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An asymptotic criterion in an explicit sequence

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Abstract
We report a novel asymptotic (large-order) behavior in an explicit sequence built out of the Bernoulli numbers and analyzed by a variant of instanton calculus or Darboux’s theorem.

We will use: $B_{2m}$ : the Bernoulli numbers; $\gamma$ : Euler’s constant; 
$k!! = k(k − 2)(k − 4)\ldots$ : double factorial (with $0!! = (−1)!! = 1$ as usual).

The real sequence explicitly spelled out for $n = 1, 2, \ldots$ as

$$u_n = (-1)^n \left[ 2^{-2n} \sum_{m=1}^{n} \frac{(-1)^m}{2m-1} \binom{2(n+m)}{n+m} \binom{n+m}{2m} \log \frac{|B_{2m}|}{(2m-3)!!} - \frac{(2n)!!}{2(2n-1)!!} \log 2\pi \right] \tag{1}$$

can be thus numerically computed (trivially to thousands of terms), and very early it satisfies (figs.)

$$u_n \approx \log n − 1.703 \tag{2}$$

This can be validated assuming the Riemann Hypothesis (= RH), as

$$u_n \sim \log n + K, \quad K = \frac{1}{2} (\gamma − \log(2\pi^2) − 1) \approx −1.70269564368. \tag{3}$$

With RH verified up to an ordinate $T_0 \gtrsim 2 \cdot 10^{12}$ currently, [5] it is plausible indeed to witness the behavior (3).
Figure 1: The coefficients $u_n$ computed by (1) up to $n = 3500$, on a logarithmic $n$-scale, vs the function $(\log n + K)$ of (3) (straight line); the first values are $u_1 = \log \pi - \frac{1}{7} \log 54 \approx -0.84976213743$, $u_2 = -\frac{2}{3} \log \pi + \frac{23}{24} \log 2 + \frac{55}{24} \log 3 - \frac{35}{24} \log 5 \approx -0.69148426053$, $u_3 \approx -0.46222439972$.

Figure 2: As fig. 1 but for the remainders $\delta u_n = u_n - (\log n + K)$, on a very dilated vertical scale. (The connecting segments between data points are only drawn for clarity.)
Now by large-order analysis through exponential asymptotics, [3][1] we found that if RH is false, \( u_n \) will also admit a clear-cut asymptotic form but of a wholly different nature, dominated by individual terms \( F_n(\rho) \) contributed by every zero \( \rho = \frac{1}{2} + t + iT \) off the critical line with \( 0 < t < \frac{1}{2} \):

\[
F_n(\rho) \sim f(\rho)(-1)^n \frac{(2n)^{\rho - 1/2}}{\log n} \quad \text{for } n \to \infty,
\]

where \( f \) is an explicit function independent of \( n \) with the main property

\[
|f(\rho)| \approx |T|^{-t-2} \text{ for } |T| \gg 1 \quad \implies \quad |F_n(\rho)| \approx |T|^{-t-2}(2n)^t/\log n.
\]

Each such \( F_n(\rho) \) will ultimately dominate (3) in the \( n \to \infty \) limit, but starts out exceedingly tiny at low \( n \), and does not approach unity until

\[
n \gtrsim \frac{1}{2}|T|^{1+2/t} \quad \text{(at best } O(|T|^{1+\epsilon}) \text{ for } t \to \frac{1}{2}^{-});
\]

yet we think that, with efficient signal-processing, the “signal” \( F_n(\rho) \) of \( \rho \) within \( u_n \) ought to be detectable much sooner than at (6) (at \( n \gtrsim O(|T|^{1+1/t}) \) or even less, but within \( n \gg |T| \)). Still, to seek a violation of RH and verify the form (4), \( |T| > T_0 \) is necessary, and \( T_0 \gtrsim 2 \cdot 10^{12} \) implies very large \( n \)-values.

The major issue is then that \( u_n \) is an alternating sum of terms which turn out to be exponentially larger by an order of \( (3 + 2\sqrt{2})^n \). Increasingly delicate cancellations thus take place, requiring a precision beyond \( \approx 0.7656 n \) decimal digits to evaluate \( u_n \) by (1). On the other hand, this purely technical demand seems to be the sole obstacle raised by the use of (1) at unlimited \( n \).

While other sequences sensitive to RH for large \( n \) are known, [6][2][7][4] we are unaware of any previous case combining a fully closed form like (1) with a practical sensitivity threshold of tempered growth \( n = O(T^\nu) \).

Details and derivations are currently under completion. [8]

References


