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Spin(7)-manifolds in compactifications to four dimensions

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Abstract:

We describe off-shell $\mathcal{N} = 1$ M-theory compactifications down to four dimensions in terms of eight-dimensional manifolds equipped with a topological $Spin(7)$-structure. Motivated by the exceptionally generalized geometry formulation of M-theory compactifications, we consider an eight-dimensional manifold $\mathcal{M}_8$ equipped with a particular set of tensors $\mathfrak{S}$ that allow to naturally embed in $\mathcal{M}_8$ a family of G_2-structure seven-dimensional manifolds as the leaves of a codimension-one foliation. Under a different set of assumptions, $\mathfrak{S}$ allows to make $\mathcal{M}_8$ into a principal S^1 bundle, which is equipped with a topological $Spin(7)$-structure if the base is equipped with a topological G_2-structure. We also show that $\mathfrak{S}$ can be naturally used to describe regular as well as a singular elliptic fibrations on $\mathcal{M}_8$, which may be relevant for F-theory applications, and prove several mathematical results concerning the relation between topological G_2-structures in seven dimensions and topological $Spin(7)$-structures in eight dimensions.

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Contents

1 Introduction **1**

2 M-theory compactifications **3**
- 2.1 The seven dimensional manifold $\mathcal{M}_7$ 5
- 2.2 The underlying eight-dimensional manifold $\mathcal{M}_8$ 7
- 2.3 Motivation for $\mathcal{M}_8$: the generalized geometric formulation 7
- 2.4 Connection to F-theory 10

3 Linear algebra of positive and admissible forms **11**
- 3.1 Positive forms on seven-dimensional vector spaces 12
- 3.2 Relation between admissible and positive forms 14

4 Decrypting the 8-dimensional manifold **16**

5 G_2-intermediate structures **18**
- 5.1 A foliation of $\mathcal{M}_8$ by seven-manifolds 19
- 5.2 G_2-structure seven-manifolds in $\mathcal{M}_8$ 21
- 5.3 Structure of the foliation 25
- 5.4 $\mathcal{M}_8$ as a S^1 principal bundle 27

6 General intermediate manifolds and elliptic fibrations **29**
- 6.1 Regular elliptic fibrations 31
- 6.2 Possibility of singular points 35

7 Conclusions **35**

A G_2-manifolds **36**

B $Spin\,(7)$-manifolds **38**

1 Introduction

Supersymmetric String-Theory compactifications have always been a beautiful source of connections between physics and differential and algebraic geometry[1]. The supersymmetry equations impose topological as well as differential conditions on the space-time manifold that can be nicely codified using the different tools existing in differential geometry, and in particular, the notion of topological and geometric G-structures [5]. For instance, in the seminal paper [6], it was proven that the internal space of a four-dimensional compactification of flux-less heterotic String Theory must be a Calabi-Yau three-fold, that is, a six-dimensional real manifold of $SU(3)$-holonomy. Likewise, the internal space of a flux-less, $\mathcal{N} = 1$ four-dimensional M-Theory compactification, must be a seven-dimensional manifold of G_2-holonomy [7], and the internal manifold of a flux-less, $\mathcal{N} = 1$ four-dimensional F-theory compactification, must be a Calabi-Yau four-fold, namely an eight-dimensional manifold of $SU(4)$-holonomy [8, 9]. As a general rule, the supersymmetry conditions

[1] See for example the reviews [1–4] for more details and further references.

on a flux-less compactification of String/M/F-Theory imply that the internal space has to be a manifold of special holonomy [10, 11]. Remarkably enough, mathematicians had started studying manifolds of special holonomy thirty years before their appearance in String Theory [12, 13], and the study of these manifolds is nowadays still an active field of research in mathematics. In order to write a lower-dimensional effective theory encoding the dynamics of the massless degrees of freedom of the compactified theory, it is necessary to know the moduli space of the compactification manifold [7]. For manifolds of special holonomy, the moduli space is relatively well understood; for example, the moduli space of a Calabi-Yau three-fold is itself a Kähler manifold that factorizes as the product of two projective Special-Kähler manifolds [14]. For the case of G_2-holonomy structures, several results are summarized in reference [11], where the moduli space is given as a finite-dimensional subspace of the infinite-dimensional space of sections of a particular bundle over the seven-dimensional internal manifold. Characterizing the moduli space of geometric structures satisfying a differential condition on a manifold as the finite subspace of the infinite-dimensional space of sections of an appropriate bundle will prove to be the right way to proceed also in more complicated examples, namely in the presence of fluxes.

Compactifications with non-trivial fluxes are considerably more complicated than its flux-less counterparts [2, 15–17]. Although the supersymmetry conditions in the presence of fluxes have been mostly worked out, starting from [18] for the case of the heterotic string, solving them is extremely complicated. This means that already the task of obtaining, or characterizing in any meaningful way, the vacuum of the compactification may be an impossible task. The first consequence of having non-trivial fluxes is that the internal manifold is not going to be in general a manifold of special holonomy. In addition, the moduli space of manifolds satisfying more cumbersome differential conditions are poorly understood, and therefore the task of obtaining an explicit effective action for the compactification theory in the presence of fluxes becomes much harder.

Still, lot of effort has been devoted in order to better understand flux compactifications, and remarkable progress has been made. Recently, two mathematical tools have been developed, to wit, generalized complex geometry (GCG) [19, 20] and exceptional generalized geometry (EGG) [21, 22], that turned out to be very powerful, among their many other applications, in order to study supersymmetric solutions and supersymmetric compactifications of String and M-Theory. The key point in order to apply GCG and EGG to String/M-Theory is to use the fact that supersymmetry conditions on the space-time manifold are sometimes better characterized not using tensors but sections of different vector bundles over the internal compactification space, which in addition imply the corresponding topological reductions on the associated principal bundles. On-shell supersymmetry can then be expressed through differential conditions on the sections of this extended bundles. In the case of GCG this new bundle is the sum of the tangent and the cotangent bundle of the internal space, whereas in EGG it is a more complicated extension of the tangent bundle, such that all the charges in String Theory (including RR ones) or in M-theory are geometrized.

M-Theory compactifications with (a priori off-shell) $\mathcal{N}=1$ supersymmetry are described using a 912-rank vector bundle $E \to \mathcal{M}_7$ where the structure group is $E_{7(7)}$ acting on the **912** representation $E_{\mathbf{912}}$ and $\mathcal{M}_7$ denotes the internal seven-dimensional manifold [22]. The **912** representation is naturally decomposed in terms of the fundamental representation of $Sl(8,\mathbb{R})$ acting on an eight-dimensional vector space V, as follows

$$E_{\mathbf{912}} = S^2V \oplus \left(\Lambda^3 V \otimes V^*\right)_0 \oplus S^2V^* \oplus \left(\Lambda^3 V^* \otimes V\right)_0 \,. \tag{1.1}$$

where S^2 denotes symmetric two-tensors and the subindex 0 denotes traceless. The eight-dimensional vector space V cannot be straightforwardly identified with the tangent space of the internal compactification manifold, being the latter seven-dimensional. It is then suggestive to try to find an *eight-dimensional structure* encoding the natural decomposition of $E_{\mathbf{912}}$ in terms of the fundamental representation of $Sl(8,\mathbb{R})$. In this paper we will pursue this idea, originally proposed in

[23], by studying an eight-dimensional manifold $\mathcal{M}_8$ equipped at every point with a set of tensors $\mathfrak{S}$ defining the decomposition of $E_{\mathbf{912}}$ in terms of $Sl(8,\mathbb{R})$, as given in (1.1). We will study the geometric properties of $\mathcal{M}_8$ relating it to the internal manifolds used in M and F-Theory compactifications. Remarkably enough, the structure $\mathfrak{S}$ prescribed by the decomposition (1.1), is precisely appropriate in order to embed in $\mathcal{M}_8$ a family of manifolds of G_2 structure, relevant in M-Theory compactifications, as well as, at the same time, proving $\mathcal{M}_8$ to have a topological $Spin(7)$ structure, which allows to relate $\mathfrak{S}$ to supersymmetry. In addition, this gives a particular relation between G_2-structure seven-dimensional and $Spin(7)$- structure eight-dimensional manifolds, which may be of physical interest in the context of String/M/F-Theory dualities. Notice that in order to conclude that $\mathcal{M}_8$ is an admissible internal space in F-Theory, it has to be elliptically fibered. It turns out that $\mathfrak{S}$ is an appropriate structure to define, under some mild assumptions, a regular as well as a singular elliptic fibration in $\mathcal{M}_8$.

In this paper we will consider exclusively off-shell supersymmetry and therefore the structures involved will always be topological. Clearly, more effort is needed in order to understand better the relevance of $\mathcal{M}_8$ and $\mathfrak{S}$ in String/M/F-Theory, the first step being to consider on-shell supersymmetry and therefore differential conditions on $\mathfrak{S}$. We leave that for a future publication. What we will unravel here is the more general question of how to build $Spin(7)$-structure eight-dimensional manifolds from G_2-structure seven-dimensional manifolds, or, more in general how G_2-structure and $Spin(7)$-structure manifolds are related. This is a question of very much physical interest, given that G_2-structure manifolds are important in M-Theory compactifications and $Spin(7)$-structure manifolds are important in F-Theory compactifications. Therefore the link between both structures should have an interpretation in terms of string dualities.

The paper is organized as follows. In section 2 we review M-Theory compactifications preserving some off-shell supersymmetry (focusing in particular on $\mathcal{N}=1$) and give the corresponding exceptional generalized geometric formulation, which is also used to motivate the set of tensors $\mathfrak{S}$ defined on $\mathcal{M}_8$. Since it will be important later in order to study the geometry of $\mathcal{M}_8$ equipped with $\mathfrak{S}$, in section 3 we give a fairly complete review of G_2-structures on seven-dimensional vector spaces and $Spin(7)$-structures on eight-dimensional vector spaces, and obtain several results about the relation between them. In section 4 we precisely define $\mathfrak{S}$ on $\mathcal{M}_8$ and state the corresponding existence obstructions. In section 5 we begin the proper study of $\mathcal{M}_8$ equipped with $\mathfrak{S}$, obtaining several results concerning $Spin(7)$-structures on $\mathcal{M}_8$ and the relation of $\mathcal{M}_8$ with seven-dimensional manifolds of G_2-structure. In section 6 we consider the relation of $\mathfrak{S}$ to regular as well as singular elliptic fibrations on $\mathcal{M}_8$, in order to evaluate the viability of $\mathcal{M}_8$ as an internal space in F-Theory compactifications. We conclude in section 7. More details on G_2 and $Spin(7)$ manifolds can be found respectively in Appendices A and B.

2 M-theory compactifications

The effective, low-energy, description of M-theory [24] is believed to be given by eleven-dimensional $\mathcal{N}=1$ Supergravity [25], whose field content is given by a Lorentzian metric g, a three-form gauge field C and a Majorana gravitino Ψ. We are interested in bosonic solutions to eleven-dimensional Supergravity, and therefore we will give from the onset a zero vacuum expectation value to the Majorana gravitino Ψ. The bosonic action of classical eleven-dimensional Supergravity reads

$$S=\int_{\mathcal{M}}\left\{\mathsf{R}\,d\mathcal{V}-\frac{1}{4}\mathsf{G}\wedge*\mathsf{G}+\frac{1}{12}\mathsf{G}\wedge\mathsf{G}\wedge\mathsf{C}\right\}, \tag{2.1}$$

where $\mathcal{M}$ denotes the eleven-dimensional space-time differentiable[2], orientable and spinnable[3] manifold, $d\mathcal{V}$ is the canonical volume form induced by the metric g, and G is the closed four-form flux associated to C, i.e. locally we can write $\mathsf{G} = d\mathsf{C}$. In a bosonic background (g, C), the only non-trivial supersymmetry transformation is the gravitino one, given by[4]

$$\delta_\epsilon \psi(v) = \nabla^S_v \epsilon + \frac{1}{6} \iota_v \mathsf{G} \cdot \epsilon + \frac{1}{12} v^\flat \wedge \mathsf{G} \cdot \epsilon \,, \tag{2.2}$$

where $\psi(v) = \iota_v \psi$, $\epsilon \in \Gamma(S)$, and $\cdot$ denotes the Clifford multiplication.

In this letter we study compactifications of M-theory down such that the resulting four dimensional theory is $\mathcal{N} = 1$ supersymmetric off-shell. Although we speak about *compactifications*, we are not going to assume that $\mathcal{M}_7$ is compact for two basic reasons. First, it may be consistent to compactify in non-compact space-times with finite volume and appropriate behavior of the laplacian operator [26]. And secondly, the results that we will obtain in the rest of the paper involve seven-dimensional manifolds that may not be necessarily compact. We we will assume that the space-time manifold $\mathcal{M}$ can be written as the direct product of four-dimensional Lorentzian space-time $\mathcal{M}_{1,3}$ and a seven dimensional, Riemannian, orientable and spinnable manifold $\mathcal{M}_7$

$$\mathcal{M} = \mathcal{M}_{1,3} \times \mathcal{M}_7 \,, \tag{2.3}$$

and we will therefore take the Lorentzian metric g_{11} on $\mathcal{M}$ to be given by

$$g_{11} = g_{1,3} \times g_7 \,. \tag{2.4}$$

Given the product structure (2.3) of the space-time manifold $\mathcal{M}$, the tangent bundle splits as follows[5]

$$T\mathcal{M} = T\mathcal{M}_{1,3} \oplus T\mathcal{M}_7 \,, \tag{2.5}$$

which allows, using (2.4), a decomposition of the structure group $\mathrm{Spin}(1,10)$ of $\mathcal{M}$ in terms of $\mathrm{Spin}(1,3) \times \mathrm{Spin}(7) \subset \mathrm{Spin}(1,10)$ representations. The corresponding branching rule is

$$\Delta^{\mathbb{R}}_{1,10} = \Delta^+_{1,3} \otimes \left(\Delta^{\mathbb{R}}_7\right)_{\mathbb{C}} \oplus \Delta^{+*}_{1,3} \otimes \left(\Delta^{\mathbb{R}}_7\right)^*_{\mathbb{C}} \,, \tag{2.6}$$

where $\Delta^+_{1,3}$ denotes the positive-chirality complex Weyl representation of $\mathrm{Spin}(1,3)$, $\Delta^{\mathbb{R}}_7$ denotes the real Majorana representation of $\mathrm{Spin}(7)$ and the subscript $\mathbb{C}$ denotes the complexification of the corresponding real representation. Let us respectively denote by $S^+_{1,3}$ and $S^{\mathbb{R}}_7$ the corresponding spin bundles over $\mathcal{M}_{1,3}$ and $\mathcal{M}_7$. Using equation (2.6) we deduce that the supersymmetry spinor $\epsilon \in \Gamma(S)$ decomposes as follows

$$\epsilon = \xi_+ \otimes \eta + \xi^c_+ \otimes \eta^c \,, \tag{2.7}$$

where $\xi_+ \in \Gamma\left(S^+_{1,3}\right)$ and

$$\eta = \eta_1 + i\eta_2 \,, \qquad \eta_1, \eta_2 \in \Gamma\left(S^{\mathbb{R}}_7\right) \,. \tag{2.8}$$

Given the decomposition (2.7), $\mathcal{M}_7$ is equipped with a globally defined no-where vanishing complex spinor η, that is a globally defined section of the complexified spin bundle $S^{\mathbb{R}}_7 \otimes \mathbb{C} \to \mathcal{M}_7$. Its

[2] By differentiable manifold we mean a Hausdorff, second-countable, topological space equipped with a *differentiable structure*.

[3] Since it is spinnable the frame bundle $F(\mathcal{M}) \to \mathcal{M}$ of $\mathcal{M}$ admits a spin structure $\tilde{F}(\mathcal{M})$ such that its associated vector bundle $S \to \mathcal{M}$ is the spin bundle over $\mathcal{M}$ manifold. In eleven dimensions with signature $(1,10)$, the spin bundle S_p, $p \in \mathcal{M}$ is the thirty-two-dimensional Majorana representation $\Delta^{\mathbb{R}}_{1,10}$ and sections $\epsilon \in \Gamma(S)$ of S are Majorana spinors.

[4] We denote by $\flat$ and $\sharp$ the musical isomorphisms defined by the manifold metric.

[5] We omit the pull-backs of the canonical projections.

real components η_1 and η_2 can a priori vanish at points or become parallel, as long as they do not simultaneously vanish.

2.1 The seven dimensional manifold $\mathcal{M}_7$

Generically, the existence of globally defined nowhere vanishing spinors implies a topological reduction of the structure group of the frame bundle from $SO(7)$ (or rather Spin(7) for a spin manifold) to a given subgroup. In the case of seven-dimensional spin manifolds, however, the reduction of the structure group is guaranteed due to the following proposition[6]

Proposition 2.1. *Let $\mathcal{M}$ be a seven-dimensional manifold. Then the following conditions are equivalent*

1. *$\mathcal{M}$ admits a topological* Spin(7)*-structure.*
2. *The first and the second Stiefel-Whitney class of $\mathcal{M}$ vanish, that is $\omega_1 = 0$ and $\omega_2 = 0$.*
3. *$\mathcal{M}$ admits a topological G_2-structure.*

Proof. The equivalence $1 \Leftrightarrow 2$ and the implication $3 \Rightarrow 1$ are obvious. We have to show therefore that the existence of a topological Spin(7)-structure implies the existence of a topological $G_2 \subset$ Spin(7) structure. Let $S^{\mathbb{R}}$ denote the real spin bundle associated to the Spin(7) structure. Since its real dimension is eight, and the real dimension of $\mathcal{M}$ is seven, there exists a global section $\psi \in \Gamma\left(S^{\mathbb{R}}\right)$ of unit length. Since the G_2^c can be defined as the isotropy group of a given real spinor of Spin(7), ψ can be used to define a topological G_2^c structure on $\mathcal{M}$. □

On a seven-dimensional spin manifold we have therefore always one globally defined no-where vanishing spinor. For compact seven dimensional manifolds, the implications of a spin structure are even stronger, namely

Proposition 2.2. *Let $\mathcal{M}$ be a compact seven-dimensional manifold. Then the following conditions are equivalent*

1. *$\mathcal{M}$ admits a topological* Spin(7)*-structure.*
2. *The first and the second Stiefel-Whitney class of $\mathcal{M}$ vanish, that is $\omega_1 = 0$ and $\omega_2 = 0$.*
3. *$\mathcal{M}$ admits a topological G_2-structure.*
4. *$\mathcal{M}$ admits a topological $SU(3)$-structure.*
5. *$\mathcal{M}$ admits a topological $SU(2)$-structure.*

Proof. See proposition 3.2 in [27]. □

Therefore, if $\mathcal{M}_7$ is orientable, compact and spin, it admits a topological $SU(2)$ structure, or equivalently three globally defined nowhere vanishing and nowhere parallel spinors.

In particular, as shown in appendix A, there is a one to one correspondence between G_2 structures, positive three-forms[7] and $Spin(7)$ real spinors on a seven-dimensional manifold. Therefore, *for every G_2-structure* there is automatically a positive three-form and a $Spin(7)$ real spinor globally defined on $\mathcal{M}$.

[6] G_2-structures will be introduced in detail in section 3. Further definitions and properties are given in Appendix A.

[7] See appendix A for the definition of positive three-form.

For compactifications preserving off-shell supersymmetry, $\mathcal{M}_7$ is equipped with two spinors $\eta_1\,,\eta_2 \in \Gamma\left(S^{\mathbb{R}}_7\right)$, which may allow for a further reduction of the structure group G_2 of $\mathcal{M}_7$. Since every orientable and spin G_2 manifold already has a globally defined section $\psi \in \Gamma\left(S^{\mathbb{R}}_7\right)$ of the spin bundle, we might have three globally defined spinors on $\mathcal{M}_7$, namely $\eta_1\,,\eta_2$ and ψ, which globally give rise to three different possibilities

- The three spinors are nowhere vanishing and nowhere linearly dependent, which implies that they define a topological $SU(2)$-structure on $\mathcal{M}_7$.
- Only two spinors are nowhere vanishing and linearly independent, which implies that they define a topological $SU(3)$-structure on $\mathcal{M}_7$.
- The three spinors are linearly dependent everywhere, which implies that they define a topological G_2-structure on $\mathcal{M}_7$, something that is always guaranteed due to proposition 2.1.

Of course, there exists the possibility that the spinors become linearly dependent only at some points in $\mathcal{M}_7$. In such situation there is no globally well-defined topological reduction of the frame bundle further than the G_2 one, but there is, in the case of two Spin(7) spinors, a well defined global reduction in the generalized bundle $\mathbb{E} = T\mathcal{M}_7 \oplus T^*\mathcal{M}_7$ from $\mathbb{R}^* \times \mathrm{Spin}(7,7)$ to $G_2 \times G_2$ [28, 29].

In the case when the three spinors are nowhere vanishing (no matter what their respective inner product is), due to the isomorphism $S^{\mathbb{R}}_7 \otimes S^{\mathbb{R}}_7 \simeq \Lambda^\bullet T^*\mathcal{M}_7$ we can write

$$\eta_a \otimes \eta_a = 1 + \phi_a + *\phi_a + \mathcal{V}_{g_a}\,, \qquad a = 1,2\,, \tag{2.9}$$

$$\psi \otimes \psi = 1 + \phi_\psi + *\phi_\psi + \mathcal{V}_{g_\psi}\,, \tag{2.10}$$

where ϕ_a and ϕ_ψ are the corresponding positive forms associated to η_a and ψ and $\mathcal{V}_{g_a}\,,\mathcal{V}_{g_\psi}$ are the volume forms associated to the metric defined by the corresponding positive three-form. In terms of spinor bilinears we have

$$\phi_a = \mathrm{i}\eta^T\gamma_{(3)}\eta\,, \quad a = 1,2\,, \qquad \phi_\psi = \mathrm{i}\psi^T\gamma_{(3)}\psi\,, \tag{2.11}$$

where $\gamma_{(3)}$ is the anti-symmetrized product of three gamma matrices. In fact, using η_a and ψ we can construct many more forms on $\mathcal{M}_7$ than those appearing in equations (2.9) and (2.10). They can be used to alternatively define the corresponding reductions of the structure group of $\mathcal{M}_7$ [30].

Backgrounds preserving $\mathcal{N} = 1$ supersymmetry (on-shell), should be invariant under a supersymmetry transformation. In the case of bosonic backgrounds, the only non-trivial one is that of the gravitino transformation, given in equation (2.2). Supersymmetry requires equation (2.2) to vanish for any vector v, and therefore implies differential equations on the supersymmetry spinors η_1 and η_2. In the case where $\eta_1 = \eta_2 = \psi$ we can only construct a single positive three-form on $\mathcal{M}_7$, ϕ_3. The holonomy of $\mathcal{M}_7$ will be G_2 if and only if

$$\nabla\phi_3 = 0\,, \tag{2.12}$$

which, from equation (2.2), is the case for supersymmetric backgrounds in the absence of fluxes (that is, when $\mathsf{G} = 0$). Equivalently, see appendix A, $\mathcal{M}_7$ will have G_2 holonomy if and only if

$$d\phi_3 = 0\,, \qquad d*\phi_3 = 0\,. \tag{2.13}$$

In the presence of fluxes, the situation is more subtle, namely there should be a connection with G_2 holonomy, but it is in general not the Levi-Civita one. In this case the manifold does not have G_2 holonomy, but still it does have a G_2-structure [30–32]. The exterior derivatives $d\phi_3$,

$d*\phi_3$ can be decomposed into G_2 representations, defining the torsion classes. In a supersymmetric compactification, the different torsion classes are related to the G_2 representations of the four-form flux (see [30–32] for details).

2.2 The underlying eight-dimensional manifold $\mathcal{M}_8$

The manifold $\mathcal{M}_7$ is of course seven-dimensional. However, as it will be explained in section 2.3, the main purpose of this work is to introduce an eight-dimensional manifold $\mathcal{M}_8$, whose existence is motivated by the Exceptional Generalized geometric formulation of $\mathcal{N}=1$ four-dimensional M-theory compactifications. We will be see that $\mathcal{M}_8$ can be related to $\mathcal{M}_7$ in a very natural way. As we did in section 2.1 for seven-dimensional manifolds, here we will present some of the properties of eight-dimensional manifolds admitting nowhere vanishing spinors.

The frame bundle of an orientable, spin, eight-dimensional manifold admits a reduction to Spin(8). In addition, if the manifold is equipped with an *admissible* four-form $\Omega \in \Gamma(\mathcal{M}_8)$ (see appendix B for more details and further references), then the structure group is reduced to Spin(7). The manifold $\mathcal{M}_8$ has Spin(7) holonomy if and only if

$$d\Omega = 0\,. \tag{2.14}$$

The failure of Ω to be closed is a measure of the deviation of $\mathcal{M}_8$ to have Spin(7) holonomy. The Spin(7) structure can be alternatively defined by a globally defined Majorana-Weyl spinor, which implies Spin(7) holonomy if and only if it is covariantly constant respect to the Levi-Civita spinor connection.

If there are two globally defined Majorana-Weyl spinors, then the structure group can be further reduced, depending on the relative properties of the spinors. If the two spinors have opposite chirality, the structure group of $\mathcal{M}_8$ is reduced to G_2, but the corresponding Riemannian metric is not irreducible. If the spinors have the same chirality and are never parallel, then the structure group is reduced to $SU(4)$. Notice that in the case of $SU(4)$ holonomy, i.e. where the spinors are covariantly constant, then the manifold os a Calabi-Yau four-fold. In the general case, where the two spinors might become parallel at some points, there is no global topological reduction further than the $Spin(7)$ one. However, as it happened in seven dimensions, there is a well-defined global reduction from the point of view of the generalized tangent space $TX \oplus T^*X$, from $\mathbb{R}^* \times Spin(8,8)$ to $Spin(7) \times Spin(7)$.

2.3 Motivation for $\mathcal{M}_8$: the generalized geometric formulation

A geometric formulation of the bosonic sector of eleven-dimensional Supergravity compactified down to four-dimensions was developed in [21, 22], extending the idea underlying a similar formulation for the NS sector of Type-II Supergravity, based on Generalized Complex Geometry. In the latter, the diffeomorphisms and the gauge transformations of the B-field, generated respectively by vectors and one-forms, are combined into sections of the generalized tangent space, which is locally the sum of the tangent plus the cotangent space. In the former, diffeomorphisms are combined with two-form gauge transformations of the three-form gauge field C. In order to complete a closed orbit under the U-duality group $E_{7(7)}$[8], one needs however to include also the gauge transformations of the dual six-form field $\tilde{C}$, given by five-forms, as well as gauge transformations for the dual graviton, parameterized by the tensor product of one-forms times seven-forms [22, 33]. The total number of degrees of freedom is 56, corresponding to the fundamental representation of $E_{7(7)}$. The exceptional generalized tangent space E is locally given by[9]

[8] $E_{7(7)}$ is the maximally non-compact real form of the complex exceptional Lie group E_7.
[9] To abbreviate we use T_7 to denote $T\mathcal{M}_7$.

$$
\begin{aligned}
E &= (\Lambda^7 T_7)^{1/2} \otimes \left(T_7 \oplus \Lambda^2 T_7^* \oplus \Lambda^5 T_7^* \oplus (T_7^* \otimes \Lambda^7 T_7^*)\right) \,, \\
\mathbf{56} &= \mathbf{7} \oplus \mathbf{21} \oplus \mathbf{21} \oplus \mathbf{7} \,,
\end{aligned}
\tag{2.15}
$$

where the overall volume factor gives the proper embedding in $E_{7(7)}$. This representation is also the one that combines the charges of the theory, namely momentum, M2 and M5-brane charge and Kaluza-Klein monopole charge. Note that the **21** and **7** representations can be combined into the **28** of $SL(8,\mathbb{R})$, corresponding to two-forms or two-vectors in eight dimensions. To be more precise, defining

$$
\begin{aligned}
T_8^* &= (\Lambda^7 T_7^*)^{-1/4} \otimes (T_7^* \oplus \Lambda^7 T_7^*) \,, \\
\mathbf{8} &= \mathbf{7} \oplus \mathbf{1} \,,
\end{aligned}
\tag{2.16}
$$

we have

$$
\begin{aligned}
E =& \Lambda^2 T_8 \oplus \Lambda^2 T_8^* \,, \\
\mathbf{56} =& \mathbf{28} \oplus \mathbf{28'} \,.
\end{aligned}
\tag{2.17}
$$

At each point over the seven-dimensional manifold $\mathcal{M}_7$, the fibre of the extended vector bundle can be naturally decomposed in terms of the eight dimensional vector space where $Sl(8,\mathbb{R})$ acts in the fundamental representation. However, this eight dimensional vector space cannot be naturally identified, at each point $p \in \mathcal{M}_7$, with the tangent vector space of $\mathcal{M}_7$, because of the obvious dimensional mismatch. Since $Sl(8,\mathbb{R})$ is the structure group of an eight-dimensional orientable manifold, we consider that is natural to propose an eight-dimensional orientable manifold $\mathcal{M}_8$ such that, at each point $p \in \mathcal{M}_8$, carries a decomposition of the $E_{7(7)}$ appropriate representation in terms of the fundamental representation of $Sl(8,\mathbb{R})$ acting on the eight-dimensional tangent space $T_p\mathcal{M}_8$. This way, the tangent space of $\mathcal{M}_8$ can be *connected* to the rank eight vector bundle, with structure group $Sl(8,\mathbb{R})$, proposed in [23]. We will elaborate later about this connection.

In order for $\mathcal{M}_8$ to carry at each point a decomposition of the appropriate $E_{7(7)}$ representation, it must be equipped, at every point, with the tensors appearing in the given decomposition. Therefore, $\mathcal{M}_8$ must be equipped with globally defined tensors, determined by the decomposition of the corresponding $E_{7(7)}$ representation in terms of $Sl(8,\mathbb{R})$ representations. There are three relevant $E_{7(7)}$ representations appearing in the Exceptional Generalized formulation of M-theory compactified to four dimensions, and thus have different possibilities depending on which $E_{7(7)}$ representation we consider, to wit

- The fundamental, symplectic, representation **56** of $E_{7(7)}$. The corresponding decomposition, in terms of $Sl(8,\mathbb{R})$ representations, is given by (2.17), namely

$$
E_{\mathbf{56}} = \Lambda^2 V \oplus \Lambda^2 V^* \,,
\tag{2.18}
$$

  where V is an eight-dimensional real vector space. Therefore, if we wanted $\mathcal{M}_8$ to carry a representation of $E_{\mathbf{56}}$ in terms of $Sl(8,\mathbb{R})$ representations, it should be equipped with a bivector β field and a two-form ω

$$
\omega \in \Gamma\left(\Lambda^2 T^* \mathcal{M}_8\right) \,, \qquad \beta \in \Gamma\left(\Lambda^2 T \mathcal{M}_8\right) \,.
\tag{2.19}
$$

- The adjoint representation **133** of $E_{7(7)}$. The corresponding decomposition, in terms of $Sl(8,\mathbb{R})$ representations, is given by

$$
E_{\mathbf{133}} = (V \otimes V^*)_0 \oplus \Lambda^4 V^* \,,
\tag{2.20}
$$

where V is an eight-dimensional real vector space and the subindex 0 denotes traceless. Therefore, if we wanted $\mathcal{M}_8$ to carry a representation of $E_{\mathbf{133}}$ in terms of $Sl(8,\mathbb{R})$ representations, it should be equipped with the following sections

$$\mu \in \Gamma\left(T\mathcal{M}_8 \otimes T^*\mathcal{M}_8\right)_0 \ , \qquad \Omega \in \Gamma\left(\Lambda^4 T^*\mathcal{M}_8\right) \ . \tag{2.21}$$

- The **912** representation of $E_{7(7)}$. The corresponding decomposition, in terms of $Sl(8,\mathbb{R})$ representations, is given by

$$E_{\mathbf{912}} = S^2V \oplus \left(\Lambda^3 V \otimes V^*\right)_0 \oplus S^2V^* \oplus \left(\Lambda^3 V^* \otimes V\right)_0 \ , \tag{2.22}$$

where V is an eight-dimensional real vector space and S^2 denotes symmetric two-tensors. Therefore, if we wanted $\mathcal{M}_8$ to carry a representation of $E_{\mathbf{912}}$ in terms of $Sl(8,\mathbb{R})$ representations, it should be equipped with the following sections

$$\begin{aligned} g &\in \Gamma\left(S^2 T\mathcal{M}_8\right) \ , \qquad \phi \in \Gamma\left(\Lambda^3 T\mathcal{M}_8 \otimes T^*\mathcal{M}_8\right)_0 \ , \\ \tilde{g} &\in \Gamma\left(S^2 T^*\mathcal{M}_8\right) \ , \qquad \tilde{\phi} \in \Gamma\left(\Lambda^3 T^*\mathcal{M}_8 \otimes T\mathcal{M}_8\right)_0 \ . \end{aligned} \tag{2.23}$$

In this letter we are going to consider the **912** representation, since it is the relevant one to describe the moduli space of $\mathcal{N}=1$ supersymmetric M-theory compactifications. In doing so, we will be able to translate the information about the moduli space contained in the generalized $E_{7(7)}$-bundle to the tangent bundle of the eight-dimensional manifold $\mathcal{M}_8$, therefore giving an *intrinsic* formulation in terms of tensor bundles instead of *extrinsic* bundles, which generically are more difficult to handel: using the $E_{7(7)}$-bundle one can characterize the moduli space using its space of sections together with the appropriate differential conditions and equivalence relation, whereas using $\mathcal{M}_8$ one can characterize the moduli space through the space of sections of several of its tensor bundles, together again with the appropriate differential conditions and equivalence relation. The space of sections needed to characterize the moduli space using $\mathcal{M}_8$ is what we will define later to be an *intermediate structure*, see definition (4.1). Therefore, we propose that the study of the moduli space of $\mathcal{N}=1$ M-theory compactifications to four-dimensional Minkowski space-time can be rephrased as the study of the moduli space of intermediate structures on the corresponding eight-dimensional manifold $\mathcal{M}_8$. The appearance of the **912** representation can be justified as follows.

The presence of the supersymmetry spinors $\eta_1, \eta_2 \in \Gamma\left(S^{\mathbb{R}}_7\right)$ in $\mathcal{M}_7$ implies a global reduction of an appropriate vector bundle which is an extension of the tangent space. In order to perform the reduction, we have to identify the complex $Spin(7)$-spinor $\eta = \eta_1 + i\eta_2$ as a complex Weyl $Spin(8)$ spinor. As explained in the previous section, with respect to $Spin(7)$, the real and imaginary parts of the spinor define each a G_2 structure and together they define a reduction of the structure group of the generalized bundle $\mathbb{E} = T\mathcal{M}_7 \oplus T^*\mathcal{M}_7$ to $G_2 \times G_2$. In $Sl(8,\mathbb{R})$, the real and imaginary parts of the complex $Spin(8)$ spinor η define a pair of $Spin(7)$ structures. In $E_{7(7)}$ the complex spinor transforms in the fundamental of $SU(8)$, and defines a single $SU(7)$ structure, since within $E_{7(7)}$ the complex spinor transforms in the **8** of $SU(8)$, and is stabilized by an $SU(7)$ subgroup. This $SU(7)$ structure can be equivalently defined by a nowhere vanishing section of the **912** representation of $E_{7(7)}$, which decomposes into the $Sl(8,\mathbb{R})$ representations as in (2.22). The following object, constructed from the internal spinor η, is indeed stabilized by $SU(7) \subset SU(8) \subset E_{7(7)}$ [22]

$$\psi = (2\eta \otimes \eta, 0, 0, 0) \, . \tag{2.24}$$

Using an $Sl(8,\mathbb{R})$ metric g_8, this object has the following $Sl(8,\mathbb{R})$ representations [23]

$$\psi = (\mathrm{Re}c\, g_8^{-1}, g_8 \cdot (\mathrm{vol}_8^{-1} \llcorner \mathrm{Re}\phi_4), \mathrm{Im}c\, g_8, g_8^{-1} \cdot \mathrm{Im}\phi_4)\,, \tag{2.25}$$

where $c = \eta^T \eta$ and the four-form ϕ_4 is

$$\phi_4 = \eta^T \gamma_{(4)} \eta\,. \tag{2.26}$$

In terms of the 7+1 split in (2.16), this is

$$\phi_4 = \rho_8 \wedge \phi_3 + *_7 \phi_3\,, \tag{2.27}$$

where ρ_8 is a one-form along the **1** in (2.16), and ϕ_3 is a complex three-form which reduces to a real three-form in the G_2-structure case (i.e. when η is Majorana), given by (2.11).
In sections 4 and on we will analyze the geometric properties of $\mathcal{M}_8$ and connect the geometric structures defined on it to the supersymmetry spinors η_1 and η_2, globally defined on $\mathcal{M}_7$ when the compactification to four-dimensions preserves some supersymmetry off-shell. Among other things, the goal of this letter is to study the possible role of $\mathcal{M}_8$ in relation to the internal manifolds appearing in $\mathcal{N} = 1$ supersymmetric M-theory compactifications and, since $\mathcal{M}_8$ is eight dimensional, also study if $\mathcal{M}_8$ is an admissible internal space for F-theory compactifications. For the latter, we give in section 2.4 a very brief review of the type of eight-dimensional manifolds that appear as internal spaces in F-theory.

2.4 Connection to F-theory

Being $\mathcal{M}_8$ eight-dimensional, the natural question is that if it is an admissible internal space for F-theory [8] compactifications down to four dimensions (see [3, 34, 35] for more details and further references). F-theory compactifications to four-dimensions are defined through M-theory compactifications to three dimensions on an eight-dimensional manifold X. In order to have $\mathcal{N} = 1$ supersymmetry in four dimensions, the compactification theory in three-dimensions must have $\mathcal{N} = 2$ supersymmetry. This imposes a constraint on the eight-dimensional internal space X, which must be a Calabi-Yau four-fold[10], that is, a $SU(4)$-holonomy eight-dimensional manifold. In order to be able to *appropriately lift* the three-dimensional effective $\mathcal{N} = 2$ Supergravity theory to four dimensions, the internal space must be in addition elliptically fibered, that is, it must be of the form

$$X \xrightarrow{\pi} \mathcal{B}\,, \tag{2.28}$$

where $\mathcal{B}$ is the base space, which should be a three-complex-dimensional Kähler manifold, and the fibre $\pi^{-1}(b)$ at every $b \in \mathcal{B}$ is an elliptic curve, possibly singular. Therefore, if we want the proposed $\mathcal{M}_8$ to be an admissible internal manifold for supersymmetric compactifications of F-theory, it must be Calabi-Yau and elliptically fibered. As mentioned before, for the structure group of an eight-dimensional manifold X to be reduced to $SU(4)$, it must be equipped with two Majorana-Weyl spinors of the same chirality and linearly independent at every point $p \in X$, which are covariantly constant if and only if X has $SU(4)$ holonomy. If X is equipped with two Majorana-Weyl spinors of the same chirality but which are parallel at some points then there is no global reduction of the structure group on the tangent space to $SU(4)$, but there is a global reduction in $TX \oplus T^*X$ from $\mathbb{R}^* \times Spin(8,8)$ to $Spin(7) \times Spin(7)$. To the best of our knowledge, F-theory compactified in manifolds with $Spin(7) \times Spin(7)$ structure structure has not been fully analyzed yet. Progress in this direction can be found in reference [36].

[10] We refer to flux-less compactifications. If we include a non-vanishing G_4 flux, then X is a *conformal* Calabi-Yau four-fold, which does not have $SU(4)$-holonomy anymore.

If we drop the requirement of $\mathcal{N} = 2$ supersymmetry in three dimensions, and demand only minimal supersymmetry in three dimensions, then X is not forced to be a Calabi-Yau four-fold but a Spin(7)-holonomy manifold, and we should not expect in principle a supersymmetric theory in for dimensions. However, and remarkably enough, [37, 38] claim that F-theory compactified on certain Spin(7)-holonomy orbifold (constructed by orbifolding a Calabi-Yau four-fold), where one dimension has the topology of an interval, give in the limit of infinite length of this interval, a supersymmetric $\mathcal{N} = 1$ theory in four dimensions. Therefore we will consider in this letter that an elliptically fibered, eight-dimensional, $Spin(7)$ manifold is an admissible internal space for F-theory compactifications. In particular, we will find that the eight-dimensional manifold $\mathcal{M}_8$ can be elliptically fibered and it is equipped, under some mild assumptions, with a $Spin(7)$ structure on the frame bundle, or more generally, a $Spin(7) \times Spin(7)$ structure on the generalized bundle. Hence, $\mathcal{M}_8$ arises as a plausible compactification space for F-theory, which in addition can be related in a precise way to G_2-structure seven-dimensional manifolds. We will leave the analysis of the holonomy of $\mathcal{M}_8$ and its preferred submanifolds to a forthcoming project [39].

3 Linear algebra of positive and admissible forms

The first part of this section is devoted to introducing some linear algebra results regarding the definition of topological G_2-structures that will be useful later on. More details can be found in appendix A. The second part of this section studies the relation, at the linear algebra level, between the differential forms associated to topological G_2-structures and topological $Spin(7)$-structures.

Definition 3.1. *Let V be an n-dimensional vector space and let $\omega \in \Lambda^q V^*$ be a q-form. ω is said to be* non-degenerate *if the following holds:*

$$\forall\ v \in V,\ i_v\omega = 0 \Rightarrow v = 0\,. \tag{3.1}$$

In other words, a non-degenerate q-form provides an injective map from V to the vector space of $(q-1)$-forms $\Lambda^{q-1}V^*$. Let $\omega \in \Omega^q(\mathcal{M})$ be a q-form defined on a differentiable manifold $\mathcal{M}$. Then ω is said to be non-degenerate if it is non-degenerate at every point $p \in \mathcal{M}$. The previous notion of non-degeneracy is extensively used in the context of multisymplectic geometry [40]. However, in references [41, 42], a different notion of non-degeneracy, called *stability*, was introduced by Hitchin. The definition goes as follows.

Definition 3.2. *Let V be an n-dimensional vector space and let $\omega \in \Lambda^q V^*$ be a q-form. ω is said to be* stable *if it lies in an open orbit of the action of the group $GL(V)$ on $\Lambda^q V^*$.*

Let $\omega \in \Omega^q(\mathcal{M})$ be a q-form defined on a differentiable manifold $\mathcal{M}$. Then ω is said to be stable if it is stable at every point $p \in \mathcal{M}$. Although the notion of stability can be defined for any form, it can be shown that stable forms only occur in certain dimensions and for certain q-forms. In particular, aside from cases $q = 1, 2$, stability can only occur for three-forms (and their Hodge-duals), in six, seven and eight dimensions. A manifold equipped with a stable form ω has its structure group reduced to the stabilizer group of ω. It is clear that the notion of stability in general is not equivalent to the notion of non-degeneracy, as defined in definition 3.1. For instance, in even dimensions, the notion of non-degenerate two-form is equivalent to the notion of stable two-form, since the general linear group has only one open orbit when acting on $\Lambda^2 V^*$, and this orbit consists exactly of the non-degenerate two forms. However, in odd dimensions a two-form can never be non-degenerate yet it can be stable.

3.1 Positive forms on seven-dimensional vector spaces

Since we are interested in seven-dimensional manifolds with G_2 structure, we will focus now on the case of three-forms in seven dimensions. The reason is explained in appendix A: a seven-dimensional manifold has the structure group of its frame bundle reduced from $GL(7,\mathbb{R})$ to the compact real form G_2 of the complex exceptional Lie group $G_2^{\mathbb{C}}$ if and only if it can be equipped with a globally defined, *positive* three-form. A positive form is a particular case of stable three-form. The three-form ϕ_0 on $\mathbb{R}^7$ we define now is positive.

Definition 3.3. *Let $(x_1,\ldots,x_7)$ be coordinates on $\mathbb{R}^7$. We define a three-form ϕ_0 on $\mathbb{R}^7$ by*

$$\phi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}\,, \tag{3.2}$$

where $dx_{ij\ldots l}$ stands for $dx_i \wedge dx_j \wedge \cdots \wedge dx_l$. The subgroup of $GL(7,\mathbb{R})$ that preserves ϕ_0 is the compact real form G_2 of the exceptional complex Lie group $G_2^{\mathbb{C}}$, which also fixes the euclidean metric $g_0 = dx_1^2 + \cdots + dx_7^2$, the orientation on $\mathbb{R}^7$ (that is, $G_2 \subset SO(7)$). Further G_2 fixes the four-form $\tilde{\phi}_0$,

$$\tilde{\phi}_0 = dx_{4567} + dx_{2367} + dx_{2345} + dx_{1357} - dx_{1346} - dx_{1256} - dx_{1247}\,. \tag{3.3}$$

Notice that $\tilde{\phi}_0 = *\phi_0$ where $*$ is the Hodge-dual operator associated to g_0.

Notice that every three-form $\phi \in \Lambda^3 V^*$ defines a symmetric bilinear form $\mathfrak{B} : V \times V \to \mathbb{R}$ as follows:

$$\mathcal{V}\cdot\mathfrak{B}(v,w) = \frac{1}{3!}\,\iota_v\phi \wedge \iota_w\phi \wedge \phi\,, \qquad v,w \in V\,, \tag{3.4}$$

where $\mathcal{V}$ is the seven-dimensional volume form in V. If ϕ is a no-where vanishing non-degenerate three-form, then (3.4) is a non-degenerate bilinear form, in the sense that

$$\mathfrak{B}(v,v) \neq 0\,, \qquad \forall\, v \in V - \{0\}\,. \tag{3.5}$$

In addition, if ϕ is positive then $\mathfrak{B} : V \times V \to \mathbb{R}$ is a positive definite symmetric bilinear form on V. If we take $\phi = \phi_0$ we obtain

$$\mathcal{V}\cdot\mathfrak{B}_0(v,w) = \frac{1}{3!}\,\iota_v\phi_0 \wedge \iota_w\phi_0 \wedge \phi_0 = g_0(v,w)\,\mathcal{V}\,, \qquad \forall\, v,w \in V\,, \tag{3.6}$$

and therefore g_0 is the metric (i.e., inner product) naturally induced on V by the three form ϕ_0.

Definition 3.4. *Let $(x_1,\ldots,x_7)$ be coordinates on $\mathbb{R}^7$. We define a three-form ϕ_1 on $\mathbb{R}^7$ by*

$$\phi_1 = -dx_{127} + dx_{145} - dx_{135} + dx_{146} + dx_{236} - dx_{245} - dx_{347} + dx_{567}\,, \tag{3.7}$$

where $dx_{ij\ldots l}$ stands for $dx_i \wedge dx_j \wedge \cdots \wedge dx_l$. The subgroup of $GL(7,\mathbb{R})$ that preserves ϕ_1 is the non-compact real form $G_2^* \subset SO(4,3)$ of the exceptional complex Lie group $G_2^{\mathbb{C}}$, which also fixes the indefinite metric $g_1 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 - dx_5^2 - dx_6^2 - dx_7^2$, the orientation on $\mathbb{R}^7$ and the four-form $*\phi_1$,

$$\tilde{\phi}_1 = -dx_{3456} + dx_{2367} - dx_{2467} + dx_{2357} + dx_{1457} - dx_{1367} - dx_{1256} + dx_{1234}\,. \tag{3.8}$$

Notice that $\tilde{\phi}_1 = *\phi_1$ where $*$ is the Hodge-dual operator associated to g_1.

In this case we have

$$\mathcal{V}\cdot\mathfrak{B}_1(v,w) = \frac{1}{3!}\iota_v\phi_1 \wedge \iota_w\phi_1 \wedge \phi_1 = g_1(v,w)\,\mathcal{V}\,, \qquad \forall\, v,w \in V\,, \tag{3.9}$$

and therefore g_1 is the metric naturally induced in V by the three form ϕ_1.

We now define positive three-forms and display some of their properties.

Definition 3.5. *Let V be an oriented seven-dimensional vector space. A three-form $\phi \in \Lambda^3 V^*$ is said to be* positive *if there exists an oriented*[11] *isomorphism $f : \mathbb{R}^7 \to V$ such that $\phi_0 = f^*\phi$. We denote by $\mathcal{P}_V$ the set of positive three-forms in $\phi \in \Lambda^3 V^*$.*

A positive form induces an inner product on V, as a consequence of $G_2 \subset SO(7)$.

Proposition 3.6. *Let V be a seven-dimensional vector space. Then the set $\mathcal{P}_V$ of positive forms is an open subset of $\Lambda^3 V^*$.*

Proof. Since $\mathcal{P}_V$ is defined as those forms in $\Lambda^3 V^*$ such that there exists an oriented isomorphism relating them to ϕ_0, and ϕ_0 is stabilized by G_2, we conclude that

$$\mathcal{P}_V \simeq \frac{GL_+(7,\mathbb{R})}{G_2}\,, \tag{3.10}$$

and thus, since G_2 is a Lie subgroup of $GL_+(7,\mathbb{R})$, $\mathcal{P}_V$ is an homogeneous manifold of dimension $\dim \mathcal{P}_V = \dim GL_+(7,\mathbb{R}) - \dim G_2 = 49-14 = 35$. $\Lambda^3 V^*$ is a vector space of dimension $\dim \Lambda^3 V^* = \frac{7!}{4!3!} = 35$. Hence, $\mathcal{P}_V$ is a submanifold of $\Lambda^3 V^*$ of the same dimension as $\Lambda^3 V^*$, and therefore it must be an open set in $\Lambda^3 V^*$. □

By the proposition above, a positive form is stable, since it belongs to an open orbit. However, the converse is not true, due to the fact there exists more than one open orbit. The description of the set of stable forms on $\mathbb{R}^7$ is the following [43–45].

- The general group $GL(7,\mathbb{R})$ acting on $\Lambda^3(\mathbb{R}^7)^*$ has exactly two open orbits Λ_0 and Λ_1, each of which is disconnected and can be characterized as follows:
  - Λ_0 contains ϕ_0, therefore any other form $\phi \in \Lambda_0$ is stabilized by a group conjugate to the real compact form G_2 of $G_2^{\mathbb{C}}$.
  - Λ_1 contains ϕ_1, therefore any other form $\phi \in \Lambda_0$ is stabilized by a group conjugate to the non-compact real form G_2^* of $G_2^{\mathbb{C}}$.
- Each open orbit consists of two connected components, namely Λ_0^+ and Λ_1^+, which are given by the action of $GL_+(7,\mathbb{R})$ on ϕ_0 and ϕ_1 respectively . Here $GL_+(7,\mathbb{R})$ and $GL_-(7,\mathbb{R})$ are the elements of $GL(7,\mathbb{R})$ of positive and negative determinant respectively.
- The set of stable three-form $\phi \in \Lambda^3 V^*$ is given by the disjoint union $\Lambda_0^+ \cup \Lambda_0^- \cup \Lambda_1^+ \cup \Lambda_1^-$. The positive forms are exactly those contained in Λ_0^+.

Proposition 3.7. *Let V be a seven-dimensional vector space and let $\mathcal{N}_V$ and $\mathcal{S}_V$ respectively denote the set of nondegenerate three-forms and the set of stable three-forms. Then, $\mathcal{S}_V \subset \mathcal{N}_V$, that is, $\mathcal{S}_V$ is a subset $\mathcal{N}_V$. In addition, $\mathcal{S}_V \neq \mathcal{N}_V$, that is, there exist non-degenerate three-forms that are not stable.*

Proof. Since every stable form is related to ϕ_0 or ϕ_1 by an isomorphism, to show $\mathcal{S}_V \subset \mathcal{N}_V$ it is enough to check that ϕ_0 and ϕ_1 are non-degenerate. This can be checked by explicit calculation. In order to see now that there are more degenerate three-forms than stable three-forms, we are going to proceed by a parameter count[12].

- Dimension of the space of three-forms: $\dim \Lambda^3 V^* = 35$.

[11] This means that f preserves the orientations.

[12] We thank Dominic Joyce for a private communication regarding this issue.

- Dimension of the space of three forms $\phi \in \Lambda^3 V^*$ such that $\iota_v \phi = 0$, where $v \in V$ is a fixed unit norm vector = Dimension of three-forms on $\mathbb{R}^6 = 20$.
- Dimension of the space of unit vectors in $V = 6$.
- Dimension of the space of three forms $\phi \in \Lambda^3 V^*$ such that $\iota_v \phi = 0$, for some unit vector $v \in V$ = Dimension of three-forms on $\mathbb{R}^6$ + Dimension of unit vectors in $V = 26$.

Therefore $\Lambda^3 V^* - \mathcal{N}_V$, the space of degenerate three-forms, has dimension 26. On the other hand, as seen above $\mathcal{S}_V$ is not connected, hence the space $\Lambda^3 V^* - \mathcal{S}_V$ of three-forms which are not stable must have a component of codimension one, i.e. dimension 34. (If $\Lambda^3 V^* - \mathcal{S}_V$ had codimension ≥ 2, then $\mathcal{S}_V$ would be connected.) Since $26 < 34$, we conclude that there must be a three-form that is not stable but which is non-degenerate. □

3.2 Relation between admissible and positive forms

Positive three-forms (definition 3.5) on seven-dimensional vector spaces are closely related to admissible four-forms (definition B.2) on eight-dimensional vector spaces. We first show how to pass from positive three-forms to admissible four-forms.

Lemma 3.8. *Let V be an eight-dimensional oriented vector space and H a seven-dimensional oriented subspace. Let ϕ be a positive three-form on H, and $v \in V$ a vector transverse to H and inducing[13] the given orientation on H. Then[14]*

$$\Omega = v^\flat \wedge \phi + * \left(v^\flat \wedge \phi \right) ,$$

is an admissible four-form on V, where $v^\flat$ and the Hodge star $*$ are taken w.r.t. the unique inner product g on V which on H agrees with the one associated to ϕ, and which satisfies $||v|| = 1$, $v \perp H$.

Proof. Denote by $e_1, \ldots, e_8$ the standard basis of $\mathbb{R}^8$. On the 7-dimensional subspace $Span\{e_2, \ldots, e_8\}$, consider the three-form ϕ_0 given as in Def. 3.3 (but shifting all the indices by one, so that they lie in the range $2, \ldots, 8$). Then $e_1^\flat \wedge \phi \in \wedge^4(\mathbb{R}^8)^*$ equals the first eight terms of the four-form Ω_0 appearing in equation (B.1), while $*_7 \phi_0 = * \left(e_1^\flat \wedge \phi_0 \right)$ equals the remaining eight terms of Ω_0, where $*_7$ denotes the Hodge-star on $Span\{e_2, \ldots, e_8\}$. Hence, $e_1^\flat \wedge \phi_0 + * \left(e_1^\flat \wedge \phi_0 \right) = \Omega_0$.

There is an orientated isometry $\tau \colon H \to \mathbb{R}^7$ identifying ϕ with ϕ_0. Denote by $f_2, \ldots, f_8$ the orthonormal basis of H corresponding to the standard basis of $\mathbb{R}^7$ under τ, and define $f_1 := v$. Then $f_1, \ldots, f_8$ is a basis of V which is orthonormal w.r.t. the inner product g and compatible with the orientation. The coordinate map $V \to \mathbb{R}^8$ is an oriented isometry which restricts to τ, and the above argument[15] on $\mathbb{R}^8$ shows that Ω is an admissible form. □

Remark 3.9. *In Lemma 3.8, the inner product on V associated to Ω is exactly g. This follows from the fact that the inner product on $\mathbb{R}^8$ associated to Ω_0 is the standard inner product. Further, in Lemma 3.8 the term $* \left(v^\flat \wedge \phi \right)$ equals $*_7 \phi$, the Hodge-dual of ϕ w.r.t. the metric on H induced by ϕ, extended to a form on V with kernel $\mathbb{R}v$.*

Conversely, we now show how to pass from admissible four-forms to positive three-forms.

[13] The orientation on H induced by the one on V and the vector v is defined as follows: a basis $w_1, \ldots, w_7$ of H is declared to be compatible with the induced orientation iff $v, w_1, \ldots, w_7$ a basis of V compatible with its orientation.

[14] Here we slightly abuse notation, denoting by the same symbol $\phi \in \Lambda^3 H^*$ and its extension to a three-form on V annihilating the vector v.

[15] Notice that the Hodge star depends on the inner product and the orientation.

Lemma 3.10. *Let V be an eight-dimensional oriented vector space and Ω an admissible four-form. For all non-zero $v \in V$, and all 7-dimensional subspaces $H \subset V$ transverse to v, the form $(\iota_v\Omega)|_H$ is a positive three-form on H (where H has the orientation induced by the orientation on V and by v).*

Proof. We may assume that $V = \mathbb{R}^8$ and that $\Omega = \Omega_0$, the four-form given in eq. (B.1). Denote by $e_1, \dots, e_8$ the standard basis of $\mathbb{R}^8$. Notice that $(\iota_{e_1}\Omega_0)|_{Span\{e_2,\dots,e_8\}}$ coincides with the three-form ϕ_0 given in Def. 3.3 (upon a shift of indices). For any 7-dimensional subspace H' transverse to e_1, the isomorphisms $H' \to Span\{e_2, \dots, e_8\}$ obtained restricting the orthogonal projection identifies $(\iota_{e_1}\Omega_0)|_{H'}$ and $(\iota_{e_1}\Omega_0)|_{Span\{e_2,\dots,e_8\}}$, therefore the former is a positive three-form.

Consider now the action of $Spin(7)$ on $\mathbb{R}^8$, obtained by restriction to $Spin(7) \subset SO(8)$ of the usual action of $SO(8)$ on $\mathbb{R}^8$. Recall that this action of $Spin(7)$ preserves Ω_0. Further, this action of $Spin(7)$ is transitive on the unit sphere S^7, as explained in [46, §3.1, Remark 3]. Hence, for every $v \in S^7$, there is $A \in Spin(7)$ with $Ae_1 = v$. The form

$$A^*(\iota_v\Omega_0) = \iota_{A^{-1}v}(A^*\Omega_0) = \iota_{e_1}\Omega_0$$

is positive once restricted to $A^{-1}(H)$, by the beginning of this proof, hence $\iota_v\Omega_0$ is positive once restricted to H.

To show that the same holds for all non-zero vectors, notice that if $r > 0$, then $\iota_v\Omega_0$ and $\iota_{rv}\Omega_0$ are $GL(8)$-related (through the dilation by $\sqrt[3]{r}$) therefore the restriction to H of the latter form is also positive. □

Remark 3.11. *In the setting of Lemma 3.10, denote by g the metric on V induced by the admissible form Ω, and assume that $||v|| = 1$ and $H = v^\perp$. Then the metric on H induced by the positive form $(\iota_v\Omega)|_H$ is the restriction of g. Indeed, since the action of $Spin(7)$ on $\mathbb{R}^8$ preserves Ω_0 as well as the metric g, and is transitive on S^7, it is enough to check this statement for $v = e_1$, for which it is clearly true.*

There is a bijective correspondence between admissible and positive forms which are compatible with a given, fixed metric.

Proposition 3.12. *Let V be an oriented eight-dimensional vector space with a fixed inner product g. Fix $v \in V$ with $||v|| = 1$. Denote by i^*g the restricted inner product on the seven-dimensional subspace $H := v^\perp$ (endowed with the orientation induced by the one on V and by v). There is a bijection*

$$\begin{aligned} A\colon \{\textit{admissible four-forms on } V \textit{ inducing } g\} &\to \{\textit{positive three-forms on } H \textit{ inducing } i^*g\} \\ \Omega &\mapsto (\iota_v\Omega)|_H \end{aligned}$$

whose inverse is given by

$$\begin{aligned} B\colon \{\textit{positive three-forms on } H \textit{ inducing } i^*g\} &\to \{\textit{admissible four-forms on } V \textit{ inducing } g\} \\ \phi &\mapsto v^\flat \wedge \phi + *\left(v^\flat \wedge \phi\right) \end{aligned}$$

Proof. Notice first that the map A is well-defined by lemma 3.10 and remark 3.11. Similarly, the map B is well-defined by lemma 3.8 and remark 3.9.

Clearly $A \circ B = Id$. Instead of showing directly that $B \circ A = Id$, we proceed as follows.

Recall from appendix B that the admissible four-forms on $\mathbb{R}^8$ are the elements of the orbit of Ω_0 (see eq. (B.1)) under the natural action of $GL_+(8, \mathbb{R})$ on $\Lambda^4(\mathbb{R}^8)^*$, and that the stabilizer of Ω_0

is the subgroup $Spin(7)$. Hence the admissible four-forms on $\mathbb{R}^8$ whose associated inner product is the standard one are given by the $SO(8)$-orbit through Ω_0. Therefore we obtain diffeomorphisms

$$\{\text{admissible four-forms on } V \text{ inducing } g\} \cong SO(8)/Spin(7) \cong \mathbb{R}P^7.$$

Similarly, by §3 and appendix A, the positive three-forms on $\mathbb{R}^7$ are the elements of the orbit of ϕ_0 (see eq. (3.3)) under the natural action of $GL_+(7,\mathbb{R})$, and the stabilizer of ϕ_0 is G_2. Hence positive three-forms on $\mathbb{R}^7$ whose associated inner product is the standard one are given by the $SO(7)$-orbit through ϕ_0. Therefore we obtain diffeomorphisms

$$\{\text{positive three-forms on } H \text{ inducing } i^*g\} \cong SO(7)/G_2 \cong \mathbb{R}P^7\,. \tag{3.11}$$

We conclude that both the domain and codomain of A are the 7-dimensional real projective space $\mathbb{R}P^7$.

The equation $A \circ B = Id$ implies that B is an injective immersion, therefore by dimension count a local diffeomorphism, and hence a diffeomorphism onto its image. The image of B is open (since B is a local diffeomorphism) and closed (since the domain of B is compact), hence it must be the whole of the codomain of B. This shows that B is surjective as well, hence bijective, with inverse A. □

Corollary 3.13. *Let V be an eight-dimensional oriented vector space with inner product g. Let $\phi \in \wedge^3 V^*$ and a unit vector $v \in V$ such that $ker(\phi) = \mathbb{R}v$. Define $H := v^\perp$.*

1. *If $v^\flat \wedge \phi + *\left(v^\flat \wedge \phi\right)$ is an admissible four-form on V (not necessarily inducing the inner product g), then $\phi|_H$ is a positive three-form on H.*
2. *$v^\flat \wedge \phi + *\left(v^\flat \wedge \phi\right)$ is an admissible four-form on V inducing g if and only if $\phi|_H$ is a positive three-form inducing i^*g.*

Proof. 1. follows from lemma 3.10, and 2) from proposition 3.12. Both use that the contraction with v of $v^\flat \wedge \phi + *\left(v^\flat \wedge \phi\right)$ is ϕ. □

4 Decrypting the 8-dimensional manifold

As explained in section 2, in exceptional generalized geometry a compactification with four-dimensional off-shell $\mathcal{N} = 1$ supersymmetry is described by a reduction of the structure group $E_{7(7)}$ of a vector bundle in the **912** representation. The decomposition of the **912** representation of $E_{7(7)}$ in terms of the fundamental representation of $Sl(8,\mathbb{R})$ (the group acting on the tangent bundle of an eight-dimensional oriented manifold $\mathcal{M}_8$) is given in (2.22). Our starting point will therefore be a differentiable manifold $\mathcal{M}_8$ equipped with the following four different global sections, not necessarily everywhere non-vanishing,

- A global section $g \in \Gamma\left(S^2 T^*\mathcal{M}_8\right)$.
- A global section $\tilde{g} \in \Gamma\left(S^2 T\mathcal{M}_8\right)$.
- A global section $\phi \in \Gamma\left(\Lambda^3 T^*\mathcal{M}_8 \otimes T\mathcal{M}_8\right)_0$.
- A global section $\tilde{\phi} \in \Gamma\left(\Lambda^3 T\mathcal{M}_8 \otimes T^*\mathcal{M}_8\right)_0$.

We will require $\mathcal{M}_8$ to be equipped with a volume form, since we want to be able to define integrals and also since we want the structure group to be reduced to $Sl(8,\mathbb{R})$, in order for $\mathcal{M}_8$ to carry the corresponding $E_{7(7)}$ representation at every point $p \in \mathcal{M}_8$. Hence $\mathcal{M}_8$ is oriented.

In addition, we want to consider compactifications with off-shell supersymmetry, and therefore we have to require $\mathcal{M}_8$ to be equipped with a Riemannian metric g in such a way that $(\mathcal{M}, \mathrm{g})$ is a spin manifold. Notice that $\mathcal{M}_8$ is equipped with a global section $g \in \Gamma\left(S^2 T^*\mathcal{M}_8\right)$. Therefore it is natural to take g to be a Riemannian metric, and $\tilde{g}$ to be the inverse of g, which is indeed the case if the **912** structure is defined from a $Spin(8)$ spinor η, see equation (2.25). We introduce thus the following definition.

Definition 4.1. *Let $\mathcal{M}_8$ be an eight-dimensional, oriented, differentiable manifold. We say that $\mathcal{M}_8$ is an* intermediate manifold *if it is equipped with the following data*

- *A Riemannian metric $g \in \Gamma\left(S^2 T^*\mathcal{M}_8\right)$ such that $(\mathcal{M}_8, g)$ is a spin manifold.*
- *A global section $\phi \in \Gamma\left(\Lambda^3 T^*\mathcal{M}_8 \otimes T\mathcal{M}_8\right)_0$.*
- *A global section $\tilde{\phi} \in \Gamma\left(\Lambda^3 T\mathcal{M}_8 \otimes T^*\mathcal{M}_8\right)_0$.*

We say then that $\mathfrak{S} = \left(g, \phi, \tilde{\phi}\right)$ is an intermediate structure on $\mathcal{M}_8$.

Notice that we do not require in principle the section ϕ to be everywhere non-vanishing nor $\mathcal{M}_8$ to be compact. This, together with the fact that any differentiable manifold can be equipped with a Riemannian metric means that there are only two obstructions for a manifold $\mathcal{M}_8$ to admit an intermediate structure $\mathfrak{S}$, namely

$$w_1\left(T\mathcal{M}_8\right) = 0\,, \qquad w_2\left(T\mathcal{M}_8\right) = 0\,, \tag{4.1}$$

where w_1 and w_2 denote respectively the first and the second Stiefel-Whitney classes. These are precisely the obstructions for a manifold to be orientable and spinnable. Therefore, every orientable and spinnable eight-dimensional manifold admits an intermediate structure $\mathfrak{S}$, which is completely fixed once we choose an orientation, a metric, and spin structure and the two global sections ϕ and $\tilde{\phi}$. Given that every intermediate manifold is equipped with a Riemannian structure, we will take the volume form to be the one induced by the Riemannian metric g.

The idea is to study now the geometry of intermediate manifolds in order to give them a physical meaning and understand if they can play any meaningful role in String/M/F-theory compactifications.

It turns out that an intermediate structure $\mathfrak{S}$ on a manifold $\mathcal{M}_8$ gives rise to many interesting different geometric situations; some of them will be studied in the next sections. More precisely:

- In section 5 we will assume the existence of a special kind of intermediate structure, called G_2-intermediate structure (definition 5.3). We consider two cases:
  - The seven-dimensional distribution on $\mathcal{M}_8$ orthogonal to the vector field v (assumed to be non-vanishing) is completely integrable. We study the foliation by seven-dimensional submanifolds $\mathcal{M}_7$ tangent to the distribution, and geometric structures on them. This is done in sections 5.1, 5.2 and 5.3.
  - The vector field v is the generator of a free S^1 action on $\mathcal{M}_8$. We study geometric structures on the seven-dimensional quotient manifold $\mathcal{M}_8/S^1$. This is done in section 5.4.
- In section 6 we consider general intermediate structures on $\mathcal{M}_8$.

We remark that most of the geometric conclusions we draw in sections 5 and 6 do not make use of the fact that $(\mathcal{M}_8, g)$ is a spin manifold.

5 G_2-intermediate structures

Let $\mathcal{M}_8$ be an eight-dimensional oriented manifold with an intermediate structure $\mathfrak{S} = \left(g, \phi, \tilde{\phi}\right)$. In this section, we will use $\mathfrak{S}$ to embed seven-dimensional G_2 structure manifolds in $\mathcal{M}_8$ as the leaves of a foliation. Interestingly enough, the eight-dimensional manifold $\mathcal{M}_8$ can then be proven, under additional assumptions that we shall enumerate, to be a manifold with a topological $Spin\,(7)$-structure. This points to the interpretation of $\mathcal{M}_8$ as a plausible internal manifold in F-theory if it is also elliptically fibered, which, as we will see in section 6, can be the case. Since $\mathcal{M}_8$ also contains in a natural way a family of seven-dimensional manifolds with G_2 structure, which are the leaves of the foliation, we find a natural correspondence between the G_2-structure manifolds and the corresponding $Spin\,(7)$-structure manifolds, which may have a physical meaning in terms of dualities in String/M/F-theory. We leave the complete study of the holonomy of the different manifolds that appear in this set-up to a separate publication [39].

Here, we will take $\tilde{\phi}$ to be the dual of ϕ by the musical isomorphisms $\flat$ and $\sharp$ given by g.[16] Let us choose ϕ as follows:

$$\phi = \phi_3 \otimes v \in \Gamma\left(\Lambda^3 T^*\mathcal{M}_8 \otimes T\mathcal{M}_8\right)_0 \,, \tag{5.1}$$

where $\phi_3 \in \Gamma\left(\Lambda^3 T^*\mathcal{M}_8\right)$ is a three-form and $v \in \Gamma\left(T\mathcal{M}_8\right)$ is a vector field. The traceless condition on ϕ reads

$$\iota_v \phi_3 = 0\,, \tag{5.2}$$

and thus we conclude that ϕ_3 is a degenerate three-form in $\mathcal{M}_8$. If v is no-where vanishing, we can without loss of generality, take it to have unit norm with respect to the metric g by a simple rescaling at every point $p \in \mathcal{M}_8$, i.e. by replacing v_p with $\frac{v_p}{\sqrt{g(v_p, v_p)}}$. In this section we are going to consider that this is the case and hence, in the following, we will take v to have unit norm:

$$||v_p|| = 1 \text{ for all } p \in \mathcal{M}_8\,. \tag{5.3}$$

Most of the definitions and results will carry over to the case where v has isolated zeros. We will come back to this point in section 6.2.

It is also natural to assume that

$$\mathrm{Ker}(\phi_{3\,p}) = \{\lambda v_p : \lambda \in \mathbb{R}\}\,, \quad p \in \mathcal{M}_8\,. \tag{5.4}$$

In other words, $f \cdot v \in \mathfrak{X}\left(\mathcal{M}_8\right)$, where $f \in C^\infty\left(\mathcal{M}_8\right)$ is any function, is the only vector field such that (5.2) holds. Condition (5.4) is the most natural situation compatible with equation (5.2), since otherwise we would artificially introduce new vector fields that would span the kernel of ϕ_3, and that are not incorporated in the intermediate structure $\mathfrak{S}$.
We will call "non-degenerate" the intermediate structures that satisfy the properties mentioned so far. The name will be justified in proposition 5.10.

Definition 5.1. *An* non-degenerate *intermediate structure is one of the form* $(g, \phi_3 \otimes v)$ *with* $||v|| = 1$ *and* $ker(\phi_3) = \mathbb{R}v$.

We define now, at every point $p \in \mathcal{M}_8$, a subset of the tangent space $T_p\mathcal{M}_8$ as follows:

$$H_{7\,p} = \{w_p \in T_{8\,p} \,|\, \xi_p\left(w_p\right) = 0\}\,, \quad \xi_p \equiv v_p^\flat \quad p \in \mathcal{M}_8\,. \tag{5.5}$$

Notice that $H_{7\,p}$ is a well defined seven-dimensional vector subspace of $T_{8\,p}$ at every point $p \in \mathcal{M}_8$. In other words,

$$H_7 = \{H_{7\,p}\,,\ p \in \mathcal{M}_8\}\,, \tag{5.6}$$

[16] In terms of the supersymmetry spinors η_1 and η_2, this is *equivalent* to taking them to be everywhere parallel.

is a globally defined, codimension one, smooth distribution on $\mathcal{M}_8$, given by the kernel of the one form ξ. Notice that H_7 is simply the distribution of vectors orthogonal to v.

Remark 5.2. *From a non-degenerate intermediate structure $(g, \phi_3 \otimes v)$, fixing a point p and restricting the three-form to H_{7p}, we obtain an element $\phi_3|_{H_{7p}} \in \Lambda^3(H_{7p})^*$. Doing so we do not lose information: the unique extension of this element of $\Lambda^3(H_{7p})^*$ to an element of $\Lambda^3 T_p{}^* \mathcal{M}_8$ that annihilates v, is exactly $(\phi_3)|_p$.*

We will be interested in a special case of the above, that gives rise to positive three forms:

Definition 5.3. *A non-degenerate intermediate structure $\mathfrak{S} = (g, \phi_3 \otimes v)$ on $\mathcal{M}_8$ is said to be a* G_2-intermediate structure *if, for all points p, $\phi_3|_{H_{7p}}$ is a positive*[17] *three-form, whose corresponding metric is the restriction of g to H_{7p}. In this case, we say that $(\mathcal{M}_8, \mathfrak{S} = (g, \phi_3 \otimes v))$ is a G_2-intermediate manifold.*

Remark 5.4. *A G_2-intermediate structure on $\mathcal{M}_8$ implies a reduction of the structure group of the manifold from $SO(8)$ to G_2, where G_2 is embedded in $SO(8)$ as*

$$\left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array} \right) : A \in G_2 \subset SO(7) \right\}.$$

More precisely, the G_2 reduction consists of

$$\cup_{p \in \mathcal{M}_8} \{ f : T_p \mathcal{M}_8 \to \mathbb{R}^8 \text{ oriented isometry such that } f(v) = e_0 \text{ and } (f|_{H_{7p}})^* \phi_0 = \phi_3|_{H_{7p}}, \}$$

where $(e_0, \ldots, e_7)$ is the canonical basis of $\mathbb{R}^8$ and ϕ_0 as in definition 3.3.

We now proceed to study various instances of G_2-intermediate structures, as outlined at the end of section 4.

5.1 A foliation of $\mathcal{M}_8$ by seven-manifolds

Let $(g, \phi_3 \otimes v)$ be a non-degenerate intermediate structure. We want to know under which conditions the distribution H_7 is completely integrable. That is, we want to know under which conditions it defines a foliation of $\mathcal{M}_8$ such that the tangent space of the leaf passing through $p \in \mathcal{M}_8$ is given by $H_{7\,p}$. The following proposition answers this question.

Proposition 5.5. *Let $\mathcal{M}$ be a differentiable manifold equipped with a non-singular one form $\xi \in \Omega^1(\mathcal{M})$. Then, the following conditions are equivalent*

- $\xi([w_1, w_2]) = 0$ *for all* $w_1, w_2 \in \mathfrak{X}(\mathcal{M})$ *such that* $\xi(w_1) = 0$ *and* $\xi(w_2) = 0$.
- $\xi \wedge d\xi = 0$.
- $d\xi = \beta \wedge \xi$ *for some* $\beta \in \Omega^1(\mathcal{M})$.

Proof. See proposition 2.1 in [47]. □

Therefore, using proposition 5.5 and the Frobrenius theorem we see that H_7 is completely integrable if and only if

$$\xi \wedge d\xi = 0, \text{ or equivalently, } d\xi = \beta \wedge \xi \text{ for some } \beta \in \Omega^1(\mathcal{M}). \tag{5.7}$$

If the distribution is completely integrable, then there exists a foliation, which we will denote by $\mathcal{F}_\xi$, of $\mathcal{M}_8$ by seven-dimensional leaves, whose tangent space at $p \in \mathcal{M}_8$ is given by $H_{7\,p}$. In other words: assuming (5.7), at each $p \in \mathcal{M}_8$ there exist a seven-dimensional submanifold $\mathcal{M}_{7\,p} \subset \mathcal{M}_8$ passing through p and tangent to H_7.

[17] Here H_{7p} has the orientation given by the one of $\mathcal{M}_8$ and by v.

Remark 5.6. *Even when v is compatible with the Riemannian metric g, in the sense the v is a Killing vector field (one whose flow consists of isometries), the orthogonal distribution H_7 may not be completely integrable. For instance, consider $\mathcal{M}_8 = S^3 \times \mathbb{R}^5$. The vector field v generating the action of S^1 on the 3-sphere S^3 (obtained restricting the action on $\mathbb{C}^2$ by simultaneous rotations) is a Killing vector field, but H_7 is not integrable. When H_7 happens to be completely integrable, its leaves are totally geodesic submanifolds*[18].

More generally, replacing the one-dimensional orbits of a Killing vector field v by submanifolds of higher dimension, we have the following. Let (M, g) be a complete Riemannian manifold. A foliation F on M is called Riemannian foliation *[48] if its holonomy acts by isometries on H, where denotes H the distribution on M orthogonal to the tangent spaces to the leaves of the foliation. The distribution H is not necessary integrable. When it is, one says that F is a* polar foliation *(see for instance [49]), and the leaves of H are totally geodesic submanifolds.*

Note that the one-form β is not uniquely defined. Indeed, let β and β' be two one-forms such that $d\xi = \beta \wedge \xi = \beta' \wedge \xi$. Then

$$(\beta - \beta') \wedge \xi = 0 \;\Rightarrow\; \beta - \beta' = f\xi\,, \tag{5.8}$$

for some function $f \in C^\infty(\mathcal{M})$.
We now go back to our setting of an intermediate manifold manifold $\mathcal{M}_8$. In that case, it is possible to write β in terms of v.

Proposition 5.7. *Let $\mathcal{M}$ be a differentiable manifold equipped with a Riemmanian metric g and a vector field v. Under the assumption that $\xi = v^\flat$ satisfies the conditions of proposition 5.5, we have:*

a) the one-form $-\mathcal{L}_v\xi$, where $\mathcal{L}_v$ is the Lie derivative along v, is a proper choice for β, that is, $d\xi = -\mathcal{L}_v\xi \wedge \xi$.

b) $d\xi = 0$ if and only if $\mathcal{L}_v\xi = 0$.

Notice that if v is a Killing vector field (i.e. $\mathcal{L}_v g = 0$) then since $\mathcal{L}_v v = [v, v] = 0$ we have $\mathcal{L}_v\xi = 0$.

Proof. For a), see proposition 2.2 in [47].

For b), notice that we have

$$d\xi = 0 \Leftrightarrow \iota_v d\xi = 0 \tag{5.9}$$

using the formula $d\xi(w_1, w_2) = w_1(\xi(w_2)) - w_2(\xi(w_1)) - \xi([w_1, w_2])$ together with the first condition in proposition 5.5. Further we have

$$\mathcal{L}_v\xi = \iota_v d\xi + d\iota_v\xi = \iota_v d\xi \tag{5.10}$$

using Cartan's identity and the fact that $\iota_v\xi = 1$. □

Remark 5.8. *The vector field $(-\mathcal{L}_v\xi)^\sharp$ is orthogonal to v. Indeed, the last computation in the proof of proposition 5.7 shows that $\mathcal{L}_v\xi = \iota_v d\xi$, and therefore*

$$\iota_v\left(\mathcal{L}_v\xi\right) = 0\,. \tag{5.11}$$

Consequently, if $H_7 := ker(\xi)$ defines a completely integrable distribution, then $(\mathcal{L}_v\xi)^\sharp$ is a vector field tangent to the leaves of the foliation.

Corollary 5.9. *Let us assume that $H_7 := ker(\xi)$ defines a completely integrable distribution. Then the foliation $\mathcal{F}_\xi$ is transversely orientable.*

[18] This means that if a geodesic of $\mathcal{M}_8$ is tangent to a leaf L, then it is contained in L at all times.

Proof. By definition, a foliation is transversely orientable if the normal bundle to the foliation is orientable. For a codimension one foliation this means exactly that there exist a vector field transverse to the foliation. This is the case in our situation, for v is a globally defined vector field transverse to $\mathcal{F}_\xi$. □

Proposition 5.10. *Let $\mathcal{M}_8$ be an eight-dimensional oriented manifold equipped with a non-degenerate intermediate structure $\mathfrak{S}$ such that equation (5.7) holds. By proposition 5.5, the kernel of $\xi \in \Omega^1(\mathcal{M}_8)$ defines an integrable distribution (5.6), whose integral manifold passing through any point $p \in \mathcal{M}_8$ is denoted by $\mathcal{M}_{7\,p} \subset \mathcal{M}_8$. If we denote by*

$$i : \mathcal{M}_{7\,p} \hookrightarrow \mathcal{M}_8 \tag{5.12}$$

the natural inclusion, then $i^*\phi_3$ is a non-degenerate three-form on $\mathcal{M}_{7\,p}$.

Proof. Fix $p \in \mathcal{M}_8$ and $w \in T_p\mathcal{M}_{7,p}$. Suppose that $\iota_w(i^*\phi_3) = 0$, i.e. that $i^*(\iota_w\phi_3) = 0$. Since $v_p \in Ker(\phi_3)$ and $T_p\mathcal{M}_{8,p} = T_p\mathcal{M}_{7,p} \oplus \mathbb{R}v_p$, we conclude that $\iota_w\phi_3 = 0$. Equation (5.4) implies that w is a multiple of v_p. Since w is tangent to $M_{7,p}$ while v_p is transverse to it, we obtain $w = 0$. □

To summarize, assuming (5.7), $\mathcal{M}_8$ is foliated by seven-dimensional manifolds which are equipped with a non-degenerate three-form $i^*\phi_3$.

5.2 G_2-structure seven-manifolds in $\mathcal{M}_8$

We are interested in connecting the seven-dimensional manifolds $\{\mathcal{M}_{7\,p}\,,\, p \in \mathcal{M}_8\}$ that form the leaves of the foliation to the seven dimensional manifolds that appear as internal spaces in off-shell $\mathcal{N} = 1$ M-theory compactifications, which have G_2-structure. Therefore, it is natural to ask whether the three-form $i^*\phi_3$ that exists on every leaf may define a G_2-structure on the leaves. In other words, we want to know if the three forms $i^*\phi_3$ can be taken to be positive. Remarkably enough, since every positive form is non-degenerate, the previous construction is consistent with taking ϕ_3 in such a way that $i^*\phi_3$ is a positive form. Of course, what we want to know is, given a vector field v on $\mathcal{M}_8$, if there is any obstruction to choose ϕ_3 in such a way that $i^*\phi_3$ is a positive three-form. The following proposition answers this question.

Proposition 5.11. *Let $\mathcal{M}_8$ be an eight-dimensional oriented Riemannian spin manifold equipped with a transversely orientable, codimension one foliation $\mathcal{F}$. Then each leaf of the foliation admits a G_2-structure, with the property that its associated Riemannian metric is the pullback of the given Riemannian metric on $\mathcal{M}_8$.*

Proof. Since $\mathcal{F}$ is transversely orientable, the normal bundle $\mathcal{N}$ to $\mathcal{F}$ is trivial, and in particular orientable, that is, $w_1(\mathcal{N}) = 0$. In addition, $w_2(\mathcal{N}) = 0$ since $2 > \operatorname{rank}\mathcal{N}$. Now, using [50, Ch.II, proposition 2.15] we deduce that the leaves of $\mathcal{F}$ are spin manifolds. We finally conclude from proposition 2.1 (and its proof) that on each leaf there exists topological G_2-structure with the above property. □

Notice that proposition 5.11 is a mere existence result, and does not address the issue of whether the G_2-structures vary smoothly[19] from leaf to leaf. When the G_2-structures do vary smoothly, we have the following.

[19] An example where this happens is the following. Let N be an oriented seven-dimensional manifold endowed with a positive three-form Φ^0, and $f\colon N \to N$ a diffeomorphism satisfying $f^*\Phi^0 = \Phi^0$. Take the mapping torus $\mathcal{M}_8 := ([0,1] \times N)/\sim$, where the equivalence relation $\sim$ identifies $(0,p)$ with $(1,f(p))$ for all $p \in N$, together with the codimension one foliation given by the fibers of the projection $\mathcal{M}_8 \to [0,1]/(0 \sim 1) = S^1$. Notice that all fibers are diffeomorphic to N.

Proposition 5.12. *Let $\mathcal{M}_8$ be an eight-dimensional oriented Riemannian spin manifold, equipped with a transversely orientable, codimension one foliation $\mathcal{F}$. Let v be a unit vector field orthogonal to $\mathcal{F}$.*

If the positive three-forms on the leaves of $\mathcal{F}$ mentioned in proposition 5.11 can be chosen so that they vary smoothly from leaf to leaf, then $\mathcal{M}_8$ is equipped with a unique three-form $\phi_3 \in \Omega^3(\mathcal{M}_8)$ such that $\iota_v \phi_3 = 0$ and ϕ_3 pulls back to the given positive three-forms on all leaves. In other words, $\mathfrak{S} = (g, \phi_3 \otimes v)$ is a G_2-intermediate structure on $\mathcal{M}_8$ such that H_7 is integrable.

Remark 5.13. *It is not always possible to choose in a globally smooth way the three-forms on the leaves mentioned in proposition 5.11. Indeed, this is possible iff there exists a topological $Spin(7)$-structure on $\mathcal{M}_8$ whose associated Riemannian metric is the given one on $\mathcal{M}_8$. This follows immediately from Thm. 5.14 and Thm. 5.15.*

A conceptual explanation for the above failure is the following. The G_2-structures on the leaves may not be unique topologically (i.e. up to continuous deformation). If the bundle of deformation classes (a discrete bundle over one dimensional the leaf space) has monodromy, it is not possible to choose the three-forms in a continuous way. We thank Dominic Joyce for pointing this out to us.

Proof. We will prove the proposition by explicitly constructing the three-form ϕ_3. Let us denote by $\mathcal{M}_{7\,p}$ the leaf of $\mathcal{F}$ passing through $p \in \mathcal{M}_8$. Every leaf $\mathcal{M}_{7,p}$ is equipped with a positive three-form $\phi_3^{(0,p)} \in \Omega^3(\mathcal{M}_{7,p})$. Let $(e_1, \ldots, e_7)$ be any basis of $T_p\mathcal{M}_{7,p}$. Now we define, for all $p \in \mathcal{M}_8$, $\phi_3|_p \in \Lambda^3 T_p^* \mathcal{M}_8$ as follows:

$$\phi_3|_p(e_i, e_j, e_k) = \phi_3^{(0,p)}(e_i, e_j, e_k)\,, \quad \phi_3|_p(e_i, e_j, v) = 0\,, \quad i, j, k = 1, \ldots, 7\,. \tag{5.13}$$

(Notice that $\phi_3|_p$ is independent of the choice of basis). By construction, ϕ_3 defined as in equation (5.13) is positive on the leaves, and since the positive forms $\{\phi_3^{(0,p)}\}$ vary smoothly from leaf to leaf, ϕ_3 is also smooth. Furthermore, by construction, $\iota_v \phi_3 = 0$. We conclude that ϕ_3 is the desired three-form. □

Notice that the seven-dimensional manifolds do not depend on ϕ_3 but only on v, and therefore they do not react to any changes in ϕ_3. However, taking ϕ_3 to be the three-form that induces a G_2-structure on the leaves is the sensible choice that allows to make contact with the rank eight bundle (2.16) introduced in [23]. According to the decomposition (2.16), the complex four-form (2.26) that is constructed from the complex Weyl $Spin(8)$ spinor η decomposes as in equation (2.27). Due to the theorem 5.14 below, we precisely obtain an analogous decomposition here but now the rank eight vector bundle is the tangent bundle of an actual manifold $\mathcal{M}_8$, which is foliated by seven-dimensional G_2-manifolds. The tangent space of the leaves is a rank seven vector bundle analogous to T_7 in equation (2.27). Furthermore, theorem 5.14 shows that $\mathcal{M}_8$ has a Spin(7)-structure. This theorem is a mild generalization of [11, Prop. 11.4.10] (we reproduce the latter in the appendix as proposition B.9).

Theorem 5.14. *Let $(\mathcal{M}_8, \mathfrak{S} = (g, \phi_3 \otimes v))$ be a G_2-intermediate manifold. Then, $\mathcal{M}_8$ has a $Spin(7)$-structure defined by the admissible four-form*

$$\Omega = v^\flat \wedge \phi_3 + *\left(v^\flat \wedge \phi_3\right)\,. \tag{5.14}$$

The Riemannian metric associated to Ω is exactly g.

Proof. Fix $p \in \mathcal{M}_8$. Applying lemma 3.8 to the vector space $V := T_p\mathcal{M}_8$ and subspace $H := (H_7)_p$ shows that $\Omega|_p$ is an admissible form. We conclude that Ω is an admissible form on $\mathcal{M}_8$. The final statement follows from remark 3.9. □

The following is a converse to theorem 5.14.

Theorem 5.15. *Suppose $\mathcal{M}_8$ is an oriented Riemannian spin manifold with a topological Spin(7)-structure (whose associated metric is the given one g), and denote by Ω the corresponding admissible four-form. Let $\mathcal{F}$ be a transversely orientable, codimension one foliation on $\mathcal{M}_8$, and let v be a vector field orthogonal to $\mathcal{F}$ with $||v|| = 1$.*

Then, for any leaf $i\colon L \hookrightarrow \mathcal{M}_8$ of $\mathcal{F}$, the three-form $i^*(\iota_v\Omega)$ is positive, varies smoothly from leaf to leaf, and the metric on L associated to this positive three-form is the restriction of g. (In other words: $\mathfrak{S} = (g, \iota_v\Omega \otimes v)$ is a G_2-intermediate structure on $\mathcal{M}_8$ such that H_7 is integrable.)

Proof. Fix a leaf L. Applying lemma 3.10 to the vector space $V := T_p\mathcal{M}_8$ and subspace $H := (H_7)_p$, for all $p \in L$, shows that $i^*(\iota_v\Omega)$ is a positive three-form on L. The statement on the metric follows from Remark 3.11. □

For the sake of clarity, we describe in coordinates the four-form Ω obtained in theorem 5.14. Fix $p \in \mathcal{M}_8$ and denote by $\mathcal{M}_{7\,p}$ the leaf of H_7 through p. There exists an orientation-preserving coordinate system $\{\mathcal{U}, \mathbf{x}_8 = (x_1, \dots, x_8)\}$ about p such that $v_p = \frac{\partial}{\partial x_1}|_p$ and

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_8}\right)\Big|_p \tag{5.15}$$

is an orthonormal basis of $T_p\mathcal{M}_8$. The metric g at p then can be written as

$$g|_p = dx_1^2|_p + \cdots + dx_8^2|_p\,. \tag{5.16}$$

The three-form ϕ_3 on $\mathcal{M}_8$ at p has coordinate expression

$$(\phi_3)|_p = \sum_{2\le i<j<k\le 8} h_{ijk}(dx_i|_p) \wedge (dx_j|_p) \wedge (dx_k|_p) \tag{5.17}$$

where h_{ijk} are real numbers. On the other hand, $\mathbf{x}_7 = (x_2, \dots, x_8)$ is a coordinate system on $\mathcal{M}_{7\,p}$ around p such that

$$\left(\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_8}\right)\Big|_p \tag{5.18}$$

is an orthonormal basis of $T_p\mathcal{M}_{7\,p}$. In particular,

$$dx_2^2|_p + \cdots + dx_8^2|_p\,, \tag{5.19}$$

is the restricted metric on $\mathcal{M}_{7\,p}$ in the coordinate system (5.18). Using that $v_p^\flat = dx_1|_p$ we finally obtain

$$\Omega = dx_1|_p \wedge \phi_3|_p + *\left(dx_1|_p \wedge \phi_3\right)\,, \tag{5.20}$$

where the Hodge dual is compute w.r.t. the standard metric (5.16).

Remark 5.16. *Notice that Thm. 5.14 does not assume that H_7 be integrable,and Thm. 5.15 does not use the integrability in an essential way either. The analogue of Thm. 5.15 without the integrability assumption on H_7 reads:*

Suppose $\mathcal{M}_8$ is an oriented Riemannian spin manifold with a topological Spin(7)-structure (whose associated metric is the given one g), and denote by Ω the corresponding admissible four-form. Let H_7 be transversely orientable, codimension one smooth distribution on $\mathcal{M}_8$, and let v be a vector field orthogonal to $\mathcal{F}$ with $||v|| = 1$. Then, at every $p \in \mathcal{M}_8$, the three-form $(\iota_v\Omega)|_{H_{7_p}}$ is positive, varies smoothly with the point p, and the metric on H_{7_p} associated to this positive three-form is the restriction of g. (In other words: $\mathfrak{S} = (g, \iota_v\Omega \otimes v)$ is a G_2-intermediate structure on $\mathcal{M}_8$.)

Remark 5.17. *Let $(\mathcal{M}_8, g)$ be an eight-dimensional oriented Riemannian spin manifold. Let $\mathcal{F}$ be a codimension-one, transversely orientable foliation, and denote by v a unit vector field orthogonal to the foliation. Denote by i^*g the restricted inner product on the each leaf $i\colon L \hookrightarrow \mathcal{F}$. Proposition 3.12 immediately implies that there is a bijection*

$$\{\textit{admissible four-forms on } \mathcal{M}_8 \textit{ inducing } g\} \leftrightarrow \{\textit{positive three-forms on the leaves of } \mathcal{F} \textit{ inducing } i^*g \textit{ and varying smoothly from leaf to leaf.}\}$$

The purpose of theorem 5.14 and theorem 5.15 above is to spell out this correspondence in terms of $Spin(7)$-structures and G_2-structures.

Corollary 5.18. *Let $(\mathcal{M}_8, g)$ be an eight-dimensional oriented Riemannian spin manifold. Let $\mathcal{F}$ be a codimension-one, transversely orientable, foliation, and denote by v a unit vector field transverse to the foliation. Denote by i^*g the restricted inner product on each leaf $i\colon L \hookrightarrow \mathcal{F}$.*

1. *If $v^\flat \wedge \phi + * \left(v^\flat \wedge \phi\right)$ is an admissible four-form on $\mathcal{M}_8$ (not necessarily inducing the inner product g), then $i^*\phi$ is a positive three-form on every leaf $i : L \hookrightarrow \mathcal{M}_8$, varying smoothly from leaf to leaf.*

2. *$v^\flat \wedge \phi + * \left(v^\flat \wedge \phi\right)$ is an admissible four-form on $\mathcal{M}_8$ inducing g if and only if $i^*\phi$ is a positive and smooth three-form on $\mathcal{F}$ inducing i^*g.*

Proof. Apply corollary 3.13 to the vector space $V := T_p\mathcal{M}_8$ and subspace $H := (H_7)_p$, for all $p \in \mathcal{M}_8$. □

Note that equation (5.14) defining a real admissible four-form is of the form (2.27), constructed in terms of a complex spinor $\eta = \eta_1 + i\eta_2$. The only difference is that equation (2.27) is obtained by a $7 + 1$ split of a rank-eight vector bundle, while equation (5.14) is obtained from a seven-dimensional foliation of an eight-dimensional manifold. Therefore, equation (5.14) can be thought as the *geometrization* of equation (2.27) in terms of manifolds and tangent spaces, instead of a rank-eight vector bundle which does not correspond to the tangent space of any manifold. We can connect both set-ups by writing Ω in terms of a real Weyl spinor $\chi \in \Gamma\left(\mathbb{S}_8^{\mathbb{R}+}\right)$, which exists since $\mathcal{M}_8$ has $Spin(7)$ structure:

$$\Omega = \chi^T \gamma_{(4)} \chi = v^\flat \wedge \phi_3 + * \left(v^\flat \wedge \phi_3\right) . \tag{5.21}$$

However χ is not yet a *supersymmetry spinor*, since it is a section the spin bundle $\mathbb{S}_8^{\mathbb{R}+} \to \mathcal{M}_8$ over $\mathcal{M}_8$, while $\eta_a\,, a = 1, 2$, are sections of the spin bundle $\mathbb{S}_7^{\mathbb{R}} \to \mathcal{M}_7$. As we have explained, $\eta_a\,, a = 1, 2$, are $Spin(7)$ real spinors, which can be identified with real $Spin(8)$-Weyl spinors. As elements of a vector space, the previous identification is perfectly fine, but if we carry it out globally we obtain that η_a is now a section of $\mathbb{S}_8^{\mathbb{R}+} \to \mathcal{M}_7$, that is, the base manifold is $\mathcal{M}_7$ instead of $\mathcal{M}_8$. Therefore, given the geometric structure defined on an intermediate manifold $(\mathcal{M}_8, \mathfrak{S})$, it is natural to identify one of the η_a (let's say η_1) with the pull-black of the real, Weyl spinor χ over $\mathcal{M}_8$, i.e.

$$i^*\chi = \eta_1\,, \tag{5.22}$$

and take $\eta_2 = 0$. Notice that $i^*\chi$ denotes a family of spinors, one spinor for each leaf, that is, a section of the vector bundle $\mathbb{S}_8^{\mathbb{R}+} \to \mathcal{M}_7$, where we denote by $\mathcal{M}_7$ a generic leaf. We have taken $\eta_2 = 0$ in order to obtain a real four-form in equation (2.26), since for the particular intermediate structure that we are considering, we only obtain a real admissible four-form. Notice that we are implicitly identifying the G_2-structure leaves as internal spaces in M-theory compactifications down to four dimensions. We will see in section 6 that when we do not consider $\tilde{\phi}$ to be the *dual* of ϕ

then we can obtain the real as well as the imaginary parts of the complex four-form constructed out of the complex spinor η, and therefore we obtain the general geometric situation as it appears from off-shell supersymmetry. Notice that (5.22) does not completely determine χ from η_1, as there are several different choices of χ such that equation (5.22) is satisfied. Loosely speaking, there is some sort of *gauge freedom* in order to choose χ. In order to fix the gauge, a natural condition is to impose

$$\nabla_v \chi = 0\,. \tag{5.23}$$

Intuitively speaking, equation (5.23) is the *covariant* version of the partial derivative being zero as a way of saying that there is no dependence on a particular coordinate, in this case the *orthogonal* coordinate to the leaves, since v is a globally defined coordinate in $\mathcal{M}_8$, perpendicular to the G_2-structure leaves that conform the foliation defined by v.

5.3 Structure of the foliation

In section 5.1 we have defined a codimension one distribution in $\mathcal{M}_8$ which is given by the kernel of the one-form ξ. The conditions under which this distribution is completely integrable are given in proposition 5.5. In the case that any of those conditions is satisfied, the eight-dimensional manifold $\mathcal{M}_8$ is foliated by seven-dimensional manifolds $\mathcal{M}_{7\,p}$, $p \in \mathcal{M}_8$. Each leaf $\mathcal{M}_{7\,p}$ admits a G_2-structure (see proposition 5.11). In some cases, as explained in section 5.2, the three-form ϕ_3 can be chosen to give, using the pull-back of the natural inclusion $i : \mathcal{M}_{7\,p} \hookrightarrow \mathcal{M}_8$, the three-form that defines the G_2-structure on the leaves. The leaves however do not have to be neither diffeomorphic nor compact and in principle very little is known about them, aside from the fact that they are smooth seven-dimensional manifolds. The purpose of this section is thus to explore in some detail the geometry of the leaves, trying to characterize them as much as possible, in order to clarify the relation between the G_2-manifolds that are the leaves and the $Spin(7)$-manifold that is the total space $\mathcal{M}_8$.

Proposition 5.19. *Let $\mathcal{M}$ be a closed manifold*[20] *equipped with a codimension one foliation $\mathcal{F}$ defined by a non-singular closed one-form ξ. Then there exists a transversal vector field, whose flow consists of diffeomorphisms preserving $\mathcal{F}$,* i.e., *mapping leaves into leaves.*

Proof. See reference [51]. □

Corollary 5.20. *In the situation of proposition 5.19, and assuming that $\mathcal{M}$ is connected, all the leaves are diffeomorphic.*

Proposition 5.21. *Let $\mathcal{M}$ be a closed manifold equipped with a non-singular closed one-form ξ. Then, there exists a fibration $f : \mathcal{M} \to S^1$ over the circle. Moreover, let $\xi' = f^* d\theta$. Then the fibration can be chosen such that $||\xi - \xi'|| < \epsilon$, where $\epsilon > 0$ is any prescribed number.*

Proof. See reference [52]. □

Here $||\cdot||$ stands for the point-wise norm on forms, defined by a Riemannian metric on $\mathcal{M}$. What proposition 5.21 states is that, if we denote by $\mathcal{F}$ the foliation defined by the closed non-singular one-form ξ, there exists arbitrarily close (in the C^0-sense) foliations which are given by the fibers of a fibration $f : \mathcal{M} \to S^1$.
To summarize, if $\mathcal{M}$ is closed and $d\xi = 0$, then the behavior of the foliation is under control: all the leaves are diffeomorphic and in addition there is a fibration $f : \mathcal{M} \to S^1$ such that the fibres are arbitrarily close to the leaves of the foliation defined by ξ. A typical example of this given by the two-torus $S^1 \times S^1$ with coordinates θ_1 and θ_2, and by $\xi = d\theta_1 + \lambda d\theta_2$ for some real number λ:

[20] A closed manifold is a compact manifold without boundary.

if λ is a rational, $\mathcal{F}$ arises from a fibration, otherwise one can approximate $\mathcal{F}$ by taking the kernel of $d\theta_1 + \lambda' d\theta_2$, where λ' is a rational number very close to λ.

Proposition 5.21 applies to closed manifolds, that is, compact manifolds without boundary. We want, however, to consider the situation where $\mathcal{M}_8$ may have a boundary, since in some M-theory/F-theory applications we expect to find manifolds with boundary. There is indeed a result concerning the case of compact manifolds with boundary.

Proposition 5.22. *Let $\mathcal{F}$ be a codimension one, C^1, transversely oriented foliation on a compact manifold $\mathcal{M}$ (possibly with boundary) with a compact leaf L such that $H^1(L, \mathbb{R}) = 0$. Then, all the leaves of $\mathcal{F}$ are diffeomorphic with L, and the leaves of $\mathcal{F}$ are the fibers of a fibration of $\mathcal{M}$ over S^1 or the interval I. We assume here that the boundary of $\mathcal{M}$ is non-empty, then $\partial\mathcal{M}$ is a union of leaves of $\mathcal{F}$.*

Proof. See theorem 1 in reference [53]. □

In order to determine if $\mathcal{M}_8$ is an admissible internal space in F-theory, since we already know that in some cases, see theorem 5.14, it is a $Spin(7)$-structure manifold, we have to study if it is elliptically fibered. That is a difficult question to answer using just with the geometric data contained in the particular case of intermediate structure that we are considering. We will see in section 6 that if we consider a more general intermediate structure then it is more natural to obtain that $\mathcal{M}_8$ is elliptically fibered. In any case, the following construction may shed some light on this issue.

- We start with a G_2-intermediate manifold $(\mathcal{M}_8, \mathfrak{S})$, where $\mathfrak{S} = (g, \phi_3 \otimes v)$. Therefore by proposition 5.5 we know that the following holds:

$$d\xi = \beta \wedge \xi \qquad \text{for some } \beta \in \Omega^1(\mathcal{M}_8)\,, \tag{5.24}$$

  where $\xi = \iota_v g$. The properties of the foliation depend now exclusively on ξ and we have thus two possibilities, namely

  1. $d\xi = 0$. In this case then, depending on the particular properties of $\mathcal{M}_8$ we can be able to apply corollary 5.20 or propositions 5.21 and 5.22. If $\mathcal{M}_8$ is taken to be a closed manifold, using proposition 5.21, we conclude that there exists a fibration over the circle

$$f : \mathcal{M}_8 \to S^1\,, \tag{5.25}$$

     with seven-dimensional fibres which are arbitrarily close to the leaves of the foliation defined by ξ. We already concluded that the leaves are seven-dimensional manifolds of G_2-structure. However, we are interested in having a G_2-structure defined also on the fibres of the fibration (5.25), which are the seven dimensional internal manifolds appearing in off-shell $\mathcal{N} = 1$ supersymmetric M-theory compactifications. Indeed, the following proposition holds.

     Proposition 5.23. *Let $\mathcal{M}_8$ be an eight-dimensional oriented closed manifold equipped with an G_2-intermediate structure $\mathfrak{S} = (g, \phi_3 \otimes v)$ such that $d\xi = 0$, where $\xi = v^\flat$. Then $\mathcal{M}_8$ admits a fibration over the circle S^1*

$$f : \mathcal{M}_8 \to S^1\,, \tag{5.26}$$

     whose fibres are seven dimensional manifolds with G_2-structure, varying smoothly from fiber to fiber.

Proof. By proposition 5.21, for any $\epsilon > 0$ there exists a fibration $f : \mathcal{M}_8 \to S^1$ over the circle such that $||\xi - \xi'|| < \epsilon$, where $\xi' := f^* d\theta$. Since v is orthogonal to $ker(\xi)$, we can choose ϵ so small that v is transverse to $ker(\xi')$. By theorem 5.14 there is an admissible four-form Ω on $\mathcal{M}_8$. Applying at every point proposition 3.10 to Ω we see that $(\iota_v \Omega)|_{ker(\xi')}$ is a positive three-form. Thus we obtain a positive three-form on the fibers of f, which clearly varies smoothly from fiber to fiber. □

Remark 5.24. *In proposition 5.23, the metric on the fibers associated to the positive three-forms on the fibers is usually not the pullback of the metric g on $\mathcal{M}_8$.*

2. $d\xi$ no-where vanishing, which implies that β is globally defined and nowhere vanishing. This means in particular that ξ and β are nowhere parallel.

   Since β and ξ are never parallel, $i^*\beta$ is a non-singular, no-where vanishing one-form on every leaf $\mathcal{M}_{7\,p}$, $p \in \mathcal{M}_8$ of the foliation. Using equation (5.24) twice we see that

   $$d\beta \wedge \xi = d\beta \wedge \xi - \beta \wedge \underbrace{d\xi}_{\beta \wedge \xi} = d(\beta \wedge \xi) = d(d\xi) = 0\,, \tag{5.27}$$

   and thus

   $$d\beta = \gamma \wedge \xi \qquad \text{for some } \gamma \in \Omega^1(\mathcal{M}_8)\,, \tag{5.28}$$

   which in turn implies

   $$di^*\beta = 0\,. \tag{5.29}$$

   Therefore we conclude that the foliation defined by $i^*\beta$ naturally satisfies the conditions of proposition 5.19, corollary 5.20 and theorem 5.21 on those leaves $\mathcal{M}_{7\,p}$ that are closed, if any. The situation is then the following: $\mathcal{M}_8$ is foliated by G_2-structure manifolds, which in principle do not need to be diffeomorphic nor compact. Each leaf, that is, each G_2-structure manifold, can in turn be foliated by six dimensional leaves. Since (5.29) says that the one-form defining such foliation is automatically closed, we deduce that if the G_2-structure manifold is closed then it is a fibration over S^1. The global structure of the eight-dimensional manifolds $\mathcal{M}_8$ cannot be determined yet, as the discussion above allows for many different, more or less complicated, possibilities.

3. $d\xi$ not identically zero but neither no-where vanishing. Therefore, $d\xi$ can be zero at some points (or sets) in $\mathcal{M}_8$. Here we find a mixed situation. In those points belonging to the support of $d\xi$[21], β is non-vanishing and not parallel to ξ, and therefore the discussion in point 2 applies. At the points where $d\xi$ is zero, β is zero over the corresponding leave, and therefore no further foliation can be defined. Hence, we find that $\mathcal{M}_8$ is foliated by G_2-structure seven-dimensional manifolds and that some of them, determined by the points where $d\xi$ is non-zero, are in turn foliated in terms of six-dimensional manifolds.

5.4 $\mathcal{M}_8$ as a S^1 principal bundle

In sections 5.1, 5.2 and 5.3 we considered the dimension seven distribution H_7 (see (5.6)) of vectors perpendicular to the vector field v, and we assumed that it be integrable. In this section we explore another natural possibility, namely, the dimension one distribution

$$H_{1p} = \{w_p \in T_p\mathcal{M}_8 \mid w_p = \lambda v_p\,, \quad \lambda \in \mathbb{R}\}\,, \quad p \in \mathcal{M}_8\,. \tag{5.30}$$

Note that the dimension one distribution $H_1 = \{H_{1p}\,,\ p \in \mathcal{M}_8\}$ is automatically integrable, the leaves being simply the integral curves of $v \in \mathfrak{X}(\mathcal{M})$. There is a one-to-one correspondence between

[21] Not taking the corresponding closure.

complete vector fields on a differentiable manifold and smooth actions of $\mathbb{R}$ on $\mathcal{M}$, which is given in terms of the standard exponential map

$$\begin{aligned} \psi : \mathbb{R} &\mapsto \mathrm{Diff}\,(\mathcal{M}) \\ t &\mapsto \psi_t = e^{tv}\,, \end{aligned} \tag{5.31}$$

and differentiation

$$\psi \mapsto v_p = \left.\frac{d\psi_t(p)}{dt}\right|_{t=0}\,, \qquad p \in \mathcal{M}\,. \tag{5.32}$$

If the orbits closes up after a fixed time, say 2π, then we actually have a S^1-action, or equivalently, a $U(1)$-action, on $\mathcal{M}$, namely

$$\begin{aligned} \psi_{S^1} : S^1 &\mapsto \mathrm{Diff}\,(\mathcal{M}) \\ e^{i\theta} &\mapsto \psi_\theta = e^{\theta v}\,. \end{aligned} \tag{5.33}$$

We will now use the following theorem to give a sufficient condition for v to define a S^1 fibration on $\mathcal{M}_8$.

Proposition 5.25. *Let G be a Lie group and $\psi_G : G \to \mathrm{Diff}\,(\mathcal{M})$ a free and proper Lie-group action on a differentiable manifold $\mathcal{M}$. Then $\mathcal{M}/G$ admits a unique differentiable structure such that $\mathcal{M} \to \mathcal{M}/G$ is a principal bundle with fibre G. In addition,* $\dim(\mathcal{M}/G) = \dim(\mathcal{M}) - \dim(G)$.

Proof. See theorem 5.119 in [54]. □

Remark 5.26. *A free action of S^1 on a manifold $\mathcal{M}$ automatically defines an S^1-principal bundle structure on $\mathcal{M}$. This follows from proposition 5.25, since an action of a compact group is always proper.*

The following theorem and its proof are analog to theorem 5.14. It essentially says that a positive three-form on the base manifold of an S^1-bundle can be pulled back to the total space, and gives rise to an admissible four-form form there.

Theorem 5.27. *Let $\mathcal{M}_8$ be an eight-dimensional manifold equipped with a free action of S^1, whose infinitesimal generator we denote by v.*

If $\mathcal{M}_8/S^1$ is endowed with a topological G_2-structure, then there exists a topological $Spin(7)$-structure on $\mathcal{M}_8$, such that the projection $\pi\colon \mathcal{M}_8 \to \mathcal{M}_8/S^1$ is a Riemannian submersion[22] w.r.t. the metrics induced by the topological structures.

Proof. Let ψ_3 be a positive three-form on the seven-dimensional manifold $\mathcal{M}_8/S^1$. Choose a codimension one distribution H on $\mathcal{M}_8$ transverse to v. Endow $\mathcal{M}_8$ with the unique Riemannian metric g that makes π into a Riemannian submersion (w.r.t. the metric on $\mathcal{M}_8/S^1$ induced by ψ_3) and which satisfies $||v|| = 1$, $v \perp H$. Define $\phi_3 := \pi^*\psi_3$. Then

$$\Omega = v^\flat \wedge \phi_3 + * \left(v^\flat \wedge \phi_3 \right)\,,$$

is an admissible four-form on $\mathcal{M}_8$, whose associated metric is exactly g. This follows applying lemma 3.8 to the vector space $V := T_p\mathcal{M}_8$ and subspace $H := H_p$, for all $p \in \mathcal{M}_8$, and by remark 3.9 (or, which is the same, it follows from theorem 5.14). □

[22] This means that at every point p, $(d_p\pi)|_{H_{7_p}} : H_{7p} \to T_{\pi(p)}(\mathcal{M}_8/S^1)$ is an isometry.

Remark 5.28. *The admissible four-form in theorem 5.27 can always be chosen to be S^1-invariant. Indeed in the proof of theorem 5.27 we can always choose the distribution H to be S^1-invariant (take H to be the kernel of a connection one-form for the S^1-principal bundle). Then the metric g will be S^1-invariant. As Ω is constructed out of v, the pullback form ϕ_3, and g, it follows that Ω is S^1-invariant.*

The next theorem is a converse to theorem 5.27.

Theorem 5.29. *Let $(\mathcal{M}_8, g)$ be an oriented Riemannian spin manifold with a topological $Spin(7)$-structure (whose associated metric is g), and denote by Ω the corresponding admissible four-form. Assume a free action of S^1 on $\mathcal{M}_8$ such that Ω is S^1-invariant.*

Then the base $\mathcal{M}_8/S^1$ is endowed with a canonical positive three-form ψ_3. In particular, $\mathcal{M}_8/S^1$ has a canonical G_2-structure. Further, the projection $\pi\colon \mathcal{M}_8 \to \mathcal{M}_8/S^1$ is a Riemannian submersion w.r.t. the metrics induced by the topological structures.

Proof. Denote by v the infinitesimal generator of the S^1-action. Since the S^1 action preserves Ω, it preserve also the metric g associated to Ω. Hence it preserves the dimension seven distribution H_7 orthogonal to v. As obviously $\mathcal{L}_v v = 0$, it follows that the S^1 action preserves $(\iota_v\Omega)|_{H_7}$. Notice that by remark 5.16, at every point p, the form $(\iota_v\Omega)|_{H_{7_p}}$ is positive, and the metric on H_{7_p} associated to this positive three-form is the restriction of g. Further, the S^1 action preserves the unique[23] three form $\phi_3 \in \Omega^3(\mathcal{M}_8)$ which annihilates v and restricts to $(\iota_v\Omega)|_{H_7}$ on H_7.

Since $\mathcal{L}_v\phi_3 = 0$ and $\iota_v\phi_3 = 0$, there exists a unique three-form ψ_3 on $\mathcal{M}_8/S^1$ such that $\pi^*(\psi_3) = \phi_3$. At every $p \in \mathcal{M}_8$, the isomorphism $(d_p\pi)|_{H_{7_p}}\colon H_{7_p} \to T_{\pi(p)}(\mathcal{M}_8/S^1)$ identifies $\phi_3|_{H_{7_p}}$ with $\psi_3|_{\pi(p)}$. Since the former is a positive three-form, the latter also is. Further, $(d_p\pi)|_{H_{7_p}}$ identifies the metric associated to $\phi_3|_{H_{7_p}}$ with the metric associated $\psi_3|_{\pi(p)}$. □

6 General intermediate manifolds and elliptic fibrations

In this section we are going to consider a more general choice of intermediate structure $\mathfrak{S} = \left(g, \phi, \tilde{\phi}\right)$ than in section 5. In particular, we are not going to take $\tilde{\phi}$ as the dual of ϕ. Nevertheless, we will still make a particular choice for ϕ and $\tilde{\phi}$. Indeed, we will take them[24] to be non-degenerate (see definition 5.1):

$$\phi = \phi_3 \otimes v\,, \qquad \tilde{\phi} = \tilde{\phi}_3 \otimes \tilde{v}\,. \tag{6.1}$$

Notice that in particular, as it happened in section 5, the trace-zero condition reads

$$\iota_v\phi_3 = 0\,, \qquad \iota_{\tilde{v}}\tilde{\phi}_3 = 0\,. \tag{6.2}$$

In order to ease the presentation, let us define from now on

$$\phi_3^1 = \phi_3\,, \qquad \phi_3^2 = \tilde{\phi}_3\,, \qquad v_1 = v\,, \qquad v_2 = \tilde{v}\,. \tag{6.3}$$

Having two non-degenerate intermediate structures on a manifold $\mathcal{M}_8$ gives rise to many geometrical interesting situations. This section is intended to give the reader just a first glance at the kind of geometric structures that appear in the presence of two non-degenerate structures. We will prove in a moment a general result about the reduction of the topological structure group of the generalized

[23] The construction of ϕ_3 is exactly the construction carried out in proposition 5.12.

[24] Here we abuse notation and denote the dual of $\tilde{\phi} \in \Gamma\left(\Lambda^3 T\mathcal{M}_8 \otimes T^*\mathcal{M}_8\right)_0$ with respect to the metric, which is a section of $\left(\Lambda^3 T^*\mathcal{M}_8 \otimes T\mathcal{M}_8\right)_0$, by the same symbol $\tilde{\phi}$.

bundle $T\mathcal{M}_8 \oplus T^*\mathcal{M}_8$ over $\mathcal{M}_8$. More importantly, in subsections 6.1 and 6.2 we explain how two non-degenerate structures can be intimately related to regular as well as singular elliptic fibrations.

Let us assume that each of (ϕ_3^a, v_a) , $a = 1, 2$, defines a G_2-intermediate structure (see definition 5.3). Then by theorem 5.14 there is a $Spin(7)$ structure on $\mathcal{M}_8$ associated to each of the admissible four-forms

$$\Omega^a = v_a^\flat \wedge \phi_3^a + * \left(v_a^\flat \wedge \phi_3^a \right) , \tag{6.4}$$

which in turn implies the existence of the corresponding $Spin(8)$ spinor $\chi_a \in \Gamma\left(S_8^{\mathbb{R}+}\right)$ on $\mathcal{M}_8$. We arrive thus at the following result.

Proposition 6.1. *Let* $\left(\mathcal{M}_8, \mathfrak{S} = \left(g, \phi_3^1 \otimes v_1, \phi_3^2 \otimes v_2\right)\right)$ *be an intermediate manifold such that* $(g, \phi_3^a \otimes v_a)$, $a = 1, 2$, *is a* G_2*-intermediate structure. Then, the structure group of the generalized bundle* $\mathbb{E} = T\mathcal{M}_8 \oplus T^*\mathcal{M}_8$ *admits a reduction from* $\mathbb{R}^* \times Spin(8,8)$ *to* $Spin(7) \times Spin(7)$*, that is, it admits a generalized* $Spin(7)$*-structure.*

Proof. Thanks to theorem 5.14, each of the G_2-intermediate structures $(g, \phi_3^a \otimes v_a)$, $a = 1, 2$, defines a topological $Spin(7)$-structure on $\mathcal{M}_8$. Let us denote by $\chi_a \in \Gamma\left(S_8^{\mathbb{R}+}\right)$ the corresponding, globally defined, spinors. By theorem 5.1 in reference [29] we obtain a generalized $Spin(7)$-structure, uniquely determined by

1. The orientation of $\mathcal{M}_8$.
2. The metric g present in the intermediate structure.
3. The vanishing two-form $B = 0$.
4. The vanishing scalar function $f = 0$.
5. Two Majorana-Weyl spinors $\chi_a \in \Gamma\left(S_8^{\mathbb{R}+}\right)$, $a = 1, 2$ such that

$$\rho = \chi_1 \otimes \chi_2 , \tag{6.5}$$

is an invariant $Spin(7) \times Spin(7)$ spinor.

□

Remark 6.2. *Theorem 5.1 in reference [29] states that a generalized* $Spin(7)$*-structure on an eight-dimensional manifold is equivalent to the following data:*

1. *An orientation.*
2. *A metric* g*.*
3. *A two-form* B*.*
4. *A scalar function* f*.*
5. *Two Majorana-Weyl spinors* χ_a , $a = 1, 2$ *either of the same or different chirality such that*

$$\rho = e^{-f} e^B \wedge \chi_1 \otimes \chi_2 , \tag{6.6}$$

is an invariant $Spin(7) \times Spin(7)$ *spinor.*

In proposition 6.1 we have considered the simplest possibility, namely we have taken $f = B = 0$ and the two spinors of the same chirality. Notice however that, given an intermediate structure $\mathfrak{S} = \left(g, \phi_3^1 \otimes v_1, \phi_3^2 \otimes v_2\right)$ we have a natural scalar function f, given in coordinates by

$$f = \phi_{3\,abc}^1 \phi_3^{2\,abc}\,, \tag{6.7}$$

as well as two two-forms B_1 and B_2 given by

$$B_1 = \iota_{v_2}\phi_3^1\,, \qquad B_2 = \iota_{v_1}\phi_3^2\,. \tag{6.8}$$

The viability of the previous choices for B and f in the definition of a generalized $Spin(7)$-structure remains yet to be understood, but it points out to a perhaps deep connection between intermediate structures and generalized $Spin(7)$-structures.

6.1 Regular elliptic fibrations

The existence of an intermediate structure $\mathfrak{S} = \left(g, \phi, \tilde{\phi}\right)$ not only implies a topological reduction of the structure group of the generalized bundle, but it may also imply a reduction on the spin bundle of $\mathcal{M}_8$. In particular, the existence of the two Majorana-Weyl spinors χ_a on $\mathcal{M}_8$ implies a reduction of the structure group of the spin bundle $S_8^{\mathbb{R}+} \to \mathcal{M}_8$ to different groups depending on whether they are linearly independent at every point or not. If they are, and they have the same chirality, then the structure group is reduced further from Spin(7) to $SU(4)$, while if they become dependent at some points in $\mathcal{M}_8$ then there exists no further global reduction. Notice that the existence of each χ_a is ensured by the existence of the corresponding $Spin(7)$ structure with associated admissible four-form Ω^a. If χ_1 and χ_2 are of different chirality, then the structure group is reduced to G_2 instead of $SU(4)$.

The combined manifold and structure $\left(\mathcal{M}_8, \mathfrak{S} = \left(g, \phi, \tilde{\phi}\right)\right)$ is thus equipped with two globally defined vector fields, v_1 and v_2. Each of them defines a codimension-one distribution which, if integrable, gives rise to a family of G_2-structure manifolds, as we have seen in section 5.2. Being $\mathcal{M}_8$ an eight-dimensional manifold which can have $SU(4)$ structure under a mild assumption, the question is if the intermediate structure $\mathfrak{S}$ may be used to define also an elliptic fibration in $\mathcal{M}_8$. Hence, the goal of this subsection is to define an elliptic fibration in $\mathcal{M}_8$ from an intermediate structure. For phenomenologically interesting F-theory applications, the elliptic fibration must be singular, meaning that the fibre at some points of the base space is not smooth torus. Here however we are going to consider only regular elliptic fibrations, commenting about how singularities can be implemented in this set up just at the very end, while leaving a complete analysis for a future publication [39]. By regular elliptic fibration we mean a principal torus bundle with total space $\mathcal{M}_8$ and six-dimensional base space that we will denote by $\mathcal{B}$. Given the structure $\mathfrak{S}$ that we have defined in our manifold $\mathcal{M}_8$, namely two three-forms ϕ_3^a and two vector fields v_a, it is natural to try to define the elliptic fibration by means of the two-dimensional distribution generated at each point by v_1 and v_2. In order to see if this is a sensible way to proceed, it is natural to ask if every elliptic fibration is equipped with two globally defined vector fields defining an integrable distribution. The following proposition answers this question in an affirmative way.

Proposition 6.3. *Let $\left(T^2, \mathcal{M}, \mathcal{B}\right)$ be a principal torus bundle, with total space $\mathcal{M}$, base $\mathcal{B}$ and fibre T^2. Then X is equipped with two globally defined vector fields v_1 and v_2 that define a completely integrable two dimensional distribution. In fact, $[v_1, v_2] = 0$.*

Proof. Since $\left(T^2, \mathcal{M}, \mathcal{B}\right)$ is a principal bundle, there is a smooth free action of $T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ on X, namely

$$\begin{aligned} \psi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} &\mapsto \mathrm{Diff}\,(\mathcal{M}) \\ ([t],[s]) &\mapsto \psi_{([t],[s])}\,, \end{aligned} \tag{6.9}$$

Therefore, for every $p \in \mathcal{M}_8$, we can define two curves $\gamma_p^1 \colon S^1 \to \mathcal{M}_8$ and $\gamma_p^2 \colon S^1 \to \mathcal{M}_8$ as follows

$$\gamma_p^1([t]) = \psi_{([t],[0])}(p), \qquad \gamma_p^2([s]) = \psi_{([0],[s])}(p). \tag{6.10}$$

They satisfy $\gamma_p^1([0]) = \gamma_p^2([0]) = p$. At every point $p \in \mathcal{M}_8$ we define then

$$v_1|_p = \frac{d}{dt}\gamma_p^1([0])\,, \qquad v_2|_p = \frac{d}{ds}\gamma_p^2([0])\,. \tag{6.11}$$

v_1 and v_2 are the infinitesimal generators of the torus action. Since the infinitesimal action is a Lie-algebra homomorphism, we must have $[v_1, v_2] = 0$. □

Therefore, it is reasonable to try to define an elliptic fibration by using two vector fields. We can even go further and completely characterize an elliptic fibration in terms of two globally defined vector fields.

Proposition 6.4. *$\mathcal{M}$ admits an elliptic fibration if and only if it is equipped with two globally defined, linearly independent vector fields v_1 and v_2, such that:*

1. *$[v_1, v_2] = 0$,*
2. *all the leaves of the foliation integrating the rank two distribution $\mathcal{H} := span\{v_1, v_2\}$ are compact,*
3. *the leaf space $\mathcal{M}/\mathcal{H}$ of the foliation is a smooth manifold,*
4. *Λ is a trivial bundle, where Λ is the bundle of isotropy groups of the $\mathbb{R}^2$-action on $\mathcal{M}$ generated by v_1,v_2.*

Remark 6.5. *a) The bundle Λ appearing in assumption 4 is a bundle of rank two lattices over $\mathcal{M}/\mathcal{H}$, contained in the trivial vector bundle $\mathbb{R}^2 \times \mathcal{M}/\mathcal{H}$. (In particular each fiber of Λ, as a group, is isomorphic to $\mathbb{Z}^2$.) If the leaf space $\mathcal{M}/\mathcal{H}$ happens to be simply connected, condition 4. is automatically satisfied.*

b) The flows of v_1 and v_2 as above are not periodic in general. Even when they are periodic of minimal period one, the principal two-torus action on $\mathcal{M}$ given by the above proposition is not the product of the S^1-action generated by v_1 with the S^1-action generated by v_2. To see this, consider the case where $\mathcal{M} = S^1 \times S^1$, and $v_1 = \frac{\partial}{\partial\theta_2}$, $v_2 = 2\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}$: the induced two-torus action is not free, for the time one flow of $\frac{1}{2}(v_1 + v_2)$ is the identity.

Proof. Given propositions 6.3, we just need to assume the existence of vector fields v_1 and v_2 as above and show that $\mathcal{M}$ admits an elliptic fibration. Since the leaves of $\mathcal{H}$ are compact, the vector fields v_1 and v_2 are complete, hence by assumption 1 they generate a $\mathbb{R}^2$-action on $\mathcal{M}$ whose orbits are exactly the leaves of $\mathcal{H}$. At every $p \in \mathcal{M}$, the isotropy group of this action is a rank two lattice in $\mathbb{R}^2$, for the orbits of the $\mathbb{R}^2$ are compact. Further, the isotropy group at p is equal to the isotropy group at any other point in the orbit through p, since $\mathbb{R}^2$ is an abelian group. We denote this isotropy group by $\Lambda_{\pi(p)}$, where $\pi \colon \mathcal{M} \to \mathcal{M}/\mathcal{H}$ is the projection to the leaf space.

At every $u \in \mathcal{M}/\mathcal{H}$ we can choose a basis (over $\mathbb{Z}$) of the lattice Λ_u, which we denote by $\{(a_1(u), a_2(u)),\ (b_1(u), b_2(u))\}$. The bundle Λ over $\mathcal{M}/\mathcal{H}$ is trivial by assumption 4. Hence this

basis can be chosen to depend smoothly on $u \in \mathcal{M}/\mathcal{H}$, that is, so that $a_1, a_2, b_1, b_2 \in C^\infty(\mathcal{M}/\mathcal{H})$. Define the following two vector fields on $\mathcal{M}$:

$$A := \pi^*(a_1)v_1 + \pi^*(a_2)v_2, \qquad B := \pi^*(b_1)v_1 + \pi^*(b_2)v_2.$$

These two vector fields commute. They generate an action of the torus $S^1 \times S^1$ on $\mathcal{M}$, since the integral curves of A and B are periodic of period one. This torus action is free, since A and B are constructed out of a basis of Λ, and its orbits are exactly the leaves of $\mathcal{H}$. Hence it defines an elliptic fibration by proposition 5.25. □

From propositions 6.3 we conclude that using the two vector fields present in an intermediate manifold to define an elliptic fibration in terms of an integrable distribution is the sensible way to proceed. Therefore, let $\left(\mathcal{M}_8, \mathfrak{S} = \left(g, \phi_3^1 \otimes v_1, \phi_3^2 \otimes v_2\right)\right)$ be an intermediate manifold. We define the following two-dimensional distribution

$$H_2 = \{H_{2\,p}\,, \quad p \in \mathcal{M}_8\}\ , \tag{6.12}$$

as follows

$$H_{2\,p} = \text{span}\left(v_1|_p, v_2|_p\right)\ . \tag{6.13}$$

By Frobenius' theorem, H_2 will be completely integrable if and only if

$$[w_1, w_2] \in \Gamma\left(H_2\right)\ , \quad \forall\, w_1, w_2 \in \Gamma\left(H_2\right)\ . \tag{6.14}$$

Now, having an elliptic fibration is equivalent to having v_1 and v_2 commuting and generating the infinitesimal, free, action of a torus on $\mathcal{M}_8$, and therefore the problem is fully characterized.

Remark 6.6. *We may wonder how many different two-dimensional foliations we have if we require the distribution (6.12) to be integrable but we do not necessarily require it to correspond to an elliptic fibration. In that case we obtain that $\mathcal{M}_8$ is foliated by parallelizable, and therefore orientable, two-dimensional manifolds. As a consequence, if the foliation is by compact leaves, then they must be oriented surfaces with zero Euler characteristic, that is, they must be elliptic surfaces.*

At this point we have completely characterized regular elliptic fibrations in terms of two vector fields on $\mathcal{M}_8$. However, we want to go further, and we want to study the possibility of having singular fibres at a set of points $S_I \subset \mathcal{B}$ on the base space $\mathcal{B}$. For F-theory applications, the space S_I has to satisfy some extra-requeriments, in particular, in the simplest case S_I must be of complex codimension one inside $\mathcal{B}$. Let A and B be the canonical basis of the homology group $H_1\left(T^2\right)$ of the torus. Then, if at every point in $b \in S_I$ there is a combination of A and B such that the cycle

$$C = pA + qB\,, \qquad p, q \in \mathbb{Z}\,, \qquad \text{m.c.d}\,(p,q) = 1\,, \tag{6.15}$$

vanishes, one can conclude then that, from the physics point of view, there is a (p, q)-seven-brane that extendes over the four non-compact dimensions and wraps the four compact dimensions of S_I. We will see in section 6.2 how to implement this kind of singularities just in terms of two vector fields, by using the map that we are going to construct now between a subset of the vector fields of $\mathcal{M}_8$ and $H_1\left(T^2\right)$.

Let us assume then that we have a manifold $\mathcal{M}_8$ equipped with an intermediate structure $\mathfrak{S} = \left(g, \phi_3^1 \otimes v_1, \phi_3^2 \otimes v_2\right)$ such that v_1 and v_2 are the infinitesimal generators of a free torus action. We thus know that there is a free torus action on $\mathcal{M}_8$

$$\psi\colon T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \text{Diff}\left(\mathcal{M}_8\right)\ , \tag{6.16}$$

with the corresponding infinitesimal action given by the Lie-algebra homomorphism

$$\begin{aligned} v_- : \mathbb{R}\times\mathbb{R} &\to \mathfrak{X}\left(\mathcal{M}_8\right) , \\ (t,s) &\mapsto v_{t,s} , \end{aligned} \tag{6.17}$$

where

$$v_{t,s}|_p = \left.\frac{d}{d\alpha}\right|_{\alpha=0} e^{\alpha(t,s)}(p), \qquad \forall p \in \mathcal{M}_8 . \tag{6.18}$$

Since $\mathbb{R}\times\mathbb{R}$ is equipped with the trivial Lie-bracket, the image of v_- is given by linear combinations of v_1 and v_2 over $\mathbb{R}$ as follows

$$v_{t,s} = t\, v_1 + s\, v_2 , \qquad t,s \in \mathbb{R} , \tag{6.19}$$

where we have identified $v_1 = v_{(1,0)}$ and $v_2 = v_{(0,1)}$. For every $p \in \mathcal{M}_8$ there is a unique curve $\gamma^p_{t,s} : \mathbb{R} \to \mathcal{M}_8$ such that

$$\gamma^p_{t,s}(0) = p , \qquad \frac{d}{d\tau}\gamma^p_{t,s}(\tau) = v_{t,s}|_{\gamma^p_{t,s}(\tau)} . \tag{6.20}$$

Let us denote by $\pi : \mathcal{M}_8 \to \mathcal{B} = \mathcal{M}_8/T^2$ the projection of the torus bundle. Then, from the standard properties of principal bundles we have

$$\gamma^p_{t,s}(\tau) \in \pi^{-1}(p) \simeq T^2 , \quad \forall p \in \mathcal{M}_8 , \quad \forall \tau \in \mathbb{R} . \tag{6.21}$$

By assumption, the flow of v_1 and v_2 is a closed curve in $\pi^{-1}(p) \simeq T^2$ for every $p \in \mathcal{M}_8$. However, not every vector field $v_{t,s}$ will give rise to a closed curve in the corresponding torus. The condition for the flow of $v_{t,s}$ to be a closed curve is given by

$$\frac{t}{s} \in \mathbb{Q} . \tag{6.22}$$

In particular, if we want the image of the closed curve $\gamma^p_{t,s}$ to be covered just one time when τ goes from zero to one then[25]

$$\text{m.c.d}\,(s,t) = 1 , \tag{6.23}$$

that is, s and t must be coprime. Let us use the notation $\mathfrak{X}^c_{s,t}\left(\mathcal{M}_8\right) = \{v_{s,t} , \mid \text{m.c.d}\,(s,t) = 1\}$. Then, for every $p \in \mathcal{M}_8$, there is a well-defined map δ^p from the set of vector fields $v_{s,t}$ such that $\text{m.c.d}\,(s,t) = 1$ to the first homology group $H_1\left(T^2\right)$ of the torus, given by

$$\begin{aligned} \delta^p : \mathfrak{X}^c_{s,t}\left(\mathcal{M}_8\right) &\to H_1\left(T^2\right) , \\ v_{s,t} &\mapsto \left[\gamma^p_{t,s}\right] , \end{aligned} \tag{6.24}$$

Let us take as a canonical basis $v_{1,0}$ and $v_{0,1}$, and therefore under δ^p we have that $v_{1,0} \mapsto A = \left[\gamma^p_{1,0}\right]$ and $v_{0,1} \mapsto B = \left[\gamma^p_{0,1}\right]$, where A and B are the standard generators of the homology group $H_1\left(T^2\right) = \mathbb{Z}\times\mathbb{Z}$.

[25] We thank Raffaele Savelli for a clarification about this point.

6.2 Possibility of singular points

So far we have considered manifolds where there is at least one globally defined, no-where vanishing vector field. As a consequence, if $\mathcal{M}_8$ is closed, we would conclude that the Euler characteristic must be zero: $\chi(\mathcal{M}_8) = 0$. This seems to be a too restrictive condition. However, we should not be concerned, for two reasons. On one hand, we do not necessarily require $\mathcal{M}_8$ to be compact, meaning that $\mathcal{M}_8$ does not have to be necessarily understood as a compactification space. In fact, if we consider $\mathcal{M}_8$ to be compact it would only be in order to make contact with eight-dimensional compact internal spaces in F-theory. On the other hand, as we have mentioned before, even if we take $\mathcal{M}_8$ to be compact, phenomenologically interesting F-theory applications are based on elliptic fibrations where the fibre becomes singular at given set of points on the base space $\mathcal{B}$, including the possibility of having a singular space $\mathcal{M}_8$. In this case we would not require the vector fields v_a to be globally non-vanishing nor $\mathcal{M}_8$ to be a smooth manifold, and therefore again the Euler number does not have to vanish. All the previous results in this paper have been derived assuming that the vector fields v_a are globally defined and nowhere vanishing. Hence, in order to include the possibility of having a singular elliptic fibration, we will study now the situation where we admit linear combinations of the vector fields v_1 and v_2 to have zeros on $\mathcal{M}_8$ (that is, v_1 and v_2 are linearly dependent at a set of points of $\mathcal{M}_8$). More precisely, let us consider the case in which the vector field

$$v_{s,t} = s\, v_1 + t\, v_2\,, \tag{6.25}$$

is zero at $p \in S_I \subset \mathcal{B}$ for the particular combination of v_1 and v_2 given by (s,t). Therefore, from equation (6.24) we deduce that the maps

$$\begin{aligned} \delta^p \colon \mathfrak{X}^c_{s,t}(\mathcal{M}_8) &\to H^1\left(T^2\right)\,, \\ v_{s,t} &\mapsto \left[\gamma^p_{t,s}\right] \end{aligned} \tag{6.26}$$

are singular at the points $v_{s,t}$. Hence, we conclude that at every point $p \in S_I$ the torus fibration is singular since the cycle $s\,A + t\,B$ of the torus $\pi^{-1}(p)$ collapses. We are thus able to see where the fibre becomes singular just from the analysis of the zeros of the vector field $v_{t,s}$. If S_I satisfy the corresponding requeriments, then it can be interpreted from the physics point of view as a set of $D7$-branes extending on the four non-compact dimensions and four compact dimensions on the base space $\mathcal{B}$.

7 Conclusions

In this paper we have studied $\mathcal{N} = 1$ M-theory compactifications down to four dimensions in terms of an eight-dimensional manifold $\mathcal{M}_8$ endowed with an intermediate structure $\mathfrak{S}$, whose presence is motivated by the exceptionally generalized geometric description of such compactifications. We have restricted our attention to off-shell supersymmetry, namely to the topological implications of $\mathfrak{S}$ on $\mathcal{M}_8$. We have found that using $\mathfrak{S}$ together with some mild assumptions, it is possible to embed in $\mathcal{M}_8$ a family of G_2-structure seven-dimensional manifolds as the leaves of a codimension-one foliation. At the same time, it is possible to prove that $\mathcal{M}_8$ is equipped with a topological $Spin(7)$-structure. This, if explored further, may give a relation between seven-dimensional manifolds that are consistent internal spaces in M-theory, and eight-dimensional manifolds in F-theory compactifications, perhaps pointing out to some kind of duality between M-theory compactified, loosely speaking, on the leaves of the foliation, and F-theory compactified on $\mathcal{M}_8$. With a different set of assumptions, it is possible to use $\mathfrak{S}$ to naturally define an S^1 principal-bundle structure on $\mathcal{M}_8$,

which turns out to be equipped with a topological $Spin(7)$-structure if the base space is equipped with a topological G_2-structure. One can go further and, considering a more general intermediate structure $\mathfrak{S}$, define in $\mathcal{M}_8$ an elliptic fibration. In that case, if the base space of the elliptic fibration has a topological $SU(3)$-structure, then $\mathcal{M}_8$ has a topological $Spin(7)$-structure[26]. In addition, we show that the elliptic fibration can be completely characterized through the pair of vector fields v_1 and v_2 that are present in $\mathfrak{S}$, and that the possible singularities on the elliptic fibration correspond to zeros of a particular combination of v_1 and v_2.

We have studied here some of the implications drawn by the existence of an intermediate structure $\mathfrak{S}$ and, more generically, we have explored the relation between eight-dimensional $Spin(7)$ and seven-dimensional G_2-manifolds. This might be of great physical interest, in particular for the study of dualities among String/M/F-Theory compactifications. For that, further work needs to be done, mainly to understand how the differential conditions on $\mathfrak{S}$ required by on-shell supersymmetry imply reductions of the holonomy group of $\mathcal{M}_8$ or its preferred submanifolds.

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A G_2-manifolds

This section is dedicated to summarize the main results concerning seven-dimensional manifolds of G_2-structure and holonomy contained in G_2. We will closely follow [11, 55]. $G_2^{\mathbb{C}}$ is a simply-connected, semisimple complex fourteen-dimensional Lie group[27]. It has two real forms, namely

- The real compact form $G_2 \subset SO(7)$.
- The real non-compact form $G_2^* \subset SO(4,3)$.

One way to characterize the real forms of G_2 is by the isotropy groups of the three-forms ϕ_0 (3.2) and ϕ_1 (3.7). More precisely, the isotropy group of ϕ_0 is the real compact form G_2 and the isotropy group of ϕ_1 is the real non-compact form G_2^*.
We recall the definition of ϕ_0 (definition 3.3): it is the three-form on $\mathbb{R}^7$ given by

$$\phi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}\,, \tag{A.1}$$

where dx_{ijl} stands for $dx_i \wedge dx_j \wedge dx_l$.

Now let $\mathcal{M}$ be an oriented seven-dimensional manifold.

Definition A.1. *For each $p \in \mathcal{M}$ we define*

$$\mathcal{P}^3_p\mathcal{M} = \left\{\phi \in \Lambda^3 T^*_p\mathcal{M} \quad | \quad \phi = f_p^*\phi_0\right\}\,, \tag{A.2}$$

where $f_p : T_p\mathcal{M} \to \mathbb{R}^7$ is an oriented[28] isomorphism.

[26] The proof of this statement will be presented elsewhere, as it involves topological $SU(3)$-structures, which were not discussed in this letter in order not to obscure the presentation.
[27] For more details see [56, 57].
[28] By this we mean: orientation preserving.

$\mathcal{P}^3_p\mathcal{M}$ is isomorphic to the quotient $GL_+(7,\mathbb{R})/G_2$ which has dimension $49-14=35$. Since the dimension of $\Lambda^3 T^*_p$ is also 35, we see that $\mathcal{P}^3_p\mathcal{M}$ is an open subset of $\Lambda^3 T^*_p$ and therefore a manifold[29]. See proposition 3.6 for more details.

Definition A.2. *We define $\mathcal{P}^3\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ to be the bundle over $\mathcal{M}$ whose fibre at $p \in \mathcal{M}$ is given by $\pi^{-1}(p) = \mathcal{P}^3_p\mathcal{M}$.*

Notice that $\mathcal{P}^3\mathcal{M}$ is an open subbundle of $\Lambda^3 T^*\mathcal{M}$ with fibre $GL_+(7,\mathbb{R})/G_2$.

Definition A.3. *A three-form $\phi \in \Omega^3(\mathcal{M})$ is said to be* positive *if $\phi_p \in \mathcal{P}^3_p\mathcal{M}$ for all $p \in \mathcal{M}$.*

Let us denote by $F_+\mathcal{M}$ the oriented frame bundle of $\mathcal{M}$, that is, the bundle over $\mathcal{M}$ whose fibre $F_+\mathcal{M}_p$ over $p \in \mathcal{M}$ is the set of ordered bases of $T_p\mathcal{M}$ inducing the given orientation:

$$F_+\mathcal{M}_p = \{e_p = (e_1, \ldots, e_7)\ , \quad | \quad e_p \quad \text{ordered oriented basis of } T_p\mathcal{M}\}\ .$$

Notice that such an ordered oriented basis e_p can be identified with an orientated isomorphism $T_p\mathcal{M} \xrightarrow{\cong} \mathbb{R}^7$, which sends the j-th basis vector of e_p to the j-th vector of the standard basis of $\mathbb{R}^7$, for all j. The Lie group $GL_+(7,\mathbb{R})$ acts freely and transitively on each fibre as follows

$$A \cdot e_p = \left(p, \sum_{j=1}^{k} A_{ij} e_j\right)\ , \quad A \in GL_+(7,\mathbb{R})\ , \quad p \in \mathcal{M}\,. \tag{A.3}$$

Hence $F_+(\mathcal{M})$ is a principal bundle with typical fibre $GL_+(7,\mathbb{R})$. Suppose now that $\mathcal{M}$ is equipped with a positive three-form ϕ. Consider

$$Q\mathcal{M}_p := \{f_p : T_p\mathcal{M} \xrightarrow{\cong} \mathbb{R}^7 \text{ oriented isomorphism such that } \phi_p = f_p^*\phi_0\}. \tag{A.4}$$

Since ϕ is positive, this is non-empty at every $p \in \mathcal{M}$. By identifying an isomorphism $T_p\mathcal{M} \to \mathbb{R}^7$ with an ordered basis of $T_p\mathcal{M}$ as above, we can regard $Q\mathcal{M}_p$ as a subset of $F_+\mathcal{M}_p$. $Q\mathcal{M}$ is a principal G_2-subbundle of $F_+\mathcal{M}$, i.e. a G_2-structure compatible with the orientation of $\mathcal{M}$, since by construction ϕ_0 is invariant under the natural action of $G_2 \subset GL_+(7,\mathbb{R})$ on $\mathbb{R}^7$. Conversely, given a G_2-subbundle $Q\mathcal{M}$ of $F_+\mathcal{M}$ we can define a positive three-form on $\mathcal{M}$ using equation (A.4). Therefore, we have found a one-to-one correspondence between positive three-forms ϕ on $\mathcal{M}$ and G_2-structures on $\mathcal{M}$ compatible with the orientation.

A Riemannian metric on an orientable manifold $\mathcal{M}$ implies a reduction of the structure group of the frame bundle from $GL_+(7,\mathbb{R})$ to $SO(7)$. Since $G_2 \subset SO(7)$, there is an associated metric g to the G_2 structure on $\mathcal{M}$. We will call a positive form ϕ on $\mathcal{M}$ together with its associated metric g a G_2-structure on $\mathcal{M}$, since, although (ϕ, g) is not a G_2-structure it uniquely defines one. Let ∇ be the Levi-Civita connection associated to g. We call $\nabla\phi$ the torsion of (ϕ, g) and we say that (ϕ, g) is torsion free if $\nabla\phi = 0$. The following proposition holds.

Proposition A.4. *Let $\mathcal{M}$ be an oriented seven-dimensional manifold and (ϕ, g) a G_2-structure on $\mathcal{M}$. Then the following are equivalent*

- *(ϕ, g) is torsion-free.*
- *$\mathrm{Hol}(g) \subseteq G_2$ and ϕ is the induced three-form.*
- *$\nabla\phi = 0$ on $\mathcal{M}$, where ∇ is the Levi-Civita connection of g.*
- *$d\phi = \delta\phi = 0$ on $\mathcal{M}$.*

[29] Notice that $\mathcal{P}^3_p\mathcal{M}$ is not a vector space.

Proof. See lemma 11.5 in [10]. □

Here δ stands for the codifferential, defined as $\delta = *d*$. Notice that none of the above conditions is linear on ϕ, since the metric g depends non-linearly on it. This implies that the operators ∇ and $d*$ depend on g, which in turn depends on ϕ and thus the equations $\nabla\phi = 0$ and $d*\phi = 0$ should not be considered as linear in ϕ.

We show now how torsion free G_2 structures arise from Calabi-Yau manifolds of complex dimension two and three respectively.

Proposition A.5. *Suppose (Y, g_Y) is a Riemannian four-dimensional manifold with holonomy $SU(2)$. Then Y admits a complex structure form J, a Kähler form ω and a holomorphic volume form $\mathcal{V}$ such that $d\omega = d\mathcal{V} = 0$. Let $\mathbb{R}^3$ have coordinates (x_1, x_2, x_3) and euclidean metric $h = dx_1^2 + dx_2^2 + dx_3^2$. Define a metric g and a three-form ϕ on $\mathbb{R}^3 \times Y$ by $g = h \times g_Y$ and*

$$\phi = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega + dx_2 \wedge \mathrm{Re}\mathcal{V} - dx_3 \wedge \mathrm{Im}\mathcal{V}\,. \tag{A.5}$$

Then (ϕ, g) is a torsion free G_2 structure on $\mathbb{R}^3 \times Y$ and

$$*\,\phi = \frac{1}{2}\omega \wedge \omega + dx_2 \wedge dx_3 \wedge \omega - dx_1 \wedge dx_3 \wedge \mathrm{Re}\mathcal{V} - dx_1 \wedge dx_2 \wedge \mathrm{Im}\mathcal{V}\,. \tag{A.6}$$

Proof. See proposition 11.1.1 in reference [11]. □

It possible to substitute $\mathbb{R}^3$ by T^3 and obtain a similar result.

Proposition A.6. *Suppose (Y, g_Y) is a Riemannian six-dimensional manifold with holonomy $SU(3)$. Then Y admits a complex structure form J, a Kähler form ω and a holomorphic volume form $\mathcal{V}$ such that $d\omega = d\mathcal{V} = 0$. Let $\mathbb{R}$ have coordinate x. Define a metric g and a three-form ϕ on $\mathbb{R} \times Y$ by $g = dx^2 \times g_Y$ and*

$$\phi = dx \wedge \omega + \mathrm{Re}\mathcal{V}\,. \tag{A.7}$$

Then (ϕ, g) is a torsion free G_2 structure on $\mathbb{R} \times Y$ and

$$*\,\phi = \frac{1}{2}\omega \wedge \omega + dx \wedge \mathrm{Im}\mathcal{V}\,. \tag{A.8}$$

Proof. See proposition 11.1.1 in reference [11]. □

It possible to substitute $\mathbb{R}$ by S^1 and obtain a similar result.

B *Spin* (7)-manifolds

This section is dedicated to summarize the main results concerning eight-dimensional manifolds with *Spin* (7)-structure and holonomy contained in *Spin* (7). We will closely follow [11, 55].

Definition B.1. *Let us consider $\mathbb{R}^8$ with coordinates $(x_1, \ldots, x_8)$. We define a four-form Ω_0 on $\mathbb{R}^8$ by*

$$\begin{aligned}\Omega_0 = {} & dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467} \\ & - dx_{2358} - dx_{2367} - dx_{2457} + dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678}\,,\end{aligned} \tag{B.1}$$

where $dx_{ij\ldots l}$ stands for the exterior form $dx_i \wedge dx_l \wedge \cdots \wedge dx_l$. The subgroup of $GL(8, \mathbb{R})$ that preserves Ω_0 is the Lie group Spin (7)[30], which also fixes euclidean metric $g_0 = dx_1^2 + \cdots + dx_8^2$ and the orientation on $\mathbb{R}^8$ (that is, Spin (7) $\subset SO(8)$). Notice that $*\Omega_0 = \Omega_0$ where $*$ is the Hodge-dual operator associated to g_0.

[30] *Spin* (7) is a compact, connected, simply-connected, semisimple an 21-dimensional Lie group, isomorphic as a Lie group to the double cover of *SO* (7).

Definition B.2. *Let V be an oriented eight-dimensional vector space. A four-form $\Omega \in \Lambda^4 V^*$ is said to be* admissible *if there exists an oriented isomorphism $f : \mathbb{R}^8 \to V$ such that $\Omega_0 = f^*\Omega$.*

Notice that an admissible form on V induces an inner product on V.

Let $\mathcal{M}$ be an oriented eight-dimensional manifold.

Definition B.3. *For each $p \in \mathcal{M}$ we define*

$$\mathcal{A}^4_p\mathcal{M} = \left\{\Omega \in \Lambda^4 T^*_p\mathcal{M} \quad | \quad \Omega = f_p^*\Omega_0\right\}, \tag{B.2}$$

where $f_p : T_p\mathcal{M} \to \mathbb{R}^8$ is an oriented isomorphism.

$\mathcal{A}^4_p\mathcal{M}$ is isomorphic to the quotient $GL_+(8,\mathbb{R})/Spin(7)$ which has dimension $64 - 21 = 43$. The dimension of $\Lambda^4 T^*_p$ is 70, and thus $\mathcal{A}^3_p\mathcal{M}$ has codimension 27 in $\Lambda^4 T^*_p$ in contrast to the case of G_2 considered in appendix A, where $\mathcal{P}^3_p\mathcal{M}$ is open in $\Lambda^3 T^*_p$. Notice that $\mathcal{A}^4_p\mathcal{M}$ is not a vector space.

Definition B.4. *We define $\mathcal{A}^4\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ to be the bundle over $\mathcal{M}$ whose fibre at $p \in \mathcal{M}$ is given by $\pi^{-1}(p) = \mathcal{A}^4_p\mathcal{M}$.*

Definition B.5. *A four-form $\Omega \in \Lambda^4 T^*\mathcal{M}$ is said to be* admissible *if $\Omega_p \in \mathcal{P}^4_p\mathcal{M}$ for all $p \in \mathcal{M}$.*

Let us denote by $F_+\mathcal{M}$ the oriented frame bundle of $\mathcal{M}$. Assume $\mathcal{M}$ is equipped with an admissible four-form ϕ. Since Ω is admissible, at every point $p \in \mathcal{M}$ there exists an oriented isomorphism

$$f_p : T_p\mathcal{M} \to \mathbb{R}^8, \tag{B.3}$$

such that $\Omega_p = f_p^*\Omega_0$. Ω can be used to define a principal subbundle $Q\mathcal{M}$ of $F_+\mathcal{M}$ as follows:

$$Q\mathcal{M}_p := \{f_p : T_p\mathcal{M} \xrightarrow{\cong} \mathbb{R}^8 \text{ oriented isomorphism such that } \Omega_p = f_p^*\Omega_0\}. \tag{B.4}$$

The structure group of $Q\mathcal{M}$ is $Spin(7)$, since the stabilizer of Ω_0 under the natural action of $GL_+(8,\mathbb{R})$ on $\mathbb{R}^8$ is $Spin(7)$. Conversely, given a $Spin(7)$-subbundle $Q\mathcal{M}$ of $F_+\mathcal{M}$ we can define an admissible four-form using equation (B.4). Therefore, we have found a one-to-one correspondence between positive four-forms Ω on $\mathcal{M}$ and $Spin(7)$-structures on $\mathcal{M}$ inducing the given orientation on $\mathcal{M}$.

A Riemannian metric on an oriented eight-dimensional manifold $\mathcal{M}$ implies a reduction of the structure group of the frame bundle from $GL(8,\mathbb{R})$ to $SO(8)$. Since $Spin(7) \subset SO(8)$, there is an associated metric g to the $Spin(7)$ structure on $\mathcal{M}$. We will call an admissible form Ω on $\mathcal{M}$ together with its associated metric g a $Spin(7)$-structure on $\mathcal{M}$, since, although (Ω, g) is not a $Spin(7)$-structure it defines uniquely defines one. Let ∇ be the Levi-Civita connection associated to g. We call $\nabla\Omega$ the torsion of (Ω, g) and we say that (Ω, g) is torsion free if $\nabla\Omega = 0$. The following proposition holds.

Proposition B.6. *Let $\mathcal{M}$ be an oriented eight-dimensional manifold and let (Ω, g) be a $Spin(7)$-structure on $\mathcal{M}$. Then the following are equivalent*

- *(Ω, g) is torsion-free.*
- *$\mathrm{Hol}(g) \subseteq Spin(7)$ and Ω is the induced four-form.*
- *$\nabla\Omega = 0$ on $\mathcal{M}$, where ∇ is the Levi-Civita connection of g.*
- *$d\Omega = 0$ on $\mathcal{M}$.*

Proof. See lemma 11.5 in [10]. □

Although $d\Omega = 0$ is a linear condition on Ω, its restriction to $\Gamma\left(\mathcal{A}^4\mathcal{M}\right)$ is non-linear.
We finish displaying cases in which a torsion-free $Spin(7)$ structure arises, for instance, from from Calabi Yau manifolds of complex dimension two, three, four and from-torsion free G_2 structures. In particular, proposition B.9 is the one we generalized in theorem 5.14, in the case with torsion.

Proposition B.7. *Suppose (Y, g_Y) is a Riemannian four-dimensional manifold with holonomy $SU(2)$. Then Y admits a complex structure form J, a Kähler form ω and a holomorphic volume form $\mathcal{V}$ such that $d\omega = d\mathcal{V} = 0$. Let $\mathbb{R}^4$ have coordinates $(x_1, \ldots, x_4)$ and euclidean metric $h = dx_1^2 + \cdots + dx_4^2$. Define a metric g and a four-form Ω on $\mathbb{R}^4 \times Y$ by $g = h \times g_Y$ and*

$$\Omega = dx_{1234} + (dx_{12} + dx_{34}) \wedge \omega + (dx_{13} - dx_{24}) \wedge \mathrm{Re}\mathcal{V} - (dx_{14} + dx_{23}) \wedge \mathrm{Im}\mathcal{V} + \frac{1}{2}\omega \wedge \omega\,. \quad \text{(B.5)}$$

Then (Ω, g) is a torsion-free $Spin(7)$ structure on $\mathbb{R}^4 \times Y$.

Proof. See proposition 13.1.1 in reference [11]. □

In the previous proposition it possible to substitute $\mathbb{R}^4$ by T^4 and obtain a similar result.

Proposition B.8. *Suppose (Y, g_Y) is a Riemannian six-dimensional manifold with holonomy $SU(3)$. Let ω be the associated Kähler form and $\mathcal{V}$ the holomorphic volume form. Let $\mathbb{R}^2$ have coordinates (x_1, x_2). Define a metric g and a four-form Ω on $\mathcal{M} = \mathbb{R}^2 \times Y$ by $g = \left(dx_1^2 + dx_2^2\right) \times g_Y$ and*

$$\Omega = dx_1 \wedge dx_2 \wedge \omega + dx_1 \wedge \Re\mathrm{e}\mathcal{V} - dx_2 \wedge \Im\mathrm{m}\mathcal{V} + \frac{1}{2}\omega \wedge \omega\,. \quad \text{(B.6)}$$

Then (Ω, g) is a torsion-free $Spin(7)$-structure on $\mathcal{M}$.

Proof. See proposition 13.1.2 in reference [11]. □

In the previous proposition it possible to substitute $\mathbb{R}^2$ by T^2 and obtain a similar result.

Proposition B.9. *Suppose (Y, g_Y) is a Riemannian seven-dimensional manifold with holonomy G_2. Let ϕ and $*\phi$ be the associated three-form and four-form. We define a metric g and a four-form Ω on $\mathcal{M} = \mathbb{R} \times Y$ by $g = dx_1^2 \times g_Y$ and*

$$\Omega = dx \wedge \phi + *\phi \quad \text{(B.7)}$$

where x is the coordinate on $\mathbb{R}$. Then (Ω, g) is a torsion-free $Spin\,(7)$-structure on $\mathcal{M}$.

Proof. See [11, Prop. 13.1.3]. □

In the previous proposition it possible to substitute $\mathbb{R}$ by S^1 and obtain a similar result.

Proposition B.10. *Suppose (Y, g_Y) is a Riemannian eight-dimensional manifold with holonomy $SU(4)$, $Sp(2)$ or $SU(2) \times SU(2)$ and associated Kähler form ω and holomorphic volume form $\mathcal{V}$. Define a four-form Ω on Y by*

$$\Omega = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}\mathcal{V}\,. \quad \text{(B.8)}$$

Then (Ω, g) is a torsion-free $Spin\,(7)$-structure on Y.

Proof. See [11, Prop. 13.1.4]. □

References

[1] B. R. Greene, *String theory on Calabi-Yau manifolds*, hep-th/9702155.

[2] M. Grana, *Flux compactifications in string theory: A Comprehensive review*, *Phys.Rept.* **423** (2006) 91–158, [hep-th/0509003].

[3] F. Denef, *Les Houches Lectures on Constructing String Vacua*, arXiv:0803.1194.

[4] P. Koerber, *Lectures on Generalized Complex Geometry for Physicists*, *Fortsch.Phys.* **59** (2011) 169–242, [arXiv:1006.1536].

[5] S. S. Chern, *The geometry of G-structures*, *Bulletin of the American Mathematical Society* **72** (1966) 167–342.

[6] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, *Vacuum Configurations for Superstrings*, *Nucl.Phys.* **B258** (1985) 46–74.

[7] M. Duff, B. Nilsson, and C. Pope, *Kaluza-Klein Supergravity*, *Phys.Rept.* **130** (1986) 1–142.

[8] C. Vafa, *Evidence for F theory*, *Nucl.Phys.* **B469** (1996) 403–418, [hep-th/9602022].

[9] K. Becker and M. Becker, *M theory on eight manifolds*, *Nucl.Phys.* **B477** (1996) 155–167, [hep-th/9605053].

[10] S. Salamon, *Riemannian Geometry and Holonomy Groups*. Longman Sc & Tech, 1989.

[11] D. D. Joyce, *Compact manifolds with special holonomy*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.

[12] E. Bonan, *Sur les variétés riemanniennes à groupe d'holonomie G_2 ou Spin(7)*, *Comptes Rendus de l'Académie des Sciences Paris - Séries I - Mathematics* **320** (1966) 127âĂŞ129.

[13] M. Berger, *Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés riemanniennes*, *Bulletin de la Société Mathématique de France* **83** (1955) 279–330.

[14] P. Candelas and X. de la Ossa, *Moduli space of Calabi-Yau manifolds*, *Nucl.Phys.* **B355** (1991) 455–481.

[15] S. Kachru, M. B. Schulz, and S. Trivedi, *Moduli stabilization from fluxes in a simple IIB orientifold*, *JHEP* **0310** (2003) 007, [hep-th/0201028].

[16] M. R. Douglas and S. Kachru, *Flux compactification*, *Rev.Mod.Phys.* **79** (2007) 733–796, [hep-th/0610102].

[17] F. Denef, M. R. Douglas, and S. Kachru, *Physics of String Flux Compactifications*, *Ann.Rev.Nucl.Part.Sci.* **57** (2007) 119–144, [hep-th/0701050].

[18] A. Strominger, *Superstrings with Torsion*, *Nucl.Phys.* **B274** (1986) 253.

[19] N. Hitchin, *Generalized Calabi-Yau manifolds*, *ArXiv Mathematics e-prints* (Sept., 2002) [math/0209].

[20] M. Gualtieri, *Generalized complex geometry*, *ArXiv Mathematics e-prints* (Jan., 2004) [math/0401221].

[21] C. Hull, *Generalised Geometry for M-Theory*, *JHEP* **0707** (2007) 079, [hep-th/0701203].

[22] P. P. Pacheco and D. Waldram, *M-theory, exceptional generalised geometry and superpotentials*, *JHEP* **0809** (2008) 123, [arXiv:0804.1362].

[23] M. Grana and H. Triendl, *Generalized N=1 and N=2 structures in M-theory and type II orientifolds*, *JHEP* **1303** (2013) 145, [arXiv:1211.3867].

[24] E. Witten, *String theory dynamics in various dimensions*, *Nucl.Phys.* **B443** (1995) 85–126, [hep-th/9503124].

[25] E. Cremmer, B. Julia, and J. Scherk, *Supergravity Theory in Eleven-Dimensions*, *Phys.Lett.* **B76** (1978) 409–412.

[26] H. Nicolai and C. Wetterich, *On the Spectrum of Kaluza-Klein Theories With Noncompact Internal Spaces*, *Phys.Lett.* **B150** (1985) 347.

[27] A. M. Th. Friedrich, I. Katha and U. Semmelmanna, *On nearly parallel G_2-structures*, *Journal of Geometry and Physics* **23** (1997) 259–286.

[28] F. Witt, *Special metric structures and closed forms*, *ArXiv Mathematics e-prints* (Feb., 2005) [math/0502443].

[29] F. Witt, *Generalised G_2 Manifolds*, *Communications in Mathematical Physics* **265** (July, 2006) 275–303, [math/0411642].

[30] P. Kaste, R. Minasian, and A. Tomasiello, *Supersymmetric M theory compactifications with fluxes on seven-manifolds and G structures*, *JHEP* **0307** (2003) 004, [hep-th/0303127].

[31] J. P. Gauntlett, D. Martelli, and D. Waldram, *Superstrings with intrinsic torsion*, *Phys.Rev.* **D69** (2004) 086002, [hep-th/0302158].

[32] T. House and A. Micu, *M-Theory compactifications on manifolds with G(2) structure*, *Class.Quant.Grav.* **22** (2005) 1709–1738, [hep-th/0412006].

[33] A. Coimbra, C. Strickland-Constable, and D. Waldram, *$E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry, Connections and M theory*, *JHEP* **1402** (2014) 054, [arXiv:1112.3989].

[34] D. R. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds. 1*, *Nucl.Phys.* **B473** (1996) 74–92, [hep-th/9602114].

[35] D. R. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds. 2.*, *Nucl.Phys.* **B476** (1996) 437–469, [hep-th/9603161].

[36] D. Tsimpis, *M-theory on eight-manifolds revisited: N=1 supersymmetry and generalized Spin(7) structures*, *JHEP* **0604** (2006) 027, [hep-th/0511047].

[37] F. Bonetti, T. W. Grimm, and T. G. Pugh, *Non-Supersymmetric F-Theory Compactifications on Spin(7) Manifolds*, *JHEP* **1401** (2014) 112, [arXiv:1307.5858].

[38] F. Bonetti, T. W. Grimm, E. Palti, and T. G. Pugh, *F-Theory on Spin(7) Manifolds: Weak-Coupling Limit*, *JHEP* **1402** (2014) 076, [arXiv:1309.2287].

[39] C. S. S. Mariana Graña and M. Zambon, *Work in progress*, .

[40] F. Cantrijn, A. Ibort, and M. De León, *On the geometry of multisymplectic manifolds*, *Journal of the Australian Mathematical Society (Series A)* **66** (5, 1999) 303–330.

[41] N. Hitchin, *The geometry of three-forms in six and seven dimensions*, *ArXiv Mathematics e-prints* (Oct., 2000) [math/0010054].

[42] N. Hitchin, *Stable forms and special metrics*, *ArXiv Mathematics e-prints* (July, 2001) [math/0107101].

[43] S. Lie and F. Engel, *Theorie de transformationsgruppen i, ii und iii*, *Leipzig* (1888).

[44] F. Engel, *Ein neues, dem linearen komplexe analoges gebilde*, *Leipz. Ber.* **52** (1900) 220–239.

[45] W. Reichel, *Über trilineare alternierende Formen in sechs und sieben Veränderlichen und die durch sie definiertengeometrischen Gebilde*. PhD thesis, Universität Greifswald, 1907.

[46] R. L. Bryant, *Some remarks on G_2-structures*, in *Proceedings of Gökova Geometry-Topology Conference 2005*, pp. 75–109, Gökova Geometry/Topology Conference (GGT), Gökova, 2006.

[47] P. Tondeur, *Geometry of foliations*. Birkhäuser, 1997.

[48] P. Molino, *Riemannian foliations*, vol. 73 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1988. Translated from the French by Grant Cairns, With appendices by Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu.

[49] G. Thorbergsson, *Singular Riemannian foliations and isoparametric submanifolds*, *Milan J. Math.* **78** (2010), no. 1 355–370.

[50] H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin geometry*, vol. 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[51] G. Reeb, *Sur la courboure moyenne des variétés intégrales d' une equation de pfaff $\omega = 0$*, *Acad. Sci. Paris 231* **231** (1950) 101–102.

[52] D. Tischler, *On fibering certain foliated manifolds over S^1*, *Topology* **9** (1970).

[53] W. P. Thurston, *A generalization of the reeb stability theorem*, *Topology* **13** (1974) 347âĂŞ352.

[54] J. M. Lee, *Manifolds and Differential Geometry*. American Mathematical Society, 2009.

[55] A. Clarke and B. Santoro, *Holonomy Groups in Riemannian Geometry*, *ArXiv e-prints* (June, 2012) [arXiv:1206.3170].

[56] J. F. Adams, *Lectures on Exceptional Lie Groups*. University of Chicago Press, 1996.

[57] I. Yokota, *Exceptional Lie groups*, *ArXiv e-prints* (Feb., 2009) [arXiv:0902.0431].