$  gauge group with flavor. We use the colors to identify the gauge and flavor symmetries associated to the D3 and D5-branes respectively.

damental fields<sup>8</sup>

$$W_{\text{bif}}^{(i)} = (-1)^i Q_{i-1,i} Q_{i,i+1} Q_{i+1,i} Q_{i,i-1}, \quad (4.1)$$

where the sign comes from the alternate signs of the adjoint masses. The interactions of the fundamental fields are

$$W_f^{(i)} = p_i Q_{i,i+1} \tilde{p}_i + \tilde{q}_i Q_{i+1,i} q_i \quad (4.2)$$

and

$$W_1 = p_1 M \tilde{q}_1 + q_G N \tilde{p}_G, \quad (4.3)$$

where  $M$  and  $N$  are gauge singlets that transform in the bifundamental representation of the first and the last pairs of flavor groups respectively.

Up to now we have considered  $x_6$  to be infinite and the D4-branes to be bounded by NS5-branes. We can allow a slightly different situation, where the direction  $x_6$  is compact and the quiver is circular. This case is shown in Figure 6. In the circular case the total number of gauge groups is  $G + 1$  and the superpotential (4.3) vanishes. These systems have a large number of Seiberg-dual phases in four dimensions, which may in principle be reduced to 3d.

At this point we can compactify the direction  $x_3$  and perform the by now familiar reduction. Observe that in the three-dimensional case we consider the baryonic symmetry as gauged, and the gauge groups are enhanced to  $U(N_i)$ . First we discuss the field theory reduction and then we show how this mechanism is realized in the brane setup. We first study the case of linear quivers and then switch to the circular ones.

<sup>8</sup> We describe in detail the case with alternating NS and NS'-branes but more general configurations are possible.



**Field theory reduction.** By putting the theory on  $\mathbb{R}^3 \times S^1$ , the additional superpotential

$$W_\eta^{(i)} = \eta_i Y_i \tilde{Y}_i, \quad (4.4)$$

is generated, where  $Y_i$  refers to the monopole of the  $i$ -th gauge group with magnetic flux  $(1, 0, \dots, 0)$ . This superpotential breaks the flavor symmetries that are anomalous in 4d. By proceeding similarly on the magnetic side we obtain a set of equivalent phases which generalize the dualities with  $\eta$ -superpotential proposed in [26].

- As in the case with one gauge group, we can flow to an Aharony-like duality for these quivers. We assign large and opposite real masses to the same amount of fundamentals and anti-fundamentals in each flavor sector. This procedure eliminates the  $\eta$ -superpotential and restores the 4d-anomalous flavor symmetries. In the dual theory the real masses are given consistently with the global symmetries; in addition one has to choose a non-trivial vacuum. This eliminates also the magnetic  $\eta'$ -superpotential and higgses the dual gauge group. The higgsing generates  $\Delta_{\text{HW}}$  superpotentials as in Eq. (2.2). This leads to the Aharony duality for this class of quiver gauge theories.
- Finally, by integrating out the remaining fundamental fields one can generate cs levels  $(k+i)$  for the  $i = 1, \dots, G$  gauge groups. This construction generalizes the GK duality [17, 20].
- For the circular quiver the  $i$ th D5-brane contributes with an opposite factor to the cs of the  $(i-1)$ st and of the  $i$ th gauge group. This implements the constraint  $\sum k_i = 0$  and we obtain the models studied in [20], the generalization to three dimensions of the  $L^{aba}$  theories [51–53]. They represent the moduli space of a stack of M2-branes probing a  $CY_4$  singularity. It has been shown in [20] that in these theories the three-dimensional Seiberg duality (*i.e.* the exchange of two consecutive five-branes) is the same as toric duality.

**Brane picture.** In this section we discuss the derivation of the dualities discussed above as seen from the brane picture.

We start with a configuration of D4-branes suspended between NS5 and NS5'-branes. For simplicity we choose the case with  $N_c$  D4-branes between each pair of five-branes even if more general configurations are possible. Then we add the D6 and D6'-branes on the NS5 and NS5'-branes. Finally we compactify the  $x_3$  direction and T-dualize, generalizing the construction of [54].

- D1-branes wrapping the circle in  $x^3$  create an  $\eta$ -superpotential for each gauge group. When doing the analogous operation in the magnetic phase, this produces the brane picture of the duality with  $\eta'$ -superpotential for the linear and circular quivers.
- Upon sending pairs of D5-branes to large distances on  $x^3$ , the D1-strings disappear and we reproduce the Aharony duality for quivers as in Section 3.2.
- As discussed in Section 4.1, we can generate cs levels from the brane picture and generalize the GK duality.
- In the circular case, if we integrate out all the flavors, the models correspond to the generalized  $L_{\{k_i\}}^{aba}$  type IIB description of M2-branes probing  $CY_4$  singularities.

These theories have been shown to enjoy a toric duality in three dimensions [20]. This connects this three-dimensional toric duality with four-dimensional Seiberg duality.

- Another class of models describing M2-branes probing  $CY_4$  singularities consists of circular vector-like quivers with bifundamental matter and chiral flavor. These models have been proposed in [55]. They can be obtained by assigning a large mass only to some of the flavors. This generates (semi-) integer cs levels and chiral matter. In the brane picture this is done by sending only half (*i.e.*  $N_f$ ) of the D5s which are broken on the NS-branes to large distances. The cs terms are due to the semi-infinite  $(1, N_f)$  fivebranes generated by this procedure.

We conclude this section but mentioning the possibility of reducing four dimensional chiral quiver gauge theories. Here the situation is more complicate, because their three dimensional counterpart, when decorated with cs levels do not have a simple interpretation in terms of M2-branes (for example the free energy does not scale as  $N^{3/2}$  [56] as expected from the conjectured gravity dual<sup>9</sup>). It would be interesting to study how the dimensional reduction of these theories can shed light on this problem.

### 4.3 Adjoint matter

In this section we study the dimensional reduction of four-dimensional Seiberg duality with adjoint matter to three dimensions. From the field theory point of view, this has been first studied in [29, 59]. Here we first review the field theoretical construction and then we discuss the mechanism from the brane perspective. In four dimensions this is called Kutasov–Schwimmer–Seiberg [4] (kss) duality.

The electric model is an  $SU(N_c)$  gauge theory with  $N_f$  fundamentals  $Q$  and  $N_f$  anti-fundamentals  $\tilde{Q}$  and one adjoint field  $X$ . There is a superpotential coupling

$$W_{\text{el}}^{\text{KSS}} = \text{Tr } X^{n+1} \quad (4.5)$$

with  $n \leq N_c$ . When  $nN_f > N_c$  there is a magnetic description. The dual field theory has an  $SU(N_c)$  gauge group with  $N_f$  dual quarks  $q$  and  $\tilde{q}$  and an adjoint  $Y$ . There are also electric mesons  $M_j$  appearing as elementary degrees of freedom in this dual phase. They have the form

$$M_j = QX^j\tilde{Q} \quad j = 0, \dots, n-1. \quad (4.6)$$

The superpotential of the dual theory is given by

$$W_{\text{m}}^{\text{KSS}} = \text{Tr } Y^{n+1} + \sum_{j=0}^{n-1} M_j q Y^{n-j-1} \tilde{q}. \quad (4.7)$$

The global symmetry group is  $SU(N_f)^2 \times U(1)_B \times U(1)_R$ . There is also an axial symmetry that is anomalous in four dimensions.

<sup>9</sup> See [57, 58] for a possible solution to this problem.

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_R$	$U(1)_I$
$Q$	$N_f$	1	1	$\Delta$	0
$\tilde{Q}$	1	$\overline{N_f}$	1	$\Delta$	0
$X$	1	1	0	$\frac{2}{n+1}$	0
$q$	$\overline{N_f}$	1	-1	$\frac{2}{n+1} - \Delta$	0
$\tilde{q}$	1	$N_f$	-1	$\frac{2}{n+1} - \Delta$	0
$Y$	1	1	0	$\frac{2}{n+1}$	0
$M_j$	$N_f$	$\overline{N_f}$	2	$2\Delta + \frac{2j}{n+1}$	0
$T_j$	1	1	$-N_f$	$N_f(1 - \Delta) - \frac{2}{n+1}(N_c - 1 - j)$	1
$\tilde{T}_j$	1	1	$-N_f$	$N_f(1 - \Delta) - \frac{2}{n+1}(N_c - 1 - j)$	-1
$t_j$	1	1	$N_f$	$N_f(\Delta - 1) + \frac{2}{n+1}(N_c + 1 + j)$	-1
$\tilde{t}_j$	1	1	$N_f$	$N_f(\Delta - 1) + \frac{2}{n+1}(N_c + 1 + j)$	1

Table 5: Global charges for the configuration in the KP duality

The three-dimensional reduction of this duality has been discussed in [29], generalizing the procedure of [26]. Here we sketch the main steps of this reduction. One first compactifies the theory on a circle of finite radius  $R_3$ . Then one realizes that in this case the analogue of the  $\eta$ -superpotential is generated in both the electric and the magnetic phases. These superpotentials are of the form

$$W_\eta = \eta \sum_{j=0}^{n-1} T_j \tilde{T}_{n-1-j}, \quad W_{\eta'} = \eta' \sum_{j=0}^{n-1} t_j \tilde{t}_{n-1-j}, \quad (4.8)$$

where  $T_j$  and  $\tilde{T}_j$  are monopole and anti-monopole operators. They can be written in terms of the original monopoles  $Y$  as  $T_j = YX^j$ . By using the same arguments of the SQCD case, the electric and the magnetic theories obtained by considering the superpotentials Eq. (4.8) are Seiberg-dual in three dimensions.

The next step consists in flowing from these dualities to the Aharony-like case. This last duality has been first introduced in three dimensions by Kim and Park in [23] (KP). This flow is similar to the one discussed for ordinary SQCD. One can first add a superpotential of the form

$$W_{\text{el}}^{P(X)} = \sum_{j=1}^n \alpha_j X^j \quad (4.9)$$

in the adjoint fields, and analogously on the magnetic side. This superpotential breaks the gauge group into a product of decoupled SQCDs, each with gauge symmetry  $U(r_i)$  with  $\sum_i r_i = N_c$ ,  $N_f$  fundamentals and anti-fundamentals and no adjoint. In each sector the reduction works like in SQCD. One can finally send the  $\alpha_i$  couplings to zero. As shown in [59], this procedure is consistent and can be applied also to the magnetic theory. Eventually the KP duality is obtained. The charges of the fields under the global symmetries in the KP duality are shown in Table 5.

Observe that one could break the gauge theory into a decoupled set of SQCDs also in the four-dimensional case and perform the reduction in the broken case.

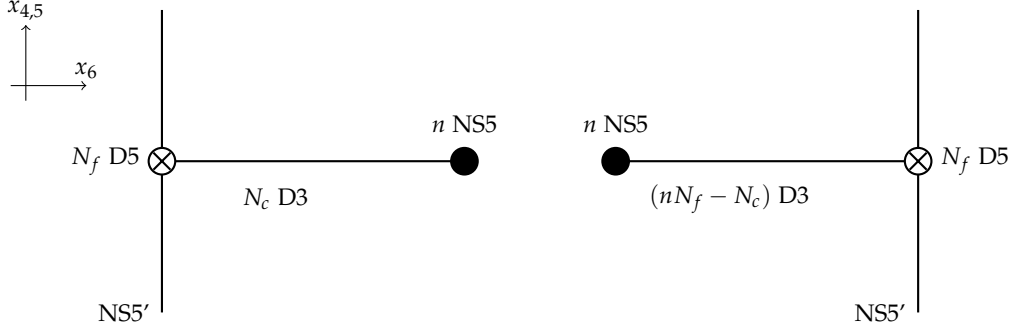


Figure 7: Electric and magnetic version of KSS from the brane picture.

This would have modified Eq. (4.8) to

$$W_\eta = \sum_{i=1}^n \eta_i Y_i \tilde{Y}_i, \quad W_{\eta'} = \sum_{i=1}^n \eta'_i y_i \tilde{y}_i, \quad (4.10)$$

where the subscript  $i$  labels the  $U(r_i)$  gauge group and  $y_i, Y_i$  are the monopoles in each phase. The relation between the superpotential in Eq. (4.8) and the one in Eq. (4.10) is obtained by the scale matching relation [4]

$$\Lambda^{2N_c - N_f} = \Lambda^{3r_i - N_f} \prod_{j \neq i} (\omega_i - \omega_j)^{r_i - 2r_j}, \quad (4.11)$$

where  $\omega_i$  parametrize the vev of the adjoint fields breaking  $U(N_c)$  to  $\prod U(r_i)$ . An analogous relation can be written for the magnetic case. This last observation is useful for the reduction of the kss duality from the brane picture.

In the brane description we modify the SQCD analysis by introducing a set of  $n$  NS5-branes instead of one. This induces the superpotential (4.5). The electric theory is represented on the left-hand side of Figure 7. The different steps of the duality are shown in Figure 8. First we separate the NS-branes, and this procedure generates the superpotential (4.9). This separation breaks the gauge symmetry to  $\prod_{i=1}^n U(r_i)$ . In the brane picture this is reflected by the separation of the  $N_c$  D3-branes along (8,9). Finally we move the NSs and the D3-branes in (8,9) and come back to the original stack. The final configuration is depicted on the right-hand side of Figure 7.

After T-duality along  $x_3$  the D4 and the D6-branes become D3s and D5s respectively, while the NS-branes are unchanged. The theory at finite radius is represented in Figure 8. Let us consider first the electric configuration (Figure 8 (a)). It consists of a set of decoupled SQCD models, each with  $N_f$  flavors. There are  $n$  of those sectors, each with gauge group  $U(r_i)$ , with  $\sum_{i=1}^n r_i = N_c$ . There is no adjoint matter anymore.

The superpotential (4.10) is generated. The same procedure can be implemented in the magnetic case. In this way we recover the duality with the  $\eta$ -superpotential discussed above. At this point we can flow to the KP duality by separating the D5-branes along  $x_3$  as done in the case of SQCD. The final duality is obtained by reconstructing the stack of  $n$  NS-branes, *i.e.* by sending the superpotential (4.9) to

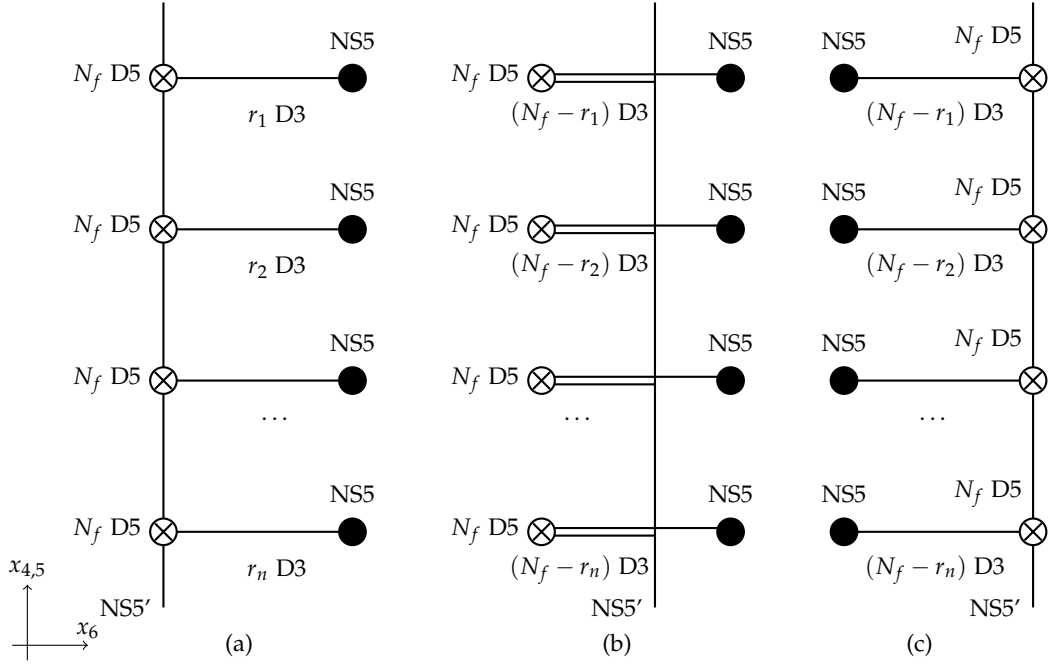


Figure 8: Duality in the KSS model

zero.

## 5 Outlook

In this section we outline further applications. A straightforward generalization involves real gauge groups or matter in more complicated gauge representations. It would also be interesting to extend the analysis to cases with higher supersymmetry or to other dimensions.

### 5.1 Orientifold planes

Real gauge groups (symplectic or orthogonal) and tensor matter (for example symmetric or antisymmetric representations) are obtained by adding orientifold planes. Here we discuss the strategy for the brane reduction for both cases.

**Real gauge groups.** The reduction of Seiberg duality for  $Sp(2N_c)$  SQCD with  $2N_f$  flavors has been discussed from the field-theoretical point of view in [26]. The  $SO(N_c)/O(N_c)$  cases have been studied in [27]. One can describe these theories in terms of the brane systems by adding O6-planes or O4-planes. Different constructions are possible, we refer to [60] for reference. The main idea is to study the reduction and the generation of the  $\eta$ -superpotential in the same way as above: one first compactifies  $x_3$  and then performs a T-duality. An  $\eta$ -superpotential is generated by the compactness of  $x_3$  because of the D1-strings stretched between

the D3-branes. We leave the detailed analysis of this case and the matching with the results of [26, 27] for future works.

**Tensor matter.** The orientifold projection can be applied to more complicated brane systems. One can for example consider a stack of D4-branes between two parallel NS-branes. If there are an NS'-brane and an orientifold plane between the two NS-branes we still have a unitary theory, but in this case it includes tensor matter (symmetric or antisymmetric with its own conjugate representation). One can also consider fundamental matter fields by including D6-branes. In this case there are Seiberg-dual phases, obtained by interchanging the NS and NS'-branes. We refer to [60] for references concerning the brane realization of these dualities. There are some cases in which the dual theory is s-confining, and the reduction with the  $\eta$ -superpotential has been applied to these systems in [30]. One can in principle describe the reduction of the s-confining theories in terms of brane systems.

Note that the theory on the circle can have both a Coulomb branch and a Higgs branch. Extra massless fields may appear at the intersection point between these two branches. In some regions of this moduli space an  $\eta$ -superpotential is generated, while in other regions it is not. It would be extremely interesting to reproduce this behavior at the level of the brane system.

## 5.2 Higher supersymmetry

Another natural generalization of the construction that we have presented consists in cases with higher supersymmetry. We can consider the system studied in Section 3, but with the NS'-brane rotated into an NS-brane resulting in a system with two parallel NS-branes. This system has  $\mathcal{N} = 2$  supersymmetry in four dimensions. By applying the same reduction discussed above one ends up with a three-dimensional system with  $\mathcal{N} = 4$  supersymmetry. This system has an  $SO(4)$  global R-symmetry corresponding to the rotations in 4589 *i.e.* the  $SU(2)_C \times SU(2)_H$  symmetry underlining three-dimensional mirror symmetry. As discussed in [61] one can in general reduce four-dimensional dualities for class-S theories [62] to three dimensions. Another possibility consists in studying the  $\mathcal{N} = 1$  dualities discussed in [63, 64], obtained by coupling two copies of the theories of [62].

## 5.3 Reduction to two dimensions

Finally we can consider the reduction of four-dimensional Seiberg dualities to two dimensions. In this case it is possible to switch on new parameters related to the compactification torus, *i.e.* by introducing twistings [65–68] or magnetic fields [69]. The resulting gauge theories inherit these parameters as couplings for the fields, *e.g.* twisted masses for the adjoints and fundamentals [65]. We are thus lead to Seiberg-like dualities for families of  $\mathcal{N} = (1, 1)$  or  $\mathcal{N} = (0, 2)$  theories in two dimensions with matching parameters that can be read off directly from brane constructions analogous to the one that we have introduced in this paper. Such dualities have been observed directly from a two-dimensional perspective in the literature [70–73].

## Acknowledgments

It is a pleasure to thank Alberto Zaffaroni, Costas Bachas, Diego Redigolo, Mario Martone and Prarit Agarwal for useful discussions and comments and Alberto Zaffaroni for insightful comments on the draft. A.A. is funded by the European Research Council (ERC-2012-ADG\_20120216) and acknowledges support by ANR grant 13-BS05-0001. D.F. is FRS-FNRS Chargé de Recherches. He acknowledges support by the FRS-FNRS, by IISN - Belgium through conventions 4.4511.06 and 4.4514.08, by the Communauté Française de Belgique through the ARC program and by the ERC through the SyDuGraM Advanced Grant. C.K. acknowledges support by ANR grant 12-BS05-003-01 and by Enhanced Eurotalents, which is co-funded by CEA and the European Commission. D.O. would like to thank the theory group at CERN for hospitality.

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