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Appendix: Maximal power of heat machines

by Roger Balian

When a motor generates work by exchanging heat with two sources at temperatures $T_1 > T_2$, Carnot's maximum efficiency $1 - T_2/T_1$ can be approached only for reversible processes; but these are too slow to yield significant power. At the other extreme, rapid processes are strongly irreversible and cannot produce much work. A compromise is needed to maximise the power output. Jacques Yvon tackled this problem in 1955, focusing on a major cause of irreversibility, the delay required for heat transport. A simple approach is presented here.

During the first step of a cycle, between the times t_0 and $t_1 = t_0 + \tau_1$, the motor M evolves at a temperature $\Theta_1(t)$, receiving the heat Q_1 from the hot source S_1 at temperature T_1 . We assume that heat transfers obey Fourier's law, so that the heat flux $\Phi_1(t)$ from S_1 to M has at times t ($t_0 \leq t \leq t_1$) the form $\Phi_1(t) = C_1[T_1 - \Theta_1(t)]$ where C_1 is supposed to be given. During the second step, between t_1 and $t_2 = t_1 + \tau_A$, M evolves adiabatically, its temperature going down from $\Theta_1(t_1)$ to $\Theta_2(t_2)$. During the third step, between t_2 and $t_3 = t_2 + \tau_2$, M evolves at a temperature $\Theta_2(t)$, giving the heat Q_2 to the cold source S_2 at temperature T_2 , through a flux $\Phi_2(t) = C_2[\Theta_2(t) - T_2]$ ($t_2 \leq t \leq t_3$). Finally, between t_3 and $t_4 = t_3 + \tau'_A$, M returns adiabatically to its initial state, with a temperature rising from $\Theta_2(t_3)$ to $\Theta_1(t_4) = \Theta_1(t_0)$. The cycle has the total duration $\tau = t_4 - t_0 = \tau_1 + \tau_A + \tau_2 + \tau'_A$. The work produced by M during a cycle is the difference between the heat $Q_1 = \int_{t_0}^{t_1} dt \Phi_1(t)$ that it received from S_1 and the heat $Q_2 = \int_{t_2}^{t_3} dt \Phi_2(t)$ that it yielded to S_2 , so that the average power output is

$$P = \frac{1}{\tau} \left\{ C_1 \int_{t_0}^{t_1} dt [T_1 - \Theta_1(t)] - C_2 \int_{t_2}^{t_3} dt [\Theta_2(t) - T_2] \right\}. \quad (1)$$

The system M evolves in a closed cycle, so that its change ΔS of entropy between the times t_0 and t_4 vanishes:

$$\Delta S = C_1 \int_{t_0}^{t_1} dt \frac{[T_1 - \Theta_1(t)]}{\Theta_1(t)} - C_2 \int_{t_2}^{t_3} dt \frac{[\Theta_2(t) - T_2]}{\Theta_2(t)} = 0. \quad (2)$$

We wish to maximise P as function of the temperatures $\Theta_1(t)$, $\Theta_2(t)$ and of the durations τ_1 , τ_A , τ_2 , τ'_A , for given values of T_1 , T_2 , C_1 , C_2 , τ , under the constraint $\Delta S = 0$. Introducing a Lagrange multiplier λ/τ , and writing that $P - \lambda\Delta S/\tau$ is stationary with respect to $\Theta_1(t)$ and $\Theta_2(t)$, we obtain

$$\Theta_1(t) = \sqrt{\lambda T_1}, \quad \Theta_2(t) = \sqrt{\lambda T_2}. \quad (3)$$

The optimal cycles for M are therefore Carnot cycles with constant temperatures $\Theta_1(t) \equiv \Theta_1$ and $\Theta_2(t) \equiv \Theta_2$. Maximising P with respect to τ_A and τ'_A provides $\tau_A/\tau \simeq 0$, $\tau'_A/\tau \simeq 0$: The adiabatic steps should be the shortest possible. It remains to write that $P - \lambda\Delta S/\tau$ is stationary with respect to $\tau_1/\tau = 1 - \tau_2/\tau$, which yields

$$C_1 (T_1 - \Theta_1) (1 - \lambda/\Theta_1) + C_2 (\Theta_2 - T_2) (1 - \lambda/\Theta_2) = 0, \quad (4)$$

or equivalently, using (3),

$$\sqrt{\lambda} = \frac{\sqrt{C_1 T_1} + \sqrt{C_2 T_2}}{\sqrt{C_1} + \sqrt{C_2}}. \quad (5)$$

From the condition $\Delta S = 0$, we get the optimal durations of the isothermal steps of the cycle:

$$\frac{\tau_1}{\tau} = \frac{\sqrt{C_2}}{\sqrt{C_1} + \sqrt{C_2}}, \quad \frac{\tau_2}{\tau} = \frac{\sqrt{C_1}}{\sqrt{C_1} + \sqrt{C_2}}. \quad (6)$$

Altogether, the maximal power is

$$P_{\max} = \frac{C_1 C_2}{(\sqrt{C_1} + \sqrt{C_2})^2} (T_1 - T_2). \quad (7)$$

This power is equivalent to a fraction, at most equal to $\frac{1}{4}$ (reached for $C_1 = C_2$), of the heat flux $C(T_1 - T_2)$ that would be transferred from S_1 to S_2 for a coefficient C equal to the average $\sqrt{C_1 C_2}$. Finally, the efficiency at maximal power is obtained from $\Delta S = Q_1/\theta_1 - Q_2/\theta_2 = 0$ as

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{\theta_2}{\theta_1} = 1 - \sqrt{\frac{T_2}{T_1}}. \quad (8)$$