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Dyson hierarchical quantum ferromagnetic Ising chain with pure or random transverse fields

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The Dyson hierarchical version of the quantum Ising chain with Long-Ranged power-law ferromagnetic couplings $J(r) \propto r^{-1-\sigma}$ and pure or random transverse fields is studied via real-space renormalization. For the pure case, the critical exponents are explicitly obtained as a function of the parameter σ, and are compared with previous results of other approaches. For the random case, the RG rules are numerically applied and the critical behaviors are compared with previous Strong Disorder Renormalization results.

I. INTRODUCTION

The quantum Ising model is the basic model in the field of zero-temperature quantum phase transitions [1]. For the short-ranged case on hypercubic lattice in dimension d [1], the dynamical exponent relating time and space via $t = L^z$ takes the simple value

$$z_{SR}^{pure} = 1 \tag{1}$$

This means that the time simply plays the role of an additional spatial dimension in the quantum-classical correspondence, so that the d-dimensional quantum Ising model is equivalent to the *classical* Ising model in $D = d + 1$ dimensions. It is also interesting to consider the long-ranged quantum Ising chain

$$H_{LR}^{pure} = -\sum_i h\sigma_i^z - \frac{1}{2}\sum_{i,j} J_{i,j}\sigma_i^x\sigma_j^x \tag{2}$$

with uniform transverse field h and power-law ferromagnetic coupling

$$J_{i,j} = \frac{1}{|i-j|^{1+\sigma}} \tag{3}$$

in order to study how the critical properties depend upon the parameter $\sigma > 0$ [2]. This quantum chain is equivalent to a $2D$ classical Ising model with nearest-neighbor coupling in the 'time' direction and long-ranged power-law coupling in the 'spatial' direction, so that the dynamical exponent $z(\sigma)$ is not given by the short-ranged isotropic value of Eq. 1 anymore, as a consequence of the strong anisotropy between time and space. In particular in the mean-field region $0 < \sigma < \sigma_u = 2/3$ [2] (see the reminder in Appendix A), the dynamical exponent z, the correlation length exponent ν and the anomalous dimension η read [2]

$$\begin{aligned} z_{MF}^{pure}(0 < \sigma < \sigma_u = 2/3) &= \frac{\sigma}{2} \\ \nu_{MF}^{pure}(0 < \sigma < \sigma_u = 2/3) &= \frac{1}{\sigma} \\ \eta_{MF}^{pure}(0 < \sigma < \sigma_u = 2/3) &= 2 - \sigma \end{aligned} \tag{4}$$

It turns out that the other problem of the *dissipative* short-ranged quantum spin chain is equivalent after integration over the bath degrees of freedom to a $2D$ classical model with nearest-neighbor coupling in the 'spatial' direction and long-ranged power-law coupling in the 'time' direction, where the parameter σ now characterizes the bath spectral function $J(\omega \to 0) \propto \omega^\sigma$ [3–6]: the value $\sigma = 1$ corresponds to the most studied Ohmic damping, whereas the region $0 < \sigma < 1$ corresponds to sub-Ohmic damping, and the region $\sigma > 1$ to super-Ohmic damping. The dissipative short-ranged chain is thus equivalent to the model of Eq. 2 after the interchange of space and time. So the dynamical exponent z', the correlation length exponent ν', and the anomalous dimension η' of the dissipative quantum chain are given by $z' = 1/z$, $\nu' = \nu z$ and $z + \eta - 1 = \frac{z'+\eta'-1}{z'}$ As a consequence, the values $z' \simeq 2$, $\nu' \simeq 0.64$ and $\eta' = 0$ measured via Monte-Carlo for the dissipative Ohmic chain [3, 5, 6] translate for the quantum chain of Eq. 2 into

$$\begin{aligned} z_{LR}^{pure}(\sigma = 1) &\simeq 0.5 \\ \nu_{LR}^{pure}(\sigma = 1) &\simeq 1.28 \\ \eta_{LR}^{pure}(\sigma = 1) &\simeq 1. \end{aligned} \tag{5}$$

Let us now consider the effects of randomness in the transverse-fields h_i

$$H_{LR}^{random} = -\sum_i h_i \sigma_i^z - \sum_{(i,j)} J_{i,j} \sigma_i^x \sigma_j^x \tag{6}$$

The relevance of a small disorder at pure quantum phase transitions needs to be discussed from two points of view [7] : on one hand the Harris criterion [8] or equivalently the Chayes *et al* inequality [9] imply that the pure fixed point can be stable only if $\nu^{pure} \geq 2/d = 2$ here in the spatial dimension $d = 1$; on the other hand, the analysis of rare regions [7] shows that it is important to compare the dimensionality d_{RR} of rare regions (here $d_{RR} = 1$ since the disorder is actually 'infinitely' correlated along the time-direction) and the lower critical dimension $d_l^{class} = 1$ sufficient to obtain ferromagnetic ordering : the case $d_{RR} = d_l^{class}$ corresponds to models where rare regions can play an essential role at criticality (see [7] for more details). The random LR chain of Eq. 6 with a finite initial disorder has been studied recently [10] via the Strong Disorder Renormalization (see [11] for a review) : the main results [10] (see also the related work [12] in arbitrary dimension d) are the following critical dynamical exponent

$$z_{SDRG}^{random}(\sigma) = 1 + \sigma \tag{7}$$

and the essential singularity of the correlation length as the control parameter θ approaches its critical value θ_c

$$\ln \xi_{SDRG}^{random} \propto \frac{1}{|\theta - \theta_c|} \tag{8}$$

corresponding formally to an infinite correlation length exponent

$$\nu_{SDRG}^{random}(\sigma) = +\infty \tag{9}$$

These properties should be contrasted with the short-ranged random Chain governed by an Infinite Disorder Fixed Point characterized by an infinite dynamical exponent

$$z_{SR}^{random}(\sigma) = +\infty \tag{10}$$

and two finite correlation length exponents $\nu_{SRtyp}^{random} = 1$ and $\nu_{SRav}^{random} = 2$ [13]. Note that the effects of disorder on the dissipative quantum chain mentioned above has been also much studied via Strong Disorder RG [14, 15] and via Monte-Carlo [16], but here the problem is not equivalent to Eq. 6, as a consequence of the columnar nature of the disorder along the time direction in each problem (the interchange of space and time discussed above for the pure case is not possible anymore).

Since the obtained critical dynamical exponent of Eq. 7 is finite and not infinite as in the short-ranged case (Eq. 10), the Strong Disorder RG approach is not exact asymptotically, but only approximate. As a consequence, it seems useful to analyze the critical properties with another approach in order to compare the results. For the random Short-Ranged chain, the seld-dual block RG procedure first introduced for the pure chain [17] has been found recently to be able to reproduce the Fisher Infinite-Disorder fixed point [18–20] : this shows that the same block RG rules can lead to conventional critical behavior for the pure chain and to Infinite disorder critical behavior for the random chain depending on the initial condition of the RG flow. The aim of the present paper is to study via some block renormalization the Dyson hierarchical analog of the pure and the random long-ranged chain of Eq. 2 and Eq. 6.

The paper is organized as follows. In section II, we introduce the Dyson hierarchical quantum Ising model and derive the renormalization rules. In section III, the RG equations for the Dyson analog of the pure long-ranged chain of Eq. 2 are solved analytically. In section IV, the RG rules for the Dyson analog of the random long-ranged chain of Eq. 6 are studied numerically to obtain the critical properties. Our conclusions are summarized in section V. Appendix A contains a reminder on the mean-field theory for the pure long-ranged chain.

II. RENORMALIZATION RULES FOR THE DYSON HIERARCHICAL MODEL

A. Dyson hierarchical version of the Long-Ranged Quantum Ising Chain

In the field of long ranged models, it is very useful to consider their Dyson hierarchical analogs, where real space renormalization procedures are usually easier to define and to solve as a consequence of the hierarchical structure. The Dyson hierarchical classical ferromagnetic Ising model [21] has been much studied by both mathematicians [22–25] and physicists [26–30]. More recently, Dyson hierarchical versions have been considered for various disordered systems, either classical like random fields Ising models [31, 32] and spin-glasses [33–37], or quantum like Anderson localization models [39–46].

Here we introduce the Dyson hierarchical analog of the Long-Ranged Quantum Ising Chain of Eq. 6 as follows. The Hamiltonian for 2^n quantum spins can be decomposed as a sum over the generations $k = 0, 1, .., n-1$

$$H_{(1,2^n)} = \sum_{k=0}^{n-1} H_{(1,2^n)}^{(k)} \tag{11}$$

The Hamiltonian of generation $k = 0$ contains the transverse fields h_i and the lowest order couplings $J^{(0)}$

$$H_{(1,2^n)}^{(k=0)} = -\sum_{i=1}^{2^n} h_i \sigma_i^z - \sum_{i=1}^{2^{n-1}} J^{(0)} [\mu_{2i-1}\sigma_{2i-1}^x][\mu_{2i}\sigma_{2i}^x] \tag{12}$$

The Hamiltonian of generation $k = 1$ reads

$$H_{(1,2^n)}^{(k=1)} = -\sum_{i=1}^{2^{n-2}} J^{(1)} \left[\mu_{4i-3}\sigma_{4i-3}^x + \mu_{4i-2}\sigma_{4i-2}^x\right]\left[\mu_{4i-1}\sigma_{4i-1}^x + \mu_{4i}\sigma_{4i}^x\right] \tag{13}$$

the Hamiltonian of generation $k = 2$ reads

$$\begin{aligned} H_{(1,2^n)}^{(k=2)} = -\sum_{i=1}^{2^{n-3}} J^{(2)} & \left[\mu_{8i-7}\sigma_{8i-7}^x + \mu_{8i-6}\sigma_{8i-6}^x + \mu_{8i-5}\sigma_{8i-5}^x + \mu_{8i-4}\sigma_{8i-4}^x\right] \\ \times & \left[\mu_{8i-3}\sigma_{8i-3}^x + \mu_{8i-2}\sigma_{8i-2}^x + \mu_{8i-1}\sigma_{8i-1}^x + \mu_{8i}\sigma_{8i}^x\right] \end{aligned} \tag{14}$$

and so on up to the last generation $k = n - 1$ that couples the two halves of the system

$$H_{(1,2^n)}^{(n-1)} = -J^{(n-1)} \left[\sum_{i=1}^{2^{n-1}} \mu_i \sigma_i^x\right] \left[\sum_{j=2^{n-1}+1}^{2^n} \mu_j \sigma_j^x\right] \tag{15}$$

The transverse fields $h_i > 0$ in Eq. 12 can be either uniform or random. The magnetic moments μ_i of the spins are set initially to unity

$$\mu_i = 1 \tag{16}$$

but we have introduced them because they will be generated by the renormalization procedure described below.

The $J^{(k)} > 0$ are given ferromagnetic couplings as a function of the generation k. To mimic the power-law behavior with respect to the distance r of Eq. 3

$$J(r) = \frac{1}{r^{1+\sigma}} \tag{17}$$

we consider the following exponential behavior with respect to the generation k

$$J^{(k)} = \frac{1}{(2^k)^{1+\sigma}} = 2^{-(1+\sigma)k} \tag{18}$$

At the classical level, the energy cost with respect to the ground state of a Domain-Wall between the first half-system having $S^x = 1$ for $1 \leq i \leq \frac{L}{2}$ and the second half-system having $S^x = -1$ for $\frac{L}{2} + 1 \leq i \leq L$ scales as

$$E^{DW}(L) \propto L^{1-\sigma} \tag{19}$$

In the region $\sigma > 1$ where it decays with the system-size L, it is clearly different from the Long-Ranged model of Eq. 2 that at least contains a constant term coming from the nearest-neighbor coupling between the two halves. On the contrary in the region $\sigma < 1$ where Eq. 19 grows with the system-size L, one may expect that the Dyson hierarchical version is an appropriate approximation of the Long-Ranged model. The case $\sigma = 1$ is at the border line, since the Domain-Wall cost remains constant for the Dyson model (Eq. 19), whereas it grows logarithmically in L for the Long-Ranged model. In the following, we will thus focus on the interval

$$0 < \sigma \leq 1 \tag{20}$$

In particular we should stress that whereas the limit $\sigma \to +\infty$ of the Long-Ranged model of Eqs 2 and 6 yields the corresponding Short-Ranged models, the limitation $\sigma \leq 1$ for the Dyson hierarchical versions does not allow to recover the Short-Ranged models in this limit.

B. Diagonalization of the lowest generation Hamiltonian $H^{(k=0)}$

The Hamiltonian $H^{(k=0)}$ of generation $k=0$ of Eq. 12 is the sum of the independent two-spin Hamiltonians

$$H_{(2i-1,2i)} = -h_{2i-1}\sigma^z_{2i-1} - h_{2i}\sigma^z_{2i} - J^{(0)}\mu_{2i-1}\mu_{2i}\sigma^x_{2i-1}\sigma^x_{2i} \tag{21}$$

1. Diagonalization in the symmetric sector

Within the symmetric sector, the diagonalization of the Hamiltonian of Eq. 21 in the σ^z basis

$$\begin{aligned} H_{(2i-1,2i)}|++> &= -(h_{2i-1}+h_{2i})|++> - J^{(0)}\mu_{2i-1}\mu_{2i}|--> \\ H_{(2i-1,2i)}|--> &= -J^{(0)}\mu_{2i-1}\mu_{2i}|++> - (h_{2i-1}+h_{2i})|--> \end{aligned} \tag{22}$$

leads to the two eigenvalues

$$\begin{aligned} \lambda^{S-}_{2i} &= -\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1}+h_{2i})^2} \\ \lambda^{S+}_{2i} &= +\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1}+h_{2i})^2} \end{aligned} \tag{23}$$

with the corresponding eigenvectors

$$\begin{aligned} |\lambda^{S-}_{2i}> &= \cos\theta^S_{2i}|++> + \sin\theta^S_{2i}|--> \\ |\lambda^{S+}_{2i}> &= -\sin\theta^S_{2i}|++> + \cos\theta^S_{2i}|--> \end{aligned} \tag{24}$$

in terms of the angle θ_S satisfying

$$\begin{aligned} \cos(\theta^S_{2i}) &= \sqrt{\frac{1+\frac{h_{2i-1}+h_{2i}}{\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2+(h_{2i-1}+h_{2i})^2}}}{2}} \\ \sin(\theta^S_{2i}) &= \sqrt{\frac{1-\frac{h_{2i-1}+h_{2i}}{\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2+(h_{2i-1}+h_{2i})^2}}}{2}} \end{aligned} \tag{25}$$

2. Diagonalization in the antisymmetric sector

Within the antisymmetric sector

$$\begin{aligned} H^{(1)}_{(2i-1,2i)}|+-> &= -(h_{2i-1}-h_{2i})|+-> - J^{(0)}\mu_{2i-1}\mu_{2i}|-+> \\ H^{(1)}_{(2i-1,2i)}|-+> &= -J^{(0)}\mu_{2i-1}\mu_{2i}|+-> + (h_{2i-1}-h_{2i})|-+> \end{aligned} \tag{26}$$

the two eigenvalues read

$$\begin{aligned} \lambda^{A-}_{2i} &= -\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1}-h_{2i})^2} \\ \lambda^{A+}_{2i} &= +\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1}-h_{2i})^2} \end{aligned} \tag{27}$$

with the corresponding eigenvectors

$$\begin{aligned} |\lambda^{A-}_{2i}> &= \cos\theta^A_{2i}|+-> + \sin\theta^A_{2i}|-+> \\ |\lambda^{A+}_{2i}> &= -\sin\theta^A_{2i}|+-> + \cos\theta^A_{2i}|-+> \end{aligned} \tag{28}$$

in terms of the angle θ_A satisfying

$$\begin{aligned} \cos(\theta^A_{2i}) &= \sqrt{\frac{1+\frac{h_{2i-1}-h_{2i}}{\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2+(h_{2i-1}-h_{2i})^2}}}{2}} \\ \sin(\theta^A_{2i}) &= \sqrt{\frac{1-\frac{h_{2i-1}-h_{2i}}{\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2+(h_{2i-1}-h_{2i})^2}}}{2}} \end{aligned} \tag{29}$$

C. Introduction of the renormalized spins $\sigma_{R(2i)}$

For each two-spin Hamiltonian $H_{2i-1,2i}$ of Eq. 21, we wish to keep the two lowest states among the four eigenstates discussed above, and to label them as the two states of some renormalized spin $\sigma_{R(2i)}$

$$\begin{aligned} |\sigma^z_{R(2i)} = + > &\equiv |\lambda^{S-}_{2i} > \\ |\sigma^z_{R(2i)} = - > &\equiv |\lambda^{A-}_{2i} > \end{aligned} \tag{30}$$

It is convenient to introduce the corresponding projector

$$P^-_{2i} \equiv |\sigma^z_{R(2i)} = + >< |\sigma^z_{R(2i)} = +| + |\sigma^z_{R(2i)} = - >< \sigma^z_{R(2i)} = -| \tag{31}$$

as well as the spin operators

$$\begin{aligned} \sigma^z_{R(2i)} &\equiv |\sigma^z_{R(2i)} = + >< |\sigma^z_{R(2i)} = +| - |\sigma^z_{R(2i)} = - >< \sigma^z_{R(2i)} = -| \\ \sigma^x_{R(2i)} &\equiv |\sigma^z_{R(2i)} = + >< |\sigma^z_{R(2i)} = -| + |\sigma^z_{R(2i)} = - >< \sigma^z_{R(2i)} = +| \end{aligned} \tag{32}$$

D. Renormalization rule for the transverse fields $h_{R(2i)}$

The projection of the Hamiltonian of Eq. 21 is given by

$$\begin{aligned} P^-_{2i} H_{(2i-1,2i)} P^-_{2i} &= \lambda^{S-}_{2i} |\lambda^{S-}_{2i} >< \lambda^{S-}_{2i}| + \lambda^{A-}_{2i} |\lambda^{A-}_{2i} >< \lambda^{A-}_{2i}| \\ &= \left(\frac{\lambda^{S-}_{2i} + \lambda^{A-}_{2i}}{2} \right) P^-_{2i} + \left(\frac{\lambda^{S-}_{2i} - \lambda^{A-}_{2i}}{2} \right) \sigma^z_{R(2i)} \\ &\equiv e_{R(2i)} P^-_{2i} - h_{R(2i)} \sigma^z_{R(2i)} \end{aligned} \tag{33}$$

where the renormalized transverse fields read

$$\begin{aligned} h_{R(2i)} &\equiv \frac{\lambda^{A-}_{2i} - \lambda^{S-}_{2i}}{2} \\ &= \frac{\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1} + h_{2i})^2} - \sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1} - h_{2i})^2}}{2} \\ &= \frac{2h_{2i-1}h_{2i}}{\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1} + h_{2i})^2} + \sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1} - h_{2i})^2}} \end{aligned} \tag{34}$$

and where the contribution to the ground-state energy of this projection reads

$$\begin{aligned} e_{R(2i)} &\equiv \frac{\lambda^{A-}_{2i} + \lambda^{S-}_{2i}}{2} \\ &= -\frac{\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1} + h_{2i})^2} + \sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1} - h_{2i})^2}}{2} \end{aligned} \tag{35}$$

E. Renormalization rule for the magnetic moments $\mu_{R(2i)}$

The projections of the σ^x operators

$$\begin{aligned} P^-_{2i} \sigma^x_{2i-1} P^-_{2i} &= \left[\sin(\theta^S_{2i})\cos(\theta^A_{2i}) + \cos(\theta^S_{2i})\sin(\theta^A_{2i})\right] \sigma^x_{R(2i)} \\ P^-_{2i} \sigma^x_{2i} P^-_{2i} &= \left[\cos(\theta^S_{2i})\cos(\theta^A_{2i}) + \sin(\theta^S_{2i})\sin(\theta^A_{2i})\right] \sigma^x_{R(2i)} \end{aligned} \tag{36}$$

yield the following renormalization rule for the magnetic moment

$$\begin{aligned} \mu_{R(2i)} &= \left[\sin(\theta^S_{2i})\cos(\theta^A_{2i}) + \cos(\theta^S_{2i})\sin(\theta^A_{2i})\right] \mu_{2i-1} + \left[\cos(\theta^S_{2i})\cos(\theta^A_{2i}) + \sin(\theta^S_{2i})\sin(\theta^A_{2i})\right] \mu_{R(2i)} \\ &= \mu_{2i-1} \sqrt{\frac{1 + \frac{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 - h^2_{2i-1} + h^2_{2i}}{\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1}+h_{2i})^2}\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1}-h_{2i})^2}}}{2}} \\ &+ \mu_{2i} \sqrt{\frac{1 + \frac{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + h^2_{2i-1} - h^2_{2i}}{\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1}+h_{2i})^2}\sqrt{(J^{(0)}\mu_{2i-1}\mu_{2i})^2 + (h_{2i-1}-h_{2i})^2}}}{2}} \end{aligned} \tag{37}$$

F. Renormalization of the Hamiltonians of generation $k \geq 1$

The Hamiltonian of generation $k=1$ of Eq. 13 is projected onto

$$\left(\prod_{i=1}^{2^{n-1}} P^-_{2i}\right) H^{(k=1)}_{(1,2^n)} \left(\prod_{i=1}^{2^{n-1}} P^-_{2i}\right) = -\sum_{i=1}^{2^{n-2}} J^{(1)} \left[\mu_{R(4i-2)}\sigma^x_{R(4i-2)}\right]\left[\mu_{R(4i)}\sigma^x_{R(4i)}\right] \tag{38}$$

The Hamiltonian of generation $k=2$ of Eq. 15 is projected onto

$$\left(\prod_{i=1}^{2^{n-1}} P^-_{2i}\right) H^{(k=2)}_{(1,2^n)} \left(\prod_{i=1}^{2^{n-1}} P^-_{2i}\right) = -\sum_{i=1}^{2^{n-3}} J^{(2)} \left[\mu_{R(8i-6)}\sigma^x_{R(8i-6)} + \mu_{R(8i-4)}\sigma^x_{R(8i-4)}\right]\left[\mu_{R(8i-2)}\sigma^x_{R(8i-2)} + +\mu_{R(8i)}\sigma^x_{R(8i)}\right] \tag{39}$$

and so on up to the Hamiltonian of last generation $k=n-1$ of Eq. 15 that is projected onto

$$\left(\prod_{i=1}^{2^{n-1}} P^-_{2i}\right) H^{(n-1)}_{(1,2^n)} \left(\prod_{i=1}^{2^{n-1}} P^-_{2i}\right) = -J^{(n-1)} \left[\sum_{i=1}^{2^{n-2}} \mu_{R(2i)}\sigma^x_{R(2i)}\right]\left[\sum_{j=2^{n-2}+1}^{2^{n-1}} \mu_{R(2j)}\sigma^x_{R(2j)}\right] \tag{40}$$

In conclusion, once the renormalization of the magnetic moments has been taken into account, the couplings $J^{(k)}$ are just translated by one generation for $k \geq 0$

$$J^{R(k)} = J^{(k+1)} \tag{41}$$

III. DYSON HIERARCHICAL CHAIN WITH UNIFORM TRANSVERSE FIELD

The pure Dyson Quantum ferromagnetic Ising model corresponds to the case where all transverse fields h_i coincide

$$h_i = h \tag{42}$$

The mean-field theory for the Long-Ranged model of Eq. 2 [2] (see the reminder in Appendix A) is expected to be also valid for the Dyson hierarchical version in the same region $0 < \sigma < \sigma_u = 2/3$. The main goal of the real-space procedure described below is to study the critical properties in the non-mean-field region $\sigma_u \geq 2/3$, but since σ is just a continuous parameter in the RG rules, we will also mention the results for $0 < \sigma < \sigma_u = 2/3$.

A. Pure renormalization rules

The renormalization rules derived in the previous section simplify as follows :
(i) the renormalized transverse field of Eq. 34 reads

$$h^R = \frac{2h^2}{J^{(0)}\mu^2 + \sqrt{(J^{(0)}\mu^2)^2 + 4h^2}} \tag{43}$$

(ii) the renormalized magnetic moment of Eq. 37 reads

$$\mu_R = \mu\sqrt{2}\sqrt{1 + \frac{J^{(0)}\mu^2}{\sqrt{(J^{(0)}\mu^2)^2 + 4h^2}}} \tag{44}$$

B. RG rules deep in the paramagnetic phase $h \to +\infty$

Deep in the paramagnetic phase $h \to +\infty$, the transverse field remains unchanged upon RG (Eq. 43)

$$h^R \underset{h \to +\infty}{\simeq} h \tag{45}$$

whereas the magnetic moment evolves according to (Eq. 44)

$$\mu_R \underset{h\to+\infty}{\simeq} \mu\sqrt{2} \tag{46}$$

For a length $L = 2^n$ obtained after n RG steps, the magnetic moment reads

$$\mu(L = 2^n) \underset{h\to+\infty}{\simeq} (\sqrt{2})^n = \sqrt{L} \tag{47}$$

As a consequence, the effective ferromagnetic coupling between two such magnetic moments scaling as

$$J^{(n)}_{eff} \equiv J^{(n)}\mu(L = 2^n)\mu(L = 2^n) \underset{h\to+\infty}{\simeq} L^{-(1+\sigma)}L = L^{-\sigma} \tag{48}$$

becomes smaller and smaller with respect to the transverse field of Eq. 45, so that the paramagnetic fixed point is attractive for all $\sigma > 0$.

C. RG rules deep in the ferromagnetic phase $h \to 0$

Deep in the ferromagnetic phase $h \to 0$, the magnetic moment evolves according to (Eq. 44)

$$\mu_R \underset{h\to 0}{\simeq} 2\mu \tag{49}$$

i.e. the magnetic moment grows linearly with the length $L = 2^n$ obtained after n RG steps

$$\mu(L = 2^n) \underset{h\to 0}{\simeq} 2^n = L \tag{50}$$

This corresponds as it should to the maximal magnetization per spin $m = \frac{\mu(L)}{L} = 1$. As a consequence, the effective ferromagnetic coupling between two such magnetic moments scales as

$$J^{(n)}_{eff} \equiv J^{(n)}\mu(L = 2^n)\mu(L = 2^n) \underset{h\to 0}{\simeq} L^{-(1+\sigma)}L^2 = L^{1-\sigma} = 2^{n(1-\sigma)} \tag{51}$$

in agreement with the classical Domain-Wall energy of Eq. 19.

Deep in the ferromagnetic phase $h \to 0$, the RG rule for the transverse field becomes (Eq. 43)

$$h^R \underset{h\to 0}{\simeq} \frac{h^2}{J^{(0)}\mu^2} = \frac{h^2}{J^{(0)}_{eff}} \tag{52}$$

After iteration over n RG steps corresponding to the length $L = 2^n$

$$h(L = 2^n) = h^{R^n} \underset{h\to 0}{\simeq} \frac{h^{2^n}}{J^{(n-1)}_{eff}(J^{(n-2)}_{eff})^2(J^{(n-3)}_{eff})^4 ... (J^{(0)}_{eff})^{2^{n-1}}} = \frac{h^{2^n}}{\prod_{k=0}^{n-1}(J^{(k)}_{eff})^{2^{n-1-k}}} \tag{53}$$

or equivalently in log-variables using $J^{(k)}_{eff} = 2^{k(1-\sigma)}$ (Eq. 51)

$$\begin{aligned} \ln h(L = 2^n) &\underset{h\to 0}{\simeq} 2^n \ln h - \sum_{k=0}^{n-1} 2^{n-1-k} \ln(J^{(k)}_{eff}) \\ &= 2^n \left[\ln h - \sum_{k=0}^{n-1} 2^{-1-k}k(1-\sigma)\ln 2\right] \\ &= 2^n \left[\ln h - (1-\sigma)\ln 2 \left(1 - (n+1)2^{-n}\right)\right] \\ &= L[\ln h - (1-\sigma)\ln 2] + (1-\sigma)(\ln L + \ln 2) \end{aligned} \tag{54}$$

At leading order, the renormalized transverse field is thus attracted exponentially in L towards zero

$$h(L) \underset{h\to 0}{\simeq} \left(\frac{h}{2^{1-\sigma}}\right)^L (2L)^{1-\sigma} \tag{55}$$

so that it becomes smaller and smaller with respect to the effective ferromagnetic coupling of Eq. 51, i.e. the ferromagnetic fixed point $h = 0$ is attractive.

D. RG rule for the control parameter K

It is clear that the important parameter of the model is the ratio between the effective ferromagnetic coupling $J^{(0)}\mu^2$ and the transverse field h

$$K \equiv \frac{J^{(0)}\mu^2}{h} \tag{56}$$

After one RG step, this control parameter is renormalized into (Eqs 43 and 44)

$$K^R \equiv \frac{J^{(1)}(\mu^R)^2}{h^R} = 2^{-1-\sigma} K \frac{\left(K+\sqrt{K^2+4}\right)^2}{\sqrt{K^2+4}} \equiv \phi_\sigma(K) \tag{57}$$

The attractive paramagnetic fixed point corresponds to $K = 0$, whereas the attractive ferromagnetic fixed point corresponds to $K = +\infty$. From now on, we focus on the unstable fixed point between them and on its critical properties.

E. Location of the critical point $K_c(\sigma)$

The critical point K_c corresponds to the non-trivial fixed point $K_c = \phi_\sigma(K_c)$ of the RG rule of Eq. 57 leading to

$$(2^\sigma - K_c)\sqrt{K_c^2+4} = K_c^2 + 2 \tag{58}$$

So keeping the condition $K_c < 2^\sigma$, we may take the square to obtain the cubic equation

$$K_c^3 - 2^{\sigma-1}K_c^2 + 4K_c - 2(2^\sigma - 2^{-\sigma}) = 0 \tag{59}$$

In terms of the standard Cardano notations for cubic equations

$$\begin{aligned} p &= 4 - \frac{4^{\sigma-1}}{3} \\ q &= 2^{1-\sigma} - \frac{2^{\sigma+2}}{3} - \frac{2^{3\sigma-2}}{27} \\ \sqrt{\frac{q^2}{4}+\frac{p^3}{27}} &= (2^{-\sigma}+2^{\sigma-1})\sqrt{1+\frac{4^\sigma}{27}} \end{aligned} \tag{60}$$

and

$$\begin{aligned} u_+ &= \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}} \\ &= \left(-2^{-\sigma} + \frac{2^{\sigma+1}}{3} + \frac{2^{3\sigma-3}}{27} + (2^{-\sigma}+2^{\sigma-1})\sqrt{1+\frac{4^\sigma}{27}}\right)^{\frac{1}{3}} \\ u_- &= -\text{sgn}(p)\left|-\frac{q}{2} - \sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right|^{\frac{1}{3}} \\ &= -\text{sgn}\left(2+\frac{\ln 3}{2\ln 2}-\sigma\right)\left|-2^{-\sigma} + \frac{2^{\sigma+1}}{3} + \frac{2^{3\sigma-3}}{27} - (2^{-\sigma}+2^{\sigma-1})\sqrt{1+\frac{4^\sigma}{27}}\right|^{\frac{1}{3}} \end{aligned} \tag{61}$$

one obtains that the only real solution of Eq. 59 reads

$$K_c(\sigma) = \frac{2^{\sigma-1}}{3} + u_+ + u_- \tag{62}$$

It is a growing function of σ with the values

$$\begin{aligned} K_c(\sigma \to 0) &= \sigma \ln 2 + O(\sigma^2) \\ K_c(\sigma = \frac{1}{4}) &= 0.177439 \\ K_c(\sigma = \frac{1}{2}) &= 0.364946 \\ K_c(\sigma = \frac{2}{3}) &= 0.497043 \\ K_c(\sigma = 1) &= 0.783243 \end{aligned} \tag{63}$$

F. Correlation length exponent $\nu(\sigma)$

The correlation length exponent ν measures the instability of the linearized RG flow around the critical point. It is determined by the derivative of the RG flow of Eq. 57

$$2^{\frac{1}{\nu}} = \phi'_\sigma(K_c) = \frac{2(2 + K_c\sqrt{K_c^2 + 4})}{(K_c^2 + 4)} \tag{64}$$

This yields that $\nu(\sigma)$ is a decaying function of σ with the values

$$\begin{aligned} \nu(\sigma \to 0) &\simeq \frac{1}{\sigma} + O(1) \\ \nu(\sigma = \frac{1}{4}) &= 4.4406 \\ \nu(\sigma = \frac{1}{2}) &= 2.45131 \\ \nu(\sigma = \frac{2}{3}) &= 1.9602 \\ \nu(\sigma = 1) &= 1.482 \end{aligned} \tag{65}$$

G. Magnetic exponent $x(\sigma)$

At criticality K_c, the RG rule for the renormalized magnetic moment of Eq. 44

$$\mu_R = \mu\sqrt{2}\sqrt{1 + \frac{K_c}{\sqrt{K_c^2 + 4}}} \tag{66}$$

yields after n RG steps corresponding to the length $L = 2^n$

$$\mu(L = 2^n) = \mu\left(\sqrt{2}\sqrt{1 + \frac{K_c}{\sqrt{K_c^2 + 4}}}\right)^n = L^{1-x} \tag{67}$$

with the magnetic exponent

$$\begin{aligned} x(\sigma) &= 1 - \frac{\ln\left(\sqrt{2}\sqrt{1 + \frac{K_c}{\sqrt{K_c^2+4}}}\right)}{\ln 2} \\ &= \frac{1-\sigma}{4} + \frac{\ln(K_c^2 + 4)}{8\ln 2} \end{aligned} \tag{68}$$

It is a decaying function of σ with the values

$$\begin{aligned} x(\sigma \to 0) &= \frac{1}{2} - \frac{\sigma}{4} + O(\sigma^2) \\ x(\sigma = \frac{1}{4}) &= 0.438914 \\ x(\sigma = \frac{1}{2}) &= 0.380907 \\ x(\sigma = \frac{2}{3}) &= 0.344141 \\ x(\sigma = 1) &= 0.275732 \end{aligned} \tag{69}$$

H. Dynamical exponent $z(\sigma)$

At criticality K_c, the RG rule for the transverse field h of Eq. 43

$$h^R = h \frac{2}{K_c + \sqrt{K_c^2 + 4}} \tag{70}$$

yields after n RG steps corresponding to the length $L = 2^n$

$$h(L = 2^n) = h \left(\frac{2}{K_c + \sqrt{K_c^2 + 4}} \right)^n = h L^{-z} \tag{71}$$

with the dynamical exponent

$$z(\sigma) = \frac{\ln \left(\frac{K_c + \sqrt{K_c^2 + 4}}{2} \right)}{\ln 2} = \frac{\sigma - 1}{2} + \frac{\ln(K_c^2 + 4)}{4 \ln 2} \tag{72}$$

By construction at criticality, the dynamical exponent z also describes the scaling of the effective ferromagnetic coupling

$$L^{-z} = J_{eff}(L) = L^{-1-\sigma} \mu^2(L) = L^{1-\sigma-2x} \tag{73}$$

i.e. it is directly related to the magnetic exponent x of Eq. 68

$$z(\sigma) = \sigma - 1 + 2x(\sigma) \tag{74}$$

with the values

$$\begin{aligned} z(\sigma \to 0) &= \frac{\sigma}{2} + O(\sigma^2) \\ z(\sigma = \frac{1}{4}) &= 0.127828 \\ z(\sigma = \frac{1}{2}) &= 0.261814 \\ z(\sigma = \frac{2}{3}) &= 0.354949 \\ z(\sigma = 1) &= 0.551463 \end{aligned} \tag{75}$$

I. Correlation exponent $\eta(\sigma)$

Within the real-space RG procedure, the two-point spatial correlation function scales as the square of the magnetization at the scale where the two spins are merged into a single renormalized spin

$$C(L = 2^n) \equiv < \sigma_1^x \sigma_L^x > \propto \left(\frac{\mu(L)}{L} \right)^2 \propto L^{-2x} \tag{76}$$

As a consequence, the exponent η of the correlation function defined by

$$C(r) \propto r^{-(d-2+z+\eta)} \tag{77}$$

after taking into account Eq. 74, takes the simple explicit value

$$\eta(\sigma) = 2 - \sigma \tag{78}$$

that coincides with the mean-field value of Eq. A21. The fact that η keeps its mean-field value $(2-\sigma)$ even in the non-mean-field region until it reaches its Short-Ranged value η_{SR} is expected for the Long-Ranged quantum Ising model [2] for the same reasons as in the classical case [47].

J. Ground state energy

In the pure case, the renormalization rule of Eq. 35 for the contribution of a single block to the ground state energy simplifies into

$$e_R = -\frac{J^{(0)}\mu^2 + \sqrt{(J^{(0)}\mu^2)^2 + 4h^2}}{2} \tag{79}$$

For a chain of $L = 2^n$ spins, the ground state energy can be obtained by summing all the contributions of the successive projections for the blocks up to the last step where the single remaining spin has for contribution $(-h_{R^n})$

$$\begin{aligned} E^{GS}(L=2^n, K) &= -\frac{L}{2}e_R - \frac{L}{4}e_{R^2} - \frac{L}{8}e_{R^3} - ... - e_{R^{n-1}} - h_{R^n} \\ &= -\sum_{k=0}^{n-1} \frac{L}{2^{1+k}} e_{R^k} - h_{R^n} \end{aligned} \tag{80}$$

In the paramagnetic limit $J = 0 = K$, where the transverse fields are not renormalized (Eq. 45), one recovers as it should the contribution $(-h)$ for each independent spin

$$E^{GS}_{para}(L=2^n, J=0) = -\sum_{k=0}^{n-1} \frac{L}{2^{1+k}} h - h = -hL \tag{81}$$

In the ferromagnetic limit $h = 0, K = +\infty$, where the effective ferromagnetic coupling between two such magnetic moments scales as Eq 51, one obtains

$$E^{GS}_{ferro}(L=2^n, h=0) = -\sum_{k=0}^{n-1} \frac{L}{2^{1+k}} 2^{k(1-\sigma)} = -\frac{1}{2(1-2^{-\sigma})}(L - L^{1-\sigma}) \tag{82}$$

as it should : the coefficient of the extensive term corresponds to the half of the sum of the couplings starting from one site, for instance the first one, i.e. $(\sum_{i=2}^{+\infty} J_{1,i})/2$, whereas the correction to extensivity in $L^{1-\sigma}$ represents the 'missing' couplings of the region $j > L$.

At criticality $K = K_c$, where the renormalized transverse-fields involve the dynamical exponent z (Eq. 71 and Eq. 72), the ground state energy reads

$$\begin{aligned} E^{GS}_{criti}(L=2^n, K_c) &= -\sum_{k=0}^{n-1} \frac{L}{2^{1+k}} h_{R^k} \frac{K_{R^k} + \sqrt{(K_{R^k})^2 + 4}}{2} - h_{R^n} \\ &= -\sum_{k=0}^{n-1} \frac{L}{2^{1+k}} h 2^{-zk} \frac{K_c + \sqrt{K_c^2 + 4}}{2} - h 2^{-zn} \\ &= -h \sum_{k=0}^{n-1} \frac{L}{2^{1+k}} 2^{-zk} 2^z - h 2^{-zn} \\ &= -h \frac{2^{2z}}{2^{1+z} - 1} L + \frac{(2^z-1)^2}{2^{1+z}-1} L^{-z} \end{aligned} \tag{83}$$

i.e. the finite-size correction to the ground state energy per spin

$$\frac{E^{GS}_{criti}(L=2^n, K_c)}{L} = -h \frac{2^{2z}}{2^{1+z}-1} + \frac{(2^z-1)^2}{2^{1+z}-1} L^{-1-z} \tag{84}$$

is of order L^{-1-z} as expected.

K. Discussion

In summary, the real-space RG procedure for the pure Dyson hierarchical quantum Ising model has the advantage of being explicitly solvable for the various observables and to yield reasonable approximated values for the critical exponents. Besides the simple anomalous dimension $\eta = 2 - \sigma$ (Eq. 78) reproduced both in the mean-field region and in the non-mean-field region as it should, it is interesting to consider some specific values of σ to compare with previous approaches :

(i) for the non-mean-field value $\sigma = 1$, the dynamical exponent $z(\sigma = 1) \simeq 0.55$, and the anomalous dimension $\eta(\sigma) = 1$ are close to the Monte-Carlo measure quoted in Eq. 5 of the Introduction, even if the correlation length $\nu(\sigma = 1) = 1.48$ is less good with respect to Eq. 5.

(ii) at the upper critical value $\sigma_u = 2/3$, the dynamical exponent $z(\sigma = \frac{2}{3}) \simeq 0.35$ and the magnetic exponent $x(\sigma = \frac{2}{3}) \simeq 0.34$ are close to the expected values of Eqs A25 and A26, even if the value for the correlation length exponent $\nu(\sigma = \frac{2}{3}) \simeq 1.96$ is again not so good with respect to Eq. A25.

(iii) at first order in the expansion near $\sigma \to 0$, the critical exponents $\nu(\sigma)$, $x(\sigma)$, $z(\sigma)$ and the critical point location $K_c(\sigma)$ found above by the real-space RG procedure actually coincide with the mean-field values recalled in Appendix A. Further work is needed to understand if the renormalization procedure can be modified to describe correctly the whole mean-field region $0 < \sigma < \sigma_u = 2/3$ and in particular its anomalous finite-size scaling properties. Indeed, the anomalous finite-size scaling properties of the mean-field region of classical statistical physics models has a long history that has been re-interpreted recently (see [48–51] and references therein), and it would be very interesting to see how it can emerge explicitly within some real-space RG. For the present case, the mean-field critical properties can actually be reproduced by the real-space RG flow for the control parameter $K = \frac{J^{(0)}\mu^2}{h}$ (instead of Eq. 57)

$$K_{MF}^R = 2^{-\sigma} \frac{K}{1-K} \tag{85}$$

with the critical point location $K_c(\sigma) = 1 - 2^{-\sigma}$ and the correlation length exponent $\nu(\sigma) = \frac{1}{\sigma}$. More precisely, the corresponding rules for the transverse field and the magnetic moment read (instead of Eqs 43 and 44)

$$\begin{aligned} h_{MF}^R &= h\sqrt{1-K} \\ \mu_{MF}^R &= \mu\sqrt{\frac{2}{\sqrt{1-K}}} \end{aligned} \tag{86}$$

corresponding to the dynamical exponent $z(\sigma) = \frac{\sigma}{2}$ and to the magnetic exponent $x(\sigma) = \frac{1}{2} - \frac{\sigma}{4}$. Within the real-space RG perspective, it is not clear to us what arguments should be used to justify that this mean-field RG flow has to be preferred to describe the critical point in the whole region $0 < \sigma \leq \sigma_u = 2/3$.

IV. DYSON HIERARCHICAL CHAIN WITH RANDOM TRANSVERSE FIELDS

In this section, we consider the Dyson hierarchical chain of Eq. 11 where the transverse fields h_i are independent random variables drawn with the flat distribution over the interval $[0, W]$

$$P(h) = \frac{\theta(0 < h \leq W)}{W} \tag{87}$$

which is the distribution previously used in the Strong Disorder RG study [10]. The parameter W which represents the typical scale of the initial transverse fields is thus the control parameter of the transition. It will be actually more convenient to use

$$\theta \equiv \ln W \tag{88}$$

as in the previous Strong Disorder RG study [10].

A. RG rules deep in the ferromagnetic phase $W \to 0$

When the transverse fields (h_{2i-1}, h_{2i}) are small with respect to the ferromagnetic term $J^{(0)}\mu_{2i-1}\mu_{2i}$, the RG rule for the magnetic moment (Eq. 37) is simply additive

$$\mu_{R(2i)} = \mu_{2i-1} + \mu_{2i} \tag{89}$$

As a consequence, after n RG steps corresponding to the length $L = 2^n$, the magnetic moment is given by the number of spins

$$\mu(L) \quad = L \tag{90}$$

and the effective ferromagnetic coupling between two such magnetic moments scales as

$$J^{(n)}_{eff} \equiv J^{(n)}\mu(L = 2^n)\mu(L = 2^n) \simeq 2^{n(1-\sigma)} = L^{1-\sigma} \tag{91}$$

in agreement with the classical Domain-Wall energy of Eq. 19.

The renormalization rule for the transverse field (Eq. 34) becomes

$$h_{R(2i)} \simeq \frac{h_{2i-1}h_{2i}}{J^{(0)}\mu_{2i-1}\mu_{2i}} = \frac{h_{2i-1}h_{2i}}{J^{(0)}_{eff}} \tag{92}$$

Upon iteration in log-variables, one can follow the same steps as in the pure case (Eqs 53 and 54) and one obtains

$$\ln h(L) \quad \simeq \sum_{i=1}^{L} \ln h_i - L(1-\sigma)\ln 2 + (1-\sigma)(\ln L + \ln 2) \tag{93}$$

For large L, the Central Limit theorem yields that the typical value

$$\begin{aligned} \ln h_{typ}(L) \quad &\equiv \overline{\ln h(L)} \\ &\simeq L\left(\overline{\ln h_i} - (1-\sigma)\ln 2\right) + (1-\sigma)(\ln L + \ln 2) \end{aligned} \tag{94}$$

becomes smaller and smaller with respect to the effective ferromagnetic coupling of Eq. 91, i.e. the ferromagnetic fixed point is attractive. In this ferromagnetic phase, the variance grows linearly (Eq. 93)

$$Var[\ln h(L)] = LVar[\ln h_i] \tag{95}$$

B. RG rules deep in the paramagnetic phase $W \to +\infty$

When the ferromagnetic term $J^{(0)}\mu_{2i-1}\mu_{2i}$ is small with respect to the transverse fields (h_{2i-1}, h_{2i}) the RG rule for the transverse field (Eq. 34) becomes

$$h_{R(2i)} \simeq \frac{2h_{2i-1}h_{2i}}{h_{2i-1} + h_{2i} + |h_{2i-1} - h_{2i}|} = \min(h_{2i-1}, h_{2i}) \tag{96}$$

and the RG rule for the magnetic moment (Eq. 37) reads

$$\begin{aligned} \mu_{R(2i)} \quad &= \sqrt{\frac{1 - sgn(h_{2i-1} - h_{2i})}{2}}\mu_{2i-1} + \sqrt{\frac{1 + sgn(h_{2i-1} - h_{2i})}{2}}\mu_{2i} \\ &= \theta(h_{2i-1} < h_{2i})\mu_{2i-1} + \theta(h_{2i-1} > h_{2i})\mu_{2i} \end{aligned} \tag{97}$$

In this paramagnetic limit, the renormalized spin is thus simply the spin with the smallest transverse field that survives with its properties, whereas the spin with the highest transverse field is decimated and disappears.

So after n RG steps corresponding to the length $L = 2^n$, the magnetic moment has simply kept its initial value

$$\mu(L = 2^n) \quad = 1 \tag{98}$$

so that the effective magnetic coupling also keeps its bare value

$$J_{eff}(L) = J^{(p)} = 2^{-(1+\sigma)p} = L^{-1-\sigma} \tag{99}$$

The renormalized transverse field is given by the minimum value among the $L = 2^n$ initial random variables

$$h(L = 2^n) = \min(h_1, h_2, .., h_L) \tag{100}$$

Its probability distribution $P_L(h)$ reads thus in terms of the initial distribution $P_1(h)$ of Eq. 87

$$\begin{aligned} P_L(h) &= L P_1(h) \left[\int_h^{+\infty} dx P_1(x) \right]^{L-1} \\ &= L \frac{\theta(0 \leq h \leq W)}{W} \left[\frac{W-h}{W} \right]^{L-1} \end{aligned} \tag{101}$$

For large L, it converges towards the exponential distribution (as a particular case of the Fréchet distribution)

$$P_L(h) \underset{L \to +\infty}{\simeq} \frac{L}{W} e^{-\frac{L}{W} h} \tag{102}$$

The characteristic scale

$$h_{typ}(L) = \frac{W}{L} \tag{103}$$

decays, but remains bigger that the effective ferromagnetic coupling of Eq. 100, so that the paramagnetic fixed point is attractive.

The approximate renormalization rules derived here in the limit $W \to +\infty$ are similar to the Strong Disorder RG rules [10]. However the numerical study described below yields that the paramagnetic phase near the critical point is described by other scaling behavior.

C. Numerical study of the Critical Point

To study the critical properties, we have used two methods :

(i) on one hand, we have generated $n_s = 4.10^3$ disordered samples of the Dyson chain containing $N = 2^{24} \simeq 1.68 \times 10^7$ spins, corresponding to $n = 24$ generations, and we have applied numerically the RG rules of Eqs 34 and 37.

(ii) on the other hand, we have used the so-called standard 'pool method' : the idea is to represent the joint probability distribution $P_n(h, \mu)$ of the renormalized transverse field h and of the magnetic moment μ of a renormalized quantum spin at generation n by a pool of M realizations $(h_n^{(i)}, \mu_n^{(i)})$ where $i = 1, 2, ..., M$. The pool at generation $(n+1)$ is then constructed as follows : each new realization $(h_{n+1}^{(i)}, \mu_{n+1}^{(i)})$ is obtained by choosing 2 ancestors spins at random from the pool of generation n and by applying the renormalization rules of Eqs 34 and 37. The numerical results presented below have been obtained with a pool of size $M = 3.10^7$ iterated up to $n = 100$ generations. The pool method allows to study much bigger sizes and statistics, but introduces some truncation related to the size of the pool in the probability distributions. We have checked that in the critical region, the results of (ii) are in agreement with (i) for the common sizes.

In both methods, for each renormalization step corresponding to the lengths $L = 2^n$, we have analyzed the statistical properties of the renormalized transverse fields h_L and of the renormalized magnetic moments μ_L. More precisely, we have measured the RG flows of the corresponding typical values

$$\begin{aligned} \ln h_L^{typ} &\equiv \overline{\ln h_L} \\ \ln \mu_L^{typ} &\equiv \overline{\ln \mu_L} \end{aligned} \tag{104}$$

as a function of the length L for various values of the control parameter θ of the initial distribution of Eq. 87. We have studied the critical properties for the two values $\sigma = 2/3$ and $\sigma = 1$ with the following critical point

$$\begin{aligned} \theta_c \left(\sigma = \frac{2}{3} \right) &\simeq 2.80725 \\ \theta_c (\sigma = 1) &\simeq 2.14875 \end{aligned} \tag{105}$$

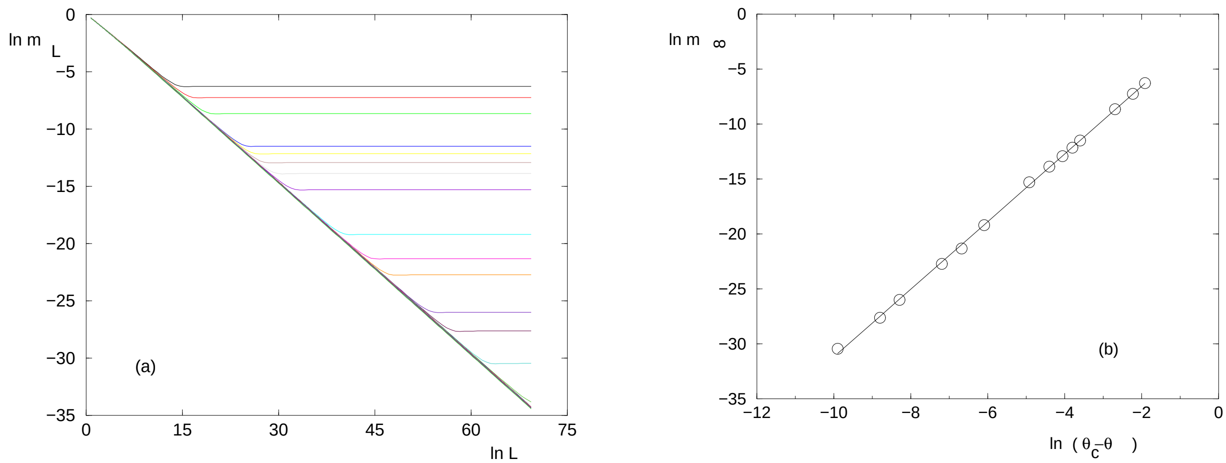

Figure 1: Case $\sigma = \frac{2}{3}$

(a) RG flow of the magnetization $m_L^{typ} \equiv \frac{\mu_L^{typ}}{L}$ in log-log plot for various control parameter θ
(i) in the ferromagnetic phase $\theta < \theta_c$, the flow converges towards a finite magnetization $m_\infty(\theta)$;
(ii) in the paramagnetic phase $\theta > \theta_c$, the slope is $(-1/2)$;
(iii) at criticality, the slope $(-x(\sigma = 2/3)) \simeq -0.5$ is not distinguishable from the paramagnetic slope.
(b) $\ln(m_\infty(\theta))$ as a function of $\ln(\theta_c - \theta)$: the slope corresponds to the exponent $\beta(\sigma = \frac{2}{3}) \simeq 3$

D. RG flows of the magnetic moment μ_L

On Fig. 1 (a), we show the RG flows of the magnetization $m_L^{typ} \equiv \frac{\mu_L^{typ}}{L}$ (Eq. 104) as a function of the length L for various control parameter θ. They can be summarized as :

$$\begin{aligned} m_L^{typ}|_{\theta<\theta_c} & \underset{L\to+\infty}{\simeq} m_\infty(\theta) \\ m_L^{typ}|_{\theta=\theta_c} & \underset{L\to+\infty}{\propto} L^{-x(\sigma)} \\ m_L^{typ}|_{\theta>\theta_c} & \underset{L\to+\infty}{\propto} A(\theta) L^{-\frac{1}{2}} \end{aligned} \tag{106}$$

On the ferromagnetic side, we obtain that the asymptotic finite magnetization vanishes as a power-law (see Fig. 1 (b))

$$m_\infty(\theta) \underset{\theta\to\theta_c^-}{\propto} (\theta_c - \theta)^{\beta(\sigma)} \tag{107}$$

and we measure the same value for $\sigma = 2/3$ and $\sigma = 1$

$$\beta(\sigma = \frac{2}{3}) \simeq 3 \simeq \beta(\sigma = 1) \tag{108}$$

On the paramagnetic side, it is not clear that the amplitude $A(\theta)$ diverges, and correspondingly at criticality, the magnetic exponent $x(\sigma)$ cannot be distinguished from the paramagnetic phase value $1/2$

$$x(\sigma = \frac{2}{3}) \simeq 0.5 \simeq x(\sigma = 1) \tag{109}$$

As a consequence, the finite-size scaling matching between Eq. 108 and 109 corresponds to the finite-size correlation length exponent $\nu_{FS} = \frac{\beta}{x}$

$$\nu_{FS}(\sigma = \frac{2}{3}) \simeq 6 \simeq \nu_{FS}(\sigma = 1) \tag{110}$$

E. RG flow of the effective coupling J_L

In the present real-space RG framework, the effective coupling J_L is just the 'slave' of the magnetic moments, and the typical value is given by

$$J_L^{typ} = L^{-1-\sigma} (\mu_L^{typ})^2 = L^{1-\sigma} (m_L^{typ})^2 \tag{111}$$

The translation of Eq. 106 corresponds to the behaviors

$$\begin{aligned} J_L^{typ}|_{\theta<\theta_c} & \underset{L\to+\infty}{\propto} L^{1-\sigma} m_\infty^2(\theta) \\ J_L^{typ}|_{\theta=\theta_c} & \underset{L\to+\infty}{\propto} L^{1-\sigma-2x} = L^{-z(\sigma)} \\ J_L^{typ}|_{\theta>\theta_c} & \underset{L\to+\infty}{\propto} L^{-\sigma} A^2(\theta) \end{aligned} \tag{112}$$

Eq 109 yields that the critical dynamical exponent is given by

$$z(\sigma) = \sigma - 1 + 2x \simeq \sigma \tag{113}$$

F. RG flow of the renormalized transverse fields

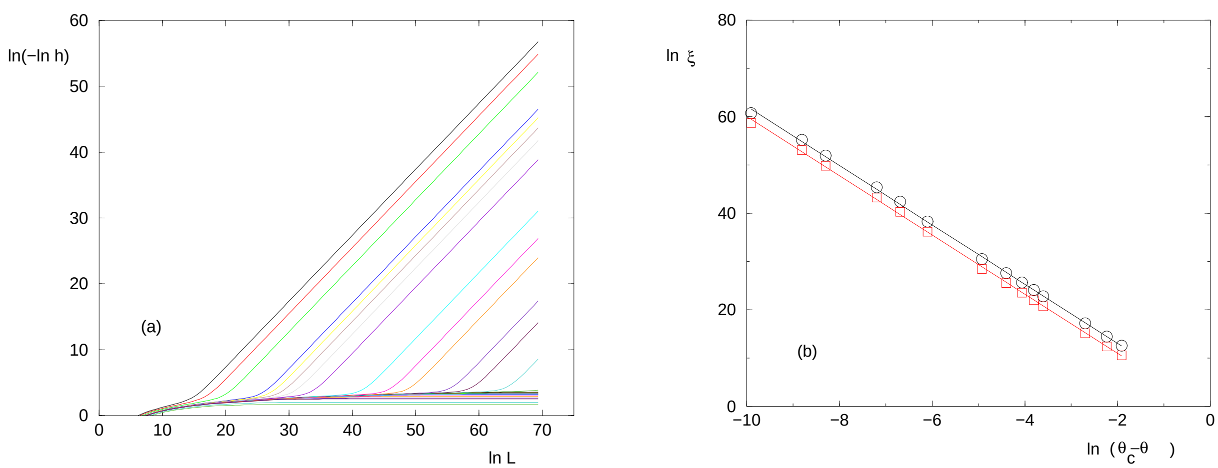

Figure 2: RG flow of the transverse fields for $\sigma = \frac{2}{3}$
(a) $\ln(-\ln h_L^{typ})$ as a function of $\ln L$:
in the ferromagnetic phase $\theta < \theta_c$, the slope unity allows to measure the correlation length $\xi_h(\theta)$ of Eq. 115.
(b) $\ln \xi_h(\theta)$ and $\ln \xi_{width}(\theta)$ as a function of $\ln(\theta_c - \theta)$: the slope corresponds to the value $\nu(\sigma = \frac{2}{3}) \simeq 6$.

At criticality, the typical renormalized transverse field involves the same dynamical exponent as in Eq. 113

$$h_L^{typ}|_{\theta=\theta_c} \underset{L\to+\infty}{\propto} L^{-z(\sigma)} \tag{114}$$

In the ferromagnetic phase, the exponential decay of the typical transverse field allows to define some correlation length $\xi_h(\theta)$ (see Fig. 2 (a))

$$\ln h_L^{typ}|_{\theta<\theta_c} \underset{L\to+\infty}{\propto} -\frac{L}{\xi_h(\theta)} \tag{115}$$

The growth of the width of the probability distribution of $\ln h$ (Eq. 95)

$$\Delta_{\ln h_L} \equiv \sqrt{\overline{(\ln h_L)^2} - (\overline{\ln h_L})^2} \propto \left(\frac{L}{\xi_{width}}\right)^{\frac{1}{2}} \tag{116}$$

also allows to define another correlation length ξ_{width}. Their divergences near criticality involve the same exponent $\nu(\sigma)$ (see Fig. 2 (b))

$$\begin{aligned} \xi_h(\theta) &\underset{\theta \to \theta_c}{\propto} (\theta_c - \theta)^{-\nu(\sigma)} \\ \xi_{width}(\theta) &\underset{\theta \to \theta_c}{\propto} (\theta_c - \theta)^{-\nu(\sigma)} \end{aligned} \tag{117}$$

Our numerical results for $\sigma = 2/3$ and $\sigma = 1$ point towards the same value

$$\nu(\sigma = \frac{2}{3}) \simeq 6 \simeq \nu(\sigma = 1) \tag{118}$$

in agreement with the finite-size scaling exponent of Eq. 110.

G. Comparison with the previous Strong Disorder RG results [10]

Let us compare the results described above with the Strong Disorder RG study [10] :

(i) here the dynamical exponent is found to be $z(\sigma) \simeq \sigma$ instead of the value $z(\sigma) \simeq 1 + \sigma$ obtained via Strong Disorder RG [10]. As a consequence of the relation of Eq. 74, this difference of unity in z is directly related to the difference between the magnetic exponent $x(\sigma) \simeq 0.5$ obtained here and $x = 1$ found in Ref [10] (where more precisely the magnetization grows as $m(L) \propto \frac{(\ln L)^2}{L}$). At the level of the renormalization rules, the origin of this difference is the following : whereas the present renormalization rules become equivalent to the Strong Disorder RG rules in the limit $W \to +\infty$ (see section IV B), they are not equivalent anymore in the critical region. Indeed in the present framework, the magnetization is already of order $m(L) \propto L^{-1/2}$ in the paramagnetic phase, which can be seen as the result of Central Limit fluctuations. Since the magnetization at criticality cannot be smaller, one obtains the bound $x \leq 1/2$.

(ii) instead of the essential singularity of Eq. 8 [10], we obtain here conventional power-law scaling laws with finite but rather large correlation length exponent $\nu \simeq 6$, and magnetic exponent $\beta \simeq 3$.

In summary, the Strong Disorder RG results obtained in [10] for the Long-Ranged chain are based on the approximation that the quantum fluctuations are negligible with respect to disorder fluctuations, whereas the present block renormalization for the Dyson version yields that quantum fluctuations are important in the critical region. Further work is needed to better understand whether the differences in the results are entirely due to the different approximations made in the renormalization rules, or whether the Dyson hierarchical version actually changes significantly the Long-ranged model and reduces the importance of rare events.

V. CONCLUSION

In this paper, we have proposed to study via real-space renormalization the Dyson hierarchical version of the quantum Ising chain with Long-Ranged power-law ferromagnetic couplings $J(r) \propto r^{-1-\sigma}$ and pure or random transverse fields. For the pure case, the RG rules have been explicitly solved as a function of the parameter σ, and we have compared the critical exponents with previous results of other approaches. For the random case, we have studied numerically the RG rules and compared the critical properties with the previous Strong Disorder Renormalization approach [10].

Our conclusion is that the Dyson hierarchical idea initially developed for classical spins is also very useful for quantum spin models, in order to derive real-space renormalization rules. In the future, we hope to apply this method to other quantum models.

Appendix A: Reminder on the Mean-Field theory for the pure Long-Ranged model

1. Thermodynamic exponents

For the pure quantum Ising chain of Eq. 2, the mean-field theory amounts to look for the uniform magnetization m

$$\begin{aligned} < \sigma_i^x > &= m \\ < \sigma_i^z > &= \sqrt{1 - m^2} \end{aligned} \tag{A1}$$

that minimizes the ground state energy per spin

$$e_{GS}(m) = \frac{E_{GS}(m)}{N} = -h\sqrt{1-m^2} - \frac{1}{2} J_{tot} m^2 \tag{A2}$$

where J_{tot} represents the sum of all couplings linked to a given spin

$$J_{tot} = \sum_{j \neq 1} J_{1,j} \tag{A3}$$

For instance for the Dyson model of Eq. 18, it is given by

$$J_{tot}^{Dyson} = \sum_{k=0}^{+\infty} 2^k J^{(k)} = \sum_{k=0}^{+\infty} 2^{-\sigma k} = \frac{1}{1-2^{-\sigma}} \tag{A4}$$

The minimization of Eq. A2

$$0 = \partial_m e_{GS}(m) = h \frac{m}{\sqrt{1-m^2}} - J_{tot} m \tag{A5}$$

yields that the critical transverse field is

$$h_c = J_{tot} \tag{A6}$$

and that the optimal magnetization reads

$$\begin{aligned} m &= 0 && \text{for } \; h > h_c \\ m &= \sqrt{1 - \left(\frac{h}{h_c}\right)^2} && \text{for } \; h < h_c \end{aligned} \tag{A7}$$

corresponding to the usual mean-field thermodynamic exponents

$$\begin{aligned} \beta_{MF} &= \frac{1}{2} \\ \alpha_{MF} &= 0 \end{aligned} \tag{A8}$$

independently of the short-range or long-ranged nature of the couplings.

2. Correlation exponents for the short-ranged case

As recalled in the Introduction, the short-ranged model is characterized by the dynamical exponent (both in mean-field and outside mean-field) [1]

$$z^{SR} = 1 \tag{A9}$$

The correlation length exponent ν and the exponent η of the critical correlation are given by the standard values

$$\begin{aligned} \nu_{MF}^{SR} &= \frac{1}{2} \\ \eta_{MF}^{SR} &= 0 \end{aligned} \tag{A10}$$

The upper critical dimension d_u above which mean-field exponents apply is the value where the mean-field exponents satisfy the hyperscaling relation

$$2 - \alpha = (d+z)\nu \tag{A11}$$

leading to

$$d_u = \frac{2-\alpha_{MF}}{\nu_{MF}^{SR}} - z^{SR} = 3 \tag{A12}$$

3. Correlation exponents for the long-ranged case

For the long-ranged model in dimension d

$$J(r) \propto r^{-d-\sigma} \tag{A13}$$

the Gaussian fixed-point of the paramagnetic phase $h > h_c$ corresponds to the correlation [2]

$$C(\vec{r},t) \equiv < S_{r,t} S_{0,0} > = \int_{-\infty}^{+\infty} \frac{d^d\vec{k}}{(2\pi)^d} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{i(\vec{k}.\vec{r}+\omega t)}}{(h-h_c)+\omega^2+|\vec{k}|^\sigma} \tag{A14}$$

where the singular term $|\vec{k}|^\sigma$ replaces the standard Short-Ranged term k^2 for $\sigma \leq 2$, with the following consequences :

(i) the dynamical exponent z is fixed by the anisotropy between ω^2 and $|\vec{k}|^\sigma$

$$z_{MF}^{LR} = \frac{\sigma}{2} \tag{A15}$$

(ii) the spatial correlation length exponent ν is fixed by the balance between the terms $|h-h_c|$ and $|k|^\sigma$

$$\nu_{MF}^{LR} = \frac{1}{\sigma} \tag{A16}$$

(ii) the temporal correlation length is fixed by the balance between the terms $(h-h_c)$ and ω^2, so that the gap (corresponding to the inverse temporal correlation length) vanishes as

$$\Delta_{MF} = (h-h_c)^{g_{MF}} \text{ with } g_{MF} = \frac{1}{2} \tag{A17}$$

as in the short-ranged case. By consistency with Eq. A15 and A16, one has of course $g_{MF} = z_{MF}^{LR} \nu_{MF}^{LR}$.

(iv) the global static susceptibility diverges

$$\chi(h) = \int_{-\infty}^{+\infty} d^d\vec{r} \int_{-\infty}^{+\infty} dt C(\vec{r},t) = \frac{1}{(h-h_c)^{\gamma_{MF}}} \text{ with } \gamma_{MF} = 1 \tag{A18}$$

(v) at criticality $h = h_c$, the correlation reads

$$C_{criti}(\vec{r},t) = \int_{-\infty}^{+\infty} \frac{d^d\vec{k}}{(2\pi)^d} e^{i\vec{k}.\vec{r}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega^2+|\vec{k}|^\sigma} = \int_{-\infty}^{+\infty} \frac{d^d\vec{k}}{(2\pi)^d} e^{i\vec{k}.\vec{r}} \frac{e^{-|\vec{k}|^{\frac{\sigma}{2}} t}}{2|\vec{k}|^{\frac{\sigma}{2}}} \tag{A19}$$

In particular, the spatial correlation at coinciding points follows the power-law

$$C_{criti}(\vec{r},t=0) = \int_{-\infty}^{+\infty} \frac{d^d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}.\vec{r}}}{2|\vec{k}|^{\frac{\sigma}{2}}} \propto r^{-(d-\frac{\sigma}{2})} \equiv r^{-(d-2+z+\eta)} \tag{A20}$$

with

$$\eta_{MF}^{LR} = 2-\sigma \tag{A21}$$

The upper critical value σ_u below which mean-field exponents apply is the value where the mean-field exponents satisfy the hyperscaling relation of Eq. A11

$$2-\alpha_{MF} = (d+z_{MF}^{LR})\nu_{MF}^{LR} \tag{A22}$$

leading to [2]

$$d_u = \frac{3\sigma_u}{2} \tag{A23}$$

In particular in dimension $d = 1$, the mean-field region $d > d_u$ corresponds to the region $\sigma < \sigma_u$ with

$$\sigma_u = \frac{2}{3} \tag{A24}$$

For comparison with the RG procedure described in the text, it is useful to mention the corresponding exponents

$$\begin{aligned} \nu(\sigma_u = \frac{2}{3}) &= \frac{1}{\sigma_u} = \frac{3}{2} \\ z(\sigma_u = \frac{2}{3}) &= \frac{\sigma_u}{2} = \frac{1}{3} \end{aligned} \tag{A25}$$

and the magnetic exponent

$$x(\sigma_u = \frac{2}{3}) = \frac{\beta_{MF}}{\nu(\sigma_u = \frac{2}{3})} = \frac{1}{3} \tag{A26}$$

Note that at the upper critical value σ_u, one can still use the standard finite-size-scaling valid in the non-mean-field region $\sigma > \sigma_u$ with the mean-field exponents valid in the mean-field region $0 < \sigma < \sigma_u$. But for $0 < \sigma < \sigma_u$, the standard finite-size-scaling does not hold anymore and is replaced by modified finite-size-scaling (see the series of recent works [48–51] and references therein).

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