Another algebraic variational principle for the spectral curve of matrix models

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Abstract

We propose an alternative variational principle whose critical point is the algebraic plane curve associated to a matrix model (the spectral curve, i.e. the large N limit of the resolvent). More generally, we consider a variational principle that is equivalent to the problem of finding a plane curve with given asymptotics and given cycle integrals. This variational principle is not given by extremization of the energy, but by the extremization of an "entropy".

1 Introduction

To a random matrix model is associated an algebraic curve, often called "spectral curve". Most often this is the Stieljes transform of the "equilibrium spectral density", although not always. That algebraic curve is either obtained from the large N limit of the loop equations, or the large N limit of the saddle point equation, see for instance the review [4]. It is a curve with some specific type of singularities and boundary conditions.

It has been known for long, in many cases, that the large N density of eigenvalues can be found by extremizing an energy functional in the space of measures, and it turns out that the extremal measure is an algebraic function.

Our goal in this article, is to present another (in fact several other) variational principle, yielding the same spectral curve, but by extremizing only a functional in the space of algebraic curves (not using measures).

2 1- matrix model

2.1 Introduction to random matrices

Consider a random hermitian matrix M of size N (see [16]), with probability law:

$$\frac{1}{Z}\,\mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\,V(M)}\;dM \tag{2-1}$$

where $dM = \prod_i dM_{i,i} \prod_{i<j} d\mathrm{Re}\, M_{i,j}\; d\mathrm{Im}\, M_{i,j}$ is the $U(N)$ invariant Lebesgue measure on H_N, and where $V(x) = \sum_k \frac{t_k}{k} x^k$ is a polynomial called the "potential", and $t > 0$ is often called "temperature". The normalization factor Z is called the partition function:

$$Z = \int_{H_N} \mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\, V(M)}\; dM. \tag{2-2}$$

One can also extend this, and replace random hermitian matrices, by random "normal matrices with eigenvalues on some contour Γ":

$$H_N(\Gamma) = \{M = U\Lambda U^\dagger \;\; | \, U \in U(N)\, ,\; \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)\, ,\; \lambda_i \in \Gamma\} \tag{2-3}$$

equipped with the measure $dM = \prod_{i<j}(\lambda_i - \lambda_j)^2\, dU\; \prod_i d\lambda_i$ where dU is the Haar measuer on $U(N)$ and $d\lambda_i$ is the curvilinear measure along Γ. For instance when $\Gamma = \mathbb{R}$ this coincides with Hermitian matrices:

$$H_N(\mathbb{R}) = H_N \;,\; dM = \text{Lebesgue measure on}\, H_N, \tag{2-4}$$

and when $\Gamma = S^1$ =unit circle in $\mathbb{C}$, this coincides with the "circular ensemble" $U(N)$ with its Haar measure:

$$H_N(S^1) = U(N)\;,\; (\det M)^{-N} dM = \text{Haar measure on}\, U(N). \tag{2-5}$$

The expectation value of the resolvent:

$$W(x) = \frac{t}{N}\, \mathbb{E}\left(\mathrm{Tr}\; (x - M)^{-1}\right) \tag{2-6}$$

plays an important role, indeed its singularities encode the information on the spectrum of M.

In many cases (depending on the choice of potential V and on the choice of contour Γ), it is known (see [12, 2, 16] for instance), that $W(x)$ has a large N limit:

$$W(x) \underset{N\to\infty}{\sim} \omega(x) \tag{2-7}$$

and in many cases (again depending on the choice of potential V and contour), it is an algebraic function of x, i.e. it satisfies an algebraic equation:

$$P(x, \omega(x)) = 0 \qquad ,\; P(x, y) = \sum_{i,j} P_{i,j} x^i\, y^j. \tag{2-8}$$

This algebraic equation has several solutions (several branches) $y = Y_k(x)$, $k = 1, \dots, d$ where $d = \deg_y P$, and $\omega(x) = Y_0(x)$ is only one branch (it has to be a branch which behaves as $\omega(x) \sim t/x$ at large x, due to eq.(2-6)). Alternatively, one can view $\omega(x)$ as a multivalued function, or alternatively, it can be viewed as a meromorphic function on the compact Riemann surface $\mathcal{C}$ defined by the algebraic equation $P(x, y) = 0$.

For the 1-matrix model, the polynomial $P(x,y)$ is always quadratic in y (the algebraic equation is said to be "hyperelliptical"), and always of the form:

$$P(x,y) = y^2 - yV'(x) + P(x) \tag{2-9}$$

Finding $\omega(x) = y$ amounts to finding the polynomial $P(x)$.

Since there is a branch of $\omega(x)$ which begaves as t/x at large x, this implies that $P(x) \sim tV'(x)/x$ at large x, i.e. $P(x)$ has degree $\deg V' - 1$.

Then we have:

$$\omega(x) = y = \frac{1}{2}\left(V'(x) \pm \sqrt{V'(x)^2 - 4P(x)}\right). \tag{2-10}$$

Branchcuts occur at the odd zeroes of $U(x) = V'(x)^2 - 4P(x)$. Since $U(x)$ has even degree, there is necessarily an even number of odd zeroes, say $2s+2$ odd zeroes.

Let us denote:

$$U(x) = V'(x)^2 - 4P(x) = M(x)^2\,\sigma(x) \tag{2-11}$$

$$\sigma(x) = \prod_{k=1}^{2s+2}(x - a_k) = \text{product of odd zeroes}\,, \quad M(x) = \sqrt{\frac{U(x)}{\sigma(x)}} = \text{product of even zeroes.} \tag{2-12}$$

The points a_k are called the branchpoints.

2.1.1 Filling fractions

Let us define for $\alpha = 1, \dots, s$:

$$\mathcal{A}_\alpha = \text{clockwise contour surrounding } [a_{2_\alpha - 1}, a_{2\alpha}]. \tag{2-13}$$

Very often, it is interesting to consider matrix models with "fixed filling fractions", i.e. where the number of eigenvalues of M in a certain region of the complex plane is held fixed. The number n_α of eigenvalues of M enclosed by a clockwise contour C_α is:

$$n_\alpha = -\frac{N}{2i\pi\, t}\oint_{C_\alpha} W(x)\, dx \tag{2-14}$$

In the large N limit, the fixed filling fraction condition amounts to fix:

$$\frac{t\, n_\alpha}{N} = -\frac{1}{2i\pi}\oint_{\mathcal{A}_\alpha} \omega(x)\, dx = \epsilon_\alpha. \tag{2-15}$$

The numbers ϵ_α are called "filling fractions", they tell the number (times t/N) of eigenvalues of M which concentrate along the segment $[a_{2_\alpha - 1}, a_{2\alpha}]$.

2.1.2 Loop equations

Our goal now is to find the polynomial $P(x)$, as a function of the potential $V(x)$, the contour Γ and the filling fractions ϵ_α's.

It is well known that this polynomial can be determined by the following equations [4, 3]:

Definition 2.1 (Loop equations) *The loop equations of the 1-matrix model with potential V and with filling fractions ϵ_α is the following set of equations:*

$$\boxed{\left\{\begin{array}{l} \exists\,\text{polynomial}\, P(x) \;\; \text{such that}\; \omega^2(x) - \omega(x)\, V'(x) + P(x) = 0 \\ \\ \exists\,\text{branch} \quad \omega(x) \sim_{x\to\infty} t/x + O(1/x^2) \\ \\ \forall\,\alpha = 1,\dots,s\,, \qquad -\frac{1}{2i\pi}\oint_{\mathcal{A}_\alpha}\omega(x)\,dx = \epsilon_\alpha \end{array}\right.} \tag{2-16}$$

Let us check that indeed this system implies as many equations as unknowns: let $d = \deg V'$. Observe that the second equation ($\omega \sim t/x$) implies that $\deg P = d-1$, and this equation also fixes the leading coefficient of $P(x)$, it gives:

$$\lim_{x\to\infty} \frac{xP(x)}{V'(x)} = t. \tag{2-17}$$

$P(x)$ has thus $d-1$ unknown coefficients. The constraint that $U(x) = V'(x)^2 - 4P(x)$ has only $2s+2$ odd zeroes, i.e. $d-s-1$ even zeroes, imposes $d-s-1$ additional constraints on $P(x)$, i.e. there are only s unknown coefficients left in $P(x)$. Those s coefficients are then determined by the s filling fraction equations.

Our goal is not to study those equations, in particular the existence and unicity or not of solutions, as there already is a large literature about them, but to show that the same equation eq. (2-16) can be obtained from a local variational principle.

2.1.3 Usual energy variational principle

In the case where M is a hermitian matrix (eigenvalues $\in \mathbb{R}$), and $V(x)$ is a real potential bounded from below on $\mathbb{R}$, there is a known variational principle to find $\omega(x)$. $\omega(x)$ is the Stieljes transform of a positive measure $d\rho(x)$ on $\mathbb{R}$, such that:

$$\omega(x) = \int_{x'\in \text{supp.}\, d\rho} \frac{d\rho(x')}{x-x'} \qquad , \quad 2\mathrm{i}\pi\, \frac{d\rho(x)}{dx} = \omega(x-i0) - \omega(x+i0). \tag{2-18}$$

It is well known that the measure $d\rho$ can be found as the **unique** minimum of the convex functional on the space of measures $d\rho$:

$$\mathcal{S}[d\rho] \;\; = \;\; \int_{x\in \text{supp.}\, d\rho} V(x)\, d\rho(x) - \int_{x\in \text{supp.}\, d\rho}\int_{x'\in \text{supp.}\, d\rho} d\rho(x)\, d\rho(x')\, \ln|x-x'|$$

$$+\sum_{\alpha} \eta_\alpha \int_{a_{2\alpha-1}}^{a_{2\alpha}} d\rho(x) \tag{2-19}$$

where η_α are Lagrange multipliers determined by requiring that

$$\int_{a_{2\alpha-1}}^{a_{2\alpha}} d\rho(x) = \epsilon_\alpha. \tag{2-20}$$

This functional is convex when V is real, supp.$d\rho \subset \mathbb{R}$ and $d\rho > 0$, so that this variational problem can be proved to have a unique minimum, and one finds that the minimum $d\rho$ is algebraic $d\rho(x) = \frac{1}{\pi}\sqrt{4P(x) - V'^2(x)}\, dx$, and is solution of the loop equations above.

In case V is not real, or $\Gamma \neq \mathbb{R}$ or $d\rho$ is not a positive measure on $\mathbb{R}$, usually the support of $d\rho$ is also unknown (free frontier problem), and the above functional is then no longer convex, instead of an extremum, it has a saddle–point, and it is not known *in general* whether saddle–points are unique or not (it might be known case by case). However, in all cases, any continuous saddle–point of the functional $\mathcal{S}$ is a solution to loop equations, and vice/versa, any solution of loop equations is a saddle–point of $\mathcal{S}$.

Our purpose here is to propose another variational principle.

2.2 New variational principle

2.2.1 Algebro geometric notations

Consider a 2-sheeted hyperelliptical Riemann surface. Its complex structure is determined by the location of its branch points a_α, $\alpha = 1, \dots, 2s+2$, as well as a choice of non–intersecting paths joining them, of the form:

$$\mathcal{A}_\alpha = \text{counter} - \text{clockwise contour around } [a_{2_\alpha - 1}, a_{2\alpha}] \qquad , \qquad \mathcal{B}_\alpha = [a_{2_\alpha}, a_{2s+1}] \tag{2-21}$$

so that

$$\mathcal{A}_\alpha \cap \mathcal{B}_\beta = \delta_{\alpha,\beta}. \tag{2-22}$$

Define:

$$\sigma(x) = \prod_{\alpha=1}^{2s+2} (x - a_\alpha) \tag{2-23}$$

Define the "Cauchy kernel":

$$dS(x) = \frac{x^s + P_{s-1}(x)}{\sqrt{\sigma(x)}}\, dx \tag{2-24}$$

where P_{s-1} is the unique polynomial of degree $s-1$, whose s coefficients are uniquely determined by:

$$\forall \alpha = 1, \dots, s, \qquad \int_{\mathcal{A}_\alpha} dS = 0 \tag{2-25}$$

Indeed, this system of equation is linear in the coefficients of P_{s-1} and admits a unique solution[1].

We define:

$$\Lambda(x) = \int_{a_{2s+2}}^{x} dS \tag{2-26}$$

and since $dS \sim \pm\frac{dx}{x}$ at $x \to \infty_\pm$ we may define:

$$\gamma = \lim_{\infty_+} x/\Lambda(x). \tag{2-27}$$

We also define the "fundamental 2nd kind form":

$$B(x,x') = \frac{dxdx'\,(\sqrt{\sigma(x)}+\sqrt{\sigma(x')})^2}{4(x-x')^2\,\sqrt{\sigma(x)}\,\sqrt{\sigma(x')}} + \frac{dxdx'\,P(x,x')}{\sqrt{\sigma(x)}\,\sqrt{\sigma(x')}} = B(x',x) \tag{2-28}$$

where $P(x,x')$ is the unique[2] symmetric polynomial in x and x' of degree $s-1$, determined by:

$$\forall\alpha = 1,\dots,s,\ \forall x, \qquad \int_{x'\in\mathcal{A}_\alpha} B(x,x') = 0 \tag{2-29}$$

We have:

$$dS(x) = -\frac{dx}{2x} + \int_{x'=\infty_-}^{\infty_+} B(x,x') \tag{2-30}$$

The holomorphic forms $du_i(x)$ are defined as:

$$du_i(x) = \frac{1}{2\imath\pi}\oint_{x'\in\mathcal{B}_i} B(x,x') = \frac{L_i(x)\,dx}{\sqrt{\sigma(x)}} \tag{2-31}$$

where $L_i(x)$ is the unique polynomial of degree $\leq s-1$ such that

$$\oint_{x\in\mathcal{A}_i} du_j(x) = \delta_{i,j}. \tag{2-32}$$

2.2.2 The Variational principle

Consider the following functional:

Definition 2.2 *Let $t>0$ and $V'(x) = \sum_{k=1}^{d} t_k x^{k-1}$ be a given potential, and let $s \leq d$ be an integer, and ϵ_i, $i = 1,\dots,s$ be given filling fractions. For any hyperelliptical surface of genus s with branch points $a_1,\dots,a_{2s+2}$, we define:*

$$\mu(\{t_k\},t;\{\epsilon\},\{a_\alpha\}) \quad := \quad -\sum_k \frac{t_k}{k}\,\mathop{\mathrm{Res}}_{\infty_-} x^k dS + \sum_{i=1}^{s}\epsilon_i\oint_{\mathcal{B}_i} dS - 2t\ln\gamma \tag{2-33}$$

[1] The fact that this linear system has a unique solution is a standard result in the theory of Riemann surfaces, see [10, 11]. It can be seen as a consequence of Riemann-Roch theorem.

[2] Again, existence and unicity of such B is a classical result of Riemannian geometry [11].

It is such that the variational principle $d\mu = 0$ is equivalent to loop equations eq. (2-16), i.e. the following theorem:

Theorem 2.1 *The set of equations*

$$\forall\, \alpha = 1, \dots, 2s+2 \ , \qquad \frac{\partial \mu}{\partial a_\alpha} = 0 \tag{2-34}$$

is equivalent to the loop equations eq. (2-16).

proof:

We have the Rauch variational formula [10, 11]:

$$\frac{\partial B(p,q)}{da_\alpha} = \mathop{\mathrm{Res}}_{\zeta \to a_\alpha} \frac{B(p,\zeta)B(q,\zeta)}{dx(\zeta)} \tag{2-35}$$

thus:

$$\frac{\partial dS(p)}{da_\alpha} = \mathop{\mathrm{Res}}_{\zeta \to a_\alpha} \frac{B(p,\zeta)dS(\zeta)}{dx(\zeta)} \tag{2-36}$$

$$\frac{\partial \ln \gamma^2}{da_\alpha} = -\mathop{\mathrm{Res}}_{\zeta \to a_\alpha} \frac{dS(\zeta)dS(\zeta)}{dx(\zeta)} \tag{2-37}$$

where ζ is a local coordinate on the Riemann surface, and residues are taken on the Riemann surface. For instance near a branchpoint a_α, a good coordinate is $\zeta = \sqrt{x - a_\alpha}$. By abuse of notation we identify the point $\zeta(a_\alpha) \equiv a_\alpha$ with its x value $a_\alpha = x(\zeta(a_\alpha))$.

The differential form dx has a zeroe at a_α, as can be seen from the choice of local coordinate $x = a_\alpha + \zeta^2$, for which $dx = 2\zeta\, d\zeta$, which vanishes at $\zeta = 0$.

Thus:

$$\frac{\partial \mu}{da_\alpha} = -\mathop{\mathrm{Res}}_{\zeta \to \zeta(a_\alpha)} \frac{dS(\zeta)}{dx(\zeta)} \left(\sum_k t_k \mathop{\mathrm{Res}}_{p \to \infty_-} x^k(p) B(p,\zeta) - 2i\pi \sum_i \epsilon_i du_i(\zeta) - t dS(\zeta) \right) \tag{2-38}$$

The equation $\frac{\partial \mu}{da_\alpha} = 0$ implies that the differential form $\sum_k t_k \,\mathrm{Res}_{p\to\infty_-} x^k(p)B(p,\zeta) - 2i\pi \sum_i \epsilon_i du_i(\zeta) - t dS(\zeta)$ (which clearly has no poles at the branch points), must vanish at all branch points, and thus is proportional to dx. Let us write it:

$$\omega(p)dx(p) = \sum_k t_k \mathop{\mathrm{Res}}_{q \to \infty_-} x^k(q) B(p,q) - 2i\pi \sum_i \epsilon_i du_i(p) - t dS(p). \tag{2-39}$$

Notice that

$$B(x,x') = \frac{dx\ dx'}{2(x-x')^2} + \frac{1}{\sqrt{\sigma(x)}} \times \text{rational function of } x \tag{2-40}$$

$$dS(x) = \frac{1}{\sqrt{\sigma(x)}} \times \text{rational function of } x \tag{2-41}$$

$$du_i(x) = \frac{1}{\sqrt{\sigma(x)}} \times \text{rational function of } x \tag{2-42}$$

so that:

$$\omega(x) = \frac{V'(x)}{2} + \sqrt{\sigma(x)} \times \text{rational function of } x. \tag{2-43}$$

This implies that $\omega(x)$ is solution of an algebraic equation of the form

$$\omega^2(x) - V'(x)\omega(x) + P(x) = 0 \tag{2-44}$$

where $P(x)$ is some rational function.

Moreover, notice that $\mathrm{Res}_{q\to\infty_-} x^k(q)B(p,q)$ has a pole only when $p \to \infty_-$, i.e. it converges when $p \to \infty_+$ in the first sheet (it diverges in the second sheet), this implies that its contribution to $\omega(x)$ is $O(1/x^2)$ as $x \to \infty$. Similarly, $du_i(x)$ has no pole, so the contribution du_i/dx to ω is $O(1/x^2)$ as $x \to \infty$. The term $dS(p)$ behaves like $\pm dx/x$ at large $p \to \infty_\pm$. All this implies that $P(x)$ has no other pole than $x = \infty$, i.e. it is a polynomial, and $\omega(x) \sim t/x$ at large x.

Moreover we have by definition $\oint_{x'\in\mathcal{A}_i} B(x,x') = 0$, $\oint_{x\in\mathcal{A}_i} dS(x) = 0$, $\oint_{x\in\mathcal{A}_i} du_j(x) = \delta_{i,j}$, so that

$$\oint_{x\in\mathcal{A}_i} \omega(x)dx = -2i\pi\epsilon_i. \tag{2-45}$$

Therefore we have proved that the equations $\partial\mu/\partial a_\alpha = 0$ imply that there exists a function $\omega(x)$ solution of

$$\left\{ \begin{array}{l} \omega(x)^2 - V'(x)\omega(x) + P(x) = 0 \\ \omega(x) \sim_\infty t/x + O(1/x^2) \\ \oint_{\mathcal{A}_i} \omega dx = -2i\pi\epsilon_i \end{array} \right. \tag{2-46}$$

i.e. $\omega(x)$ is a solution to the loop equation eq. (2-16).

Converse:

Now assume that ω is solution to loop equations, then it is of the form

$$\omega = \frac{V'(x)}{2} + \sqrt{\sigma(x)} \times \text{polynomial of } x. \tag{2-47}$$

One thus sees that

$$r(x) = \omega(p)dx(p) - \sum_k t_k \mathop{\mathrm{Res}}_{q\to\infty_-} x^k(q)B(p,q) + 2i\pi\sum_i \epsilon_i du_i(p) + t dS(p). \tag{2-48}$$

is a meromorphic differential form on the Riemann surface of the form $C(x)/\sqrt{\sigma(x)}\,dx$ where $C(x)$ is some polynomial of x. It is easy to see that this polynomial $C(x)$ must behave at most like $O(x^{s-1})$ so that

$$C(x) = \sum_i c_i\, L_i(x), \tag{2-49}$$

i.e.

$$r(x) = \sum_i c_i du_i(x) \tag{2-50}$$

and one has $\oint_{\mathcal{A}_\alpha} r(x) = 0$ so that $c_i = 0$, and thus

$$r(x) = 0. \tag{2-51}$$

This implies that

$$\frac{\partial \mu}{da_\alpha} = -\operatorname*{Res}_{\zeta \to \zeta(a_\alpha)} \frac{dS(\zeta)}{dx(\zeta)} (\omega(\zeta) dx(\zeta)) = -\operatorname*{Res}_{\zeta \to \zeta(a_\alpha)} dS(\zeta) \omega(\zeta) = 0 \tag{2-52}$$

since there is no pole at $\zeta(a_\alpha)$.

This proves the theorem.

□

2.3 Example: 1-cut case, $s = 0$

The previous variational problem can be further simplified in the genus zero case (1 cut, $s = 0$). For any α and γ, consider the function $x : \mathbb{C}^* \to \mathbb{C}$ defined as:

$$x(p) = \alpha + \gamma \left(p + \frac{1}{p} \right) \tag{2-53}$$

and consider the function:

$$\mu(\{t_i\}, t; \alpha, \gamma) = \operatorname*{Res}_{p \to \infty} V(x(p)) \frac{dp}{p} - 2t \ln \gamma \tag{2-54}$$

We have

$$\frac{\partial \mu}{\partial \alpha} = \operatorname*{Res}_{p \to \infty} V'(x(p)) \frac{dp}{p} \tag{2-55}$$

$$\frac{\partial \mu}{\partial \gamma} = \operatorname*{Res}_{p \to \infty} V'(x(p)) \left(p + \frac{1}{p} \right) \frac{dp}{p} - \frac{2t}{\gamma} \tag{2-56}$$

Let us write:

$$V'(x(p)) = \sum_{k=0}^{\deg V'} u_k (p^k + p^{-k}) \tag{2-57}$$

The equations $\partial \mu / \partial \alpha = 0$ and $\partial \mu / \partial \gamma = 0$ imply:

$$u_0 = 0 \qquad , \qquad u_1 = \frac{t}{\gamma} \tag{2-58}$$

Then, the function:

$$\omega(p) := \sum_{k=1}^{p} u_k p^{-k} \tag{2-59}$$

is such that

$$V'(x(p)) - \omega(p) = \sum_{k=1}^{p} u_k p^k \tag{2-60}$$

and thus

$$(V'(x(p)) - \omega(p))\omega(p) \tag{2-61}$$

is a polynomial of p and $1/p$ which is symmetric when $p \to 1/p$, i.e. it is a polynomial of $p + 1/p$, and so can be written as a polynomial of $x(p)$:

$$(V'(x(p)) - \omega(p))\omega(p) = P(x(p)). \tag{2-62}$$

Moreover the condition $u_1 = t/\gamma$ implies that at $p \to \infty$ one has

$$\omega(p) \sim t/x(p) + O(1/x(p)^2). \tag{2-63}$$

I.e. we get the loop equations of the 1-matrix model.

2.4 Link with the free energy

The free energy is the limit

$$F_0 = \lim_{N\to\infty} \frac{t^2}{N^2} \ln Z \tag{2-64}$$

where Z is the partition function eq.(2-2). It is well known [4] that it is worth

$$F_0 = \frac{1}{2}\Big(\mathop{\mathrm{Res}}_{p\to\infty_+} V(x(p))\, \omega(p) + t\mu^* + \sum_\alpha \epsilon_\alpha \oint_{\mathcal{B}_\alpha} \omega \Big) \tag{2-65}$$

where ω is the solution of loop equations, and μ^* is the value of the functional μ at its extremum. It is also well known that:

$$\frac{\partial F_0}{\partial t} = \mu^*. \tag{2-66}$$

so that μ^* is the value of the derivative of the free energy with respect to t. It can be called the "entropy".

When the eigenvalues are real and V is real, i.e. when ω is the Stieljes transform of a positive measure $d\rho$ on $\mathbb{R}$, extremum of $\mathcal{S}[d\rho]$ it is known that we have

$$F_0 = -\,\mathcal{S}[d\rho^*]. \tag{2-67}$$

2.4.1 Extremal filling fractions

Often the filling fractions ϵ_α are not fixed, and one determines the filling fractions by requiring:

$$\frac{\partial \mathrm{Re}\, F_0}{\partial \epsilon_\alpha} = 0 \tag{2-68}$$

i.e.

$$\mathrm{Re} \oint_{\mathcal{B}_\alpha} \omega = 0 \tag{2-69}$$

Then notice that if $\epsilon_\alpha \in \mathbb{R}$ one has

$$\mathrm{Re} \oint_{\mathcal{A}_\alpha} \omega = \mathrm{Re}\, 2\mathrm{i}\pi\, \epsilon_\alpha = 0 \tag{2-70}$$

and if t is real one has

$$\mathrm{Re} \oint_{\infty\pm} \omega = \pm \mathrm{Re}\, 2\mathrm{i}\pi\, t = 0 \tag{2-71}$$

This implies that for any closed cycle C on the Riemann surface one has

$$\mathrm{Re} \oint_C \omega = 0 \tag{2-72}$$

This is the "Boutroux property".

Definition 2.3 *An algebraic curve has the Boutroux property, iff there exists a one-form ω, such that for all closed contour C one has*

$$\mathrm{Re} \oint_C \omega = 0. \tag{2-73}$$

In this case, the primitive $h(x) = \mathrm{Re} \int_{.}^{x} \omega$, is a harmonic function globally defined on the algebraic curve (indeed the value of h is independent of the choice of integration contour).

An important property of F_0 is that:

$$\frac{\partial^2 F_0}{\partial \epsilon_\alpha \partial \epsilon_\beta} = 2\mathrm{i}\pi \oint_{\mathcal{B}_\alpha} du_\beta := 2\mathrm{i}\pi\, \tau_{\alpha,\beta} \tag{2-74}$$

and the $s \times s$ matrix τ, called the Riemann matrix of periods, has the well known property [10, 11] that:

$$\tau = \tau^t \qquad , \qquad \mathrm{Im}\, \tau > 0. \tag{2-75}$$

Since the imaginary part is positive definite, we have that:

$$\mathrm{Re}\, \frac{\partial^2 F_0}{\partial \epsilon_\alpha \partial \epsilon_\beta} = -2\pi \mathrm{Im}\, \tau < 0 \tag{2-76}$$

i.e. $\mathrm{Re}\, F_0$ is a concave[3] function of filling fractions, and thus it has a unique maximum.

So, in case the filing fractions were not fixed at the beginning, they are chosen as the ones which maximize $\mathrm{Re}\, F_0$.

[3] Here we have a concave function because we defined $Z = \mathrm{e}^F$ instead of the usual Gibbs convention $Z = \mathrm{e}^{-\mathcal{F}}$ with which $\mathcal{F} = -F$ is convex.

3 The 2 matrix model

A similar variational principle can be found for the loop equations of the 2-matrix model [13].

3.1 Introduction 2-matrix model

Consider two random hermitian matrices (or two random normal matrices with eigenvalues on some contours) M_1, M_2 of size N, with probability law:

$$\frac{1}{Z}\,\mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\,(V_1(M_1)+V_2(M-2)-M_1 M_2)}\,dM_1\,dM_2 \tag{3-1}$$

where $V_1(x) = \sum_k \frac{t_k}{k} x^k$, and$V_2(y) = \sum_k \frac{\tilde{t}_k}{k} y^k$ are polynomials called the potentials, and $t > 0$ is often called "temperature", and Z is the partition function:

$$Z = \int_{H_N \times H_N} \mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\,(V_1(M_1)+V_2(M-2)-M_1 M_2)}\,dM_1\,dM_2. \tag{3-2}$$

The expectation value of the resolvent of matrix M_1:

$$W(x) = \frac{t}{N}\,\mathbb{E}\left(\mathrm{Tr}\,\,(x - M_1)^{-1}\right) \tag{3-3}$$

plays an important role, indeed it encodes the information on the spectrum of M_1.

In many cases (depending on the choice of potentials V_1, V_2, and on the choices of contours), it is known or conjectured (see [4] for instance), that $W(x)$ has a large N limit, which we write:

$$\exists \lim_{N\to\infty} V_1'(x) - W(x) = \omega(x) \tag{3-4}$$

and in many cases (again depending on the choice of potentials V_1, V_2 and contours), it is an algebraic function of x, i.e. it satisfies an algebraic equation [13, 14, 17, 5]:

$$P(x, \omega(x)) = 0 \qquad , \; P(x,y) = \sum_{i,j} P_{i,j} x^i\, y^j. \tag{3-5}$$

For the 2-matrix model, the polynomial $P(x, y)$ is in general not quadratic in y, instead it takes the form [5]:

$$P(x,y) = (y - V_1'(x))\,(x - V_2'(y)) + Q(x,y) \tag{3-6}$$

where $Q(x, y)$ is a polynomial such that:

$$\deg_x Q < \deg V_1' \qquad , \quad \deg_y Q < \deg V_2'. \tag{3-7}$$

3.1.1 Some algebraic geometry

The equation $P(x, y) = 0$ is an algebraic equation, it defines a compact Riemann surface $\mathcal{C}$. This Riemann surface has a certain genus $\mathfrak{g}$.

3.1.2 Filling fractions

Let us define for $\alpha = 1, \dots, \mathfrak{g}$, a basis of $2\mathfrak{g}$ non–contractible cycles on $\mathcal{C}$:

$$\mathcal{A}_{\alpha=1,\dots,\mathfrak{g}} \qquad , \qquad \mathcal{B}_{\alpha=1,\dots,\mathfrak{g}}, \tag{3-8}$$

with canonical symplectic intersections

$$\mathcal{A}_\alpha \cap \mathcal{B}_\beta = \delta_{\alpha,\beta} \qquad , \qquad \mathcal{A}_\alpha \cap \mathcal{A}_\beta = \emptyset \qquad , \qquad \mathcal{B}_\alpha \cap \mathcal{B}_\beta = \emptyset. \tag{3-9}$$

Such a canonical basis always exists but is not unique.

Very often, it is interesting to consider matrix models with "fixed filling fractions", i.e. where the number of eigenvalues of M_1 or M_2 in a certain region of the complex plane is held fixed. The number n_α of eigenvalues of M_1 enclosed by a clockwise contour C_α is:

$$n_\alpha = -\frac{N}{2i\pi\, t} \oint_{C_\alpha} W(x)\, dx \tag{3-10}$$

In the large N limit, the fixed filling fraction condition amounts to fix:

$$\frac{t\, n_\alpha}{N} = \frac{1}{2i\pi} \oint_{\mathcal{A}_\alpha} \omega(x)\, dx = \epsilon_\alpha \tag{3-11}$$

The numbers ϵ_α are called "filling fractions", they tell the number (times t/N) of eigenvalues of M_1 which concentrate in regions enclosed by the $\mathcal{A}_\alpha$'s.

3.1.3 Loop equations

Our goal now is to find the polynomial $Q(x, y)$.

It is well known [5, 14] that this polynomial can be determined by the following equations:

Definition 3.1 (Loop equations) *The loop equations of the 2-matrix model with potentials V_1, V_2 and with filling fractions ϵ_α is the following set of equations [5, 14, 6]:*

$$\boxed{\left\{\begin{array}{l} \exists\, \text{polynomial}\, Q(x,y) \;\; \text{such that} \;\; (\omega(x) - V_1'(x))\,(x - V_2'(\omega(x))) + Q(x, \omega(x)) = 0 \\[2ex] \omega(x) \sim_{\infty_+} V_1'(x) - t/x + O(1/x^2) \\[2ex] x \sim_{\infty_-} V_2'(\omega(x)) - t/\omega(x) + O(1/\omega(x)^2) \\[2ex] \forall\, \alpha = 1, \dots, \mathfrak{g}\,, \qquad -\frac{1}{2i\pi} \oint_{\mathcal{A}_\alpha} \omega(x)\, dx = \epsilon_\alpha \end{array}\right.} \tag{3-12}$$

Let us check that this system implies as many equations as unknowns. The 2 equations regarding the behaviors at $\infty_\pm$ imply that $\deg_x Q < \deg V_1'$ and $\deg_y Q < \deg V_2'$, and they also imply that the leading term (largest power of both x and y) is of the form:

$$Q(x,y) \sim t\, \frac{V_1'(x)\, V_2'(y)}{xy}. \tag{3-13}$$

This implies that the number of unknown coefficients of $Q(x,y)$ is $\deg V_1' \times \deg V_2' - 1$, which is also[4] the genus $\mathfrak{g}$ of the Riemann surface of equation $P(x,y)=0$. Therefore the number of unknown coefficients of $Q(x,y)$ matches the number of filling fraction conditions.

Our goal is not to study those equations, in particular their number of solutions (existence or unicity questions), as there already is a large literature about them, but to show that the same set of equations eq. (3-12) can be obtained from a variational principle.

3.2 Algebro-geometric notations

Let $\mathcal{C}$ be a compact Riemann surface of genus $\mathfrak{g}$, defined by an algebraic equation $P(x,y)=0$.

This means that every point $p\in\mathcal{C}$ corresponds to a point $(x(p),y(p))\in\mathbb{C}^2$ such that $P(x(p),y(p))=0$. In other words there exists two analytical meromorphic functions $x:\mathcal{C}\to\mathbb{C}$, $y:\mathcal{C}\to\mathbb{C}$

$$\left\{\begin{array}{ll} x: & \mathcal{C}\to\mathbb{C} \\ & p\mapsto x(p) \end{array}\right. \qquad , \qquad \left\{\begin{array}{ll} y: & \mathcal{C}\to\mathbb{C} \\ & p\mapsto y(p) \end{array}\right. \tag{3-14}$$

such that

$$\{(x,y)\in\mathbb{C}^2 \mid P(x,y)=0\} \quad\equiv\quad \{(x(p),y(p))\,|\,p\in\mathcal{C}\}. \tag{3-15}$$

3.2.1 Branchpoints

We define branchpoints as the zeroes of the differential dx on $\mathcal{C}$:

$$dx(e_\alpha)=0. \tag{3-16}$$

Their x–projection is denoted:

$$a_\alpha = x(e_\alpha). \tag{3-17}$$

We assume that, generically, those zeroes are simple zeroes, i.e. a good local coordinate on $\mathcal{C}$ near e_α is:

$$\zeta=\sqrt{x-a_\alpha} \qquad , \qquad x=a_\alpha+\zeta^2 \quad , dx = 2\zeta d\zeta. \tag{3-18}$$

[4]classical result of algebraic geometry, the genus is the number of interior points of the Newton's polygon. And here the Newton's polygon has $\deg V_1' \times \deg V_2' - 1$ interior points.

3.2.2 Holomorphic forms

There exists [11, 10]) a unique basis of holomorphic forms $du_i(p)$ on $\mathcal{C}$ normalized on $\mathcal{A}$-cycles such that:

$$\oint_{\mathcal{A}_i} du_j(p) = \delta_{i,j} \qquad , \qquad i,j = 1,\dots,\mathfrak{g}. \tag{3-19}$$

One can always write:

$$du_i(p) = \frac{R_i(x(p),y(p))\ dx(p)}{P'_y(x(p),y(p))} \tag{3-20}$$

where $R_i(x,y) \in \mathbb{C}[x,y]$ is the unique polynomial of degree $\deg_x R_i < \deg V'_1$ and $\deg_y R_i < \deg V'_2$, chosen such that $du_i(p)$ has no pole on $\mathcal{C}$ and $\oint_{\mathcal{A}_i} du_j(p) = \delta_{i,j}$.

3.2.3 2nd kind form

Similarly, there exists a unique symmetric bi–differential form $B(x,y) \in T^*(\mathcal{C})\otimes T^*(\mathcal{C})$, having a double pole on the diagonal, and no other pole, and normalized on $\mathcal{A}$-cycles:

$$B(p,p') \underset{p\to p'}{\sim} \frac{d\zeta(p)\otimes d\zeta(p')}{(\zeta(p)-\zeta(p'))^2} + \text{analytical at}\, p=p' \tag{3-21}$$

$$\forall\, i=1,\dots,\mathfrak{g}\,,\ \forall\, p\in\mathcal{C} \qquad \oint_{p'\in\mathcal{A}_i} B(p,p') = 0 \tag{3-22}$$

$B(p,p')$ is called the "fundamental form of the second kind" or (derivative of) "Green–function" or "heat kernel" on $\mathcal{C}$.

It has the property [11] that:

$$\oint_{p'\in\mathcal{B}_i} B(p,p') = 2\mathrm{i}\pi du_i(p). \tag{3-23}$$

We also define the 3-rd kind differential:

$$dS(p) = \int_{p'=\infty_-}^{\infty_+} B(p,p') \tag{3-24}$$

where the integration path is chosen[5] such that it doesn't intersect any $\mathcal{A}$-cycle or $\mathcal{B}$-cycle.

Then, let p_0 be an arbitrary basepoint and define

$$\Lambda(p) = \exp\int_{p_0}^{p} dS \tag{3-25}$$

where again the integration contour avoids $\mathcal{A}$-cycles and $\mathcal{B}$-cycles. Let

$$\gamma = \lim_{p\to\infty_+} x(p)^{1/\deg_{\infty_+}(x)}/\Lambda(p) \tag{3-26}$$

$$\tilde\gamma = \lim_{p\to\infty_-} \Lambda(p)/\, y(p)^{1/\deg_{\infty_-}(y)} \tag{3-27}$$

Notice that the product $\gamma\tilde\gamma$ is independent of the choice of p_0.

[5] Notice that $\mathcal{C}\setminus\cup_\alpha\mathcal{A}_\alpha\cup_\alpha\mathcal{B}_\alpha$ is simply connected, and thus dS is well defined.

3.3 The variational principle

Definition 3.2 *Consider the following functional:*

$$\begin{aligned}\mu(\{t_k\},\{\tilde{t}_k\},t;(\mathcal{C},x,y)) \quad := \quad & \sum_k t_k \mathop{\mathrm{Res}}_{p\to\infty_+} x(p)^k dS(p) - \sum_k \tilde{t}_k \mathop{\mathrm{Res}}_{p\to\infty_-} y^k dS(p) \\ & -c \mathop{\mathrm{Res}}_{p\to\infty_+} x(p)y(p)\, dS(p) + \sum_i \epsilon_i \oint_{\mathcal{B}_i} dS(p) - t\,\ln\gamma\tilde{\gamma}\end{aligned} \qquad (3-28)$$

where $(\mathcal{C},x,y)$ is a compact Riemann surface of genus $\mathfrak{g}$ with 2 distinct marked points called ∞_+ and ∞_-, and x and y any two meromorphic functions on $\mathcal{C}\to\mathbb{P}^1$.

It is such that an extremum of μ, i.e. $d\mu=0$ is a solution of the loop equation eq. (3-12).

Theorem 3.1 *The set of equations (differential with respect to variations of $(\mathcal{C},x,y)$)*

$$d\mu = 0 \qquad (3\text{-}29)$$

is equivalent to the loop equations eq. (3-12).

proof:

Let $(\mathcal{C},x,y)$ be a compact Riemann surface of genus $\mathfrak{g}$, with 2 marked points $\infty_\pm$, and x and y any two meromorphic functions on $\mathcal{C}\to\mathbb{P}^1$.

The tangent (infinitesimal variations) of the moduli space of $(\mathcal{C},x,y)$ is isomorphic to the space of meromorphic forms on $\mathcal{C}$. Notice that one can vary at the same time the complex structure of $\mathcal{C}$, as well as the functions x and y.

Let δ denote a tangent direction, i.e.

$$\delta(y)dx - \delta(x)dy = \Omega \qquad (3\text{-}30)$$

Ω is a meromorphic form.

The Rauch variational formula gives:

$$\delta\ B(p,q)|_{x(p),x(q)} = \sum_\alpha \mathop{\mathrm{Res}}_{s\to e_\alpha} \frac{B(p,s)B(q,s)\Omega(s)}{dx(s)dy(s)} \qquad (3\text{-}31)$$

thus:

$$\delta\ dS(p)|_{x(p)} = \sum_\alpha \mathop{\mathrm{Res}}_{s\to e_\alpha} \frac{B(p,s)dS(s)\Omega(s)}{dx(s)dy(s)} \qquad (3\text{-}32)$$

$$\delta\ \ln\Lambda(p)|_{x(p)} = \sum_\alpha \mathop{\mathrm{Res}}_{s\to e_\alpha} \frac{dE_p(s)dS(s)\Omega(s)}{dx(s)dy(s)} \qquad (3\text{-}33)$$

$$\delta\ \ln\gamma = -\mathop{\mathrm{Res}}_{s\to e_\alpha} \frac{dS_{\infty_x,o}(s)dS(s)\Omega(s)}{dx(s)dy(s)} \qquad (3\text{-}34)$$

By the chain rule we have:

$$\delta\ dS(p)|_{y(p)} \quad = \quad \delta\ dS(p)|_{x(p)} - d\left(\frac{\Omega(p)dS(p)}{dx(p)dy(p)}\right) \tag{3-35}$$

$$\delta\ dS(p)|_{y(p)} = \mathop{\mathrm{Res}}_{s\to e_\alpha} \frac{B(p,s)dS(s)\Omega(s)}{dx(s)dy(s)} \tag{3-36}$$

$$\delta\ \ln\Lambda(p)|_{y(p)} = \mathop{\mathrm{Res}}_{s\to e_\alpha} \frac{dE_p(s)dS(s)\Omega(s)}{dx(s)dy(s)} \tag{3-37}$$

$$\delta\ \ln\tilde{\gamma} = \mathop{\mathrm{Res}}_{s\to e_\alpha} \frac{dS_{\infty_y,o}(s)dS(s)\Omega(s)}{dx(s)dy(s)} \tag{3-38}$$

and

$$\delta\ \ln(\gamma\tilde{\gamma}) = -\mathop{\mathrm{Res}}_{s\to e_\alpha} \frac{dS(s)dS(s)\Omega(s)}{dx(s)dy(s)} \tag{3-39}$$

Thus:

$$\begin{aligned}\delta\mu \quad = \quad & \mathop{\mathrm{Res}}_{e} \frac{\Omega dS}{dxdy}(-cydx + \sum t_k \mathop{\mathrm{Res}}_{\infty_+} x^k B - \sum \tilde{t}_k \mathop{\mathrm{Res}}_{\infty_y} y^k B + c \mathop{\mathrm{Res}}_{\infty_y} xyB \\ & + \sum \epsilon_i du_i + tdS)\end{aligned} \tag{3-40}$$

$\delta\mu = 0$ for any meromorphic 1-form Ω implies that

$$cydx = \sum t_k \mathop{\mathrm{Res}}_{\infty_x} x^k B - \sum \tilde{t}_k \mathop{\mathrm{Res}}_{\infty_y} y^k B + c \mathop{\mathrm{Res}}_{\infty_y} xyB + \sum \epsilon_i du_i + tdS \tag{3-41}$$

This expression of ydx implies that near ∞_+ one has

$$cy \sim V_1'(x) - \frac{t}{x} + O(1/x^2) \tag{3-42}$$

Doing the same computation with fixed y instead of fixed x yields:

$$cxdy = \sum \tilde{t}_k \mathop{\mathrm{Res}}_{\infty_-} y^k B - \sum t_k \mathop{\mathrm{Res}}_{\infty_+} x^k B + c \mathop{\mathrm{Res}}_{\infty_+} xyB - \sum \epsilon_i du_i - tdS \tag{3-43}$$

which gives that near ∞_- one has

$$cx \sim V_2'(y) - \frac{t}{y} + O(1/y^2) \tag{3-44}$$

and moreover

$$\oint_{\mathcal{A}_i} ydx = 2i\pi\,\epsilon_i. \tag{3-45}$$

The reverse proposition is obvious, this concludes the proof.

□

3.4 Example: Genus zero curves

Genus 0 curves can be parametrized by rational functions. Consider $(\mathcal{C}, x, y)$ where $\mathcal{C}$ is a genus zero curve with 2 marked points, i.e. it is the Riemann sphere $\mathbb{P}^1$, and we can chose the 2 marked points to be $\infty_+ = \infty$ and $\infty_- = 0$, and x and y are 2 rational functions. Let us assume that x has a simple pole at $p = \infty$ and an arbitrary pole at $p = 0$, and y has a simple pole at $p = 0$ and an arbitrary pole at $p = \infty$:

$$\begin{aligned} x(p) &= \sum_{k=-1}^{d_2} \alpha_k p^{-k} \\ y(p) &= \sum_{k=-1}^{d_1} \beta_k p^k \end{aligned} \tag{3-46}$$

Consider the following function:

$$\mu(\{t_i\}, \{\tilde{t}_i\}, c, t; \{\alpha_k\}, \{\beta_k\}) \tag{3-47}$$

$$\begin{aligned} \mu \qquad := \qquad & \sum_k t_k \operatorname*{Res}_{\infty} x(p)^k \frac{dp}{p} + \sum_k \tilde{t}_k \operatorname*{Res}_{\infty} y(p)^k \frac{dp}{p} \\ & -c \operatorname*{Res}_{\infty} x(p) y(p) \frac{dp}{p} - t \ln(\alpha_{-1}\beta_{-1}) \\ (3-48) \end{aligned}$$

We have:

$$\frac{\partial \mu}{\partial \alpha_j} = \operatorname*{Res}_{\infty} (\sum_k k t_k x(p)^{k-1} - c y(p)) \frac{p^{-j} dp}{p} - t \frac{\delta_{j,-1}}{\alpha_{-1}} \tag{3-49}$$

$\frac{\partial \mu}{\partial \alpha_j} = 0$ implies:

$$\forall\, j = -1, \dots, d_2 \quad \frac{\partial \mu}{\partial \alpha_j} = 0 \quad \longrightarrow \quad c y(p) = \sum_k k t_k x(p)^{k-1} - \frac{t}{x(p)} + O(1/p^2) \tag{3-50}$$

and similarly with the β_j's

$$\forall\, j = -1, \dots, d_1 \quad \frac{\partial \mu}{\partial \beta_j} = 0 \quad \longrightarrow \quad c x(p) = \sum_k k \tilde{t}_k y(p)^{k-1} - \frac{t}{y(p)} + O(p^2) \tag{3-51}$$

i.e. we obtain the loop equations, for instance as written in [7].

4 Generalization: algebraic plane curve with fixed behaviors at poles

The 1-matrix and 2-matrix loop equations are special cases of the following problem (related to the Witham hierarchy [15, 1]):

Problem: Let $\mathfrak{g}$, m, $\{t_{k,j}\}_{k=1,\dots,m,\, j=1,\dots,d_k}$, $\{\epsilon_i\}_{i=1,\dots,\mathfrak{g}}$, $\{X_j\}_{j=1,\dots,m}$ be given.

Find $(\mathcal{C}, x, y)$ where $\mathcal{C}$ is a compact Riemann surface of genus $\mathfrak{g}$, with m marked points $\{\infty_k\}_{k=1,\dots,m}$, and with $2\mathfrak{g}$ closed cycles whose homology class form a symplectic basis of cycles $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$, and x and y are 2 meromorphic functions on $\mathcal{C}$, such that:

- y and x are holomorphic on $\mathcal{C} \setminus \{\infty_k\}_{k=1,\dots,m}$,
-

$$\forall\, k = 1, \dots, m \qquad x(\infty_k) = X_k. \tag{4-1}$$

If $X_k = \infty$ we define the local coordinate $\zeta_k(p) = x(p)^{-1/\deg_{\infty_k}(x)}$, and if $X_k \neq \infty$ we define $\zeta_k(p) = x(p) - X_k$.

- the 1-form ydx has a prescribed negative part of its Laurent series expansion near ∞_k:

$$y(p)dx(p) \underset{\infty_k}{\sim} \sum_{j=0}^{d_k} t_{k,j}\, \zeta_k(p)^{-j-1}\, d\zeta_k(p) + \text{analytical at}\, \infty_k \tag{4-2}$$

- one has prescribed filling fractions

$$\frac{1}{2i\pi} \oint_{\mathcal{A}_i} ydx = \epsilon_i. \tag{4-3}$$

Here we shall not consider the question of existence and/or unicity of a solution. We just mention that a necessary condition for a solution to exist is that the sum of residues of a meromorphic form vanishes i.e.

$$\sum_k t_{k,0} = 0. \tag{4-4}$$

From now on, we assume that this condition is fulfilled, and we shall merely reformulate the question as a variational principle.

4.1 Variational principle

Definition 4.1 *Let $(\mathcal{C}, x)$ be a Hurwitz space, where $\mathcal{C}$ is a Riemann surface of genus $\mathfrak{g}$, with marked points ∞_k, and with a given symplectic basis of cycles $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$, and x is a meromorphic function on $\mathcal{C}$, used as a projection on the base Riemann sphere: $x : \mathcal{C} \to \overline{\mathbb{C}}$.*

We define $\forall\, i, i'$ any two distinct $\infty_i \neq \infty_{i'}$:

$$\begin{aligned}\mu_{i,i'}(\{t_{k,j}\}; (\mathcal{C}, x, y)) \qquad = \qquad & \sum_k \mathop{\text{Res}}_{p \to \infty_k} \sum_{j=1}^{d_k} \frac{t_{k,j}}{j}\, \zeta_j(p)^{-j}\, dS_{\infty_i, \infty_{i'}}(p) \\ & + \sum_k t_{k,0} \ln \gamma_k + \sum_\alpha \epsilon_\alpha \oint_{p \in \mathcal{B}_\alpha} dS_{\infty_i, \infty_{i'}}(p)\end{aligned} \tag{4-5}$$

where

$$dS_{\infty_i,\infty_{i'}}(p) = \int_{\infty_i}^{\infty_{i'}} B(.,p) \tag{4-6}$$

and if o is an arbitrary generic point of $\mathcal{C}$

$$\gamma_k = \frac{E(\infty_i,\infty_k)\, E(\infty_{i'},o)}{E(\infty_{i'},\infty_k)\, E(\infty_i,o)}. \tag{4-7}$$

notice that since $\sum_k t_{k,0} = 0$, we have that $\sum_k t_{k,0} \ln \gamma_k$ is independent of the choice of $o \in \mathcal{C}$.

Theorem 4.1 *For any i, i', let $\mu = \mu_{i,i'}$, then a solution of $d\mu = 0$ is a solution to the problem above.*

proof:

The tangent space to the moduli space of $(\mathcal{C}, x, y)$, is the space of meromorphic forms Ω on $\mathcal{C}$ such that:

$$\delta y\, dx - x \delta y = \Omega \tag{4-8}$$

Moreover, if we consider that x and y have poles only at the ∞_k's, we require that Ω can have poles only at the ∞_k's.

As before, we use Rauch formula and get:

$$\begin{aligned} \frac{\partial \mu}{\partial a_\alpha} &= \operatorname*{Res}_{p\to e_\alpha} \frac{dS_{\infty_i,\infty_{i'}}(p)}{dx(p)} \Big(\sum_k \sum_{j\geq 1} \frac{t_{k,j}}{j} \operatorname*{Res}_{q\to\infty_k} B(p,q)\zeta_k(q)^{-j} \\ &\quad + \sum_k t_{k,0} dS_{\infty_k,o}(p) + 2\mathrm{i}\pi \sum_\alpha \epsilon_\alpha du_\alpha(p) \Big) \end{aligned} \tag{4-9}$$

(notice again that since $\sum_k t_{k,0} = 0$, then $\sum_k t_{k,0} dS_{\infty_k,o}$ is independent of the choice of $o \in \mathcal{C}$).

Notice that the quantity inside the bracket has no pole at e_α, and thus the fact that the residue vanishes implies that the quantity in the bracket vanishes at e_α, and thus can be divided by dx:

$$\begin{aligned} y &= \frac{1}{dx} \Big(\sum_k \sum_{j\geq 1} \frac{t_{k,j}}{j} \operatorname*{Res}_{q\to\infty_k} B(p,q)\zeta_k(q)^{-j} \\ &\quad + \sum_k t_{k,0} dS_{\infty_k,o}(p) + 2\mathrm{i}\pi \sum_\alpha \epsilon_\alpha du_\alpha(p) \Big) \end{aligned} \tag{4-10}$$

is a meromorphic function with the required Laurent series behavior near poles and filling fractions, it is thus a solution to the problem.

□

5 Conclusion

We have seen that the loop equations of various matrix models, which consist in finding a plane curve with prescribed asymptotic behaviors at poles and prescribed filling fractions on $\mathcal{A}$-cycles, are equivalent to a local variational principle.

Contrarily to the energy functional $\mathcal{S}$ or F_0, the functional μ doesn't have convexity properties, so one cannot easily conclude to the existence of a solution of the variational principle. However, the functional μ is in fact easier to compute, and the loop equations easier to derive from μ. Also, the geometric meaning of that μ needs to be understood, in particular the equation eq. (2-66).

In this article we have explicitly considered only the 1 and 2-matrix models, although section 4 guarantees that it also applies to the "chain of matrices" [7, 8] matrix model, and possibly more. Also, we have written the explicit proof for 1 and 2 matrix model only for polynomial potentials, and again section 4 guarantees that the same works for potentials whose derivative is a rational function (called semi-classical potentials [1]), or also for matrix models with hard edges [1, 9].

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