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Simplifications of the Keiper/Li approach to the Riemann Hypothesis

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Abstract

The Keiper/Li constants $\{\lambda_n\}_{n=1,2,\dots}$ are asymptotically ($n \rightarrow \infty$) sensitive to the Riemann Hypothesis, but highly elusive analytically and difficult to compute numerically. We present quite explicit variant sequences that stay within the abstract Keiper–Li frame, and appear simpler to analyze and compute.

The present work develops results that we announced in 2015. [26]

1 Generalities and notations

We use the standard basic notions (e.g., [8, chap. 8]):

$\zeta(x)$: the Riemann zeta function (analytic over $\mathbb{C} \setminus \{+1\}$);

$2\xi(x)$: a completed zeta function, with its Riemann's Functional Equation:

$$2\xi(x) \stackrel{\text{def}}{=} x(x-1)\pi^{-x/2}\Gamma(x/2)\zeta(x) \equiv 2\xi(1-x) \quad (1)$$

(this *doubled* Riemann's ξ -function is better normalized: $2\xi(0) = 2\xi(1) = 1$).

$\{\rho\}$: the set of zeros of ξ (i.e., the nontrivial zeros of ζ , or *Riemann zeros*, counted with multiplicities if any, and grouped in pairs $(\rho, 1-\rho)$ in the sums that we write as $\sum_{\langle\rho,1-\rho\rangle}$); they all lie in the strip $\{0 < \text{Re } x < 1\}$.

The **Riemann Hypothesis (RH)** : *all* the Riemann zeros lie on the *critical line* $L \stackrel{\text{def}}{=} \{\text{Re } x = \frac{1}{2}\}$.

$k!!$: the double factorial, used here for odd integers k only, in which case

$$\begin{aligned} k!! &\stackrel{\text{def}}{=} k(k-2)\cdots 1 && \text{for odd } k > 0, \\ &\stackrel{\text{def}}{=} 2^{(k+1)/2} \Gamma(\frac{1}{2}k + 1)/\sqrt{\pi} && \text{for odd } k \geq 0 \quad (\text{e.g., } (-1)!! = 1). \end{aligned} \quad (2)$$

B_{2m} : the Bernoulli numbers; γ : Euler's constant.

1.1 The Keiper and Li coefficients

In 1992 Keiper [15] considered a real sequence $\{\lambda_n\}$ of generating function

$$f(z) \stackrel{\text{def}}{=} \log 2\xi(M(z)) \equiv \sum_{n=1}^{\infty} \lambda_n^K z^n, \quad M(z) \stackrel{\text{def}}{=} \frac{1}{1-z}, \quad (3)$$

(λ_n^K : our notation for *Keiper's* λ_n), deduced that

$$\lambda_n^K \equiv n^{-1} \sum_{\langle \rho, 1-\rho \rangle} [1 - (1 - 1/\rho)^n], \quad (4)$$

and argued that, under RH, $\lambda_n^K > 0$ ($\forall n$) and moreover “if [...] the zeros are very evenly distributed, we can show that” [this without proof]

$$\lambda_n^K \approx \frac{1}{2} \log n + c, \quad c = \frac{1}{2}(\gamma - \log 2\pi - 1) \approx -1.130330700754. \quad (5)$$

In (3), the conformal mapping $M : x = (1 - z)^{-1}$ acts to pull back the critical line L to the unit circle $\{|z| = 1\}$, with the fundamental consequence:

$$\text{RH} \iff f \text{ regular in the whole open unit disk } \{|z| < 1\}. \quad (6)$$

Then, (3) specifies the sequence $\{\lambda_n\}$ as a particular encoding of the germ of $\log 2\xi(x)$ at the “basepoint” $\underline{x} = M(0)$ (here: $\underline{x} = 1$).

In 1997 Li [17] independently introduced another sequence λ_n , through

$$\lambda_n^L = \frac{1}{(n-1)!} \frac{d^n}{dx^n} [x^{n-1} \log 2\xi(x)]_{x=1}, \quad n = 1, 2, \dots \quad (\lambda_n^L \stackrel{\text{def}}{=} \text{Li's } \lambda_n), \quad (7)$$

deduced that

$$\lambda_n^L \equiv \sum_{\langle \rho, 1-\rho \rangle} [1 - (1 - 1/\rho)^n], \quad (8)$$

and proved the *sharp equivalence*: $\text{RH} \iff \lambda_n^{\text{L}} > 0$ for all n (*Li's criterion*).
 Actually, by comparing (4) vs (8) for instance,

$$\lambda_n^{\text{L}} \equiv n \lambda_n^{\text{K}} \quad \text{for all } n = 1, 2, \dots; \quad (9)$$

our superscripts K vs L will disambiguate λ_n whenever the factor n matters.

1.2 Probing RH through the Keiper–Li $\{\lambda_n\}$

In 2000 Oesterlé proved (but left unpublished) [21] that RH alone implies

$$\lambda_n^{\text{L}} = n(\tfrac{1}{2} \log n + c) + o(n), \quad \text{with } c = \tfrac{1}{2}(\gamma - \log 2\pi - 1) \text{ as in (5)}. \quad (10)$$

In 2004–2006, using the saddle-point method on an integral form of λ_n , we gave an *asymptotic criterion* for RH [23][24] in the form of this alternative:

- RH false: $\lambda_n^{\text{L}} \sim - \sum_{\text{Re } \rho > 1/2} (1 - 1/\rho)^{-n} \pmod{o(r^n) \forall r > 1}; \quad (11)$

- RH true: $\lambda_n^{\text{L}} \sim n(\tfrac{1}{2} \log n + c) \pmod{o(n)} \quad (12)$

(erratum: we had the sign wrong in the case RH false, which did not affect the purely qualitative consequences we drew at the time; correction in [25]).

In 2007 Lagarias [16] strengthened (10) by improving $o(n)$ to $O(\sqrt{n} \log n)$.
 In 2011 Arias de Reyna [1] further improved $o(n)$ to $n y_n$ with $\{y_n\} \in \ell^2$.

To assess how the above criteria may advance the testing of RH, one must bring in the height T_0 up to which RH is confirmed by direct means:

$$T_0 \approx 2.4 \cdot 10^{12} \quad \text{currently (since 2004)}. \quad [12] \quad (13)$$

It is then known that: first, no λ_n can go negative as long as $n < T_0^2$ [21][4, § 2.3]; and more broadly, if a zero $\rho = \tfrac{1}{2} \pm t \pm iT$ violates RH (with $t > 0$, $T > T_0$), then no effect of that will be detectable upon the λ_n unless [24]

$$n \gtrsim T^2/t > 2T_0^2 \quad (\text{since } t < \tfrac{1}{2}), \text{ currently implying } n \gtrsim 10^{25} \quad (14)$$

($n \gtrsim T^2/t$ actually states the *uncertainty principle* in the relevant geometry).

At the same time, the λ_n are quite elusive analytically [5][7], and also numerically (see Maślanka [18][19] and Coffey [6]) as their evaluation requires a recursive machinery, whose intricacy grows very rapidly with n , and which

moreover destroys ca. 0.2 decimal place of precision per step n (if done *ex nihilo* - i.e., using no Riemann zeros as input) [19, fig. 6]; only λ_n -values up to $n \approx 4000$ were thus accessed. Upon posting this ms, we learnt from F. Johansson that he recently attained $n = 10^5$ [14, § 4.2] (seeing a loss of 1 bit ≈ 0.3 decimal place per step n). Now even this big improvement stays well below the range (14) needed for up-to-date testing of RH.

2 An *explicit* variant sequence $\{\Lambda_n\}$

We propose to deform the $\{\lambda_n^K\}$ (in Keiper’s normalization (3)) into a simpler sequence $\{\Lambda_n\}$ having a *totally closed form*. The original λ_n appeared rigidly specified, but only inasmuch as the pole $x = 1$ of $\zeta(x)$ was invariably made *the* basepoint. Now while this choice can make sense, it is by no means compulsory. On the contrary, other conformal mappings than M in (3) realize the Keiper–Li *idea* (RH-sensitivity, embodied in (6)) just as well: the key condition is that all Riemann zeros on L must pull back to $\{|z| = 1\}$, achieving (6), while nothing binds the basepoints \underline{x} to which $z = 0$ can map; the resulting λ_n will just vary with \underline{x} as functions of the derivatives $\xi^{(m)}(\underline{x})$. As such, Sekatskii’s “generalized Li’s sums” [22] have $\underline{x} = (1 - a) \in \mathbb{R} \setminus \{\frac{1}{2}\}$, whereas our “centered” λ_n^0 were tailored to have $\underline{x} = \frac{1}{2}$, the symmetry center for $\xi(x)$ ([25, § 3.4], and Appendix). Our next construction will push this idea of deformation even further, and have no single distinguished basepoint (except, loosely, $\underline{x} = \infty$?): we will substitute selected *finite differences* for the derivatives of $\log \xi$ that enter the original λ_n (and, in the Appendix, our centered $\lambda_n^{(0)}$), to attain more explicit sequences.

2.1 Construction of the new sequence

The original definition (3) is equivalent, by the residue theorem, to the contour integral formula

$$\lambda_n^K = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} f(z), \quad f(z) \equiv \log 2\xi\left(\frac{1}{1-z}\right), \quad (15)$$

with a positive contour in the unit disk around $z = 0$ excluding all other singularities (i.e., those of f). Derivatives of $\log 2\xi(x)$ up to order n occur in λ_n because the denominator z^{n+1} has all its zeros degenerate (at $z = 0$).

Now at given n , if we *split those zeros apart* as $0, z_1, \dots, z_n$ (all distinct, and still inside the contour), then the so modified integral evaluates to a linear combination of the $f(z_m)$: derivatives become *finite differences*. To split the zeros, instead of plain shifts of the factors $z \mapsto z - z_m$ which fail to preserve the all-important unit disk, we use *hyperbolic* translates

$$z \mapsto B_{z_m}(z) = (z - z_m)/(1 - z_m^* z) \quad (\text{Möbius transformations}). \quad (16)$$

The point $z = 0$ has now lost its special status, hence so does the particular mapping M (picked for pulling back the pole $x = 1$ to $z = 0$), so that the variable x , natural for the ζ -function, also becomes the simplest to use. Then (15) expresses as

$$\lambda_n^K = \frac{1}{2\pi i} \oint \frac{dx}{x(x-1)} \left(\frac{x}{x-1}\right)^n \log 2\xi(x) \quad (\text{integrated around } x = 1), \quad (17)$$

and the deformations as above read as

$$\frac{1}{2\pi i} \oint_{\mathcal{C}_n} \frac{dx}{x(x-1)} \frac{1}{b_{x_1}(x) \dots b_{x_n}(x)} \log 2\xi(x), \quad b_{\tilde{x}}(x) \equiv \frac{\tilde{x}^*}{\tilde{x}} \frac{x - \tilde{x}}{x + \tilde{x}^* - 1}, \quad (18)$$

where the contour \mathcal{C}_n encircles the points $1, x_1, \dots, x_n$ positively (and may as well depend on n). Now the integral in (18) readily evaluates to

$$\sum_{m=1}^n \frac{1}{x_m(x_m - 1)} \frac{1}{[b_{x_1} \dots b_{x_n}]'(x_m)} \log 2\xi(x_m) \quad (19)$$

by the residue theorem ($x = 1$ contributes zero since $\log 2\xi(1) = 0$).

Finally, for each n we select $x_m \equiv 2m$ for $m = 1, 2, \dots$ (independently of n) to benefit from the known values $\zeta(2m)$, and a contour \mathcal{C}_n just encircling the real interval $[1, 2n]$ positively (encircling the subinterval $[2, 2n]$ would suffice, however here it will always be of interest to dilate, not shrink, \mathcal{C}_n). All that fixes the sequence

$$\Lambda_n \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_{\mathcal{C}_n} \frac{dx}{x(x-1)} G_n(x) \log 2\xi(x), \quad (20)$$

$$G_n(x) \stackrel{\text{def}}{=} \prod_{m=1}^n \frac{x + 2m - 1}{x - 2m} \equiv \frac{\Gamma(\frac{1}{2}x - n) \Gamma(\frac{1}{2}(x+1) + n)}{\Gamma(\frac{1}{2}x) \Gamma(\frac{1}{2}(x+1))} \quad (21)$$

$$\equiv g(x)(-1)^n \frac{\Gamma(\frac{1}{2}(x+1) + n)}{\Gamma(1 - \frac{1}{2}x + n)}, \quad g(x) \stackrel{\text{def}}{=} \frac{\sqrt{\pi} 2^{x-1}}{\sin(\pi x/2) \Gamma(x)} \quad (22)$$

(by the duplication and reflection formulae for Γ). For this case, (19) yields

$$\Lambda_n \equiv (-1)^n \sum_{m=1}^n (-1)^m A_{nm} \log 2\xi(2m), \quad n = 1, 2, \dots, \quad (23)$$

with

$$\begin{aligned} A_{nm} &= \frac{2^{-2n}}{2m-1} \binom{2(n+m)}{n+m} \binom{n+m}{2m} \equiv \frac{2^{m-n} (2(n+m)-1)!!}{(2m-1)(n-m)!(2m)!} \\ &\equiv \frac{2^{2m} \Gamma(n+m+1/2)}{(2m-1)(n-m)!(2m)! \sqrt{\pi}} \quad (\text{for } m = 0, 1, 2, \dots), \quad (24) \end{aligned}$$

$$2\xi(2m) = \frac{|B_{2m}|}{|(2m-3)!!|} (2\pi)^m \equiv \frac{2|B_{2m}|}{|\Gamma(m-\frac{1}{2})|} \pi^{m+1/2} \quad (25)$$

(the absolute values in the last two denominators only act for $m = 0$, resulting in $\log 2\xi(0) = 0$ which vanishes thereafter).

So, this particular deformation $\{\Lambda_n\}$ of Keiper's $\{\lambda_n^K\}$ is specified by (23) in a *totally explicit* form (and fairly uniquely dictated as above). With no recursion involved, any single Λ_n can be computed straight away and by itself, in welcome contrast to the original λ_n .

Remarks.

1) $\sum_{m=1}^n (-1)^m A_{nm} m$ is computable by the second sum rule (30) below (with $A_{n0} \equiv -2^{-n}(2n-1)!!/n!$ by (24)); the $(\log 2\pi)$ -contributions to (23) from the first expression (25) can thereby be summed, resulting in $\Lambda_n \equiv \frac{1}{2} \log 2\pi + u_n$ with

$$u_n \stackrel{\text{def}}{=} (-1)^n \left[\sum_{m=1}^n (-1)^m A_{nm} \log \frac{|B_{2m}|}{(2m-3)!!} + \frac{1}{2A_{n0}} \log 2\pi \right]: \quad (26)$$

it was through this sequence $\{u_n\}$ that we earlier announced our results [26]. Likewise, the last expression (25) leads to the partially summed form

$$\begin{aligned} \Lambda_n &\equiv \frac{1}{2} \log \pi + (-1)^n \left[\sum_{m=1}^n (-1)^m A_{nm} \log \frac{|B_{2m}|}{\Gamma(m-\frac{1}{2})} \right. \\ &\quad \left. + \left(\frac{1}{A_{n0}} - A_{n0} \right) \log 2 + \left(\frac{1}{A_{n0}} - \frac{A_{n0}}{2} \right) \log \pi \right]. \quad (27) \end{aligned}$$

2) If in place of (25) we use (1) and the expanded logarithm of the Euler product: $\log \zeta(x) \equiv \sum_p \sum_{r=1}^{\infty} p^{-rx}/r$ ($x > 1$) where p runs over the *primes*, then

(23) yields an *arithmetic* form for Λ_n , in analogy with Bombieri–Lagarias’s Thm 2 for λ_n^L . [5, § 3]

3) Báez-Duarte’s sequential criterion for RH [2] is similarly explicit in terms of the Bernoulli numbers, but there, any effect of RH-violating zeros seems hopelessly tiny until inordinately large $n \gtrsim e^{\pi T_0}$ [20, § 4][11, § 7] (the latter quotes $n \gtrsim 10^{600,000,000}$).

4) With L-functions for real primitive Dirichlet characters χ in place of ζ , [8, chaps. 5, 6, 9] the whole argument carries over, essentially unchanged for χ even, whereas

$$\Lambda_{\chi,n} = (-1)^n \sum_{m=1}^n (-1)^m \frac{2^{m-n} (2(n+m)+1)!!}{(2m+1)(n-m)!(2m+1)!} \log \xi_{\chi}(2m+1) \quad (28)$$

for χ odd, where $\xi_{\chi}(x)$ is the completed L_{χ} -function (normalized to $\xi_{\chi}(0) \equiv \xi_{\chi}(1) = 1$, like $2\xi(x)$ for ζ in (1)), whose values at $x = 2m+1$ are explicit.

2.2 Expression of Λ_n in terms of the Riemann zeros

Let the primitive

$$\begin{aligned} F_n(x) &\stackrel{\text{def}}{=} \int_{\infty}^x \frac{G_n(y)}{y(y-1)} dy \\ &\equiv (-1)^n \left[-\frac{1}{A_{n0}} \log(x-1) + \sum_{m=0}^n (-1)^m A_{nm} \log(x-2m) \right] \end{aligned} \quad (29)$$

be defined as single-valued from a neighborhood of $x = \infty$ to the whole x -plane minus the cut $[0, 2n]$. E.g., $F_1(x) = \frac{1}{2} \log [x(x-2)^3/(x-1)^4]$; and for general n , (29) follows from, e.g., [13, §2.102] using the A_{nm} from (24).

For $x \rightarrow \infty$, $F_n(x) \sim \int_{\infty}^x dy/y^2 = -1/x$; the consistency of this with (29) imposes the identities

$$\sum_{m=0}^n (-1)^m A_{nm} \equiv \frac{1}{A_{n0}}, \quad 2 \sum_{m=1}^n (-1)^m A_{nm} m \equiv (-1)^n + \frac{1}{A_{n0}}. \quad (30)$$

In terms of (29), the Λ_n result by summing the following series over the zeros (converging like $\sum_{\langle \rho, 1-\rho \rangle} 1/\rho$ for any n):

$$\Lambda_n \equiv \sum_{\langle \rho, 1-\rho \rangle} F_n(\rho), \quad n = 1, 2, \dots \quad (31)$$

(For the original λ_n^K , (29) uses $[x/(x-1)]^n$ in place of G_n by (17), exceptionally yielding *rational* functions: $n^{-1}[1 - (1 - 1/(1-x))^n]$, for which (31) restores (4).)

Proof of (31) (condensed, see fig. 1): first stretch the contour \mathcal{C}_n in (20) to \mathcal{C}'_n fully enclosing the cut $[0, 2n]$ of F_n (as allowed by $\log 2\xi(0) = 0$). Since F_n is single-valued on \mathcal{C}'_n , the so modified (20) can be integrated by parts,

$$\Lambda_n \stackrel{\text{def}}{=} -\frac{1}{2\pi i} \oint_{\mathcal{C}'_n} F_n(x) \left[\frac{\xi'}{\xi} \right] (x) dx, \quad (32)$$

then the contour \mathcal{C}'_n can be further deformed into a sum of an outer anti-clockwise circle \mathcal{C}_R centered at $\frac{1}{2}$ of radius $R \rightarrow \infty$ (not drawn), and of small clockwise circles around the poles of the meromorphic function ξ'/ξ inside \mathcal{C}_R ; these poles are the Riemann zeros ρ , and each contributes $F_n(\rho)$. By the Functional Equation (1), the integral on \mathcal{C}_R is also $\oint_{\mathcal{C}_R} \frac{1}{2} [F_n(x) + F_n(1-x)] \left[\frac{\xi'}{\xi} \right] (x) dx$, which tends to 0 if $R \rightarrow \infty$ staying far enough from ordinates of Riemann zeros in a classic way (so that $|\zeta'/\zeta|(s+iR) < K \log^2 R$ for all $s \in [-1, +2]$ [8, p. 108]), hence (31) results. \square

3 Criterion for RH based on the new sequence

We will sketch why the totally explicit sequence $\{\Lambda_n\}$ largely shares the sensitivity to RH of the highly elusive Keiper–Li sequence.

3.1 Asymptotic criterion

We will mainly argue an *asymptotic* sensitivity to RH as $n \rightarrow \infty$, through this alternative for $\{\Lambda_n\}$ which parallels (11)–(12) for $\{\lambda_n\}$:

- RH false: $\Lambda_n \sim \left\{ \sum_{\text{Re } \rho > 1/2} F_n(\rho) \right\} \pmod{o(n^\varepsilon) \forall \varepsilon > 0}, \quad (33)$

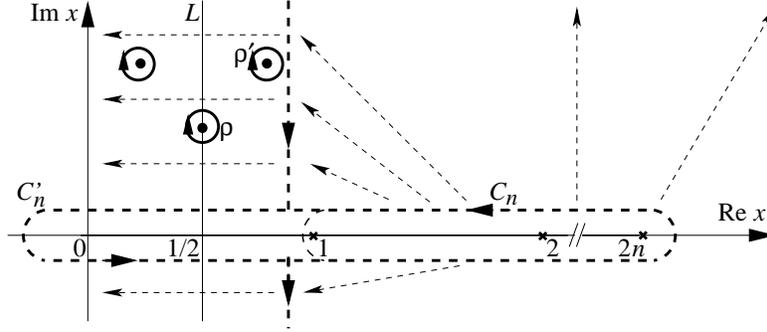


Figure 1: Deformation of the integration path for the integral (32) against the meromorphic function ξ'/ξ whose poles are the Riemann zeros, here exemplified - not on scale - by ρ (on the critical line), and ρ' (off the line, putative, shown with its partner across the critical line). A symmetrical lower half-plane is implied.

$$\text{and } F_n(\rho) \sim \frac{g(\rho)}{\rho(\rho-1)} (-1)^n \frac{n^{\rho-1/2}}{\log n} \quad (n \rightarrow \infty), \quad (34)$$

$$\implies |F_n(\rho)| \approx \frac{1}{|\text{Im } \rho|^2 \log n} \left(\frac{2n}{|\text{Im } \rho|} \right)^{\text{Re } \rho} \quad \text{for } n \gg |\text{Im } \rho| \gg 1. \quad (35)$$

- RH true: $\Lambda_n \sim \log n + C$, $C = \frac{1}{2}(\gamma - \log \pi - 1) \approx -0.783757110474$, (36)

the latter to be compared to (10), with $C = c + \frac{1}{2} \log 2$. As for (33), the summation converges if the terms with ρ and ρ^* are grouped together (as symbolized by the curly brackets), and more caveats are issued in § 3.2.

We give a condensed derivation. Past some common generalities, we will separate the cases RH true/false (short of a unified method as in [23]).

The general idea is nowadays known as large-order perturbative analysis or instanton calculus, but initially we just follow the pioneering Darboux's theorem [9, §7.2][3] to get the large-order behavior of Taylor series like (3) out of the integral form (15) or more simply, its integration by parts $\lambda_n^L = (2\pi i)^{-1} \oint z^{-n} f'(z) dz$ because f' is meromorphic whereas f has branch cuts. Then this integrand has the large- n form $e^{\Phi_n(z)}$ where Φ_n tends to ∞ with n ($\Phi_n(z) \sim -n \log z$), hence the steepest-descent method applies: [10, § 2.5] we deform the integration contour C toward decreasing $\text{Re } \Phi_n$, i.e. here, into a circle of radius growing toward 1 (fig. 2); then, each of the encountered singularities of f' , here simple poles $M^{-1}(\rho')$ for RH-violating zeros ρ' , yields an asymptotic contribution $-z_{\rho'}^{-n}$, all of which add up to (11). [23] If on

the other hand RH is true, then the contour can arbitrarily approach the unit circle, (11) stays empty, and only a finer analysis of the limiting integral ([21], recalled in § 3.3.1 below) leads to a definite asymptotic form, as (10).

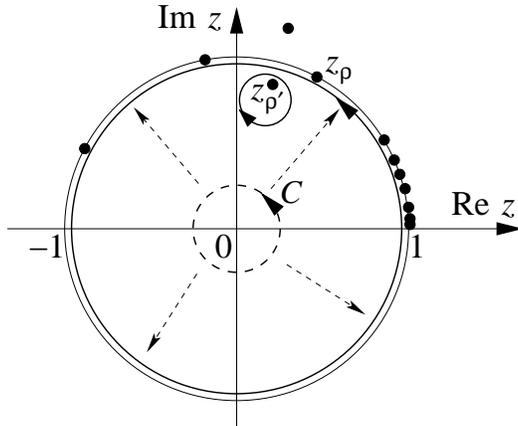


Figure 2: Contour deformation for λ_n^L , $n \rightarrow \infty$, in the steepest-descent method. (The symmetrical poles in the lower half-plane are not drawn.)

We then wish to do the same with an (x -plane) integral form for Λ_n , be it (20) (with the function $G_n(x)$ defined by (21)–(22)), or (32) (with $F_n(x)$ defined by (29)). Now (22) at once implies

$$G_n(x) \sim g(x)(-1)^n n^{x-1/2} \sim g(x)(-1)^n e^{\log n(x-1/2)} \quad \text{for } n \rightarrow \infty \text{ at fixed } x, \quad (37)$$

hence now the large asymptotic parameter is $\log n$ and the large- n level lines of the integrand are $\{\operatorname{Re} x = \frac{1}{2} + t_0\}$. For the steepest-descent method, $|z| \rightarrow 1^-$ in fig. 2 thus becomes $t_0 \rightarrow 0^+$. A new complication is that these level lines now all terminate at ∞ , an essential singularity. Temporarily ignoring this, we note that the contour deformation on (32) for Λ_n has already yielded (31), so we simply have to extract the asymptotically relevant part of $\sum_\rho F_n(\rho)$. For $n \rightarrow \infty$, $F_n(\rho)$ is to be expressed using a steepest-descent path [10, § 2.5], as

$$F_n(\rho) = \int_{-\infty}^{\rho} \frac{G_n(x)}{x(x-1)} dx \sim \int_{-\infty}^{\rho} \frac{g(x)}{x(x-1)} (-1)^n n^{x-1/2} dx : \quad (38)$$

a Laplace transform in the variable $\log n$, of asymptotic form [10, eq. 2.2(2)]

$$F_n(\rho) \sim \frac{g(\rho)}{\rho(\rho-1)} (-1)^n \frac{n^{\rho-1/2}}{\log n}. \quad (39)$$

Consequently, the removal of all $o(n^{t_0+\varepsilon})$ terms from (31) unconditionally leaves us with

$$\Lambda_n = \left\{ \sum_{\operatorname{Re} \rho > t_0} F_n(\rho) \right\} + o(n^{t_1}) \quad \text{for all } t_1 > t_0 \geq 0, \quad (40)$$

where the summation converges if the terms with ρ and ρ^* are grouped together (as symbolized by the curly brackets).

However, in the RH true case, (40) with $t_0 = 0$ delivers no better than $\Lambda_n = o(n^\varepsilon) \forall \varepsilon > 0$, and only a finer analysis of the limiting integral on the critical line L will lead to a definite asymptotic form, in § 3.3.2. Hence we pursue the case RH false first.

3.2 Details for the case RH false

If RH-violating zeros exist, we cannot ensure that they are finitely many, nor that they otherwise can be enumerated according to non-increasing real parts. Then, unlike (11), the series (40) ought not to be directly readable as an explicit asymptotic expansion, to whatever order $t_0 < \frac{1}{2}$. Instead, any closed-form asymptotic statement on Λ_n would have to involve the detailed $2D$ distribution of RH-violating zeros toward ∞ , currently unknown. Indeed, for no $t_0 < \frac{1}{2}$ can we perform or describe the sum of the series (40) explicitly (barring the purely hypothetical case of finitely many terms). In particular, it ought to be unlawful to substitute the individual asymptotic forms (39) in bulk into any of the series (40); we can only interpret the latter as a total of individual RH-violating zeros' contributions to the large- n behavior of Λ_n .

Moreover, any such zero $\rho = \frac{1}{2} + t \pm iT$ with $t > 0$ must have $T > T_0$ hence $T \gg 1$, which implies

$$|g(\rho)| \approx \left(\frac{2}{T}\right)^t \implies |F_n(\rho)| \approx \frac{1}{T^2 \log n} \left(\frac{2n}{T}\right)^t. \quad (41)$$

All in all, letting $t_0 = 0$ we obtain (33)–(35) in the case RH false. \square

The upshot of (34) is that each RH-violating zero ρ imparts Λ_n with a growing $n^{\rho-1/2}$ -like oscillation; one consequence (in view of § 3.4 below) is that it would take improbable cancellations to have $\Lambda_n > 0$ forever, if RH was false.

3.3 Details for the case RH true

Here our quickest path is to adapt:

3.3.1 Oesterlé's argument for λ_n^L

(as reworded by us). We start from this real integral giving λ_n^K : [21][23]

$$\lambda_n^K = \int_0^\pi 2 \sin n\theta N\left(\frac{1}{2} \cot\left(\frac{1}{2}\theta\right)\right) d\theta; \quad (42)$$

here $N(T) = \#\{\rho \in [\frac{1}{2}, \frac{1}{2} + iT] \subset L\}$ is the zeros' staircase counting function; $T \equiv \frac{1}{2} \cot(\frac{1}{2}\theta)$ where $\theta \in (0, \pi]$ is the angle subtended by the real segment $[0, 1]$ from the point $\frac{1}{2} + iT$, $dT \equiv -(\frac{1}{4} + T^2) d\theta$, and the integrand is actually the reduction of

$$2 \operatorname{Im} \left(\frac{x}{x-1} \right)^n \log 2\xi(x) \frac{dx}{x(x-1)} \quad (43)$$

once the integration path in (20) has reached $\{x = \frac{1}{2} + 0 + iT\}$ (under RH) and θ reparametrizes T .

Then $\lambda_n^K \bmod o(1)$ will stem from the Riemann–von Mangoldt theorem: [8, chaps. 8, 15]

$$N(T) = \frac{T}{2\pi} \left(\log \frac{T}{2\pi} - 1 \right) + \delta N(T), \quad \delta N(T) = O(\log T) \text{ as } T \rightarrow +\infty. \quad (44)$$

Proof: (42) mod $o(1)$ evaluates as follows:

1) in $N(\cdot)$, the term $\delta N(\cdot)$ is integrable *up to* $\theta = 0$ *included*, then its integral against $\sin n\theta$ is $o(1)$ (Riemann–Lebesgue lemma) hence negligible;

2) change to the variable $\Theta_n \equiv n\theta$; then, change the resulting upper integration bound $n\pi$ to $+\infty$ and use $T \sim 1/\theta = n/\Theta_n$ to get, mod $o(1)$,

$$\lambda_n^K \sim \int_0^\infty 2 \sin \Theta_n \frac{n}{2\pi\Theta_n} \left[\log \frac{n}{2\pi\Theta_n} - 1 \right] \frac{d\Theta_n}{n}. \quad (45)$$

Now the classic formulae $\int_0^\infty \sin \Theta d\Theta/\Theta = \pi/2$ and $\int_0^\infty \sin \Theta \log \Theta d\Theta/\Theta = -\pi\gamma/2$ [13, eqs. (3.721(1)) and (4.421(1))] yield the result (amounting to (10))

$$\lambda_n^K = \frac{1}{2} \log n + c + o(1) \quad \text{under RH true.} \quad (46)$$

□

3.3.2 Parallel treatment for Λ_n

Basically for Λ_n , $\left(\frac{x}{x-1}\right)^n$ in (43) is to be replaced by $G_n(x)$ from (21), hence (42) changes to

$$\Lambda_n = \int_0^\pi 2 \sin \Theta_n(\theta) N\left(\frac{1}{2} \cot\left(\frac{1}{2}\theta\right)\right) d\theta, \quad (47)$$

where $\Theta_n \in (0, n\pi]$ (previously $\equiv n\theta$) is now the sum of the n angles subtended by the real segments $[1 - 2m, 2m]$ from the point $\frac{1}{2} + iT$, for $m = 1, 2, \dots, n$. The two *endpoint slopes* of the function $\Theta_n(\theta)$ will mainly matter (independently):

$$\Theta'_n(0) = \sum_{m=1}^n (4m - 1) \equiv n(2n + 1), \quad (48)$$

$$\Theta'_n(\pi) = \sum_{m=1}^n (4m - 1)^{-1} \equiv \frac{1}{4}[(\Gamma'/\Gamma)(n + \frac{3}{4}) + \gamma + 3 \log 2 - \pi/2]. \quad (49)$$

We then follow the same steps as with λ_n^K just above.

1) $\int_0^\pi 2 \sin \Theta_n(\theta) \delta N\left(\frac{1}{2} \cot\left(\frac{1}{2}\theta\right)\right) d\theta = o(1)$ if a nonstationary-phase principle can apply for the oscillatory function $\sin \Theta_n(\theta)$, i.e., if the minimum slope of $\Theta_n(\theta)$ ($\theta \in [0, \pi]$) goes to ∞ with n : previously (with $\Theta_n \equiv n\theta$) that slope was n , now it is $\Theta'_n(\pi) \sim \frac{1}{4} \log n$ which still diverges for $n \rightarrow \infty$ therefore gives the $o(1)$ bound; but due to $\Theta'_n(\pi) \ll n$, this $o(1)$ may decay much slower than the corresponding $o(1)$ for λ_n^K .

2) In this step (i.e., $T \rightarrow +\infty$), only $\theta \rightarrow 0$ behaviors enter; here $\Theta_n \sim \Theta'_n(0)\theta$, vs $n\theta$ previously, so it suffices to substitute $\Theta'_n(0)$ for n in the asymptotic result (46) for λ_n^K , to get

$$\Lambda_n \sim \frac{1}{2} \log \Theta'_n(0) + c = \frac{1}{2} \log[n(2n + 1)] + c \sim \log n + (c + \frac{1}{2} \log 2). \quad (50)$$

□

3.4 Asymptotic or full-fledged Li's criterion?

We do not control well enough the function F_n in (29) or for that matter, the primitive $\int \sin \Theta_n(\theta) d\theta$ in (47), to be able to infer that RH implies $\Lambda_n > 0$ for all n , as was the case for λ_n straightforwardly from (4).

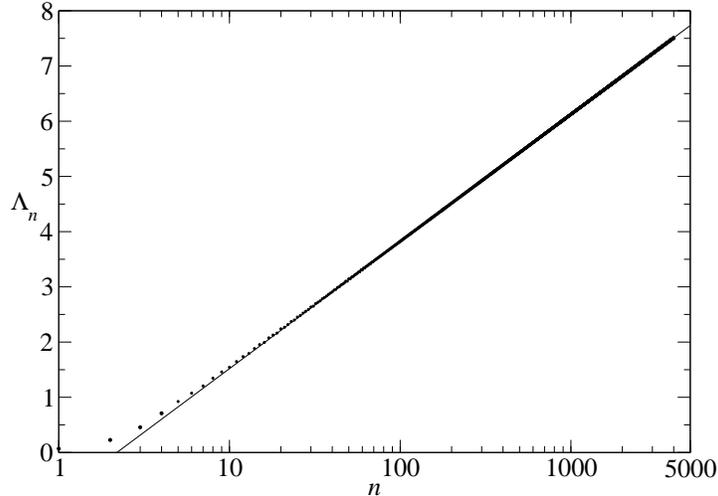


Figure 3: The coefficients Λ_n computed by (23)–(25) up to $n = 4000$, on a logarithmic n -scale (straight line: the function $(\log n + C)$ of (36)).

On the other hand, our criterion (33)–(36) is synonymous of large- n positivity for Λ_n if and only if RH holds (invoking the last sentence of § 3.2), while low- n positivity is numerically patent (see next §).

All in all, as an aside we then also *conjecture* that: *Li's criterion works for the sequence Λ_n* (RH $\iff \Lambda_n > 0$ for all n).

4 Quantitative aspects

4.1 Numerical data

Low- n calculations of Λ_n (fig. 3) agree very early with the *logarithmic* behavior (36), just as they agreed for λ_n with its leading behavior under RH [15][18]. The remainder term $\delta\Lambda_n = \Lambda_n - (\log n + C)$ looks compatible with an $o(1)$ bound (fig. 4), albeit much less neatly than $\delta\lambda_n^K$ [15, fig. 1][18, fig. 6b], (note: both of these plot $\delta\lambda_n^L = n \delta\lambda_n^K$). For the record,

$$\begin{aligned} \Lambda_1 &= \frac{3}{2} \log \frac{\pi}{3} \approx 0.069176395771, & \Lambda_2 &\approx 0.22745427267, & \Lambda_3 &\approx 0.45671413349; \\ \Lambda_{10000} &\approx 8.428662659671506 & (\delta\Lambda_{10000} &\approx +0.0020794), & & (51) \\ \Lambda_{20000} &\approx 9.120189975922122 & (\delta\Lambda_{20000} &\approx -0.000485565), & & \end{aligned}$$

It would be interesting to comprehend the bumpy fine structure of $\delta\Lambda_n$.

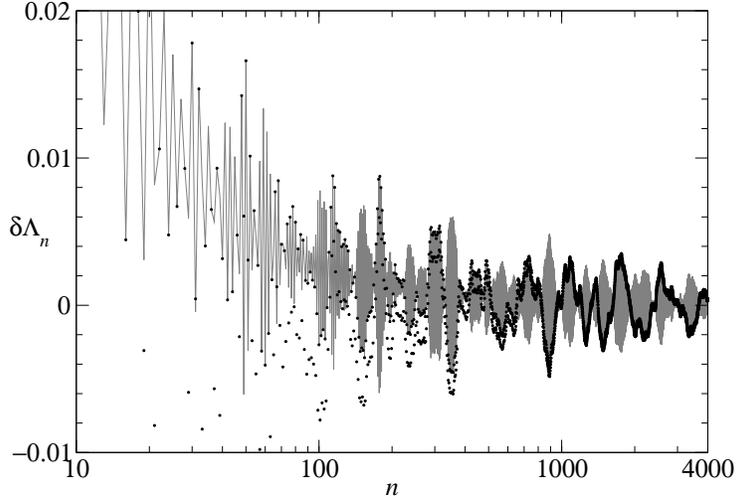


Figure 4: The remainder sequence $\delta\Lambda_n = \Lambda_n - (\log n + C)$ (in gray: the connecting segments are drawn for visual clarity only), and a rectified form $(-1)^n \delta\Lambda_n$ (black dots) to cancel the period-2 oscillations.

4.2 Imprints of putative zeros violating RH

RH-violating zeros ρ (if any) seem to enter the picture just as for the λ_n : their contributions (34) will asymptotically dominate $\log n$, but numerically they will emerge and take over extremely late. For such a zero $\rho = \frac{1}{2} + t + iT$, with $0 < t < \frac{1}{2}$ and $T \gtrsim 2.4 \cdot 10^{12}$ [12], its contribution sizes like $T^{-2}(2n/T)^t / \log n$ in modulus, by (35). We then get its crossover threshold (in order of magnitude, neglecting logarithms and constants relative to powers) by solving

$$T^{-2}(n/T)^t \approx 1 \quad (52)$$

$$\implies n \gtrsim T^{1+2/t} \quad (\text{best case: } O(T^{5+\epsilon}) \text{ for } t = \frac{1}{2} - 0). \quad (53)$$

This is worse than (14) for λ_n , all the more if a negativity test is pursued (the right-hand side of (52) must then be $\log^2 n$). There is however room for possible improvement: the core problem is to filter out a weak ρ -signal from the given background (36), therefore any predictable structure in the latter is liable to boost the gain. For instance, the hyperfine structure of $\delta\Lambda_n$ is oscillatory of period 2 (fig. 4); this suggests to average over that period, which *empirically* discloses a rather neat $(1/n)$ -decay trend (fig. 5):

$$\overline{\delta\Lambda}_n \stackrel{\text{def}}{=} \frac{1}{2}(\delta\Lambda_n + \delta\Lambda_{n-1}) \approx 0.25/n. \quad (54)$$

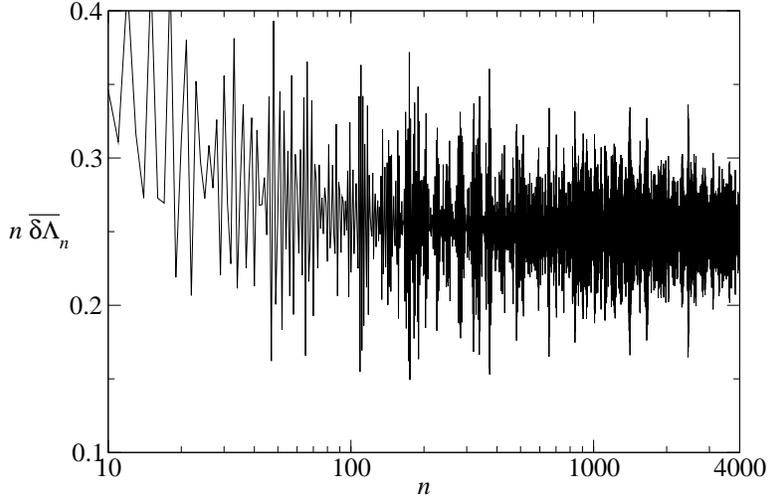


Figure 5: The averaged remainder sequence (54) rescaled by n , namely: $n \overline{\delta\Lambda}_n$. (Some further values: 0.27027 for $n = 10000$, 0.23970 for $n = 20000$.)

The same operation on a ρ -signal $F_n(\rho)$ in (33) roughly applies $\frac{1}{2}(d/dn)$ to the factor n^T therein (again neglecting $t \ll T$ and $\log n$), i.e., multiplies it by $\frac{1}{2}(T/n)$. Thus heuristically, i.e., conjecturing the truth of (54) for $n \rightarrow \infty$ under RH, the crossover condition improves from (52) to

$$\begin{aligned} (T/n) T^{-2} (n/T)^t &\approx \overline{\delta\Lambda}_n \approx 1/n \\ \implies n &\gtrsim T^{1+1/t} \text{ (best case: } O(T^{3+\epsilon}) \text{ for } t = \frac{1}{2} - 0). \end{aligned} \quad (55)$$

We can hope that efficient signal-analysis techniques may still lower this detection threshold. And an empirical attitude may suffice here: once a violating zero would be suspected and roughly located, other rigorous algorithms exist to find it accurately (or disprove it). [12]

4.3 *The hitch*

A major computational issue is that, according to (23), the $(\log n)$ -sized values Λ_n result from alternating summations of *much faster-growing* terms: this entails a loss of precision increasing with n . Thus in our case (sums $\sum s_m$ of order comparable to unity), to reach the *slightest* end accuracy we must use each summand s_m up to $\approx \log_{10} |s_m|$ significant digits (in base 10 throughout); plus uniformly D more to obtain $\sum s_m$ accurate to D digits.

We quantify the precision loss in (23) at large fixed n by using the Stirling formula, to find that $m_* \approx n/\sqrt{2}$ is where the largest summand occurs and the minimum required precision $\log_{10} |s_m|$ peaks, reaching $\log_{10} |A_{nm_*} \log 2\xi(2m_*)| \sim \log_{10}(3 + 2\sqrt{2}) n \approx 0.76555 n$ digits, see fig. 6 (vs $(0.2 \text{ to } 0.3) n$ digits for λ_n [19, fig. 6][14, § 4.2]). Even then, a crude feed of (23), (26) or (27) into a mainstream arbitrary-precision system (Mathematica 10 [27]) suffices to readily output the Λ_n -values of § 4.1. Computing times varied erratically but could go down to ca. 4 min for Λ_{10000} , 43 min for Λ_{20000} using (27) (CPU times on an Intel Xeon E5-2670 0 @ 2.6 GHz processor).

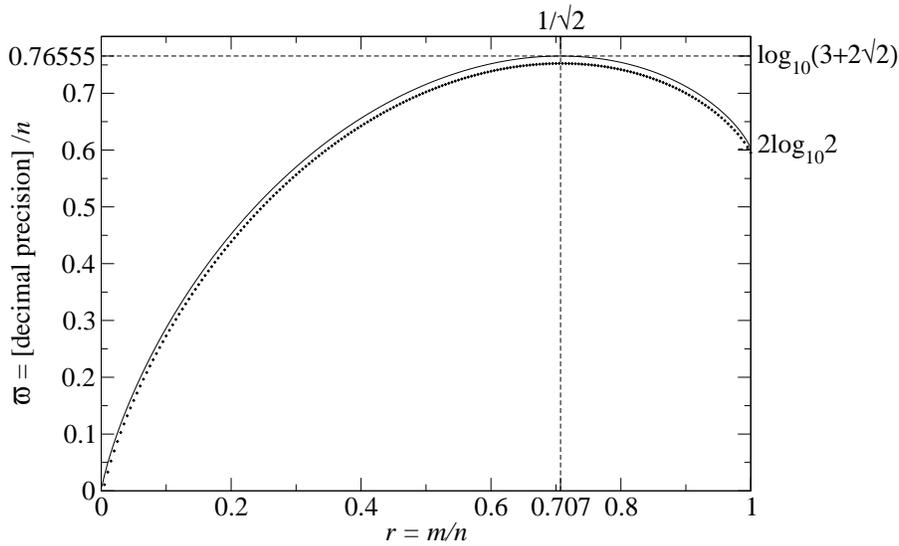


Figure 6: Minimum decimal precisions needed for the summands of Λ_n in (23), as estimated by $\log_{10} |A_{nm} \log 2\xi(2m)|$ which is plotted against m in axes rescaled by $1/n$. Dotted curve: the case $n = 200$; continuous curve: the $n \rightarrow \infty$ limiting form $\varpi = -2r \log_{10} r + (1+r) \log_{10}(1+r) - (1-r) \log_{10}(1-r)$ ($r = m/n$).

Now with $|T| \gtrsim 2.4 \cdot 10^{12}$ currently, the challenge is to probe $n \gtrsim 2 \cdot 10^{36}$ (if the more favorable estimate (55) holds, 10^{60} otherwise), which then needs a working precision $\gtrsim 1.6 \cdot 10^{36}$ decimal places at times. This need for a huge precision already burdened the original λ_n but somewhat less and amidst several steeper complexities, now for the Λ_n the ill-conditioning worsened while the other difficulties waned.

As advantages of $\{\Lambda_n\}$ over $\{\lambda_n\}$, inversely: the Λ_n are fully explicit; their

evaluations are recursion-free, thus *very few* samples (at high enough n , for sure) might suffice to signal that RH is violated *somewhere*; and the required working precision peaking at $\approx 0.766 n$ stands as the *only* stumbling block, and as a *purely logistic* problem, which might still be eased if (23) came to admit better conditioned variants. Thus in (26), a much lower precision (growing like $\frac{1}{2} \log_{10} n$) suffices for $\log 2\pi$ with its factor $(2A_{n0})^{-1} \sim -\sqrt{\pi n}/2$ which grows negligibly, compared to the $A_{nm} \log(|B_{2m}|/(2m-3)!!)$: only these *simpler* expressions demand maximal precision, and only for $m \approx n/\sqrt{2}$.

While other sequences sensitive to RH for large n are known [2][11], not to mention Keiper–Li again, we are unaware of any previous case combining a fully *closed form* like (23) with a practical sensitivity-threshold of *tempered growth* $n = O(T^\nu)$.

Appendix: Centered variant

We sketch a treatment parallel to the main text for our Li-type sequences having the alternative basepoint $x = \frac{1}{2}$ (the *center* for the ξ -function).

We recall that the Functional Equation $\xi(1-x) \equiv \xi(x)$ allows us, in place of the mapping $z \mapsto x = (1-z)^{-1}$ within ξ as in (3), to use the double-valued one $y \mapsto x_{\tilde{w}}(y) = \frac{1}{2} \pm \sqrt{\tilde{w}} y^{1/2}/(1-y)$ on the unit disk (parametrized by $\tilde{w} > 0$). That still maps the unit circle $\{|y| = 1\}$ to the completed critical line $L \cup \infty$, but now minus its interval $\{|\operatorname{Im} x| < \frac{1}{2}\sqrt{\tilde{w}}\}$. As before, all Riemann zeros on L have to pull back to $\{|y| = 1\}$ which then imposes $\tilde{w} < 4 \min_\rho |\operatorname{Im} \rho|^2 \approx 799.1618$. We thus define the sequence $\{\lambda_n^0(\tilde{w})\}$ by

$$\log 2\xi\left(\frac{1}{2} \pm \frac{\sqrt{\tilde{w}} y^{1/2}}{1-y}\right) \equiv \log 2\xi\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{\lambda_n^0(\tilde{w})}{n} y^n \quad (56)$$

([25, §3.4], where only the case $\tilde{w} = 1$ is detailed), or

$$\frac{\lambda_n^0(\tilde{w})}{n} \equiv \frac{1}{2\pi i} \oint \frac{dy}{y^{n+1}} \log 2\xi(x_{\tilde{w}}(y)), \quad n = 1, 2, \dots \quad (57)$$

We now build an *explicit* variant for this sequence (57), similar to $\{\Lambda_n\}$ for $\{\lambda_n^K\}$. First, the deformations of (57) analogous to those in §2.1 read as

$$\frac{1}{2\pi i} \oint \frac{dy}{B_{y_0}(y) \cdots B_{y_n}(y)} \log 2\xi(x) \quad (\text{here } x \equiv x_{\tilde{w}}(y)), \quad (58)$$

for which the simplest analytical form we found, similar to (18), is now

$$\frac{1}{2\pi i} \oint \frac{2 dr}{(r+1)^2} \prod_{m=0}^n \frac{r+r_m}{r-r_m} \log 2\xi(x), \quad r_m \stackrel{\text{def}}{=} \frac{1+y_m}{1-y_m}, \quad (59)$$

all in terms of the new variable

$$r \stackrel{\text{def}}{=} \frac{1+y}{1-y} \equiv [1 + (2x-1)^2/\tilde{w}]^{1/2} \quad (\text{Re } r > 0). \quad (60)$$

Then with $x_m \equiv 2m$ as before (but now including $m=0$), the integral (59) evaluated by the residue theorem yields the explicit result (akin to (23)–(25))

$$\Lambda_n^0(\tilde{w}) \stackrel{\text{def}}{=} \sum_{m=1}^n \frac{2}{(r_m+1)^2} \frac{\prod_{k=0}^n (r_m+r_k)}{\prod_{k \neq m} (r_m-r_k)} \log 2\xi(2m), \quad r_m \equiv \sqrt{1 + (4m-1)^2/\tilde{w}}. \quad (61)$$

This result is, however, algebraically less simple and less analyzable than for Λ_n before. A potential asset is that it openly relies on the Functional Equation, but we saw no practical benefit accruing from that yet.

The corresponding *asymptotic alternative* for RH analogous to (33)–(36) reads as

- RH false: $\Lambda_n \sim \left\{ \sum_{\text{Re } \rho > 1/2} \Delta_\rho \Lambda_n^0(\tilde{w}) \right\} \pmod{o(n^\varepsilon) \forall \varepsilon > 0}$ (62)

$$\text{with } \log |\Delta_\rho \Lambda_n^0(\tilde{w})| \sim (\rho - 1/2) \log n, \quad (63)$$

- RH true: $\Lambda_n \sim \sqrt{\tilde{w}} (\log n + C)$, $C = \frac{1}{2}(\gamma - \log \pi - 1) \approx -0.78375711$. (64)

The latter is proved by extending Oesterlé's method just as with Λ_n ; whereas the former needs large- n estimations of the product in (59), but the ones we have remain crude compared to the full Stirling formula available for (21); that precludes us from reaching the absolute scales of the $\Delta_\rho \Lambda_n^0(\tilde{w})$ and the values of n from which any such terms might become detectable.

As for numerical tests, all results are very close to those shown above for Λ_n , aside from the overall factor $\sqrt{\tilde{w}}$ in (64) (but nothing about the case RH false can be tested: that is still way beyond numerical reach).

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