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We study the entanglement entropy between the two outgoing particles in an elastic scattering process. It is formulated within an S-matrix formalism using the partial wave expansion of two-body states, which plays a significant role in our computation. As a result, we obtain a novel formula that describes the entanglement entropy in a high energy scattering by the use of physical observables, namely the elastic and total cross sections and a physical bound on the impact parameter range, related to the elastic differential cross-section.

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Quantum entanglement is a significant concept which appears in various subjects of physics. Recently quantum entanglement has been attracting many attentions of theoretical physicists. When ρ is a density matrix for a state in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, we can define a reduced density matrix as $\rho_A = \operatorname{tr}_B \rho$. Then the entanglement entropy is given by $-\operatorname{tr}_A \rho_A \ln \rho_A$. Calabrese and Cardy [1] developed the replica method in order to calculate the Rényi entropy and the entanglement entropy in a quantum field theory. If one can obtain the Rényi entropy, $S_{\mathrm{RE}}(n) = (1-n)^{-1} \ln \operatorname{tr}_A(\rho_A)^n$, the entanglement entropy is given by $S_{\mathrm{EE}} = \lim_{n\to1} S_{\mathrm{RE}}(n) = -\lim_{n\to1} \frac{\partial}{\partial n} \operatorname{tr}_A(\rho_A)^n$. From the point of view of holography, Ryu and Takayanagi [2] related the entanglement entropy with the area of an extremal surface in an anti-de Sitter space, while Maldacena and Susskind [3] proposed the ER=EPR conjecture, which claims that a pair of entangled objects are connected by a wormhole. For examples supporting this conjecture, Ref. [4] studied a pair of accelerating quark and anti-quark, and Ref. [5] studied a scattering gluon-gluon pair in the AdS/CFT correspondence. Then one can naturally ask the following question: How does the entanglement entropy of a pair of particles change from an initial state to a final one in an elastic scattering process? Ref. [6] analyzed such change of entanglement entropy perturbatively [7] in a field theory with a weak coupling by the use of an S-matrix [8]. In this letter we exploit the S-matrix formalism further in order for a non-perturbative understanding of the entanglement entropy in a scattering process in a context where inelastic scattering also takes place. This is especially required in the context of strong interaction scattering at high energies where inelastic multi-particle scattering plays an important role which can be analyzed without refereeing explicitly to the underlying quantum field theory [9, 10].

Following Refs. [9, 10], we consider a scattering process of two incoming particles, A and B, whose masses are m_A and m_B respectively, in 1+3 dimensions. This process is divided [9] into the following two channels:

$$\begin{aligned} &\text{Two-particle “elastic” channel: A + B} \to \text{A + B} \\ &\text{Many-particle “inelastic” channel: A + B} \to \text{X} \end{aligned}$$

where X means any possible multi-particle states. We postpone the study extended to a matrix including more varieties of two-particle channels [10] to a further publication. The Hilbert space of two-particle states is not usually factorized as $\mathcal{H}_A \otimes \mathcal{H}_B$ in an interacting system. However, in the S-matrix formalism, the Hilbert space of both the initial and final states are factorizable, and thus can be expressed as the simple product of their two Hilbert spaces of particle states. Indeed, in the S-matrix formalism, one considers only asymptotic initial and final states long before and after the interaction. Therefore we can use the two-particle Fock space $\{|\vec{p}\rangle_A\} \otimes \{|\vec{q}\rangle_B\}$ as the Hilbert space and the two-particle state which consists of particle A with momentum $\vec{p}$ and B with $\vec{q}$ is denoted by $|\vec{p},\vec{q}\rangle = |\vec{p}\rangle_A \otimes |\vec{q}\rangle_B$. We define an inner product of the two-particle states by $\langle \vec{p},\vec{q}|\vec{k},\vec{l}\rangle = 2E_{A\vec{p}}\delta^{(3)}(\vec{p}-\vec{k})\, 2E_{B\vec{q}}\delta^{(3)}(\vec{q}-\vec{l})$, where $E_{I\vec{p}} = \sqrt{p^2+m_I^2}$ $(I=A,B)$ and $p = |\vec{p}|$.

We introduce the S-matrix, $\mathcal{S}$, between the overall set of initial and final states. Once we fix an initial state $|\mathrm{ini}\rangle$, the final state $|\mathrm{fin}\rangle$ is determined by the S-matrix. In this letter we are interested in the entanglement between two particles in a final state of elastic scattering in the presence of a non-negligible fraction of open inelastic final states. Therefore we additionally introduce a projection operator Q onto the two-particle Hilbert space. Then the final elastic state is described as $|\mathrm{fin}\rangle = Q\mathcal{S}|\mathrm{ini}\rangle$.

Partial wave expansion — The partial wave expansion is often useful to analyze a scattering process. Before starting to study the entanglement entropy, let us recall what Refs. [9, 10] studied.

We employ a center-of-mass frame. The state of the two particles, A + B, which have momenta $\vec{p}$ and $-\vec{p}$ is denoted by $|\vec{p}\rangle\rangle := |\vec{p},-\vec{p}\rangle$, while the many-particle state of X is denoted by $|X\rangle$. Since the complete set of states is given by the orthogonal basis, $\{|\vec{p}\rangle\!\rangle, |X\rangle\}$, one can describe the identity matrix as

$$\mathbf{1} = \int \frac{d^3\vec{p}}{2E_{A\vec{p}}\,2E_{B\vec{p}}\,\delta^{(3)}(0)} |\vec{p}\rangle\!\rangle\langle\!\langle\vec{p}| + \int dX\,|X\rangle\langle X|\,. \quad (1)$$

We notice that $\delta^{(3)}(0)$ comes from $\langle\!\langle\vec{k}|\vec{l}\rangle\!\rangle = 2E_{A\vec{k}}\,2E_{B\vec{k}}\,\delta^{(3)}(\vec{k}-\vec{l})\,\delta^{(3)}(0)$, due to our definition of the inner product of states.

One can expand the S-matrix elements in term of partial waves. Let us consider the S-matrix and T-matrix defined by $\mathcal{S} = \mathbf{1} + 2i\mathcal{T}$. The unitarity condition is $\mathcal{S}^\dagger\mathcal{S} = \mathbf{1}$, which is equivalent to $i(\mathcal{T}^\dagger - \mathcal{T}) = 2\mathcal{T}^\dagger\mathcal{T}$. Extracting the factor of energy-momentum conservation, we describe the T-matrix elements as $\langle\!\langle\vec{p}|\mathcal{T}|\vec{q}\rangle\!\rangle = \delta^{(4)}(P_{\vec{p}} - P_{\vec{q}})\langle\!\langle\vec{p}|\mathbf{t}|\vec{q}\rangle\!\rangle$ and $\langle\!\langle\vec{p}|\mathcal{T}|X\rangle = \delta^{(4)}(P_{\vec{p}} - P_X)\langle\!\langle\vec{p}|\mathbf{t}|X\rangle$. $P_{\vec{p}}$ and P_X are the total energy-momenta of $|\vec{p}\rangle\!\rangle$ and $|X\rangle$ respectively, that is, $P_{\vec{p}} = (E_{A\vec{p}} + E_{B\vec{p}}, 0, 0, 0)$.

One introduces the overlap matrix $F_{\vec{p}\,\vec{k}}(k, \cos\theta)$,

$$F_{\vec{p}\,\vec{k}} = \frac{2\pi k}{E_{A\vec{k}} + E_{B\vec{k}}} \int dX\,\langle\!\langle\vec{p}|\mathbf{t}^\dagger|X\rangle\delta^{(4)}(P_X - P_{\vec{k}})\langle X|\mathbf{t}|\vec{k}\rangle\!\rangle\,, \quad (2)$$

where k and θ are defined by $\vec{p}\cdot\vec{k} = pk\cos\theta$ and $k = p$. In terms of partial-wave decomposition, one introduces

$$\frac{\pi k}{E_{A\vec{k}} + E_{B\vec{k}}}\langle\!\langle\vec{p}|\mathbf{t}|\vec{k}\rangle\!\rangle = \sum_{\ell=0}^{\infty}(2\ell+1)\tau_\ell(k)P_\ell(\cos\theta)\,, \quad (3)$$

$$F_{\vec{p}\,\vec{k}}(k, \cos\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\theta)\,, \quad (4)$$

where $P_\ell(\cos\theta)$ are the Legendre polynomials. Then one can rewrite the unitarity condition as

$$\mathrm{Im}\,\tau_\ell = |\tau_\ell|^2 + \frac{f_\ell}{2}\,. \quad (5)$$

Introducing $s_\ell := 1 + 2i\tau_\ell$, which imply the components of partial wave expansion of the S-matrix element, the unitarity condition is equivalent to $s_\ell^* s_\ell = 1 - 2f_\ell$. A comment in order [9, 10] is that we can define a pseudo-unitary two-body S-matrix with partial wave components, $\omega_\ell^*\omega_\ell = 1$, by rescaling s_ℓ as $\omega_\ell := s_\ell/\sqrt{1-2f_\ell}$.

Entanglement entropy of two particles — We consider two unentangled particles, A and B, with momenta $\vec{k}$ and $\vec{l}$ as incident particles. That is to say, we choose a single state as an initial state;

$$|\mathrm{ini}\rangle = |\vec{k}, \vec{l}\rangle = |\vec{k}\rangle_A \otimes |\vec{l}\rangle_B\,. \quad (6)$$

Here we have not taken the center-of-mass frame yet. Of course the entanglement entropy of the initial state vanishes. In terms of the S-matrix, the final state of two particles, $|\mathrm{fin}\rangle = Q\mathcal{S}|\mathrm{ini}\rangle$, is described as

$$|\mathrm{fin}\rangle = \left(\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{d^3\vec{q}}{2E_{B\vec{q}}}|\vec{p}, \vec{q}\rangle\langle\vec{p}, \vec{q}|\right)\mathcal{S}|\vec{k}, \vec{l}\rangle\,. \quad (7)$$

Then we can define the total density matrix of the final state by $\rho := \mathcal{N}^{-1}|\mathrm{fin}\rangle\langle\mathrm{fin}|$. The normalisation factor $\mathcal{N}$ will be determined later so that ρ satisfies $\mathrm{tr}_A\,\mathrm{tr}_B\,\rho = 1$. Tracing out ρ with respect to the Hilbert space of particle B, we obtain the reduced density matrix, $\rho_A := \mathrm{tr}_B\,\rho$, namely,

$$\rho_A = \frac{1}{\mathcal{N}}\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{d^3\vec{q}}{2E_{B\vec{q}}}\frac{d^3\vec{p}\,'}{2E_{A\vec{p}\,'}} \times \left(\langle\vec{p}, \vec{q}|\mathcal{S}|\vec{k}, \vec{l}\rangle\langle\vec{k}, \vec{l}|\mathcal{S}^\dagger|\vec{p}\,', \vec{q}\rangle\right)|\vec{p}\rangle_{A\,A}\langle\vec{p}\,'|\,. \quad (8)$$

Now let us adopt the center-of-mass frame, which leads to $\vec{k} + \vec{l} = 0$. Then the initial state is $|\mathrm{ini}\rangle = |\vec{k}\rangle\!\rangle$, and the reduced density matrix becomes

$$\rho_A = \frac{1}{\mathcal{N}}\int \frac{d^3\vec{p}}{2E_{A\vec{p}}}\frac{\delta(0)\delta(p-k)}{4k(E_{A\vec{k}} + E_{B\vec{k}})}\left|\langle\!\langle\vec{p}|\mathbf{s}|\vec{k}\rangle\!\rangle\right|^2|\vec{p}\rangle_{A\,A}\langle\vec{p}|\,, \quad (9)$$

where $\mathbf{s} := \mathbf{1} + 2i\mathbf{t}$, and $\delta(0)$ stems from the modulus equality of the initial and final particles' momenta. Since the normalization factor $\mathcal{N}$ is determined so that $\mathrm{tr}_A\,\rho_A = 1$, we obtain $\mathcal{N} = \delta^{(4)}(0)\mathcal{N}'$ with $\mathcal{N}' = (\pi k)^{-1}(E_{A\vec{k}} + E_{B\vec{k}})\sum_{\ell=0}^{\infty}(2\ell+1)|s_\ell|^2$, where we used the partial wave expansion (3).

Since $\mathrm{tr}_A(\rho_A)^n$ straightforwardly provides us the Rényi and entanglement entropy, we calculate

$$\mathrm{tr}_A(\rho_A)^n = \int_{-1}^{1} d\zeta\,\mathcal{P}(\zeta)G^{n-1}(\zeta)\,, \quad \zeta := \cos\theta\,, \quad (10)$$

$$\mathcal{P}(\zeta) = \frac{1}{2}\frac{\left|\sum_\ell(2\ell+1)s_\ell P_\ell(\zeta)\right|^2}{\sum_\ell(2\ell+1)|s_\ell|^2}\,, \quad (11)$$

$$G(\zeta) = \frac{\left|\sum_\ell(2\ell+1)s_\ell P_\ell(\zeta)\right|^2}{\sum_\ell(2\ell+1)\cdot\sum_\ell(2\ell+1)|s_\ell|^2}\,, \quad (12)$$

where we used the three-dimensional Dirac delta function in spherical coordinates with azimuthal symmetry, $\delta^{(3)}(\vec{p}-\vec{k}) = (4\pi k^2)^{-1}\delta(p-k)\sum_\ell(2\ell+1)P_\ell(\cos\theta)$. Due to $s_\ell = 1 + 2i\tau_\ell$ and the unitarity condition (5), one can rewrite Eq. (11) as

$$\mathcal{P}(\zeta) = \delta(1-\zeta)\frac{V - 4\sum_\ell(2\ell+1)\,\mathrm{Im}\,\tau_\ell}{V - 2\sum_\ell(2\ell+1)f_\ell} + \frac{2|\sum_\ell(2\ell+1)\tau_\ell P_\ell(\zeta)|^2}{V - 2\sum_\ell(2\ell+1)f_\ell}\,, \quad (13)$$

where $V := \sum_\ell(2\ell+1)$. Here $\sum_\ell(2\ell+1)\,\mathrm{Im}\,\tau_\ell$, $\sum_\ell(2\ell+1)f_\ell$ and $|\sum_\ell(2\ell+1)\tau_\ell P_\ell(\zeta)|^2$ correspond to physical observables and thus are necessarily finite, while the infinite sum V diverges. Therefore (13) leads to $\mathcal{P}(\zeta) = \delta(1-\zeta)$. Then one can easily proceed the integration in (10) and gets finally

$$\mathrm{tr}_A(\rho_A)^n = K^{n-1}\,, \quad (14)$$

$$K := G(1) = \frac{\left|\sum_\ell(2\ell+1)s_\ell\right|^2}{\sum_\ell(2\ell+1)\cdot\sum_\ell(2\ell+1)|s_\ell|^2}\,. \quad (15)$$

Obviously Eq. (14) for $n = 1$ correctly reproduces the normalization condition, $\mathrm{tr}_A\,\rho_A = 1$.

From Eq. (14) the Rényi entropy is $S_{\rm RE} = -\ln K$ and equal to the entanglement entropy,

$$S_{\rm EE} = -\lim_{n\to 1}\frac{\partial}{\partial n}\,\mathrm{tr}_A(\rho_A)^n = -\ln K\,. \tag{16}$$

Using a Cauchy-Schwarz inequality applied to (15), K satisfies $0 \le K \le 1$, that is to say, $0 \le S_{\rm EE} \le \infty$.

When there is no interaction, s_ℓ is equal to one for all ℓ and K becomes one, that is to say, the entanglement entropy $S_{\rm EE}$ vanishes. This is natural, because the final state is same as the initial state without interaction and the initial state (6) is not entangled. On the other hand, if the system has interaction, the entanglement entropy is expected to increase in scattering processes.

We have a comment on the elastic case without the inelastic channel, *i.e.*, $f_\ell = 0$ for all ℓ. In this case, one has $s_\ell = \exp(2i\delta_\ell)$, where δ_ℓ are the phase shifts. Hence one obtain the expression of Eq. (15) in terms of the phase shifts, $K = V^{-2}(|\sum_\ell(2\ell+1)\cos 2\delta_\ell|^2 + |\sum_\ell(2\ell+1)\sin 2\delta_\ell|^2)$.

Let us rewrite Eq. (15) in terms of τ_ℓ and f_ℓ as

$$K = 1 - \frac{4\sum_\ell(2\ell+1)|\tau_\ell|^2 - \frac{4}{V}|\sum_\ell(2\ell+1)\tau_\ell|^2}{V - 2\sum_\ell(2\ell+1)f_\ell}\,. \tag{17}$$

Formally the full Hilbert space extends over all partial waves, and thus one has $V = \sum_{\ell=0}^{\infty}(2\ell+1) = \infty$. It causes $K = 1$, in other words, the entanglement entropy vanishes. However, in physical elastic processes, the Hilbert space is essentially limited by energy-momentum conservation, so that the physical Hilbert space provides a meaningful entanglement entropy as we shall see further.

In all generality, the integrated elastic cross section, the integrated inelastic cross section and the total cross section are given by $\sigma_{\rm el} = (4\pi/k^2)\sum_{\ell=0}^{\infty}(2\ell+1)|\tau_\ell|^2$, $\sigma_{\rm inel} = (2\pi/k^2)\sum_{\ell=0}^{\infty}(2\ell+1)f_\ell$ and $\sigma_{\rm tot} = (4\pi/k^2)\sum_{\ell=0}^{\infty}(2\ell+1)\,\mathrm{Im}\,\tau_\ell$ respectively. The differential elastic cross section is $d\sigma_{\rm el}/dt = (\pi/k^4)\sum_{\ell,\ell'}(2\ell+1)(2\ell'+1)\tau_\ell\tau_{\ell'}^* P_\ell(\cos\theta)P_{\ell'}(\cos\theta) = |A|^2/64\pi s k^2$, where $A(s,t)$ is the scattering amplitude, s and t the Mandelstam variables, and the scattering angle $\cos\theta = 1 + t/2k^2$. Then K can be described in terms of those cross sections as

$$K = 1 - \frac{\sigma_{\rm el} - \frac{4k^2}{V}\frac{d\sigma_{\rm el}}{dt}\big|_{t=0}}{\frac{\pi V}{k^2} - \sigma_{\rm inel}}\,. \tag{18}$$

By a power expansion of $S_{\rm EE}$ with respect to $1/V \ll 1$, we obtain $S_{\rm EE} = (k^2/\pi V)\sigma_{\rm el} + \mathcal{O}(1/V^2)$. The leading term is proportional to the elastic cross section. It is consistent with the result shown in the perturbative analysis of field theory with a weak coupling by Ref. [6]. When $k^2\sigma_{\rm el}$ is finite, K is almost equal to one, in other words, the entanglement entropy is negligibly small for elastic scattering at weak coupling. A physical example of such a context may be provided by "hard" (*e.g.* at high momentum transfer) elastic scattering which is governed by a weak coupling constant in QCD.

Physical Hilbert space — In an actual scattering process at a given momentum k, too high angular momentum modes are strongly depleted and negligible in the elastic scattering amplitude. In a semi-classical picture using the impact parameter $b = \ell/k$ representation, the limitation can be depicted as a maximal sizable value $b/2 \le R$, where R is interpreted as the mean of incident particle effective radii. In this context the largest relevant angular momentum $\ell_{\rm max}$ is

$$\ell_{\rm max} \sim 2kR\,. \tag{19}$$

In practice, we shall consider (19) as the maximal value of the angular momentum beyond which the summation over partial wave amplitudes τ_ℓ can be neglected. We thus approximate by truncation the sum over ℓ of the Hilbert space states. Note that reasonable values of R may be obtained from experimental determination of the impact-parameter profile of the scattering amplitude, which can be inferred [11] from the elastic differential cross-section $d\sigma_{\rm el}/dt$.

At high energy, *i.e.*, large momentum k with maximal impact parameter $2R$, $\ell_{\rm max}$ is large. Therefore our derivation of the entropy, $S_{\rm RE} = S_{\rm EE} = -\ln K$, is approximately valid in replacing $\sum_{\ell=0}^{\infty}$ with $\sum_{\ell=0}^{\ell_{\rm max}}$. The Hilbert space volume becomes $V = \sum_{\ell=0}^{\ell_{\rm max}}(2\ell+1) = (1+\ell_{\rm max})^2 \sim \ell_{\rm max}^2 \gg 1$. Then Eq. (18) remains a good approximation with the parameter $V/k^2 \sim 4R^2$. Finally K is obtained as

$$K \sim 1 - \frac{\sigma_{\rm el} - \frac{1}{R^2}\frac{d\sigma_{\rm el}}{dt}\big|_{t=0}}{4\pi R^2 - \sigma_{\rm inel}}\,, \tag{20}$$

so that one gets a finite value for the Renyi and entanglement entropy. Note that $4\pi R^2$ can be considered as the classical "geometric" cross-section of the scattering. Formula (20) implies that, if we measure the cross sections and get an evaluation of the impact parameter profile in a collider experiment, one can give a reliable approximate estimate of the entanglement entropy of the final elastic state of the two outgoing particles.

It is instructive to examine the limiting values of (20) in $0 \le K \le 1$. The value $K = 1$, corresponding to zero entanglement entropy, can be met when R^2 reaches its minimal value $(d\sigma_{\rm el}/dt)|_{t=0}/\sigma_{\rm el}$, which is nothing else than the average size of the elastic diffraction peak. The limit $K \to 0$, *i.e.* $S_{\rm EE} \to \infty$, may be reached only at a zero of the expression $4\pi R^2 - \sigma_{\rm tot} + (d\sigma_{\rm el}/dt)|_{t=0}/R^2$, whose only solution is $\mathrm{Re}\,A(s,0) = 0$ and $\sigma_{\rm tot} = 8\pi R^2$, that is twice the geometric cross section. An exception, leaving K depending on sub-leading terms, is when both numerator and denominator in (20) tend simultaneously to zero, namely $\sigma_{\rm el} = \sigma_{\rm inel} = \sigma_{\rm tot}/2 = 4\pi(d\sigma_{\rm el}/dt)|_{t=0}/\sigma_{\rm el} =$

$4\pi R^2$. Interestingly enough it corresponds to the so-called "black disk" limit, which happens to be phenomenologically relevant for the high energy asymptotics [12].

Conclusion and comments — We have studied the entanglement entropy between two outgoing particles, A and B, in an elastic scattering at high energy, where many inelastic channels are also opened. In the derivation of the entanglement entropy, we used the unitarity condition on the S-matrix (5). As a result, we obtained the formula for the entanglement entropy (16), $S_{\rm EE} = -\ln K$, with Eq. (15).

The Rényi entropy is same as the entanglement entropy, *i.e.*, $S_{\rm RE} = -\ln K$. This implies that the outgoing particles are maximally entangled. This is caused by the fact that the reduced density matrix (9) is diagonal due to the momentum conservation of two scattering particles in the center-of-mass frame.

Eq. (16) is reminiscent of Boltzmann's entropy formula with the Boltzmann constant $k_B = 1$. In this sense, one can regards $1/K$ as a kind of micro-canonical ensemble of final states. Indeed it can be recast in the following form derived from (10):

$$S_{\rm EE} = \ln \frac{V}{2} - \int_{-1}^{1} d\zeta \; \mathcal{P}(\zeta) \ln \mathcal{P}(\zeta) \,, \qquad (21)$$

due to $\mathcal{P}(\zeta) = \frac{V}{2} G(\zeta)$. $\mathcal{P}(\zeta)$ is positive and of norm one in both cases of the full and physical Hilbert spaces, because $\int_{-1}^{1} d\zeta \, \mathcal{P}(\zeta) = 1$ thanks to the orthogonality of Legendre polynomials. Hence one can identify $\mathcal{P}(\zeta)$ with a well-defined probability measure over the interval $\zeta \in [-1, +1]$. We also see that $\mathcal{P}(\zeta)$ originates from the probability $|\langle\langle\vec{p}|\mathbf{s}|\vec{k}\rangle\rangle|^2$ in Eq. (9). Since V can be interpreted as the total number of final two-body quantum states ($2\ell + 1$ at level ℓ), the second term in Eq. (21) can be understood of the correction to the total entropy due to entanglement.

The result for K is described as Eq. (17). The subspace volume of elastic states is small in size with respect to the volume of the overall Hilbert space, K is almost equal to zero, formally equivalent to $V \to \infty$ in formula (17). In other words the entanglement entropy is negligibly small. This is realized, *e.g.*, for elastic scattering at weak coupling corresponding to "hard scattering" contributions.

However for scattering at high energy, conveniently called "soft scattering", we can employ the physical truncation of the Hilbert space given by Eq. (19). We take the limit of large momentum k with fixed impact parameter $b \leq 2R$. Then K becomes Eq. (20). This implies that the entanglement entropy is described in terms of the cross sections and the maximal impact parameter. Since it is possible to measure these parameters in experiments, *e.g.*, a proton-proton scattering in a collider, the entanglement entropy can be evaluated using (20). It would be interesting, in order to confirm the validity of our formula, to confront this result obtained within the S-matrix framework of strong interactions, to a microscopic derivation of the entanglement entropy in a gauge field theory at strong coupling using, *e.g.*, the holographic formalism [2]. It would require the holographic study of a QCD-like theory.

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