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► **To cite this version:**

| André Voros. An asymptotic criterion in an explicit sequence. 2015. cea-01166324

**HAL Id: cea-01166324**

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Preprint submitted on 30 Jul 2015

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# An asymptotic criterion in an explicit sequence

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June 28, 2015

## Abstract

We report a novel asymptotic (large-order) behavior in an explicit sequence built out of the Bernoulli numbers and analyzed by a variant of instanton calculus or Darboux's theorem.

We will use:  $B_{2m}$  : the Bernoulli numbers;  $\gamma$  : Euler's constant;  $k!! = k(k-2)(k-4)\dots$  : double factorial (with  $0!! = (-1)!! = 1$  as usual).

The real sequence explicitly spelled out for  $n = 1, 2, \dots$  as

$$u_n = (-1)^n \left[ 2^{-2n} \sum_{m=1}^n \frac{(-1)^m}{2m-1} \binom{2(n+m)}{n+m} \binom{n+m}{2m} \log \frac{|B_{2m}|}{(2m-3)!!} - \frac{(2n)!!}{2(2n-1)!!} \log 2\pi \right] \quad (1)$$

can be thus numerically computed (trivially to thousands of terms), and very early it satisfies (figs.)

$$u_n \approx \log n - 1.703 . \quad (2)$$

This can be validated *assuming the Riemann Hypothesis* (= RH), as

$$u_n \sim \log n + K, \quad K = \frac{1}{2}(\gamma - \log(2\pi^2) - 1) \approx -1.70269564368 . \quad (3)$$

With RH verified up to an ordinate  $T_0 \gtrsim 2 \cdot 10^{12}$  currently, [5] it is plausible indeed to witness the behavior (3).

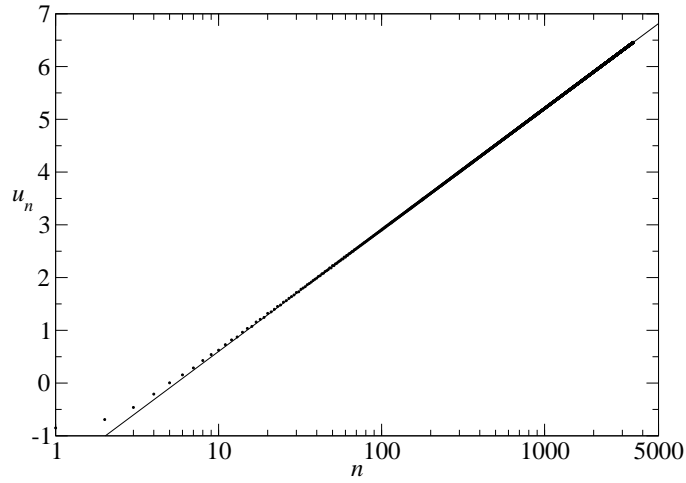


Figure 1: The coefficients  $u_n$  computed by (1) up to  $n = 3500$ , on a logarithmic  $n$ -scale, vs the function  $(\log n + K)$  of (3) (straight line); the first values are  $u_1 = \log \pi - \frac{1}{2} \log 54 \approx -0.84976213743$ ,  $u_2 = -\frac{4}{3} \log \pi + \frac{23}{24} \log 2 + \frac{55}{24} \log 3 - \frac{35}{24} \log 5 \approx -0.69148426053$ ,  $u_3 \approx -0.46222439972$ .

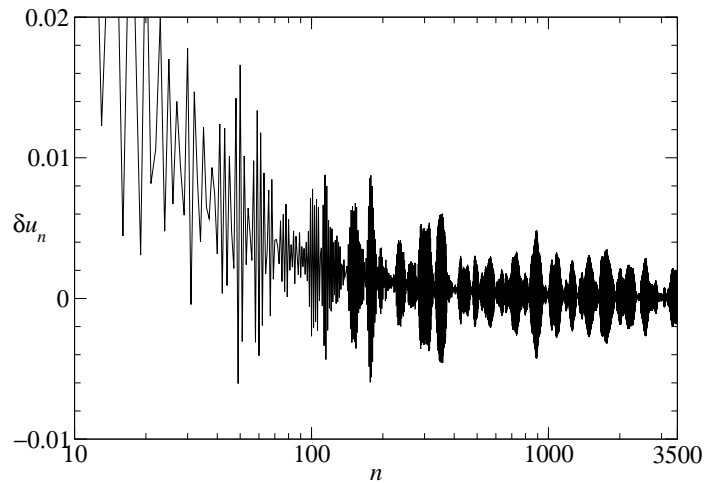


Figure 2: As fig. 1 but for the remainders  $\delta u_n = u_n - (\log n + K)$ , on a very dilated vertical scale. (The connecting segments between data points are only drawn for clarity.)

Now by large-order analysis through exponential asymptotics, [3][1] we found that *if RH is false*,  $u_n$  will also admit a clear-cut asymptotic form but of a wholly different nature, dominated by individual terms  $F_n(\rho)$  contributed by every zero  $\rho = \frac{1}{2} + t + iT$  off the critical line with  $0 < t (< \frac{1}{2})$  :

$$F_n(\rho) \sim f(\rho)(-1)^n \frac{(2n)^{\rho-1/2}}{\log n} \quad \text{for } n \rightarrow \infty, \quad (4)$$

where  $f$  is an explicit function independent of  $n$  with the main property

$$|f(\rho)| \approx |T|^{-t-2} \text{ for } |T| \gg 1 \implies |F_n(\rho)| \approx |T|^{-t-2}(2n)^t / \log n. \quad (5)$$

Each such  $F_n(\rho)$  will ultimately dominate (3) in the  $n \rightarrow \infty$  limit, but starts out exceedingly tiny at low  $n$ , and does not approach unity until

$$n \gtrsim \frac{1}{2}|T|^{1+2/t} \quad (\text{at best } O(|T|^{5+\epsilon}) \text{ for } t \rightarrow \frac{1}{2}^-); \quad (6)$$

yet we think that, with efficient signal-processing, the “signal”  $F_n(\rho)$  of  $\rho$  within  $u_n$  ought to be detectable much sooner than at (6) (at  $n \gtrsim O(|T|^{1+1/t})$  or even less, but within  $n \gg |T|$ ). Still, to seek a violation of RH and verify the form (4),  $|T| > T_0$  is necessary, and  $T_0 \gtrsim 2 \cdot 10^{12}$  implies very large  $n$ -values.

The major issue is then that  $u_n$  is an alternating sum of terms which turn out to be exponentially larger by an order of  $(3 + 2\sqrt{2})^n$ . Increasingly delicate cancellations thus take place, requiring a precision beyond  $\approx 0.7656 n$  decimal digits to evaluate  $u_n$  by (1). On the other hand, this purely technical demand seems to be the *sole* obstacle raised by the use of (1) at unlimited  $n$ .

While other sequences sensitive to RH for large  $n$  are known, [6][2][7][4] we are unaware of any previous case combining a fully *closed form* like (1) with a practical sensitivity threshold of *tempered growth*  $n = O(T^\nu)$ .

Details and derivations are currently under completion. [8]

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