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Completeness relations are associated through Mercer's theorem to complete orthonormal basis of square integrable functions, and prescribe how a Dirac delta function can be decomposed into basis of eigenfunctions of a Sturm-Liouville problem. We use Gegenbauer's addition theorem to prove a relation very close to a completeness relation, but for a set of Bessel functions not known to form a complete basis in $L^2[0,1]$.

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I. INTRODUCTION

If one studies Helmholtz equation on the interior of a sphere of unitary radius subjected to condition of regularity of the solution at $r = 0$, and such that the solution vanishes at $r = 1$, the Green's function can be constructed in terms of the eigenfunctions of the Sturm-Liouville problem using Mercer's formula, from which follows the *completeness relation* for spherical Bessel functions:

$$\sum_{n=1}^{\infty} \frac{j_l(a_n^l x) j_l(a_n^l y)}{[j_{l+1}(a_n^l)]^2} = \frac{1}{2}\delta(x-y) \tag{1}$$

where $0 \leq x, y \leq 1$, and a_n^l is the n-th zero of the spherical Bessel function of order l. The set of functions $j_l(a_n^l x)$ forms a complete basis of orthogonal functions on $L^2[0,1]$, and this kind of relations are very useful in quantum mechanics, for example.

Since Bessel functions are associated to representations of Euclidean group, they appear abundantly in physical problems, and in particular on the problem of random flights, that are the random motions in a D-dimensional euclidean space performed by a particle that, always with constant speed, change the direction of its motion after a set of instants of time distributed accordingly to some law[1–9]. The probability for the walker to be at a distance r from the origin of the motion after n changes of directions is related to an integral of $n + 1$ Bessel functions[10]. We can study random flights that start at a subspace D_1-dimensional of a D_2-dimensional space and, after a given number of steps on the space of smaller dimensions, accesses the space of larger dimension. Such problems can be easily imagined in physical situations and is particularly realized in Cosmic Microwave Background (CMB) physics[11,12]. The completeness-like relation that we shall prove here plays the role of a consistency relation for the decomposition of such flights, and was first found in the context of CMB physics[13].

II. COMPLETENESS-LIKE RELATIONS FOR BESSEL FUNCTIONS

Theorem II.1 (Completeness-like relation for Bessel functions) *Let $0 \leq x, y \leq 1$, $D_2 \geq D_1$, $D_2 - D_1 > 0$ an even number, and $\lambda_k^{D_2/2-1}$ the k-th zero of the Bessel function*

$J_{D_2/2-1}(\,.\,)$. *Then:*

$$\sum_{k=1}^{\infty} \frac{J_{D_1/2-1}\left(\lambda_k^{D_2/2-1} x\right) J_{D_1/2-1}\left(\lambda_k^{D_2/2-1} y\right)}{J^2_{D_2/2}\left(\lambda_k^{D_2/2-1}\right)} = \frac{1}{2x}\delta\left(x-y\right) . \tag{2}$$

Proof In order to establish our result, we shall look at the integral

$$\mathcal{I} := \int dq\, q^{D_2-1} \frac{J_{D_2/2-1}(qr)}{(qr)^{D_2/2-1}} \frac{J_{D_2/2-1}(qs_1)}{(qs_1)^{D_2/2-1}} \frac{J_{D_1/2-1}(qr_1)}{(qr_1)^{D_1/2-1}} \tag{3}$$

in three different ways. First, we can contract the two Bessel functions of order $D_2/2-1$ using Eq. (A2), what yields:

$$\begin{aligned} \mathcal{I} = & \frac{2^{D_1/2-1}}{(rs_1)^{\Delta/2}} \frac{(\Delta/2)!\Gamma(D_1/2-1)}{2\pi\Gamma(D_1-2+\Delta/2)} \int_0^{\pi} d\alpha \sin^{D_1-2}\alpha\, C^{D_1/2-1}_{\Delta/2}(\cos\alpha) \\ & \times \frac{1}{(\rho r_1)^{D_1/2-1}} \int dq\, q\, J_{D_1/2-1}(q\rho)\, J_{D_1/2-1}(qr_1)\,, \end{aligned} \tag{4}$$

with $\rho^2 := r^2 + s_1^2 - 2rs_1\cos\alpha$, and

$$\int dq\, q\, J_{D_1/2-1}(q\rho)\, J_{D_1/2-1}(qr_1) = \frac{1}{r_1}\delta(\rho - r_1)\,. \tag{5}$$

Secondly, we can regard the integral $\int dq\, q\, J_{D_1/2-1}(q\rho)\, J_{D_1/2-1}(qr_1)$ as a function of ρ, with r_1 fixed. Since Eq. (3) is a discontinuous Weber-Schafheitlin integral[10], we know that it vanishes identically if r_1, s_1, and r do not form a triangle. We must have, therefore, $|s_1 - r| \le \rho \le s_1 + r =: S$. Since the domain of ρ is contained in $[0, S]$, we can construct the Fourier-Bessel decomposition[10,14]:

$$\int dq\, q\, J_{D_1/2-1}(q\rho)\, J_{D_1/2-1}(qr_1) = \sum_{k=1}^{\infty} c_k J_{D_1/2-1}\left(\lambda_k^{D_1/2-1}\frac{\rho}{S}\right) \tag{6}$$

with

$$c_k = \frac{2}{S^2 J^2_{D/2}\left(\lambda_k^{D/2-1}\right)} \int_0^S d\rho\, \rho \int dq\, q\, J_{D_1/2-1}(q\rho)\, J_{D_1/2-1}(qr_1)\, J_{D/2-1}\left(\lambda_k^{D/2-1}\frac{r}{S}\right) , \tag{7}$$

and λ_k^{ν} being the k-th zero of the Bessel function $J_\nu(\,\cdot\,)$. The integral $\int dq\, q\, J_{D_1/2-1}(q\rho)\, J_{D_1/2-1}(qr_1)$ vanishes identically if $\rho > S$, what allows us to extend the limit of integration to infinity. Applying the Fourier-Bessel integral[15]

$$f(x) = \int_0^{\infty} J_n(tx)\, t \left[\int_0^{\infty} f(x') J_n(tx') x' dx'\right] dt \tag{8}$$

to Eq. (7), and inserting the result into Eq. (6), we obtain:

$$\int dq\, q\, J_{D_1/2-1}(q\rho)\, J_{D_1/2-1}(qr_1) = \sum_{k=1}^{\infty} \frac{2}{S^2 J^2_{D_1/2}\left(\lambda_k^{D_1/2-1}\right)} \times J_{D_1/2-1}\left(\lambda_k^{D_1/2-1}\frac{\rho}{S}\right) J_{D_1/2-1}\left(\lambda_k^{D_1/2-1}\frac{r_1}{S}\right) . \qquad (9)$$

Collecting the partial results presented in Eqs. (4), and (9), we write integral (3) as:

$$\mathcal{I} = \frac{2^{D_1/2-1}}{(rs_1)^{\Delta/2}} \frac{(\Delta/2)!\Gamma(D_1/2-1)}{2\pi\Gamma(D_1-2+\Delta/2)} \int_0^{\pi} d\alpha \sin^{D_1-2}\alpha\, C_{\Delta/2}^{D_1/2-1}(\cos\alpha) \times \frac{1}{(\rho r_1)^{D_1/2-1}} \sum_{k=1}^{\infty} \frac{2}{S^2 J^2_{D_1/2}\left(\lambda_k^{D_1/2-1}\right)} J_{D_1/2-1}\left(\lambda_k^{D_1/2-1}\frac{\rho}{S}\right) J_{D_1/2-1}\left(\lambda_k^{D_1/2-1}\frac{r_1}{S}\right) . \qquad (10)$$

We can now consider the third way of treating Eq. (3). Since r, r_1, and s_1 must be related to sides of a triangle, the inequality $|r_1 - s_1| \leq r \leq r_1 + s_1 := \tilde{S}$ must hold. Hence the integral $\int dq\, q\, J_{D_2/2-1}(qr)\, J_{D_2/2-1}(qs_1)\, \frac{J_{D_1/2-1}(qr_1)}{(qr_1)^{D_1/2-1}}$ can be expressed in terms of a Fourier-Bessel series:

$$\int dq\, q\, J_{D_2/2-1}(qr)\, J_{D_2/2-1}(qs_1)\, \frac{J_{D_1/2-1}(qr_1)}{(qr_1)^{D_1/2-1}} = \sum_{k=1}^{\infty} \frac{2}{\tilde{S}^2 J^2_{D_2/2}\left(\lambda_k^{D_2/2-1}\right)} \times J_{D_2/2-1}\left(\lambda_k^{D_2/2-1}\frac{r}{\tilde{S}}\right) J_{D_2/2-1}\left(\lambda_k^{D_2/2-1}\frac{s_1}{\tilde{S}}\right) \frac{J_{D_1/2-1}\left(\lambda_k^{D_2/2-1}\frac{r_1}{\tilde{S}}\right)}{\left(\lambda_k^{D_2/2-1}\frac{r_1}{\tilde{S}}\right)^{D_1/2-1}} . \qquad (11)$$

We can use Eq. (A2) to contract the two Bessel functions of order $D_2/2-1$, obtaining:

$$\mathcal{I} = \frac{2^{D_1/2-1}}{(rs_1)^{\Delta/2}} \frac{(\Delta/2)!\Gamma(D_1/2-1)}{2\pi\Gamma(D_1-2+\Delta/2)} \int_0^{\pi} d\alpha \sin^{D_1-2}\alpha\, C_{\Delta/2}^{D_1/2-1}(\cos\alpha) \times \frac{1}{(\rho r_1)^{D_1/2-1}} \sum_{k=1}^{\infty} \frac{2}{\tilde{S}^2 J^2_{D_2/2}\left(\lambda_k^{D_2/2-1}\right)} J_{D_1/2-1}\left(\lambda_k^{D_2/2-1}\frac{\rho}{\tilde{S}}\right) J_{D_1/2-1}\left(\lambda_k^{D_2/2-1}\frac{r_1}{\tilde{S}}\right) , \qquad (12)$$

where ρ is also given by the relation $\rho^2 = r^2 + s_1^2 - 2rs_1\cos\alpha$.

Since the function $\frac{\sin^{D_1-2}\alpha\, C_{\Delta/2}^{D_1/2-1}(\cos\alpha)}{\rho^{D_1/2-1}}$ does not vanish for all $\alpha \in [0,\pi]$ for any D_1, Δ, we conclude, comparing Eqs. (10) and (12), that:

$$\sum_{k=1}^{\infty} \frac{2}{S^2 J^2_{D_1/2}\left(\lambda_k^{D_1/2-1}\right)} J_{D_1/2-1}\left(\lambda_k^{D_1/2-1}\frac{\rho}{S}\right) J_{D_1/2-1}\left(\lambda_k^{D_1/2-1}\frac{r_1}{S}\right) = \sum_{k=1}^{\infty} \frac{2}{\tilde{S}^2 J^2_{D_2/2}\left(\lambda_k^{D_2/2-1}\right)} J_{D_1/2-1}\left(\lambda_k^{D_2/2-1}\frac{\rho}{\tilde{S}}\right) J_{D_1/2-1}\left(\lambda_k^{D_2/2-1}\frac{r_1}{\tilde{S}}\right) . \qquad (13)$$

The left hand side of this equation is equal to $\frac{1}{r_1}\delta(\rho - r_1)$ because of the orthogonality of Bessel functions, as stated in Eq. (5), and therefore

$$\sum_{k=1}^{\infty} \frac{J_{D_1/2-1}\left(\lambda_k^{D_2/2-1}\frac{\rho}{\tilde{S}}\right) J_{D_1/2-1}\left(\lambda_k^{D_2/2-1}\frac{r_1}{\tilde{S}}\right)}{J^2_{D_2/2}\left(\lambda_k^{D_2/2-1}\right)} = \frac{1}{2\left(\frac{r_1}{\tilde{S}}\right)}\delta\left(\frac{\rho}{\tilde{S}} - \frac{r_1}{\tilde{S}}\right), \tag{14}$$

as we wanted to demonstrate. ∎

The form of the equation Eq. (14) resembles the well known completeness relation (1) with the striking difference that in Eq. (1) the zeros of the Bessel function appearing in the sum belong the to the Bessel function of that order. In Eq. (14), however, we sum over Bessel functions of order $D_1/2 - 1$ with zeroes in their argument belonging to the Bessel function of order $D_2/2 - 1$.

The integral $\mathcal{I}$ is directly associated, on the theory of random flights, to the probability density of finding the walker at a distance r from the origin after realizing one step of length r_1 in a space of dimension D_1, and one step of length s_1 on a D_2-dimensional space. These two steps can actually be effective steps associated to contraction of all steps executed on the spaces of each dimension[11]. The completeness-like relation proved here assures that contraction of steps in any possible way leads to consistent results.

III. DISCUSSION

We have demonstrated an identity formally close to the completeness relation for Bessel functions, that we called completeness-like relation. The curious fact about this relation is that sequences of products of Bessel functions evaluated at the zeroes of a Bessel function of different order converges to a Dirac distribution. This may constitute an step into the generalization of this result in the context of Schlömilch's series.

The identity (14) is direct consequence of the Gegenbauer addition theorem and Fourier-Bessel series expansions, and holds for all $D_1, D_2 > 0$ such that $D_2 - D_1 = \Delta \in \mathtt{N}$, Δ even. $D_1, D_2 > 0$ must be positive because the set $\{\sqrt{x}J_\nu(\lambda_k^\nu x)\}_{k\in\mathtt{N}}$ only forms a complete set in $L^2(0,1)$ for $\nu > -1$, which is necessary for definiteness of the Fourier-Bessel series.

A theorem demonstrated in Ref. 16 states: *If $\nu > -1/2$, the set $\{x^{1/2}J_\nu(\lambda_n x)\}$ forms a complete (hence total) sequence in $L^p(0,1)$, $1 \leq p < \infty$, if for all sufficiently large n we* *have:*

$$0 < \lambda_n \leq \pi \left(n + \frac{1}{4} + \frac{\nu}{2} - \frac{1}{2p} \right) .$$

Completeness, in the sense of this theorem, means that if $\int_0^1 x^{1/2} J_\nu(\lambda_n x) g(x) dx = 0$ for all $\nu > -1/2$, with $g(x) \in L^p[0,1]$, then $g(x)$ vanishes almost everywhere on $(0,1)$[17]. For large values of the argument, the Bessel functions $J_\nu(z)$ behave like $\cos(z - \nu\pi/2 - \pi/4)$[10], and the theorem assures that the set $\{x^{1/2} J_\nu(\lambda_n x)\}$ is complete in the Hilbert space $L^2[0,1]$ when λ_n are taken to be the positive real zeros of $J_\mu(z)$, but only if $\mu \leq \nu$. Since in our case $D_2 \geq D_1$ (which corresponds to $\mu \geq \nu$), what we have shown is a completeness-like relation for a set of functions whose completeness cannot be discussed inside the scope of the aforementioned theorem. It is not known to the authors whether there are extensions of this theorem that include the case presented here, nor is it clear what is the meaning of the completeness-like relation that we obtained in the general context of the completeness of sets of Bessel functions in $L^p[0,1]$.

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Appendix A: Gegenbauer addition theorem

For Bessel functions of first kind, the Gegenbauer addition theorem states that[10]:

$$\int_0^\pi \frac{J_\nu(Z^2 + z^2 - 2zZ\cos\alpha)}{(Z^2 + z^2 - 2zZ\cos\alpha)^{\nu/2}} C_n^\nu(\cos\alpha) \sin^{2\nu}\alpha \, d\alpha = \frac{\pi\,\Gamma(2\nu + n)}{2^{\nu-1}\, n!\, \Gamma(\nu)} \frac{J_{\nu+n}(Z)}{Z^\nu} \frac{J_{\nu+n}(z)}{z^\nu} \tag{A1}$$

where $\nu > -1/2$, $\nu \in \mathtt{R}$, $m \in \mathtt{N}$, and $C_n^\nu(\cos\alpha)$ are Gegenbauer polynomials, defined by the relation:

$$\frac{1}{(1 - 2t\cos\alpha + t^2)^\nu} = \sum_{n=0}^\infty C_n^\nu(\cos\alpha)\, t^n .$$

As immediate consequence we can construct the contraction of two Bessel function of a given order into a Bessel function of smaller order:

$$\frac{J_{D_2/2-1}(ql_1)}{(ql_1)^{D_2/2-1}}\frac{J_{D_2/2-1}(ql_2)}{\left(\frac{ql_2}{2}\right)^{D_2/2-1}} = \frac{2^{D_1/2-1}}{q^{\Delta}\left(\frac{l_1 l_2}{2}\right)^{\Delta/2}}\frac{(\Delta/2)!\Gamma(D_1/2-1)}{2\pi\Gamma(D_1-2+\Delta/2)} \times \int_0^{\pi} d\alpha\, \frac{J_{D_1/2-1}(q\rho)}{\left(\frac{q\rho}{2}\right)^{D_1/2-1}}\, C_{\Delta/2}^{D_1/2-1}(\cos\alpha)\,\sin^{D_1-2}\alpha\,, \qquad \text{(A2)}$$

where $D_2 = D_1 + \Delta$, $\Delta \in \mathbb{N}$ is an even number, and $\rho^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos\alpha$.

Here we see that the Gegenbauer addition theorem allows us to contract the product of two Bessel functions of order $D_2/2-1$ into a marginalization over one Bessel function of order $D_1/2-1$ *if* Δ is an even number, since the lower index of the Gegenbauer polynomials must be an integer. Even if the Gegenbauer polynomials can be defined (through its hypergeometrical representation) for non-integer lower indexes, extensions of the Gegenbauer addition theorem to these cases are not known by the authors.

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