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Abstract

We study U($N|M$) character expectation value with the supermatrix Chern–Simons theory, known as the ABJM matrix model, with emphasis on its connection to the knot invariant. This average just gives the half BPS circular Wilson loop expectation value in ABJM theory, which shall correspond to the unknot invariant. We derive the determinantal formula, which gives U($N|M$) character expectation values in terms of U(1|1) averages for a particular type of character representations. This means that the U(1|1) character expectation value is a building block for all the U($N|M$) averages, and in particular, by an appropriate limit, for the U(N) invariants. In addition to the original model, we introduce another supermatrix model obtained through the symplectic transform, which is motivated by the torus knot Chern–Simons matrix model. We obtain the Rosso–Jones-type formula and the spectral curve for this case.

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1 Introduction

Since it was shown by Witten [1] that a knot invariant is realized by using the Wilson loop operators in Chern–Simons gauge theory, knot theory has been providing various kinds of interesting topics not only for mathematicians, but also for physicists. In particular the most important example of the knot invariant, which is called Jones polynomial, is obtained from Chern–Simons theory with the Wilson loop in a fundamental representation

$$J(K;q) = \Big\langle W_\Box(K;q) \Big\rangle \Big/ \Big\langle W_\Box(\bigcirc;q) \Big\rangle , \tag{1.1}$$

where the Wilson loop in a representation R is given by

$$W_R(K;q) = \operatorname{Tr}_R \mathrm{P} \exp \left(\oint_K A \right) , \tag{1.2}$$

and its expectation value is taken with respect to Chern–Simons theory with SU(2) gauge group on a three-sphere S^3

$$S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{S^3} \operatorname{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) . \tag{1.3}$$

In this case the parameter q is associated with the level of Chern–Simons theory as $q = \exp(2\pi i/(k+2))$. This prescription to derive the knot invariant is quite general: when the fundamental representation $R = \Box$ is replaced by a generic representation, one obtains the colored Jones polynomial, and SU(N) and SO(N)/Sp(N) generalizations provide HOMFLY and Kauffman polynomials, respectively.

The knot polynomial is usually defined by the Skein relation with a proper normalization of the unknot invariant. Although it is in principle computable for any knots based on that definition, their expressions get much complicated as the number of crossings in knots increases, or the representation of the knot polynomial becomes highly involved. For a particular class of knots, the unknot and also torus knots, there is a useful integral representation of the knot invariant [2, 3, 4, 5], which is applicable to a generic gauge group and representations. This is based on the matrix integral formula for the partition function of Chern–Simons theory, especially defined on a three-sphere S^3, and then given as the expectation value of the character in the corresponding group.

In this paper we consider a supergroup character average with the supermatrix Chern–Simons theory, known as the ABJM matrix model [6], towards a supersymmetric generalization of the knot invariant. This supermatrix model is derived from $\mathcal{N} = 6$ superconformal Chern–Simons–matter theory with gauge group $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$, which is so-called ABJM theory [7], by implementing the localization technique for the path integral. In this theory the Wilson loop, in particular the half BPS operator, is described by a holonomy with a superconnection taking a value in U($N|N$), which is written as a supergroup character [8]. Thus it is expressed in terms of the supersymmetric Schur function [9, 10]. In this sense the average we compute here is expected to be an unknot invariant for U($N|N$) theory.

The knot invariant and the character average have an analogous structure to a matrix integral in the presence of external fields, which is dual to a correlation function of characteristic polynomials. Especially in the case of supermatrix models, the corresponding correlation function is that for the characteristic polynomial ratio. There is an interesting determinantal formula for this correlation function, which consists of a single pair correlation function as a kernel [11, 12, 13]. In this way we shall expect that a similar determinantal structure can be found in U($N|N$) theory, and the U($1|1$) expectation value plays a role as the kernel function there. We will show in this paper such a determinantal formula for the average with a particular type of the character representation.

According to the general scheme of the topological recursion [14], one can determine all order perturbation series for the correlation function from the spectral curve. In the case of the knot invariant, there are basically two kinds of perturbative expansions. The first is based on the spectral curve, which is obtained from the A-polynomial, and its expansion is from the large representation limit [15, 16]. Although this A-polynomial was originally introduced to the Jones polynomial, namely SU(2) Chern–Simons theory, it can be now extended to more generic theories. See for example [17]. The other expansion comes from the spectral curve arising in the large rank limit of the knot invariant. This kind of spectral curves is quite analogous to that discussed in random matrix theory, because for some kinds of knots we have matrix integral-like expressions for the knot invariant, and in this case the matrix size N corresponds to the rank of Chern–Simons gauge group $\mathrm{SU}(N)$. This large N limit plays an important role in topological string theory, because it describes the geometric transition of the corresponding Calabi–Yau threefold [18]. Such a duality is also available for the situation even in the presence of a knot [19], which gives a brane on a proper Lagrangian submanifold of the Calabi–Yau threefold. In the sense of topological strings, the corresponding spectral curve provides the mirror Calabi–Yau threefold. In this paper we will discuss the large N spectral curve for the supermatrix Chern–Simons theories.

The supergroup character average discussed in this paper is based on ABJM theory. As pointed out in [20], it is perturbatively equivalent to Chern–Simons theory on the lens space $L(2,1) = S^3/\mathbb{Z}_2 \cong \mathbb{RP}^3$, which is dual to the topological string on the local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry, and the associated spectral curve becomes a genus-one curve. Therefore the unknot spectral curve is just given by the mirror curve of the local $\mathbb{P}^1 \times \mathbb{P}^1$. In the case of the torus knot, the spectral curve is obtained by applying the symplectic transformation of the unknot [5]. As well as the ordinary HOMFLY polynomial for $\mathrm{SU}(N)$ Chern–Simons theory, we will introduce the (P,Q)-deformed supermatrix model through the symplectic transform of the original matrix model, which is motivated by the torus knot Chern–Simons matrix model. Since the Adams operation works well even for the Schur function associated with the supergroup $\mathrm{U}(N|N)$, we can derive the Rosso–Jones formula for the torus knot average. This means that the torus knot character average can be represented as a linear combination of the fractionally framed unknot averages. We will also derive the spectral curve from the saddle point equations of the (P,Q)-deformed supermatrix model, and then obtain a consistent result with the symplectic transform of the unknot curve.

This paper is organized as follows. In Sec. 2 we study the supergroup character average with the ABJM matrix model, which is just given as the half BPS circular Wilson loop operator in ABJM theory. We especially focus on the representation, corresponding to the partition such that the number of its diagonal components is given by N. We will show the determinantal formula factorizing the $\mathrm{U}(N|N)$ character average into the $\mathrm{U}(1|1)$ expectation values. In Sec. 3 we then consider the (P,Q)-deformation of the supermatrix model, which is obtained by the $\mathrm{SL}(2,\mathbb{Z})$ transform of the original ABJM matrix model. We will show that the Rosso–Jones formula holds even for the $\mathrm{U}(N|N)$ theoy, and the $\mathrm{U}(1|1)$ average plays a role of a building block for the (P,Q)-deformed $\mathrm{U}(N|N)$ character expectation value with a particular kind of representations. In Sec. 4 we extend the argument to $\mathrm{U}(N|M)$ theory, and we will see that the determinantal formula for the character average can be derived even in this case. We obtain the expression which interpolates $\mathrm{U}(N)$ and $\mathrm{U}(N|N)$ theories. In Sec. 5 we discuss the spectral curve for the (P,Q)-deformed matrix model. We show that there are two consistent ways of obtaining the spectral curve: One is the symplectic transform of the original matrix model and the other is the saddle point analysis of the (P,Q)-deformed matrix model itself. In Sec. 6 we give some comments on the relations to topological string and random matrix theory. Sec. 7 is devoted to a summary and discussions.

2 Unknot matrix model

Let us start with the matrix model description of Chern–Simons theory at level k defined on a three-sphere S^3. When we take the gauge group $G = \mathrm{U}(N)$, the partition function is given as the matrix model-like integral

$$\mathcal{Z}_{\mathrm{CS}}(S^3;q) = \frac{1}{N!}\int \prod_{i=1}^{N} \frac{dx_i}{2\pi}\, e^{-\frac{1}{2g_s}x_i^2} \prod_{i<j}^{N} \left(2\sinh\frac{x_i - x_j}{2}\right)^2 , \tag{2.1}$$

where the parameter q is related to the coupling constant as $q = \exp g_s$ with

$$g_s = \frac{2\pi i}{k+N} . \tag{2.2}$$

We then consider the expectation value of the circular Wilson loop operator with representation R, which corresponds to the unknot invariant,

$$\left\langle W_R(\bigcirc)\right\rangle = \frac{1}{\mathcal{Z}_{\mathrm{CS}}}\frac{1}{N!}\int \prod_{i=1}^{N} \frac{dx_i}{2\pi}\, e^{-\frac{1}{2g_s}x_i^2} \prod_{i<j}^{N} \left(2\sinh\frac{x_i - x_j}{2}\right)^2 \mathrm{Tr}_R\, U(x) . \tag{2.3}$$

The matrix $U(x)$ is given by

$$U(x) = \begin{pmatrix} e^{x_1} & & \\ & \ddots & \\ & & e^{x_N} \end{pmatrix} , \tag{2.4}$$

and $\mathrm{Tr}_R\, U$ is indeed the character of $G = \mathrm{U}(N)$ in the representation R. This character is written as a Schur polynomial with the corresponding partition λ to the representation R

$$\mathrm{Tr}_R\, U = s_\lambda(e^{x_1}, \cdots, e^{x_N}) . \tag{2.5}$$

Although we can deal with only the unknot Wilson loop based on this matrix model, it is possible to obtain the torus knot invariants by applying a slightly different matrix model, as discussed in Sec. 3.

We now consider a supersymmetric extension of this Chern–Simons theory. In this paper we especially apply the supermatrix generalization of the Chern–Simons matrix model (2.1), namely the ABJM matrix model [8]. It was shown in [6] that ABJM theory [7], which is the three-dimensional superconformal Chern–Simons–matter theory with gauge group $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$, can be similarly reduced to the matrix model-like integral

$$\begin{aligned}\mathcal{Z}_{\mathrm{ABJM}}(S^3;q) \quad &= \quad \frac{1}{N!^2}\int \prod_{i=1}^{N} \frac{dx_i}{2\pi}\frac{dy_i}{2\pi}\, e^{-\frac{1}{2g_s}(x_i^2 - y_i^2)} \prod_{i,j}^{N} \left(2\cosh\frac{x_i - y_j}{2}\right)^{-2} \\ &\times \prod_{i<j}^{N} \left(2\sinh\frac{x_i - x_j}{2}\right)^2 \left(2\sinh\frac{y_i - y_j}{2}\right)^2 . \end{aligned} \tag{2.6}$$

In this case there is no level shift in the coupling constant

$$g_s = \frac{2\pi i}{k} \,. \tag{2.7}$$

We can insert the Wilson loop operator into this matrix model as well as Chern–Simons theory with the classical group. Although there are some possibilities for the operators in this case, the relevant choice to this study is the half BPS circular Wilson loop operator, which is given by a character of the supergroup $\mathrm{U}(N|N)$ [8, 20],

$$\begin{aligned} \Big\langle W_R(\bigcirc)\Big\rangle &= \frac{1}{\mathcal{Z}_{\mathrm{ABJM}}}\frac{1}{N!^2}\int\prod_{i=1}^{N}\frac{dx_i}{2\pi}\frac{dy_i}{2\pi}\,e^{-\frac{1}{2g_s}(x_i^2-y_i^2)}\prod_{i,j}^{N}\left(2\cosh\frac{x_i-y_j}{2}\right)^{-2} \\ &\quad\times\prod_{i<j}^{N}\left(2\sinh\frac{x_i-x_j}{2}\right)^2\left(2\sinh\frac{y_i-y_j}{2}\right)^2\,\mathrm{Str}_R\,U(x;y)\,, \end{aligned} \tag{2.8}$$

where the matrix $U(x;y)$ is of the size $2N\times 2N$

$$U(x;y) = \begin{pmatrix} U(x) & \\ & -U(y) \end{pmatrix} . \tag{2.9}$$

Let us call this expectation value the unknot Wilson loop average as an analogy with the knot invariant. The supergroup character for $\mathrm{U}(N|N)$ is obtained by replacing the power sum polynomial $\mathrm{Tr}\,U^n$ in the $\mathrm{U}(N+N)$ character with the supertrace $\mathrm{Str}\,U^n$ [9].

As well as the $\mathrm{U}(N)$ representation theory, the supergroup character can be also expressed as the Schur polynomial, but with a prescribed symmetry [10]

$$\mathrm{Str}_R\,U(x;y) = s_\lambda(e^x;e^y)\,. \tag{2.10}$$

For $\mathrm{U}(N|N)$ theory, we have a useful determinantal formula in terms of the Frobenius coordinate of the partition $\lambda=(\alpha_1,\cdots,\alpha_{d(\lambda)}|\beta_1,\cdots,\beta_{d(\lambda)})$ with $\alpha_i=\lambda_i-i$ and $\beta_i=\lambda_i^t-i$ [21]

$$s_\lambda(u;v) = \det_{1\le i,j\le d(\lambda)}\left(\sum_{k,l=1}^{N} u_k^{\alpha_i}\left(C^{-1}\right)_{kl} v_l^{\beta_j}\right) , \tag{2.11}$$

where the matrix C^{-1} is the inverse of the Cauchy matrix

$$C = \left(\frac{1}{u_k+v_l}\right)_{1\le k,l\le N} . \tag{2.12}$$

We remark that the supersymmetric Schur polynomial is identically zero when $d(\lambda)>N$, or equivalently $\lambda_{N+1}>N$. The formula (2.11) also implies that it can be written only in terms of the hook representations

$$s_\lambda(u;v) = \det_{1\le i,j\le d(\lambda)} s_{(\alpha_i|\beta_j)}(u;v)\,. \tag{2.13}$$

This is just a supersymmetric version of the Giambelli formula. Actually this relation is useful to study the Wilson loop operators with various representations in ABJM theory [22].

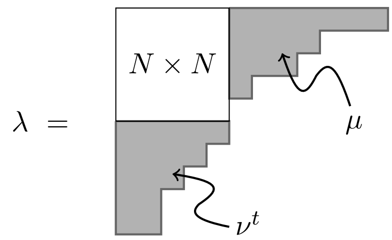

Figure 1: The partition $\lambda = (12,9,8,6,5,5,4,3,2,2)$, which is also represented as $\lambda = (11,7,5,2,0|9,8,5,3,1)$ in the Frobenius coordinate with $d(\lambda) = N (= 5)$. We obtain sub diagrams $\mu = (7,4,3,1,0)$ and $\nu^t = (5,4,3,2,2)$ involved in this partition.

Especially for the most generic situation with $d(\lambda) = N$, on which we focus in this paper, the supersymmetric Schur polynomial is decomposed into the ordinary ones [21]

$$\begin{aligned} s_\lambda(u;v) &= \det_{1\le i,j\le N} u_i^{\alpha_j} \cdot \det_{1\le i,j\le N} v_i^{\beta_j} \Big/ \det_{1\le i,j\le N} C_{ij} \\ &= s_\mu(u)\, s_\nu(v) \prod_{i,j=1}^{N} (u_i + v_j)\,, \end{aligned} \tag{2.14}$$

where the partitions μ and ν are given by

$$\mu_i = \lambda_i - N\,, \qquad \nu_i = \lambda_i^t - N\,, \qquad i = 1, \cdots, N\,, \tag{2.15}$$

or equivalently, $\mu_i^t = \lambda_{i+N}^t$, $\nu_i^t = \lambda_{i+N}$, as shown in Fig. 1. The determinant of the Cauchy matrix is given by

$$\det C = \Delta(u)\Delta(v) \prod_{i,j=1}^{N} (u_i + v_j)^{-1}\,, \tag{2.16}$$

with the Vandermonde determinant

$$\Delta(u) = \prod_{i<j}^{N} (u_i - u_j)\,. \tag{2.17}$$

2.1 U(1|1) theory

Let us consider the simplest example with the supergroup U(1|1), which plays a fundamental role in this study. In this case only the hook representation is possible, which is written as $\lambda = (\alpha|\beta)$ with the Frobenius coordinate. The corresponding Schur function reads

$$s_{(\alpha|\beta)}(e^x; e^y) = (e^x + e^y)\ e^{\alpha x + \beta y}\,. \tag{2.18}$$

The unknot Wilson loop expectation value is given by

$$\Big\langle W_{(\alpha|\beta)}(\bigcirc) \Big\rangle = \frac{1}{\mathcal{Z}_{\rm ABJM}} \int \frac{dx}{2\pi} \frac{dy}{2\pi}\, e^{\frac{ik}{4\pi}(x^2 - y^2)} \left(2\cosh \frac{x-y}{2} \right)^{-2} (e^x + e^y)\, e^{\alpha x + \beta y}\,. \tag{2.19}$$

We can compute this integral explicitly by applying the Fourier transform formula

$$\frac{1}{2\cosh w} = \int \frac{dz}{2\pi} \frac{e^{2iwz/\pi}}{\cosh z} \,. \tag{2.20}$$

Thus we have

$$\int \frac{dx}{2\pi}\frac{dy}{2\pi}\frac{dz}{2\pi}\, \frac{1}{\cosh z}\, e^{\frac{ik}{4\pi}(x^2-y^2)+\left(\alpha+\frac{1}{2}\right)x+\left(\beta+\frac{1}{2}\right)y+\frac{i}{\pi}(x-y)z} = \frac{1}{k} \frac{q^{\frac{1}{2}(\alpha+\beta+1)(\alpha-\beta)}}{q^{\frac{1}{2}(\alpha+\beta+1)}+q^{-\frac{1}{2}(\alpha+\beta+1)}} \,. \tag{2.21}$$

Since the partition function becomes $\mathcal{Z}_{\mathrm{ABJM}} = (4k)^{-1}$ for $N=1$, we obtain the expectation value as follows,

$$\left\langle W_{(\alpha|\beta)}(\bigcirc) \right\rangle = \frac{4\, q^{\frac{1}{2}(\alpha+\beta+1)(\alpha-\beta)}}{q^{\frac{1}{2}(\alpha+\beta+1)}+q^{-\frac{1}{2}(\alpha+\beta+1)}} \,. \tag{2.22}$$

This depends only on the total and relative lengths of the partition, given by $\alpha+\beta+1$ and $\alpha-\beta$, respectively. We will see that the numerator can be seen as the framing factor in the following section.

2.2 U($N|N$) theory

We then consider the character expectation value for U($N|N$) theory, in particular with a representation with $d(\lambda) = N$, $\lambda = (\alpha_1, \cdots, \alpha_N | \beta_1, \cdots, \beta_N)$. Let us first rewrite the partition function (2.6)

$$\mathcal{Z}_{\mathrm{ABJM}}(S^3; q) = \frac{1}{N!^2} \int [dx]^N [dy]^N \, \det \left(\frac{1}{2\cosh \frac{x_i - y_j}{2}} \right)^2 , \tag{2.23}$$

with shorthand notations

$$[dx] = \frac{dx}{2\pi}\, e^{-\frac{1}{g_s}x^2} \,, \qquad [dy] = \frac{dy}{2\pi}\, e^{\frac{1}{g_s}y^2} \,, \tag{2.24}$$

where we have used the Cauchy formula

$$\det_{1\le i,j\le N} \left(\frac{1}{2\cosh\frac{x_i-y_j}{2}} \right) = \prod_{i,j=1}^{N} \left(2\cosh\frac{x_i-y_j}{2} \right)^{-1} \prod_{i<j}^{N} \left(2\sinh\frac{x_i-x_j}{2} \right) \left(2\sinh\frac{y_i-y_j}{2} \right) . \tag{2.25}$$

In this case the Schur function has a simple expression as shown in (2.14). Therefore the unnormalized unknot expectation value is now given by

$$\left\langle W_R(\bigcirc) \right\rangle = \frac{1}{N!^2} \int [dx]^N [dy]^N \, \det \left(\frac{1}{2\cosh\frac{x_i-y_j}{2}} \right) \prod_{i=1}^{N} e^{x_i\xi_i + y_i\eta_i} \,, \tag{2.26}$$

where the parameters ξ_i and η_i, defined as $\xi_i = \lambda_i - i + 1/2 = \alpha_i + 1/2$ and $\eta_i = \lambda^t_i - i + 1/2 = \beta_i + 1/2$, play a similar role to external fields in matrix models, as discussed in Sec. 6. Since all the x_i and y_i are not distinguishable, this integral can be expressed as a size N determinant

$$\frac{1}{N!^2} \det_{1\le i,j\le N} \left[\int \frac{dx}{2\pi}\frac{dy}{2\pi}\, e^{\frac{ik}{4\pi}(x^2-y^2)+x\xi_i+y\eta_j} \left(2\cosh\frac{x-y}{2} \right)^{-1} \right] . \tag{2.27}$$

At this moment the computation is almost reduced to that for U(1|1) theory. Again using the formula (2.20), we obtain the determinantal formula for the character expectation value

$$\left\langle W_R(\bigcirc)\right\rangle = \frac{1}{N!^2\, k^N} \prod_{i=1}^N q^{\frac{1}{2}(\xi_i^2-\eta_i^2)} \det_{1\le i,j\le N} \left(\frac{1}{q^{\frac{1}{2}(\xi_i+\eta_j)} + q^{-\frac{1}{2}(\xi_i+\eta_j)}} \right) . \tag{2.28}$$

This shows that the U($N|N$) unknot character average is factorized into that for U(1|1) theory (2.22), and thus the measure of this matrix integral is *Giambelli compatible* in the sense of [12].

To see that the U($N|N$) theory contains the U(N) knot invariant, it is convenient to rewrite the expression (2.28) by applying the Cauchy formula,

$$\begin{aligned}\left\langle W_R(\bigcirc)\right\rangle = \frac{1}{N!^2\, k^N}\, q^{\frac{1}{2}(C_2(\mu)-C_2(\nu))} \prod_{i,j=1}^N \left(q^{\frac{1}{2}(\alpha_i+\beta_j+1)} + q^{-\frac{1}{2}(\alpha_i+\beta_j+1)} \right)^{-1} \\ \times \prod_{i<j}^N \left(q^{\frac{1}{2}(\alpha_i-\alpha_j)} - q^{-\frac{1}{2}(\alpha_i-\alpha_j)} \right) \left(q^{\frac{1}{2}(\beta_i-\beta_j)} - q^{-\frac{1}{2}(\beta_i-\beta_j)} \right) ,\end{aligned} \tag{2.29}$$

where the 2nd Casimir operator is defined by $C_2(\lambda) = \sum_{i=1}^{\infty} \left(\left(\lambda_i - i + \frac{1}{2} \right)^2 - \left(-i + \frac{1}{2} \right)^2 \right)$. Thus in this case we see that the standard framing factor is given by $q^{\frac{1}{2}(C_2(\mu)-C_2(\nu))}$. Up to the normalization constants, the factors in the second line coincide with the Wilson loop expectation value for U(N) theory, which is given by the quantum dimension of the representation R,

$$\begin{aligned}\left\langle W_R(\bigcirc)\right\rangle_{\mathrm{U}(N)} &= \prod_{i<j}^N \frac{[\lambda_i - \lambda_j - i + j]_q}{[-i+j]_q} \\ &\equiv \dim_q R ,\end{aligned} \tag{2.30}$$

with

$$[x]_q = q^{x/2} - q^{-x/2} . \tag{2.31}$$

The denominator in (2.30) corresponds to the partition function of U(N) Chern–Simons theory

$$\mathcal{Z}_{\mathrm{CS}}(S^3; q) = \frac{e^{\frac{\pi i}{8}N(N-1)}}{N^{\frac{1}{2}}(k+N)^{\frac{N-1}{2}}} \prod_{i<j}^N \left(q^{\frac{1}{2}(-i+j)} - q^{-\frac{1}{2}(-i+j)} \right) . \tag{2.32}$$

Let us now comment on the situation such that the representation of the character obeys $d(\lambda) < N$. In this case we do not obtain a simple determinantal formula, since the Schur function with such a representation is not simply factorized any more. For example, the character expectation value for the hook representation, corresponding to the simplest situation $d(\lambda) = 1$, is given by

$$\left\langle W_{(\alpha|\beta)}(\bigcirc)\right\rangle = \frac{1}{\mathcal{Z}_{\mathrm{ABJM}}} \frac{1}{N!^2} \int [dx]^N [dy]^N \det \tilde{C}^2 \sum_{i,j=1}^N e^{\left(\alpha+\frac{1}{2}\right)x_i + \left(\beta+\frac{1}{2}\right)y_j} \left(\tilde{C}^{-1}\right)_{ij} , \tag{2.33}$$

where we have

$$\tilde{C} = \left(\frac{1}{2\cosh\frac{x_i - y_j}{2}} \right)_{1\le i,j\le N} . \tag{2.34}$$

Although it is difficult to find an explicit formula for this integral, we expect that its asymptotic behavior is obtained from the determinantal formula (2.28) by taking the limit of $\alpha_1, \beta_1 \to \infty$. Let us comment that we can apply the Fermi gas method even to this case. Actually if we consider the grand canonical partition function, it turns out to be written as a Fredholm determinant, due to the Giambelli formula. See, for example, [22].

3 Torus knot matrix model

In addition to the unknot invariant, there is a similar integral formula for the torus knot Wilson loop based on Chern–Simons theory [2, 3, 4, 5],

$$\Big\langle W_R(K_{P,Q}) \Big\rangle = \frac{1}{\mathcal{Z}_{\mathrm{CS}}^{(P,Q)}} \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi} e^{-\frac{1}{2\hat{g}_s}x_i^2} \prod_{i<j}^{N} \left(2\sinh\frac{x_i - x_j}{2P}\right)\left(2\sinh\frac{x_i - x_j}{2Q}\right) \mathrm{Tr}_R U(x) , \tag{3.1}$$

where the coupling constant is now rescaled $\hat{g}_s = PQg_s$, and the corresponding partition function is then given by

$$\mathcal{Z}_{\mathrm{CS}}^{(P,Q)}(S^3; q) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi} e^{-\frac{1}{2\hat{g}_s}x_i^2} \prod_{i<j}^{N} \left(2\sinh\frac{x_i - x_j}{2P}\right)\left(2\sinh\frac{x_i - x_j}{2Q}\right) . \tag{3.2}$$

This is obtained from the ordinary matrix model by applying the $\mathrm{SL}(2,\mathbb{Z})$ transformation [5]. Note that it is also seen as the biorthogonal generalization of the Chern–Simons matrix model [23].

For the torus knot invariant there is a useful formula, which is called Rosso–Jones formula [24]

$$\Big\langle W_R(K_{P,Q}) \Big\rangle = \sum_V c_{R,Q}^V \Big\langle W_V(K_{1,f}) \Big\rangle , \tag{3.3}$$

with $f = P/Q$. This means that the (P,Q) torus knot invariant can be expressed as a linear combination of the fractionally framed unknot invariant. This formula is easily derived from the integral formula (3.1) by using the Adams operation

$$s_\lambda(u^Q) = \sum_\mu c_{\lambda,Q}^\mu \, s_\mu(u) . \tag{3.4}$$

The coefficient $c_{\lambda,Q}^\mu$ can be determined by the Frobenius formula for the Schur and power sum polynomials,

$$s_\lambda = \sum_\mu \frac{1}{z_\mu} \chi_\lambda(C_\mu) \, p_\mu , \qquad p_\mu = \sum_\nu \chi_\nu(C_\mu) \, s_\nu , \tag{3.5}$$

where χ_λ and C_μ are the character and the conjugacy class for the symmetric group, and the coefficient z_μ is given by $z_\mu = \prod_j \mu_j!\, j^{\mu_j}$. Thus we have

$$c^{\mu}_{\lambda,Q} = \sum_{\nu} \frac{1}{z_\nu} \chi_\lambda(C_\nu) \chi_\mu(C_{Q\nu}) \,. \tag{3.6}$$

Since the Rosso–Jones formula (3.3) is obtained from the representation theoretical point of view, it is natural to think that there is a similar formula even for supergroup theories. Actually supersymmetric polynomials obey the same Frobenius formula (3.5) by definition [9]. Therefore we obtain the Adams operation for U$(N|N)$ theory with the same coefficient (3.6),

$$s_\lambda(u^Q; v^Q) = \sum_{\mu} c^{\mu}_{\lambda,Q}\, s_\mu(u; v) \,. \tag{3.7}$$

Following the above discussions, we now introduce a supermatrix version of the torus knot matrix model (3.2)

$$\mathcal{Z}^{(P,Q)}_{\rm ABJM} = \frac{1}{N!^2} \int [dx]^N [dy]^N \det \left(\frac{1}{2\cosh \frac{x_i - y_j}{2P}} \right) \det \left(\frac{1}{2\cosh \frac{x_i - y_j}{2Q}} \right) , \tag{3.8}$$

with the rescaled coupling constant $\hat{g}_s = PQ g_s$.[1] Then we consider the character expectation value with respect to this partition function

$$\Big\langle W_R(K_{P,Q}) \Big\rangle = \frac{1}{\mathcal{Z}^{(P,Q)}_{\rm ABJM}} \frac{1}{N!^2} \int [dx]^N [dy]^N \det \left(\frac{1}{2\cosh \frac{x_i - y_j}{2P}} \right) \det \left(\frac{1}{2\cosh \frac{x_i - y_j}{2Q}} \right) s_\lambda(e^x; e^y) \,. \tag{3.9}$$

Let us call this expectation value the torus knot character average. We can easily show that this (P,Q)-deformed U$(N|N)$ character average also satisfies the Rosso–Jones formula (3.3) by applying the Adams operation for U$(N|N)$ theory (3.7). In this sense it is enough to compute the framed unknot average to obtain the torus knot average. In fact, when we start with a generic partition $\lambda = (\alpha_1, \cdots, \alpha_N | \beta_1, \cdots, \beta_N)$ for a particular torus knot, representations appearing in the expansion (3.3) only provide partitions satisfying $d(\lambda) = N$.[2] Therefore we now focus on the torus knot and framed unknot with a representation with $d(\lambda) = N$. In this case we can show that the framed unknot average has the same expression as (2.29), up to the framing factor,

$$\begin{aligned} \Big\langle W_R(K_{1,f}) \Big\rangle &= \frac{1}{\mathcal{Z}^{(1,1)}_{\rm ABJM}} \frac{1}{N!^2\, k^N}\, q^{\frac{f}{2}(C_2(\mu) - C_2(\nu))} \det_{1 \le i,j \le N} \left(\frac{1}{q^{\frac{1}{2}(\alpha_i + \beta_j + 1)} + q^{-\frac{1}{2}(\alpha_i + \beta_j + 1)}} \right) \\ &= \frac{1}{\mathcal{Z}^{(1,1)}_{\rm ABJM}} \frac{1}{N!^2\, k^N}\, q^{\frac{f}{2}(C_2(\mu) - C_2(\nu))} \prod_{i,j=1}^{N} \left(q^{\frac{1}{2}(\alpha_i + \beta_j + 1)} + q^{-\frac{1}{2}(\alpha_i + \beta_j + 1)} \right)^{-1} \\ &\qquad \times \prod_{i<j}^{N} \left(q^{\frac{1}{2}(\alpha_i - \alpha_j)} - q^{-\frac{1}{2}(\alpha_i - \alpha_j)} \right) \left(q^{\frac{1}{2}(\beta_i - \beta_j)} - q^{-\frac{1}{2}(\beta_i - \beta_j)} \right) . \end{aligned} \tag{3.10}$$

[1] We can derive the so-called mirror description for this partition function, as well as the ordinary ABJM matrix model [25]. It depends on the parameters (P,Q) in a trivial way (A.3). See Appendix A for details.

[2] Although we do not have an explicit proof of this statement, we check it with a number of examples by numerical calculations.

This means that the U(1|1) expectation value plays a role of the building block for the torus knot average at least with thid kind of representations.

For U(1|1) theory we can compute the character expectation value (3.9) explicitly, as well as the unknot average (2.22). In this case we have

$$\left\langle W_R(K_{P,Q}) \right\rangle = \frac{4\, q^{\frac{PQ}{2}(\alpha+\beta+1)(\alpha-\beta)} \left(q^{\frac{PQ}{2}(\alpha+\beta+1)} + q^{-\frac{PQ}{2}(\alpha+\beta+1)} \right)}{\left(q^{\frac{P}{2}(\alpha+\beta+1)} + q^{-\frac{P}{2}(\alpha+\beta+1)} \right) \left(q^{\frac{Q}{2}(\alpha+\beta+1)} + q^{-\frac{Q}{2}(\alpha+\beta+1)} \right)} . \tag{3.11}$$

Then we obtain "U(1|1) knot invariant" for the torus knot from this expectation value by implementing the normalization with the unknot contribution (2.22), as shown in (1.1). Removing the framing factor, we have

$$J_R(K_{P,Q}) = \frac{\left(q^{\frac{1}{2}(\alpha+\beta+1)} + q^{-\frac{1}{2}(\alpha+\beta+1)} \right) \left(q^{\frac{PQ}{2}(\alpha+\beta+1)} + q^{-\frac{PQ}{2}(\alpha+\beta+1)} \right)}{\left(q^{\frac{P}{2}(\alpha+\beta+1)} + q^{-\frac{P}{2}(\alpha+\beta+1)} \right) \left(q^{\frac{Q}{2}(\alpha+\beta+1)} + q^{-\frac{Q}{2}(\alpha+\beta+1)} \right)} . \tag{3.12}$$

This is a generic formula for the (P, Q) torus knot with the representation $\lambda = (\alpha|\beta)$. This expression is not a polynomial of q and q^{-1} in general, but it is manifestly invariant under the exchange of $P \leftrightarrow Q$, and also the inversion $q \leftrightarrow q^{-1}$.

4 U($N|M$) theory

The argument shown above can be straightforwardly extended to U($N|M$) theory. The U($N|M$) supermatrix Chern–Simons model is obtained from the Chern–Simons–matter theory with gauge group $\mathrm{U}(N)_k \times \mathrm{U}(M)_{-k}$, which is called ABJ theory [26],

$$\mathcal{Z}_{\mathrm{ABJ}}(S^3; q) = \frac{1}{N!M!} \int [dx]^N [dy]^M \prod_{i=1}^{N} \prod_{j=1}^{M} \left(2\cosh \frac{x_i - y_j}{2} \right)^{-2} \times \prod_{i<j}^{N} \left(2\sinh \frac{x_i - x_j}{2} \right)^2 \prod_{i<j}^{M} \left(2\sinh \frac{y_i - y_j}{2} \right)^2 . \tag{4.1}$$

The matrix measure in this model can be also expressed as a determinant using the generalized Cauchy determinant formula [27]

$$\Delta_N(u)\Delta_M(v) \prod_{i=1}^{N} \prod_{j=1}^{M} (u_i + v_j)^{-1} = \det \begin{pmatrix} u_i^{k-1} \\ (u_i + v_j)^{-1} \end{pmatrix} , \tag{4.2}$$

where we assume $N \geq M$, and the indices run as $i = 1, \cdots, N$, $j = 1, \cdots, M$ and $k = 1, \cdots, N - M$. Thus we have

$$\prod_{i=1}^{N} \prod_{j=1}^{M} \left(2\cosh \frac{x_i - y_j}{2} \right)^{-1} \prod_{i<j}^{N} \left(2\sinh \frac{x_i - x_j}{2} \right) \prod_{i<j}^{M} \left(2\sinh \frac{y_i - y_j}{2} \right) = \prod_{i=1}^{N} e^{\frac{-N+M+1}{2} x_i} \prod_{j=1}^{M} e^{\frac{N-M+1}{2} y_j} \det \begin{pmatrix} e^{x_i (k-1)} \\ (e^{x_i} + e^{y_j})^{-1} \end{pmatrix} . \tag{4.3}$$

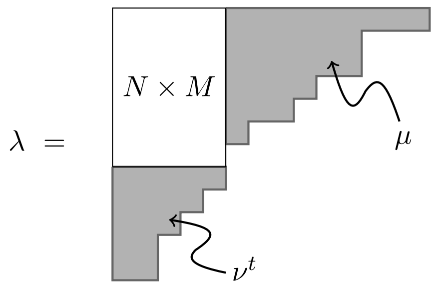

Figure 2: The partition $\lambda = (14, 11, 11, 9, 8, 6, 5, 5, 4, 3, 2, 2)$ satisfying $\lambda_N \geq M$ with $N = 7$ and $M = 5$, which includes $\mu = (9, 6, 6, 4, 3, 1, 0)$ and $\nu^t = (5, 4, 3, 2, 2)$.

We then consider the expectation value of the Wilson loop operator with respect to this partition function. As in the case of $\mathrm{U}(N|N)$ theory, if the partition, corresponding to the representation of the Wilson loop, satisfies $\lambda_N \geq M$ as shown in Fig. 2, the Schur function is factorized into the ordinary ones [21]

$$
\begin{aligned}
s_\lambda(u; v) &= s_\mu(u)\, s_\nu(v) \prod_{i=1}^{N} \prod_{j=1}^{M} (u_i + v_j) \\
&= \prod_{i=1}^{N} u_i^{\lambda_i + N - M - i} \prod_{j=1}^{M} v_j^{\lambda_j^t + M - N - j} \det \begin{pmatrix} u_i^{k-1} \\ (u_i + v_j)^{-1} \end{pmatrix}, \qquad (4.4)
\end{aligned}
$$

where the partitions μ and ν are defined as $\mu_i = \lambda_i - M$ for $i = 1, \cdots, N$ and $\nu_j = \lambda_j^t - N$ for $j = 1, \cdots, M$, or equivalently $\mu_i^t = \lambda_{i+M}^t$ and $\nu_j^t = \lambda_{j+N}$. Therefore the unknot character expectation value is now given by

$$
\begin{aligned}
\Big\langle W_R(\bigcirc) \Big\rangle &= \frac{1}{\mathcal{Z}_{\mathrm{ABJ}}} \frac{1}{N!M!} \int [dx]^N [dy]^M \det \begin{pmatrix} e^{x_i(k-1)} \\ (e^{x_i} + e^{y_j})^{-1} \end{pmatrix} \prod_{i=1}^{N} e^{x_i(\lambda_i - i + 1)} \prod_{j=1}^{M} e^{y_j(\lambda_j^t - j + 1)} \\
&= \frac{1}{\mathcal{Z}_{\mathrm{ABJ}}} \frac{1}{N!M!} \det \begin{pmatrix} \displaystyle\int \frac{dx}{2\pi}\, e^{-\frac{1}{2g_s} x^2 + x\left(\xi_i + k - \frac{1}{2}\right)} \\ \displaystyle\int \frac{dx}{2\pi} \frac{dy}{2\pi}\, e^{-\frac{1}{2g_s}(x^2 - y^2) + x\xi_i + y\eta_j} \left(2\cosh\frac{x-y}{2}\right)^{-1} \end{pmatrix},
\end{aligned}
\qquad (4.5)
$$

with $\xi_i = \lambda_i - i + \frac{1}{2}$ for $i = 1, \cdots, N$ and $\eta_j = \lambda_j^t - j + \frac{1}{2}$ for $j = 1, \cdots, M$. This yields the determinantal formula for $\mathrm{U}(N|M)$ theory

$$
\Big\langle W_R(\bigcirc) \Big\rangle = \frac{1}{\mathcal{Z}_{\mathrm{ABJ}}} \frac{1}{N!M!} \frac{i^{\frac{N-M}{2}}}{k^{\frac{N+M}{2}}} \prod_{i=1}^{N} q^{\frac{1}{2}(\xi_i^2 + \xi_i)} \prod_{j=1}^{M} q^{\frac{1}{2}(-\eta_j^2 + \eta_j)} \prod_{k=1}^{N-M} q^{\frac{1}{2}\left(k - \frac{1}{2}\right)^2} \det \begin{pmatrix} q^{\xi_i(k-1)} \\ \left(q^{\xi_i} + q^{\eta_j}\right)^{-1} \end{pmatrix}. \qquad (4.6)
$$

This formula is also represented as follows,

$$\begin{aligned}&\Big\langle W_R(\bigcirc)\Big\rangle\\&=\frac{1}{\mathcal{Z}_{\mathrm{ABJ}}}\frac{1}{N!M!}\frac{i^{\frac{N-M}{2}}}{k^{\frac{N+M}{2}}}q^{\frac{1}{6}(N-M)\left(N-M+\frac{1}{2}\right)\left(N-M-\frac{1}{2}\right)}\prod_{i=1}^{N}q^{\frac{1}{2}\left(\xi_i^2+(N-M)\xi_i\right)}\prod_{j=1}^{M}q^{-\frac{1}{2}\left(\eta_j^2+(N-M)\eta_j\right)}\\&\times\prod_{i<j}^{N}\left(q^{\frac{1}{2}(\xi_i-\xi_j)}-q^{-\frac{1}{2}(\xi_i-\xi_j)}\right)\prod_{i<j}^{M}\left(q^{\frac{1}{2}(\eta_i-\eta_j)}-q^{-\frac{1}{2}(\eta_i-\eta_j)}\right)\prod_{i=1}^{N}\prod_{j=1}^{M}\left(q^{\frac{1}{2}(\xi_i+\eta_j)}+q^{-\frac{1}{2}(\xi_i+\eta_j)}\right)^{-1}.\end{aligned} \tag{4.7}$$

It is easy to see that this expression is reduced to the U($N|N$) average (2.29) and the U(N) invariant (2.30) by taking $N=M$ and $M=0$, respectively.

We can similarly introduce the U($N|M$) supermatrix model for torus knots

$$\begin{aligned}\mathcal{Z}_{\mathrm{ABJ}}^{(P,Q)}(S^3;q)=\frac{1}{N!M!}\int[dx]^N[dy]^M\prod_{i=1}^{N}\prod_{j=1}^{M}\left(2\cosh\frac{x_i-y_j}{2P}\right)^{-1}\left(2\cosh\frac{x_i-y_j}{2Q}\right)^{-1}\\\times\prod_{i<j}^{N}\left(2\sinh\frac{x_i-x_j}{2P}\right)\left(2\sinh\frac{x_i-x_j}{2Q}\right)\prod_{i<j}^{M}\left(2\sinh\frac{y_i-y_j}{2P}\right)\left(2\sinh\frac{y_i-y_j}{2Q}\right).\end{aligned} \tag{4.8}$$

The torus knot average is obtained from this matrix model by inserting the U($N|M$) character, and thus satisfies the Rosso–Jones formula (3.3) thanks to the Adams operation (3.7). In the next section we will discuss the spectral curve for this matrix model arising in the large N limit.

5 Spectral curve

We then provide the spectral curve for the ABJ(M) matrix model for (P,Q) torus knots, which is introduced in Sec. 3 and Sec. 4. In order to obtain the spectral curve, we have to solve the saddle point equations arising in the large N limit of the corresponding matrix model. It is well known that the ABJM matrix model is perturbatively equivalent to the Chern–Simons theory on the lens space $L(2,1)=S^3/\mathbb{Z}_2\cong\mathbb{RP}^3$, which is regarded as the two-cut solution of the Chern–Simons matrix model [20]. As well as the ordinary ABJM matrix model, we can obtain the spectral curve for the torus knot ABJM model from the (P,Q)-modified Chern–Simons theory on the lens space $L(2,1)$

$$\begin{aligned}\mathcal{Z}_{\mathrm{CS}}^{(P,Q)}(L(2,1);q)=\frac{1}{N!^2}\int\prod_{i=1}^{N}\frac{dx_i}{2\pi}\frac{dy_i}{2\pi}\,e^{-\frac{1}{2\hat{g}_s}(x_i^2+y_i^2)}\prod_{i,j=1}^{N}\left(2\cosh\frac{x_i-y_j}{2P}\right)\left(2\cosh\frac{x_i-y_j}{2Q}\right)\\\times\prod_{i<j}^{N}\left(2\sinh\frac{x_i-x_j}{2P}\right)\left(2\sinh\frac{x_i-x_j}{2Q}\right)\left(2\sinh\frac{y_i-y_j}{2P}\right)\left(2\sinh\frac{y_i-y_j}{2Q}\right).\end{aligned} \tag{5.1}$$

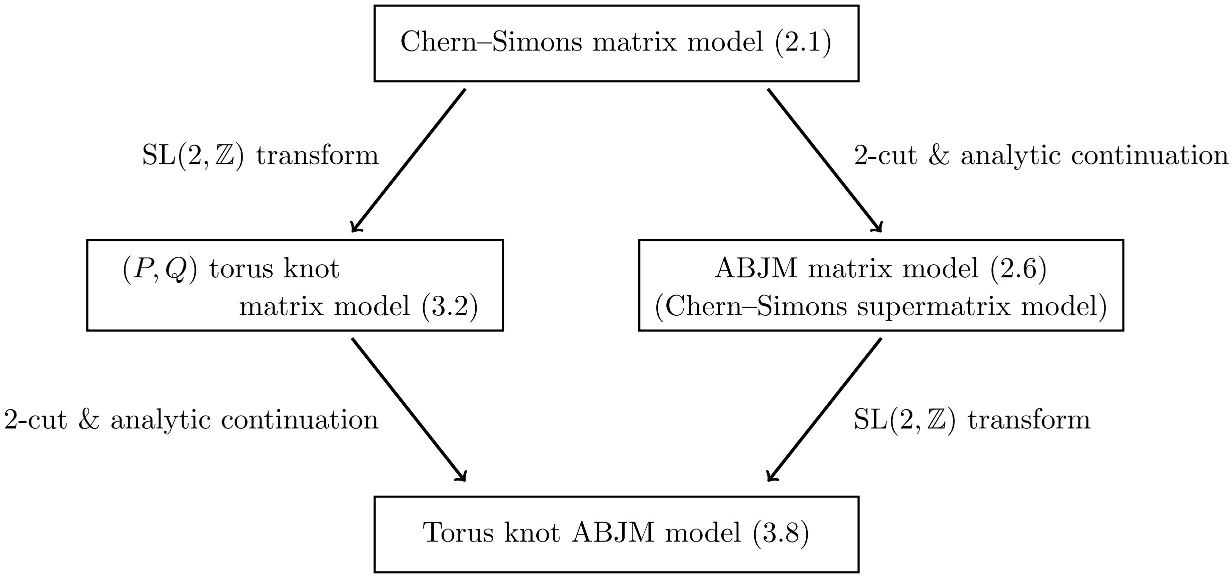

Figure 3: Two ways of obtaining the (P, Q) torus knot ABJM (supermatrix) model.

This is interpreted as the two-cut matrix model of the original torus knot model (3.2), which breaks the gauge group as $\mathrm{U}(N + N) \to \mathrm{U}(N) \times \mathrm{U}(N)$.

As shown in Fig. 3, we have two ways of obtaining the spectral curve for the torus knot ABJM matrix model. The first is to apply the $\mathrm{SL}(2, \mathbb{Z})$ transformation to the spectral curve for ABJM theory, which is given by that for the lens space Chern–Simons theory, through the analytic continuation [28, 29]. We note that this kind of symplectic transformation is considered to discuss the knot invariant for the lens space [30, 31]. The second is directly solving the saddle point equation for the matrix model (5.1). In this case its saddle point analysis is similar to the ordinary one-cut solution, which is discussed in [5]. We will show that these two methods provide a consistent result.

5.1 Symplectic transformation

Let us start with the spectral curve for the Chern–Simons theory on the lens space $S^3/\mathbb{Z}_2 \cong \mathbb{RP}^3$, which is essentially equivalent to that for ABJM theory. The spectral curve $\mathcal{C}$ is defined as the zero locus of the two-parameter function

$$\mathcal{C} = \left\{ (U, V) \in \mathbb{C}^* \times \mathbb{C}^* \,\middle|\, H(U, V) = 0 \right\}, \tag{5.2}$$

where the function $H(U, V)$ for the lens space $L(2, 1)$ is now given by [29]

$$H(U, V) = c \left(U + \frac{V^2}{U} \right) - V^2 + \zeta V - 1 \,. \tag{5.3}$$

The parameter ζ is to be determined, and c is related to the 't Hooft coupling constant as $c = \exp g_s(N+M)/2$ for the two-cut model with $\mathrm{U}(N)\times\mathrm{U}(M)$ symmetry.[3]

In order to compare with the expression obtained in [5], we modify the curve (5.4) by replacing the variables $(U,V)\to(U^2, c^{-\frac{1}{2}}U^{\frac{1}{2}}V)$,

$$H(U,V) = cU^2 + V^2 - c^{-1}V^2U^2 + c^{-\frac{1}{2}}\zeta VU - 1 = 0\,. \tag{5.5}$$

This gives

$$V = \frac{1}{1-c^{-1}U^2}\left(-\frac{\zeta}{2c^{\frac{1}{2}}}U \pm \sqrt{\frac{\zeta^2}{4c}U^2 + (1-cU^2)(1-c^{-1}U^2)}\right)\,. \tag{5.6}$$

By taking the limit $\zeta\to 0$, this solution is just reduced to the one-cut solution, and it reproduces the previous result [5], up to a proper replacement of the variable $V\to V^2$,

$$V \overset{\zeta\to 0}{\longrightarrow} \frac{1-cU^2}{1-c^{-1}U^2}\,. \tag{5.7}$$

If we write the discriminant of (5.6) as

$$\frac{\zeta^2}{4c}U^2 + (1-cU^2)(1-c^{-1}U^2) = \left(U^2-\alpha\right)\left(U^2-\frac{1}{\alpha}\right)\,, \tag{5.8}$$

the end point of the cut is determined by

$$\alpha + \frac{1}{\alpha} = c + \frac{1}{c} - \frac{\zeta^2}{4c}\,. \tag{5.9}$$

Thus the parameter ζ is seen as the blow-up parameter for the spectral curve, which is determined by requiring that the filling fractions of the cuts are:

$$\frac{1}{2\pi i}\oint_{[\alpha^{-1/2},\alpha^{1/2}]} \log V\,\frac{dU}{U} = g_s N\,, \tag{5.10}$$

$$\frac{1}{2\pi i}\oint_{[-\alpha^{1/2},-\alpha^{-1/2}]} \log V\,\frac{dU}{U} = g_s M\,, \tag{5.11}$$

where the integration contour surround the corresponding segments counterclockwise.

We can obtain the spectral curve for the (P,Q) torus knot from (5.6) through the symplectic transformation, which is characterized by the $\mathrm{SL}(2,\mathbb{Z})$ matrix [5]

$$M_{P,Q} = \begin{pmatrix} Q & P \\ \gamma & \delta \end{pmatrix}\,, \tag{5.12}$$

[3] The spectral curve for Chern–Simons theory on the lens space $L(r,1) = S^3/\mathbb{Z}_r$ (the r-cut Chern–Simons matrix model) is given by [29]

$$H(U,V) = c\left(U + \frac{V^r}{U}\right) - p_r(V) = 0\,, \tag{5.4}$$

where $p_r(V)$ is a degree r polynomial such that the coefficients of V^r and V^0 are given by one, $p_r(V) = V^r + \cdots + 1$. When the Chern–Simons gauge group is broken as $\mathrm{U}(N_1+\cdots+N_r)\to\mathrm{U}(N_1)\times\cdots\times\mathrm{U}(N_r)$, the parameter c corresponds to the total 't Hooft coupling as $c = \exp g_s(N_1+\cdots+N_r)/2$. Since the polynomial $p_r(V)$ has $r-1$ parameters, the total number of the parameters becomes $1+(r-1)=r$, which is consistent with that of the subgroups, $\mathrm{U}(N_i)$ with $i=1,\cdots,r$.

where these integer entries satisfy the condition

$$Q\delta - P\gamma = 1\,. \tag{5.13}$$

We then apply the following choice of variables corresponding to the matrix (5.12),

$$X = U^Q V^P\,, \tag{5.14}$$

$$Y = U^\gamma V^\delta\,. \tag{5.15}$$

In this case by substituting the original expression (5.6) and rescaling the variable $U^2 \to c^{\frac{P}{Q}} U^2$, we obtain the spectral curve for the (P, Q) torus knot Chern–Simons theory on the lens space

$$X = \frac{U^Q}{\left(1 - c^{\frac{P}{Q}-1}U^2\right)^P} \left(-\frac{\zeta}{2} c^{(\frac{P}{Q}-1)/2} U \pm \sqrt{\frac{\zeta^2}{4} c^{\frac{P}{Q}-1} U^2 + \left(1 - c^{\frac{P}{Q}-1}U^2\right)\left(1 - c^{\frac{P}{Q}+1}U^2\right)}\right)^P, \tag{5.16}$$

$$V = \frac{1}{1 - c^{\frac{P}{Q}-1}U^2} \left(-\frac{\zeta}{2} c^{(\frac{P}{Q}-1)/2} U \pm \sqrt{\frac{\zeta^2}{4} c^{\frac{P}{Q}-1} U^2 + \left(1 - c^{\frac{P}{Q}-1}U^2\right)\left(1 - c^{\frac{P}{Q}+1}U^2\right)}\right). \tag{5.17}$$

It is shown in [30, 31] that the new curve obtained through this kind of symplectic transformation gives the topological invariants for torus knots in the lens space, which is dual to the topological string on the local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry.

5.2 Saddle point analysis

We study the spectral curve for the torus knot from the large N limit of the matrix model (5.1), and then check its consistency with the result obtained through the symplectic transformation, (5.16) and (5.17). We now rewrite the matrix integral (5.1) with another set of variables, $u_i = e^{x_i/(PQ)}$ and $v_i = e^{y_i/(PQ)}$,

$$\begin{aligned} \mathcal{Z}_{\text{CS}}^{(P,Q)}(L(2,1);q) = \frac{1}{N!^2} \int \frac{d^N u}{(2\pi)^N} \frac{d^N v}{(2\pi)^N} \exp\Bigg[&\sum_{i,j=1}^N \Big(\log(u_i^P + v_j^P) + \log(u_i^Q + v_j^Q)\Big) \\ &+ \sum_{i<j}^N \Big(\log(u_i^P - u_j^P) + \log(u_i^Q - u_j^Q) + \log(v_i^P - v_j^P) + \log(v_i^Q - v_j^Q)\Big) \\ &- \sum_{i=1}^N \left(\frac{PQ}{2g_s}\left(\log^2 u_i + \log^2 v_i\right) + \left(\frac{P+Q}{2}(2N-1) + 1\right)(\log u_i + \log v_i)\right)\Bigg]. \end{aligned} \tag{5.18}$$

The matrix integral has a convex potential (see [32]), and this implies that the integrand has a unique minimum.

In this case we have two saddle point equations with u_i and v_i variables,

$$\sum_{j(\neq i)}^{N}\left[\frac{Pu_i^P}{u_i^P-u_j^P}+\frac{Qu_i^Q}{u_i^Q-u_j^Q}\right]+\sum_{j=1}^{N}\left[\frac{Pu_i^P}{u_i^P+v_j^P}+\frac{Qu_i^Q}{u_i^Q+v_j^Q}\right]=\frac{PQ}{g_s}\log u_i+\frac{P+Q}{2}(2N-1)+1\,, \tag{5.19}$$

$$\sum_{j(\neq i)}^{N}\left[\frac{Pv_i^P}{v_i^P-v_j^P}+\frac{Qv_i^Q}{v_i^Q-v_j^Q}\right]+\sum_{j=1}^{N}\left[\frac{Pv_i^P}{v_i^P+u_j^P}+\frac{Qv_i^Q}{v_i^Q+u_j^Q}\right]=\frac{PQ}{g_s}\log v_i+\frac{P+Q}{2}(2N-1)+1\,. \tag{5.20}$$

Let us now rewrite these equations in terms of the resolvent [5]. Using the formula

$$\frac{Px^{P-1}}{x^P-y^P}=\sum_{k=0}^{P-1}\frac{1}{x-\omega^{-kQ}y}\,, \tag{5.21}$$

$$\frac{Px^{P-1}}{x^P+y^P}=\sum_{k=0}^{P-1}\frac{1}{x-\omega^{-(k+\frac{1}{2})Q}y}\,, \tag{5.22}$$

with the primitive PQ-th root of unity $\omega=\exp 2\pi i/(PQ)$, we have

$$\begin{aligned}&\sum_{j(\neq i)}^{N}\frac{Pu_i^P}{u_i^P-u_j^P}+\sum_{j=1}^{N}\frac{Pu_i^P}{u_i^P+v_j^P}\\&=\sum_{j(\neq i)}^{N}\frac{u_i}{u_i-u_j}+\sum_{k=1}^{P-1}\left(\frac{1}{g_s}W_0^{(1)}(u_i\,\omega^{kQ})-\frac{1}{1-\omega^{-kQ}}\right)+\sum_{k=0}^{P-1}\frac{1}{g_s}W_0^{(2)}(u_i\,\omega^{(k+\frac{1}{2})Q)})\,,\end{aligned} \tag{5.23}$$

where $W_0^{(i)}(u)$ is the leading contribution to the resolvents

$$W^{(1)}(u)=\left\langle g_s\sum_{i=1}^{N}\frac{u}{u-u_i}\right\rangle=\sum_{g=0}^{\infty}g_s^{2g-1}W_g^{(1)}(u)\,, \tag{5.24}$$

$$W^{(2)}(u)=\left\langle g_s\sum_{i=1}^{N}\frac{u}{u-v_i}\right\rangle=\sum_{g=0}^{\infty}g_s^{2g-1}W_g^{(2)}(u)\,. \tag{5.25}$$

These resolvents are analytic in the complex plane except for a finite set of cuts. We now assume that each resolvent has only one cut in the complex plane $\mathbb{C}$. We can check later that this assumption is correct, using unicity of the exremum guaranteed by [32].

We introduce the 't Hooft coupling constants for this matrix model. If we consider a generic situation with the gauge group $\mathrm{U}(N_1+N_2)\to\mathrm{U}(N_1)\times\mathrm{U}(N_2)$, we have two constants

$$t^{(i)}=g_sN_i\,,\qquad i=1,\,2\,. \tag{5.26}$$

The summation of them is denoted by $t=t^{(1)}+t^{(2)}$. ABJM theory corresponds to the situation such that N_2 is analytically continuated as $N_2\to -N_2$, and then the gauge group ranks are chosen to be $N_{1,2}\to N$. This means that the total 't Hooft coupling becomes zero $t=0$. Then the boundary conditions for the resolvents are given by

$$W_0^{(i)}(u)\ \longrightarrow\ \begin{cases}0 & (u\to 0)\\ t^{(i)} & (u\to\infty)\end{cases}\,, \tag{5.27}$$

and the saddle point equations in the 't Hooft limit with $N \to \infty$ and $g_s \to 0$ yield

$$PQ \log u + \frac{P+Q}{2} t = W_0^{(1)}(u+i0) + W_0^{(1)}(u-i0) + \sum_{k=1}^{P-1} W_0^{(1)}(u\,\omega^{kQ}) + \sum_{k=1}^{Q-1} W_0^{(1)}(u\,\omega^{kP}) + \sum_{k=0}^{P-1} W_0^{(2)}(u\,\omega^{(k+\frac{1}{2})Q}) + \sum_{k=0}^{Q-1} W_0^{(2)}(u\,\omega^{(k+\frac{1}{2})P}) \,, \tag{5.28}$$

$$PQ \log u + \frac{P+Q}{2} t = W_0^{(2)}(u+i0) + W_0^{(2)}(u-i0) + \sum_{k=1}^{P-1} W_0^{(2)}(u\,\omega^{kQ}) + \sum_{k=1}^{Q-1} W_0^{(2)}(u\,\omega^{kP}) + \sum_{k=0}^{P-1} W_0^{(1)}(u\,\omega^{(k+\frac{1}{2})Q}) + \sum_{k=0}^{Q-1} W_0^{(1)}(u\,\omega^{(k+\frac{1}{2})P}) \,. \tag{5.29}$$

In order to deal with these equations, it is convenient to introduce the exponentiated resolvents

$$y^{(a)}(u) = -u \exp \frac{P+Q}{PQ} \left(\frac{t}{2} - W_0^{(1)}(u) - W_0^{(2)}(u\,\omega^{\frac{1}{2}Q}) \right) \,, \tag{5.30}$$

$$y^{(b)}(u) = -u \exp \frac{P+Q}{PQ} \left(\frac{t}{2} - W_0^{(1)}(u) - W_0^{(2)}(u\,\omega^{\frac{1}{2}P}) \right) \,, \tag{5.31}$$

with the boundary behavior

$$y^{(a,b)}(u) \;\longrightarrow\; \begin{cases} -u\, e^{\frac{P+Q}{2PQ} t} & (u \to 0) \\ -u\, e^{-\frac{P+Q}{2PQ} t} & (u \to \infty) \end{cases} . \tag{5.32}$$

The saddle point equations are now written as follows,

$$y^{(a)}(u+i0) y^{(b)}(u-i0) \prod_{k=1}^{P-1} y^{(a)}(u\,\omega^{kQ}) \prod_{k=1}^{Q-1} y^{(b)}(u\,\omega^{kP}) = 1 \,, \tag{5.33}$$

$$y^{(a)}((u+i0)\omega^{\frac{1}{2}Q}) y^{(b)}((u-i0)\omega^{\frac{1}{2}P}) \prod_{k=1}^{P-1} y^{(a)}(u\,\omega^{(k+\frac{1}{2})Q}) \prod_{k=1}^{Q-1} y^{(b)}(u\,\omega^{(k+\frac{1}{2})P}) = 1 \,. \tag{5.34}$$

Because it is converted to each other under the exchange of $W^{(1)}(u)$ and $W^{(2)}(u)$, these equations imply an equivalent condition for the resolvents, which is essentially the same as the one-cut Chern–Simons matrix model for the (P,Q) torus knot [5]. Thus we can apply a similar approach to solve these equations.

We consider the products of the resolvents

$$F_k(u) = \prod_{l=0}^{P-1} y^{(a)}(u\,\omega^{kP+lQ}) \,, \quad 0 \le k \le Q-1 \,, \tag{5.35}$$

$$F_{Q+l}(u) = \prod_{k=0}^{Q-1} \frac{1}{y^{(b)}(u\,\omega^{kP+lQ})} \,, \quad 0 \le l \le P-1 \,. \tag{5.36}$$

Since we have assumed that the original resolvents $W^{(i)}(u)$ have a single cut, these functions $F_k(u)$ and $F_{Q+l}(u)$ have $2P$ and $2Q$ cuts, obtained by rotating the original one with integer

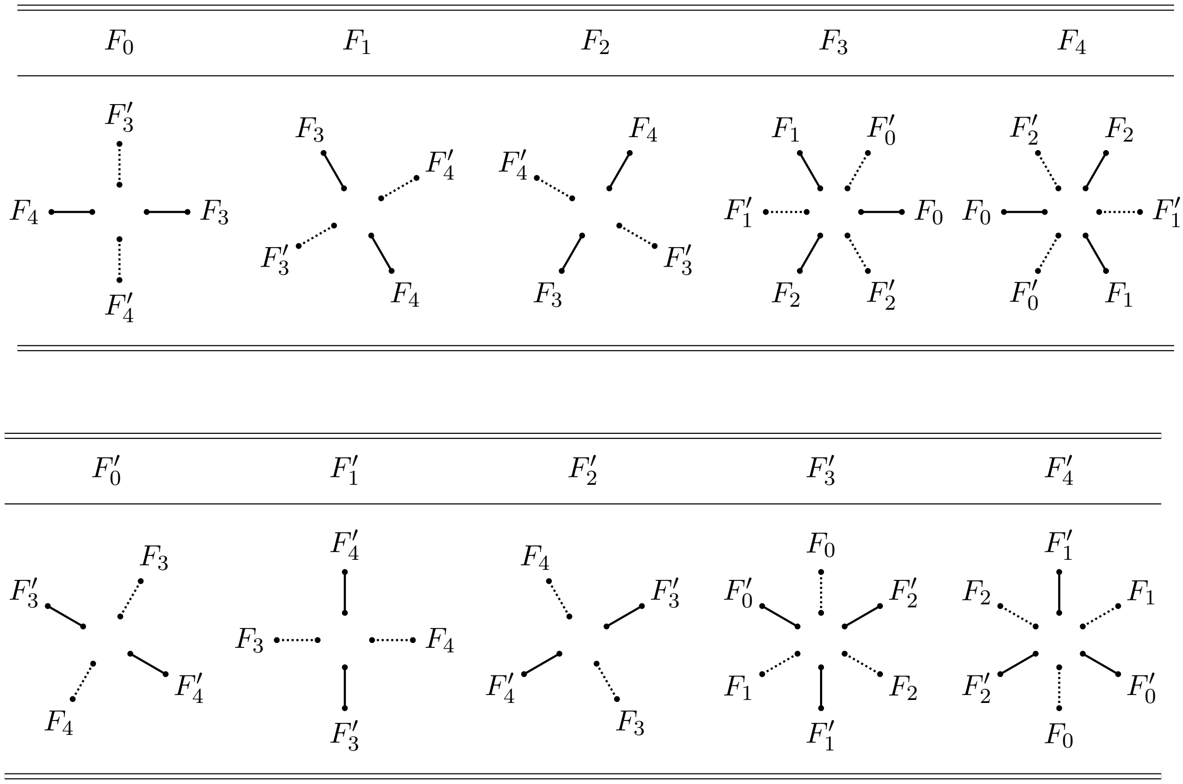

Table 1: The cuts of the functions $F_k(u)$ and $F_{Q+l}(u)$ for $(P,Q)=(2,3)$, where $F_k=F_k(u)$, $F'_k=F_k(u\,\omega^{\frac{1}{2}(P-Q)})$ and $F'_{Q+l}=F_{Q+l}(u\,\omega^{\frac{1}{2}(Q-P)})$. The solid and dotted lines correspond to the cuts from the first resolvent $W^{(1)}(u)$ and the second resolvent $W^{(2)}(u)$. For example, we can see $F_0=F_3$, F_4 under crossing the corresponding cut of $W^{(1)}(u)$, and $F_0=F'_3$, F'_4 through the cut from $W^{(2)}(u)$.

multiple angles of $2\pi/(PQ)$. The total number of the cuts is thus $2PQ$. Due to the saddle point equations they satisfy

$$\begin{aligned} F_k(u-i0) &= F_{Q+l}(u+i0) && \text{for} \quad W^{(1)}(u)\,, \\ F_k(u-i0) &= F_{Q+l}((u+i0)\omega^{\frac{1}{2}(Q-P)}) && \text{for} \quad W^{(2)}(u)\,, \\ F_k((u-i0)\omega^{\frac{1}{2}(P-Q)}) &= F_{Q+l}((u+i0)\omega^{\frac{1}{2}(Q-P)}) && \text{for} \quad W^{(1)}(u)\,, \\ F_k((u-i0)\omega^{\frac{1}{2}(P-Q)}) &= F_{Q+l}(u+i0) && \text{for} \quad W^{(2)}(u)\,. \end{aligned} \tag{5.37}$$

This means that $F_k(u-i0)=F_{Q+l}(u+i0)$ under crossing the cut from the first resolvent $W^{(1)}(u)$, $F_k(u-i0)=F_{Q+l}((u+i0)\omega^{\frac{1}{2}(Q-P)})$ for the cut from the second $W^{(2)}(u)$, and so on. See Table 1 for the case with $(P,Q)=(2,3)$.

Using these functions we define a function

$$\mathcal{S}(u,f)=\mathcal{S}_1(u,f)\,\mathcal{S}_2(u,f)\,, \tag{5.38}$$

where

$$\mathcal{S}_1(u,f) = \prod_{k=0}^{Q-1}\Big(f - F_k(u)\Big)\prod_{l=0}^{P-1}\Big(f - F_{Q+l}(u)\Big)\,, \tag{5.39}$$

$$\mathcal{S}_2(u,f) = \prod_{k=0}^{Q-1}\Big(f - F_k(u\,\omega^{\frac{1}{2}(P-Q)})\Big)\prod_{l=0}^{P-1}\Big(f - F_{Q+l}(u\,\omega^{\frac{1}{2}(Q-P)})\Big)\,. \tag{5.40}$$

This function has no cut in the complex plane

$$\mathcal{S}(u-i0,f) = \mathcal{S}(u+i0,f)\,, \tag{5.41}$$

and the only singularities at $u=0$ or $u=\infty$ as poles. This implies that $S(u,f)$ is an entire function of u in $\mathbb{C}^*$ with polynomial behavior at 0 and at ∞, therefore it must be a Laurent polynomial of u. Moreover since this function satisfies $\mathcal{S}(u,f)=\mathcal{S}(u\,\omega,f)$, it depends only on u^{PQ}, and must be a Laurent polynomial of u^{PQ}. We remark that $\mathcal{S}_1(u,f)$ and $\mathcal{S}_2(u,f)$ still have the cuts, and thus they are not analytic functions.

By definition, $\mathcal{S}(u,f)$ vanishes when $f = F_k(u)$, and thus the spectral curve is the algebraic equation:

$$\mathcal{S}(u,f) = 0\,. \tag{5.42}$$

As shown in [5] we can determine coefficients of the polynomial $\mathcal{S}(u,f)$ by the asymptotic behavior of $F_k(u)$ and $F_{Q+l}(u)$,

$$F_k(u) \;\longrightarrow\; \begin{cases} -\omega^{kP^2}\,e^{\frac{(P+Q)}{2Q}t}\,u^P & (u\to 0)\\ -\omega^{kP^2}\,e^{-\frac{(P+Q)}{2Q}t}\,u^P & (u\to\infty)\end{cases}\,, \tag{5.43}$$

$$F_{Q+l}(u) \;\longrightarrow\; \begin{cases} -\omega^{-lQ^2}\,e^{-\frac{(P+Q)}{2P}t}\,u^{-Q} & (u\to 0)\\ -\omega^{-lQ^2}\,e^{\frac{(P+Q)}{2P}t}\,u^{-Q} & (u\to\infty)\end{cases}\,. \tag{5.44}$$

If we write

$$\mathcal{S}_i(u,f) = \sum_{k=0}^{P+Q}(-1)^k\,\mathcal{S}_{i,k}(u)\,f^{P+Q-k}\,,\quad i=1,\,2\,, \tag{5.45}$$

this behavior implies

$$\begin{cases} \mathcal{S}_{i,k}(u) = \mathcal{O}(u^{-kQ})\,, & 1\le k\le P-1\,,\\ \mathcal{S}_{1,k}(u) = (-1)^{P+Q+PQ}e^{-\frac{P+Q}{2}t}u^{-PQ}\left(1+\mathcal{O}(u)\right)\,, & k=P\,,\\ \mathcal{S}_{2,k}(u) = (-1)^{PQ}e^{-\frac{P+Q}{2}t}u^{-PQ}\left(1+\mathcal{O}(u)\right)\,, & k=P\,,\\ \mathcal{S}_{i,k}(u) = \mathcal{O}(u^{-PQ}u^{(k-P)P})\,, & P+1\le k\le P+Q-1\,,\end{cases} \quad \text{at}\quad u\to 0\,, \tag{5.46}$$

$$\begin{cases} \mathcal{S}_{i,k}(u) = \mathcal{O}(u^{kP}) \,, & 1 \le k \le Q-1 \,, \\ \mathcal{S}_{1,k}(u) = (-1)^{P+Q+PQ} e^{-\frac{P+Q}{2}t} u^{PQ} \left(1 + \mathcal{O}(1/u)\right) \,, & k = Q \,, \\ \mathcal{S}_{2,k}(u) = (-1)^{PQ} e^{-\frac{P+Q}{2}t} u^{PQ} \left(1 + \mathcal{O}(1/u)\right) \,, & k = Q \,, \\ \mathcal{S}_{i,k}(u) = \mathcal{O}(u^{PQ} u^{-(k-Q)Q}) \,, & Q+1 \le k \le P+Q-1 \,, \end{cases} \quad \text{at} \quad u \to \infty \,. \tag{5.47}$$

Thus we obtain that $\mathcal{S}(u,f)$ is a polynomial of f and of u^{PQ}

$$\mathcal{S}(u,f) = f^{2(P+Q)} + 1 + (-1)^{P+Q} e^{-(P+Q)t} \left(u^{-2PQ} f^{2Q} + u^{2PQ} f^{2P}\right) + \cdots \,. \tag{5.48}$$

where $\cdots$ means terms within the Newton polygon, and which are not determined by asymptotic behaviors. The spectral curve for the two-cut matrix model is now given by the following polynomial relation,

$$\mathcal{S}(u,f) = 0 \,. \tag{5.49}$$

It is shown in [32, 33] that if the potential has convexity property, then the solution of the saddle point analysis is unique. This is the case here. We are looking for a two cut solution, i.e. a genus 1 spectral curve. A generic $\mathcal{S}(u,f)$ with the given Newton's polygon (5.48), would have genus the number of interior points of the Newton's polygon, i.e. $4(P+Q)-3$. However we are here looking for a genus 1 curve, which implies that all but 1 coefficients of $\mathcal{S}(u,f)$ must be fixed by vanishing of some discriminants. The last coefficient is fixed by the filling fraction condition for $\mathrm{U}(N|M)$, corresponding to (5.10) and (5.11):

$$\frac{1}{2\pi i} \oint_{\mathcal{A}} W(u)\, du = g_s N \,, \tag{5.50}$$

where $W(u)$ is the sum of the resolvents, and $\mathcal{A}$ is a contour around the first cut. One could determine $\mathcal{S}(u,f)$ by solving all vanishing discriminant equations, so as to impose that the curve have genus 1. However there is a short-cut: one can exhibit an $\mathcal{S}(u,f)$ with the correct Newton's polygon, the correct filling fraction condition and the correct asymptotic behaviors at 0 and ∞ and which is guaranteed to be genus 1. By unicity according to [32, 33] it must be the correct spectral curve.

The $\mathcal{S}(u,f)$ which has genus 1 and the correct asymptotic behavior is simply the symplectic transform of the unknot of section 5.1, which we write parametrically as [5]

$$f = \frac{V}{U} \,, \qquad u^{-PQ} = X = U^Q V^P \,, \tag{5.51}$$

we obtain another expression of the spectral curve (5.49) in terms of U and V,

$$\mathcal{S}(U,V) = U^{2(P+Q)} + V^{2(P+Q)} + (-1)^{P+Q} e^{-(P+Q)t} (UV)^{2(P+Q)} + (-1)^{P+Q} e^{-(P+Q)t} + \cdots = 0 \,, \tag{5.52}$$

which is equivalent to

$$\mathcal{S}(X,V) = V^{2(P+Q)} + X^2 + (-1)^{PQ} e^{-(P+Q)t} X^2 V^{2Q} + (-1)^{PQ} e^{-(P+Q)t} V^{2P} + \cdots = 0 \,. \tag{5.53}$$

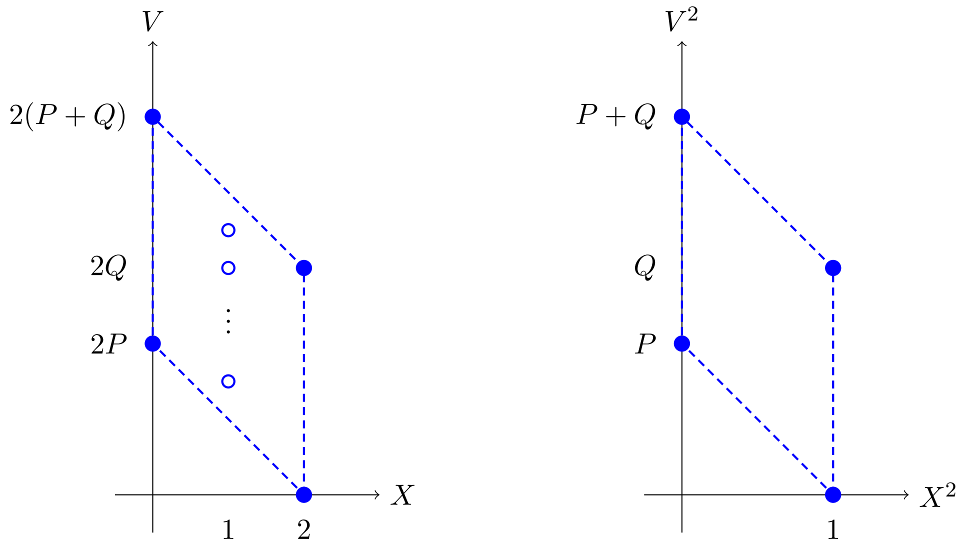

Figure 4: Newton's polygons for the spectral curve. The polygon in the left panel corresponds to the curve (5.53). There are cetain lattice points inside the polygon, which are fixed by the condition that the curve should have genus 1, and by the filling fraction condition. The right panel shows its singular limit, where the two resolvents $W^{(1)}(u)$ and $W^{(2)}(u)$ coincide with each other. In this case there is no lattice point inside the polygon.

We depict the Newton's polygon for this spectral curve in Fig. 4.

Let us then comment on the singular limit of the spectral curve (5.53), which is realized when the two resolvents $W^{(1)}(u)$ and $W^{(2)}(u)$ coincide with each other. In this case the analytic function obeys $\mathcal{S}(u,f) = \mathcal{S}(u\,\omega^{\frac{1}{2}}, f)$. This implies it depends only on u^{2PQ}. Therefore the spectral curve (5.53) has to be written only in terms of X^2 and V^2. The Newton's polygon corresponding to this situation is shown in the right panel of Fig. 4. Since there is no lattice point inside the polygon, the spectral curve has no free parameter. This means that it is just reduced to the genus-zero curve, which corresponds to the limit $\zeta \to 0$ of the curve (5.3).

5.3 Asymptotic expansion and topological recursion

The matrix integral (3.8) is of the type discussed in [32], therefore it guarantees that it has an asymptotic expansion of the type

$$\log \mathcal{Z}_{\text{ABJM}}^{(P,Q)} = \sum_{g=0}^{\infty} g_s^{2g-2} F_g \tag{5.54}$$

which obeys the topological recursion of [14], i.e. F_g is the g^{th} symplectic invariant of the spectral curve as defined by the topological recursion [14]. Similarly, all expectation values have a g_s expansion, whose coefficients are given by the topological recursion.

The character expectation values $\left\langle \mathrm{Tr}_R U \right\rangle$ can be decomposed on the basis of power sums. We have:

$$\left\langle \prod_{i=1}^{n} \mathrm{Tr}\, U^{p_i} \right\rangle = \mathrm{Res}_{x_1\to 0} \dots \mathrm{Res}_{x_n \to 0} W_n(x_1, \dots, x_n)\; x_1^{p_1} \dots x_n^{p_n} \prod_{i=1}^{n} \frac{dx_i}{x_i} \tag{5.55}$$

and W_n has a topological expansion:

$$W_n(x_1, \dots, x_n) = \sum_{g=0}^{\infty} g_s^{2g-2+n} W_{g,n}(x_1, \dots, x_n) \tag{5.56}$$

and $W_{g,n}$ is computed by the topological recursion.

6 Comments on related topics

6.1 Topological A-model

Let us comment on the realization of the knot invariant in topological string theory, especially in the topological A-model. It was shown in [19] that the knot invariant for K is obtained in the topological string by adding a brane with a proper Lagrangian submanifold of the Calabi–Yau threefold L_K. This insertion of the brane corresponds to the expectation value of the characteristic polynomial with respect to Chern–Simons theory

$$\begin{aligned} \mathcal{Z}_{\mathrm{top}}(K;x) &= \left\langle \det \left(1 \otimes 1 - U \otimes e^{-x} \right) \right\rangle_{\mathrm{CS}} \\ &= \sum_{n=0}^{\infty} \left\langle \mathrm{Tr}_{R_n} U \right\rangle_{\mathrm{CS}} e^{-nx}\,, \end{aligned} \tag{6.1}$$

where the matrix U is the holonomy along the knot K, $U = \mathrm{P}\exp\left(\oint_K A\right)$, and R_n is the totally symmetric representation with n boxes. This means that this topological string partition function is the discrete Fourier (Laplace) transform of the HOMFLY polynomial, since the expectation value of the holonomy $\left\langle \mathrm{Tr}_{R_n} U \right\rangle$ just gives the knot invariant. If we consider a multi-point correlator of the characteristic polynomials, the knot invariant with more generic representations is obtained. Note that this knot invariant is analogous to the matrix integral with the external source, as discussed in Sec. 6.3. This kind of relation between the characteristic polynomial and the external source is naturally interpreted from the viewpoint of the topological expansion of spectral curves.

In our case it is natural to consider a supermatrix version of the partition function (6.1), corresponding to ABJM theory

$$\mathcal{Z}_{\mathrm{top}}(K;x,y) = \left\langle \mathrm{Sdet} \left(1 \otimes 1 - U \otimes \begin{pmatrix} e^{-x} & \\ & e^{-y} \end{pmatrix} \right) \right\rangle_{\mathrm{ABJM}} . \tag{6.2}$$

Actually one can obtain this partition function by applying both of bosonic and fermionic modes, describing strings stretching between the three-sphere S^3 and the Lagrangian L_K.

We can similarly expand (6.2) with the corresponding expectation value $\left\langle \mathrm{Str}_{R_n} U \right\rangle_{\mathrm{ABJM}}$. This partition function is also expanded with the string coupling constant in the sense of the WKB expansion [19]. Using the identity

$$\mathrm{Sdet}\left(1 \otimes 1 - U \otimes \begin{pmatrix} e^{-x} & \\ & e^{-y} \end{pmatrix}\right) = \exp\left[\mathrm{Str}\,\log\left(1 \otimes 1 - U \otimes \begin{pmatrix} e^{-x} & \\ & e^{-y} \end{pmatrix}\right)\right], \tag{6.3}$$

we have

$$\mathcal{Z}_{\mathrm{top}}(K; x, y) \;\sim\; \exp\left(\frac{1}{g_s}\int_y^x p(x)dx\right), \tag{6.4}$$

where the integrand is given by

$$p(x) = \lim_{g_s \to 0} \sum_{n=0}^{\infty} g_s \left\langle \mathrm{Str}\, U^n \right\rangle_{\mathrm{ABJM}} e^{-nx}. \tag{6.5}$$

In this way we can show that its leading contribution is given by the disc amplitude as well as the ordinary knot invariant. In this case, since the ABJM matrix model is obtained from Chern–Simons theory on the lens space $L(2,1)$ through the analytic continuation, the mirror curve, on which the one-form is defined, is replaced with that for the local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry. Actually the Wilson loop expectation value is evaluated based on this spectral curve [20, 34]. Furthermore, in (6.4), both of the initial and end points of the integral have physical meanings, as positions of brane and anti-brane. Thus the partition function describes pair creation of branes in the topological string. If we take the limit $y \to \infty$, it goes back to the usual one, including either of bosonic or fermionic modes.

6.2 Topological B-model

In the B-model description of the topological strings, the n-point function defined in (5.56) plays an important role. As well as the one-point function $W_1 = W$ which is given as the resolvent, we also have a series expansion of the multi-point function with respect to the coupling constant. If once a spectral curve is obtained, one can determine higher order terms by using the topological recursion [14]. In this study the spectral curve is given by the genus-one mirror curve for the local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry and its symplectic transform. Indeed this expansion has a natural interpretation in terms of the B-model topological strings. The multi-point correlation function corresponds to multiple insertion of the Wilson loop operators into the Chern–Simons matrix model. Thus a set of variables in the multi-point function (5.56) provides the boundary condition for topological strings. This implies that it computes the open string sector of the B-model, and we can obtain the corresponding open Gromov–Witten invariants based on the mirror symmetry in a perturbative way [35, 36, 37].

As in the case of the topological A-model, the knot invariant can be investigated also in the B-model through the mirror symmetry. In addition to the unknot invariants [38, 39], the torus knot invariant is formulated by using the symplectic transformation acting on the B-model open string moduli [5].[4] Moreover this prescription for giving the torus knot invariant is now applied to the knot invariant not only in the three-sphere S^3, but also in the lens space $L(2,1)$ [30, 31]. Since the ABJM matrix model is perturbatively equivalent to Chern–Simons theory on the lens space $L(2,1)$, the perturbative analysis of the knot invariant for $L(2,1)$, which is based on the topological recursion, is apparently relevant to our supermatrix model for the torus knot introduced in Sec. 3. Although we can obtain a systematic expansion of the correlation functions, we have to also take into account non-perturbative contributions, which play an important role in knot theory [15, 16] and ABJM theory [41]. The study of such a non-perturbative effect on the torus knots is an interesting and important issue to be clarified in the future.

6.3 Matrix models

We comment on the idea underlying this work, which comes from random matrices. In random matrix theory, the wave functions are expectation values of characteristic polynomials

$$\psi(x) = \Big\langle \det(x - M) \Big\rangle \,, \tag{6.6}$$

where the expectation value is taken with respect to a certain measure, which is specified later. The Hamiltonian associated with this wave function is given by the non-commutative Riemann surface, which is obtained through quantization of the spectral curve. In fact, a more important object is the kernel

$$\mathcal{K}(x;y) = \frac{1}{y-x} \left\langle \frac{\det(x-M)}{\det(y-M)} \right\rangle \,. \tag{6.7}$$

Out of that kernel one can reconstruct every other observable. For instance, if the matrix size is given by N, the wave function is obtained by sending $y \to \infty$,

$$\psi(x) = \lim_{y \to \infty} y^{N+1} \, \mathcal{K}(x;y) \,. \tag{6.8}$$

Every other correlations of characteristic polynomials is obtained by the Fay identity [11] (See also [12, 13])

$$\left\langle \prod_{i=1}^{k} \frac{\det(x_i - M)}{\det(y_i - M)} \right\rangle = \frac{\prod_{i,j}(y_i - x_j)}{\prod_{i<j}(x_i - x_j)\,(y_i - y_j)} \det_{1 \le i,j \le k} \mathcal{K}(x_i; y_j) \,, \tag{6.9}$$

which is also seen as Plücker or Hirota equation. The factor in front of the determinant of the kernel can be written as the Cauchy determinant (2.16).

Wave functions and kernels can be seen themselves as partition functions. Indeed let

$$\mathcal{Z} = \int d\mu(M) \tag{6.10}$$

[4] Let us note that another kind of approach to the B-model description, which is in principle applicable to any knots, is discussed based on the A-polynomial [40].

be a matrix integral with some measure $d\mu(M)$ depending on a coupling constant g_s. In many cases, there is a small g_s expansion of the type

$$\log \mathcal{Z} = \sum_{g=0}^{\infty} g_s^{2g-2}\, F_g(\mathcal{C}) \tag{6.11}$$

where $\mathcal{C}$ is the spectral curve associated to the measure $d\mu$, i.e. the $g_s \to 0$ limit of the eigenvalue distribution.

An expectation value of characteristic polynomials can be written

$$\mathcal{K}(x_1,\ldots,x_k;y_1,\ldots,y_k) = \frac{1}{\mathcal{Z}} \int d\mu(M) \prod_{i=1}^{k} \frac{\det(x_i - M)}{\det(y_i - M)}\,, \tag{6.12}$$

i.e. it is the partition function with a new measure

$$d\mu_{\boldsymbol{x};\boldsymbol{y}}(M) = \exp \sum_{i=1}^{k} \Big(\mathrm{Tr}\,\log(x_i - M) - \mathrm{Tr}\,\log(y_i - M) \Big)\, d\mu(M)\,. \tag{6.13}$$

There is also a g_s expansion with a new spectral curve

$$\log \mathcal{K}(x_1,\ldots,x_k;y_1,\ldots,y_k) = \sum_{g=0}^{\infty} g_s^{2g-2}\, (F_g(\mathcal{C}_{\boldsymbol{x};\boldsymbol{y}}) - F_g(\mathcal{C}))\,. \tag{6.14}$$

The way to find the new spectral curve is as follows. The spectral curve $\mathcal{C}$ is a Riemann surface equipped with two analytic functions u and v as $\mathcal{C} = \{(u,v) \in \mathbb{C} \times \mathbb{C} \,|\, H(u,v) = 0\}$. The new spectral curve $\mathcal{C}_{\boldsymbol{x};\boldsymbol{y}}$ is a Riemann surface such that vdu has additional simple poles at the x_i's (residue $+1$) and at the y_i's (residue -1).

Instead of characteristic polynomials one may also be interested in external fields interactions

$$\left\langle \mathrm{e}^{\mathrm{Tr}\, MA} \right\rangle = \frac{1}{\mathcal{Z}} \int d\tilde{\mu}(M)\ \mathrm{e}^{\mathrm{Tr}\, MA}\,. \tag{6.15}$$

Again, this has (under some assumptions) a g_s expansion with a new spectral curve $\mathcal{C}_A$

$$\log \left\langle \mathrm{e}^{\mathrm{Tr}\, MA} \right\rangle = \sum_{g=0}^{\infty} g_s^{2g-2}\ (F_g(\mathcal{C}_A) - F_g(\mathcal{C}))\ . \tag{6.16}$$

The way to find the new spectral curve is as follows. The spectral curve $\mathcal{C}$ is a Riemann surface equipped with two analytic functions u and v. The new spectral curve $\mathcal{C}_A$ is a Riemann surface such that udv has additional simple poles at the a_i's (the eigenvalues of A) with a residue equal to the multiplicity of a_i.

We see that under the exchange $u \leftrightarrow v$ and by identifying the a_i's with the x_i's (multiplicity $\alpha_i = 1$) and the y_i's (multiplicity $\alpha_i = -1$), this would be the same spectral curve and thus

$$\left\langle \mathrm{e}^{\mathrm{Tr}\, MA} \right\rangle_{\tilde{\mu}} \propto \left\langle \prod_{i=1}^{k} \det(x_i - M)^{\alpha_i} \right\rangle_{\mu} \tag{6.17}$$

where the measures μ and $\tilde{\mu}$ are obtained by exchanging the role of u and v in the spectral curves, and A is a matrix with eigenvalues x_i with multiplicity α_i [14, 42, 43]. This raises the question of how can a multiplicity be negative? This is why supermatrix models are needed. Negative multiplicities correspond to the fermionic side of the supermatrix [44]. This duality between external field and expectation values of characteristic polynomials appeared in [45].

As shown in (2.26), the half BPS Wilson loop expectation value in ABJM theory has a quite similar expression to that of matrix integral with the external source. The Gaussian matrix model with the external source is given as follows,

$$\begin{aligned} \left\langle e^{\operatorname{Tr} MA} \right\rangle &= \frac{1}{\mathcal{Z}} \int dM \, e^{-\frac{1}{2g_s}\operatorname{Tr} M^2 + \operatorname{Tr} MA} \\ &= \frac{1}{\mathcal{Z}} \frac{1}{\Delta(a)} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi} \, e^{-\frac{1}{2g_s}x_i^2 + x_i a_i} \prod_{i<j}^{N} (x_i - x_j) \,, \end{aligned} \tag{6.18}$$

where the U(N) part is integrated out using the Harish-Chandra–Itzykson–Zuber formula [46, 47]. Writing the U(N) character in terms of the Schur function

$$s_\lambda(e^x) = \frac{1}{\Delta(e^x)} \det_{1\le i,j\le N} e^{x_i(\lambda_j + N - j)} \,, \tag{6.19}$$

we obtain the Wilson loop expectation value with respect to the U(N) Chern–Simons theory

$$\Big\langle W_R(\bigcirc) \Big\rangle_{\mathrm{U}(N)} = \frac{1}{\mathcal{Z}_{\mathrm{CS}}} \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi} \, e^{-\frac{1}{2g_s}x_i^2 + x_i \tilde{\xi}_i} \prod_{i<j}^{N} \left(2 \sinh \frac{x_i - x_j}{2} \right) , \tag{6.20}$$

with $\tilde{\xi}_i = \lambda_i - i + \frac{N+1}{2}$. In this sense the insertion of the Wilson loop operator corresponds to the external fields for the matrix integral.

This kind of interpretation can be possible even for the supergroup character average (2.26), at least as long as the partition λ, corresponding to the representation R, satisfies $d(\lambda) = N$.[5] All this means that characters of U($N|N$) are dual to expectation values of characteristic polynomials of another matrix model with some measure μ,

$$\Big\langle W_R(K) \Big\rangle_{\mathrm{U}(N|N)} = \left\langle \prod_{i=1}^{N} \frac{\det(\alpha_i - M)}{\det(\beta_i - M)} \right\rangle_{\mu} , \tag{6.22}$$

[5]When the representation does not satisfy this condition, there is no simple analogy between the matrix integral with the external fields and the Wilson loop average (2.33), because such an external field has to consist of $N + N$ parameters for U($N|N$) theory. On the other hand, this analogy holds for arbitrary representations in the ordinary U(N) Chern–Simons theory as (6.20). It is because even if the number of non-zero elements in the partition is less than N at the first place, it can be made N by the constant shift, since the average (2.30) is invariant under the shift of the partition

$$(\lambda_1, \lambda_2, \cdots, \lambda_N) \longrightarrow (\lambda_1 + c, \lambda_2 + c, \cdots, \lambda_N + c) \,. \tag{6.21}$$

It is obvious that the U($N|N$) invariant (2.26) is not invariant under such a constant shift of the partition, since the corresponding external field is characterized by the Frobenius coordinates of the partition.

and where α_i, β_i are the Frobenius coordinates for the representation R. Then the Fay identity says that

$$\Big\langle W_R(K) \Big\rangle_{\mathrm{U}(N|N)} = \det_{1\le i,j\le N} \Big\langle W_{(\alpha_i|\beta_j)}(K) \Big\rangle_{\mathrm{U}(1|1)} . \tag{6.23}$$

This is what we have checked in this article, especially as the determinantal formula for the unknot average shown in (2.28). This means that those supergroup character expectation values are Giambelli compatible [12].

7 Discussion

In this paper we have considered the supergroup character expectation value based on the ABJM matrix model as the supermatrix Chern–Simons theory, with emphasis on its connection to the knot invariant. We have explicitly computed the U(1|1) expectation value for the unknot and torus knot matrix models. We have obtained the determinantal formula, where the U(1|1) average plays a role of a building block for $\mathrm{U}(N|N)$ theory, and shown the Rosso–Jones-type formula for the supergroup character average. We have also discussed the $\mathrm{U}(N|M)$ theory, and found its determinantal formula which interpolates $\mathrm{U}(N)$ and $\mathrm{U}(N|N)$ theories. We have derived the spectral curve for the torus knot as the symplectic transform of the unknot curve, and by analyzing the saddle point equations for the torus knot matrix model itself. We have shown these two methods to obtain the spectral curve are consistent with each other. We have then commented on how to realize the knot invariant in topological string theory, and the underlying idea coming from random matrix theory.

Let us comment on some open issues to be investigated in the future. The most attractive one is to check wheather the supergroup character average can be a knot invariant or not. We have certain evidence on this point. First of all, the $\mathrm{U}(N|N)$ character expectation value contains $\mathrm{U}(N)$ part as shown for the unknot Wilson loop (2.29). It is expected that the $\mathrm{U}(N|N)$ average can be reduced to the HOMFLY polynomial in this way. Secondly the ABJM matrix model is obtained from the lens space Chern–Simons theory through the analytic continuation. Since the knot invariant in the lens space is given as the character average with the corresponding matrix model, we can expect the character expectation value with the supermatrix model is also a knot invariant, at least up to some non-perturbative contributions. Moreover, the construction of the knot invariant shown in Sec. 6.1 seems natural from the viewpoint of the topological string, and possibly applied to an arbitrary knot. These supporting facts suggest that we can obtain a knot invariant from ABJM theory. For this purpose, for example, it is interesting to see whether the torus knot supermatrix model can be directly obtained from ABJM theory. In the case of classical group theory, it is shown by [48] that the torus knot matrix model is obtained from $\mathcal{N} = 2$ Chern–Simons theory on the ellipsoid-type squashed three-sphere S_b^3. In this sense, it is expected that the supergroup torus knot character average is given by the half BPS Wilson loop for the $\mathcal{N} = 6$

ABJM theory on S_b^3. However the integral formula (3.8) is not obtained naively using the localization method for S_b^3, because the matter contribution cannot be written as a simple cosh function for such a case [49]. Probably we have to turn on the flux coupled to the matter sector, or modify the supersymmetry transformation.

If we can have the supergroup knot invariant, it is also interesting to provide another definition of the U($N|N$) knot invariant, for example, based on the Skein relation, which enable us to compute the invariant for any knots in principle. It is naively expected that such a relation can be related to the supergroup WZW model. However, it is known that the U(1|1) WZW model describes the Skein relation for the Alexander–Conway polynomial [50, 51, 52], while the torus knot average obtained in this paper (3.12) is not consistent with that. This reflects the fact that ABJM theory is not just Chern–Simons theory with the supergroup. We have to explore other possibility of the two-dimensional CFT model, which shall live on the boundary of ABJM theory, if it exists. Furthermore, as the recent progress in knot theory, it is definitely interesting to study the volume conjecture and its generalization [53, 54, 55], the AJ conjecture [56], and also the knot homology [57, 58, 59, 60, 61] corresponding to the supergroup knot invariant. In order to discuss the volume conjecture, we have to deal with the hyperbolic knot and the volume of its complement in S^3. Therefore our construction, which is so far available only for the unknot and torus knots, is not yet enough to study this conjecture. Also from this point of view, the definition of the knot invariant based on the Skein relation is highly desirable, as commented above. The AJ conjecture claims that the knot invariant satisfies some integrable equations, which are obtained through quantization of the A-polynomial. In other words, this implies that the knot invariant is associated with the corresponding τ-function. As pointed out in Sec. 6, the knot invariant is closely related to the matrix integral with the external fields, which can be seen as a certain τ-function. For example, the determinantal formula for the character expectation value given in this paper can be one of the supporting results for such a suggestive relation. The determinantal formula also gives an insight into the knot homology. Using the Jacobi identity for determinants, one can obtain some relations between the knot invariants for different rank groups. Such a relation could provide a natural differential on the knot homology.

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A Mirror of the torus knot ABJM partition function

In this appendix we derive the mirror description of the torus knot ABJM partition function [25]. We first expand the determinants in (3.8) as summation over permutations

$$\mathcal{Z}^{(P,Q)}_{\rm ABJM}=\sum_{\sigma,\sigma'\in\mathfrak{S}_N}(-1)^{\sigma+\sigma'}\frac{1}{N!^2}\int[dx]^N[dy]^N\prod_{i=1}^N\left(2\cosh\frac{x_i-y_{\sigma(i)}}{2P}\,2\cosh\frac{x_i-y_{\sigma'(i)}}{2Q}\right)^{-1}. \tag{A.1}$$

Applying the formula (2.20), we have

$$\begin{aligned}&\sum_{\sigma,\sigma'\in\mathfrak{S}_N}(-1)^{\sigma+\sigma'}\frac{1}{N!^2}\int\frac{d^Nx}{(2\pi)^N}\frac{d^Ny}{(2\pi)^N}\frac{d^Nz}{(2\pi)^N}\frac{d^Nw}{(2\pi)^N}\prod_{i=1}^N(\cosh z_i\,\cosh w_i)^{-1}\\&\qquad\times\exp\left[\frac{ik}{4PQ\pi}\sum_{i=1}^N(x_i^2-y_i^2)+\frac{i}{\pi}\sum_{i=1}^N\left(x_i\left(\frac{z_i}{P}+\frac{w_i}{Q}\right)-y_i\left(\frac{z_{\sigma^{-1}(i)}}{P}+\frac{w_{\sigma'^{-1}(i)}}{Q}\right)\right)\right]\\&=\sum_{\sigma,\sigma'\in\mathfrak{S}_N}(-1)^{\sigma+\sigma'}\frac{(PQ)^N}{N!^2\,k^N}\int\frac{d^Nz}{(2\pi)^N}\frac{d^Nw}{(2\pi)^N}\prod_{i=1}^N\frac{\exp\left[-\frac{2i}{k\pi}\left(z_iw_i-z_{\sigma^{-1}(i)}w_{\sigma'^{-1}(i)}\right)\right]}{\cosh z_i\,\cosh w_i}.\end{aligned} \tag{A.2}$$

At this moment it is obvious that the partition function depends only on the composition of permutations $\sigma\cdot\sigma'^{-1}$. Thus, by fixing either of them σ' as the trivial permutation, we obtain

$$\begin{aligned}\mathcal{Z}^{(P,Q)}_{\rm ABJM}&=\sum_{\sigma\in\mathfrak{S}_N}(-1)^N\frac{(PQ)^N}{N!\,k^N}\int\frac{d^Nz}{(2\pi)^N}\frac{d^Nw}{(2\pi)^N}\prod_{i=1}^N\frac{\exp\left[-\frac{2i}{k\pi}\left(z_i-z_{\sigma(i)}\right)w_i\right]}{\cosh z_i\,\cosh w_i}\\&=\sum_{\sigma\in\mathfrak{S}_N}(-1)^N\frac{(PQ)^N}{N!\,k^N}\int\frac{d^Nz}{(2\pi)^N}\prod_{i=1}^N\left(\cosh z_i\cdot 2\cosh\frac{z_i-z_{\sigma(i)}}{k}\right)^{-1}\\&=\frac{(PQ)^N}{N!(2k)^N}\int\frac{d^Nz}{(2\pi)^N}\prod_{i<j}\left(\tanh\frac{z_i-z_j}{2k}\right)^2\prod_{i=1}^N\left(2\cosh\frac{z_i}{2}\right)^{-1}\\&=(PQ)^N\mathcal{Z}^{(1,1)}_{\rm ABJM}.\end{aligned} \tag{A.3}$$

This is the mirror expression of the partition function (3.8). Especially for $k=1$, the mirror theory turns out to be $\mathcal{N}=4$ SYM theory with a single fundamental and a single adjoint hypermultiplet. The dependence on the parameters (P,Q) becomes obvious in this mirror representation.

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