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Abstract In these notes we review the field-theoretical approach to the computation of the scalar product of multi-magnon states in the Sutherland limit where the magnon rapidities condense into one or several macroscopic arrays. We formulate a systematic procedure for computing the $1/M$ expansion of the on-shell/off-shell scalar product of M-magnon states in the generalised integrable model with $SU(2)$-invariant rational R-matrix. The coefficients of the expansion are obtained as multiple contour integrals in the rapidity plane.

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Ivan Kostov
Institut de Physique Théorique, CNRS-URA 2306, C.E.A.-Saclay, F-91191 Gif-sur-Yvette, France
Associate member of the Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussée, 1784 Sofia, Bulgaria

1 Introduction

In many cases the calculation of form factors and correlation functions within quantum integrable models solvable by the Bethe Ansatz reduces to the calculation of scalar products of Bethe vectors. The best studied case is that of the models based on the $SU(2)$-invariant R-matrix. A determinant formula for the the norm-squared of an on-shell state has been conjectured by Gaudin [1], and then proved by Korepin in [2]. Sum formulas for the scalar product between two generic Bethe states were obtained by by Izergin and Korepin [2–4]. Furthermore, the scalar product between an on-shell and off-shell Bethe vector was expressed in determinant form by Slavnov [5]. This representation proved to be very useful in the computation of correlation functions of the XXX and XXZ models [6]. Although the Slavnov determinant formula is, by all evidence, not generalisable for higher rank groups, compact and potentially useful expressions of the scalar products as multiple contour integrals of (products of) determinants were proposed in [7–10].

The above-mentioned sum and determinant formulas are efficient for states compsed if few magnons. In order to evaluate scalar products of multi-magnon states, new semi-classical methods specific for the problem need to be developed.

Of particular interest is the evaluation of the scalar product of Bethe wave functions describing the lowest excitations above the ferromagnetic vacuum composed of given (large) number of magnons. The magnon rapidities for such excitations organise themselves in a small number of macroscopically large bound complexes [15, 16]. It is common to refer this limit as a thermodynamical, or semi-classical, or Sutherland limit. In the last years the thermodynamical limit attracted much attention in the context of the integrability in AdS/CFT [17], where it describes "heavy" operators in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM), dual to classical strings embedded in the curved $AdS_5 \times S^5$ space-time [18, 19]. It has been realised that the computation of some 3-point functions of such heavy operators boils down to the computation of the scalar product of the corresponding Bethe wave functions in the thermodynamical limit [11, 12, 20–22].

In this notes, based largely on the results obtained in [11–14], we review the field-theoretical approach developed by E. Bettelheim and the author [14], which leads to a systematical semi-classical expansion of the on-shell/off-shell scalar product. The field-theoretical representation is not sensitive to the particular representation of the monodromy matrix and we put it in the context of the generalised integrable model with $SU(2)$ invariant rational R-matrix.

The text is organised as follows. In Section 2 we remind the basic facts and conventions concerning the Algebraic Bethe Ansatz for rational $SU(2)$-invariant R-matrix. In Section 3 we give an alternative determinant representation of the on-shell/off-shell scalar product of two M-magnon Bethe vectors in spin chains with rational $SU(2)$-invariant R-matrix. This representation, which has the form of an $2M \times 2M$ determinant, possesses an unexpected symmetry: it is invariant under the group S_{2M} of the permutations of the *union* of the magnon rapidities of the left and the right states, while the Korepin sum formulas and the Slavnov determinant have a smaller $S_M \times S_M$ symmetry. We refer to the symmetric expression in question as $\mathscr{A}$-functional to underline the relation with a similar quantity, previously studied in the papers [20, 31] and denoted there by the same letter. In the generalised $SU(2)$-invariant integrable model the $\mathscr{A}$-functional depends on the ratio of the eigenvalues of the diagonal elements of the monodromy matrix on the pseudo-vacuum, considered as a free functional variable. In Section 4 we write the $\mathscr{A}$-functional as an expectation value in the Fock space of free chiral fermions. The fermionic representation implies that the $\mathscr{A}$-functional is a KP τ-function, but we do not use this fact explicitly. By two-dimensional bosonization we obtain a formulation of the $\mathscr{A}$-functional in terms of a chiral bosonic field with exponential interaction. The bosonic field describes a Coulomb gas of dipole charges. The thermodynamical limit $M \gg 1$ is described by an effective $(0 + 1)$-dimensional field theory, obtained by integrating the fast-scale modes of the original bosonic field. In terms of the dipole gas the effective theory contains composite particles representing bound states of any number of dipoles. The Feynman diagram technique for the effective field theory for the slow-scale modes is expected to give the perturbative $1/M$ expansion of the scalar product. We evaluate explicitly the first two terms of this expansion. The leading term reproduces the known expression as a contour integral of a dilogarithm, obtained by different methods in [31] and [11, 12], while the subleading term, given by a double contour integral, is a new result reported recently in [14].

2 Algebraic Bethe Ansatz for integrable models with $su(2)$ R-matrix

We remind some facts about the ABA for the $su(2)$-type models and introduce our notations. The monodromy matrix $M(u)$ is a 2×2 matrix [23, 24]

$$M(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{1}$$

The matrix elements A, B, C, D are operators in the Hilbert space of the model and depend on the complex spectral parameter u called rapidity. The monodromy matrix obeys the RTT-relation (Yang-Baxter equation)

$$R(u-v)(M(u) \otimes I)(I \otimes M(v)) = (I \otimes M(u))(M(v) \otimes I)R(u-v). \tag{2}$$

Here I denotes the 2×2 identity matrix and the 4×4 matrix $R(u)$ is the $SU(2)$ rational R-matrix whose entries are c-numbers. The latter is given, up to a numerical factor, by

$$R_{\alpha\beta}(u) = u\, I_{\alpha\beta} + i\varepsilon\, P_{\alpha\beta}, \tag{3}$$

with the operator $P_{\alpha\beta}$ acting as a permutation of the spins in the spaces α and β. In the standard normalization $\varepsilon = 1$.

The RTT relation determines the algebra of the monodromy matrix elements, which is the same for all $su(2)$-type models. In particular, $[B(u), B(v)] = [C(u), C(v)] = 0$ for all u and v.

The trace $T = A + D$ of the monodromy matrix is called transfer matrix. Sometimes it is useful to introduce a twist parameter κ (see, for example, [25]). The twist preserves the integrability: the twisted transfer matrix

$$T(u) = \mathrm{tr}\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & \kappa \end{smallmatrix}\right) M(u)\right] = A(u) + \kappa\, D(u) \tag{4}$$

satisfies $[T(u), T(v)] = 0$ for all u and v.

To define a quantum-mechanical system completely, one must determine the action of the elements of the monodromy matrix in the Hilbert space. In the framework of the ABA the Hilbert space is constructed as a Fock space associated with a cyclic vector $|\Omega\rangle$, called pseudovacuum, which is an eigenvector of the operators A and D and is annihilated by the operator C:

$$A(u)|\Omega\rangle = a(u)|\Omega\rangle,\ D(u)|\Omega\rangle = d(u)|\Omega\rangle,\ C(u)|\Omega\rangle = 0. \tag{5}$$

The dual pseudo-vacuum satisfies the relations

$$\langle\Omega|A(u) = a(u)\langle\Omega|,\ \langle\Omega|D(u) = d(u)\langle\Omega|,\ \langle\Omega|B(u) = 0. \tag{6}$$

Here $a(u)$ and $d(u)$ are are complex-valued functions whose explicit form depends on the choice of the representation of the algebra (2). We will not need the specific form of these functions, except for some mild analyticity requirements. In other words, we will consider the generalized $SU(2)$ model in the sense of [2], in which the functions $a(u)$ and $d(u)$ are considered as free functional parameters.

The vectors obtained from the pseudo-vacuum $|\Omega\rangle$ by acting with the 'raising operators' $B(u)$,

$$|\mathbf{u}\rangle = B(u_1) \dots B(u_M)|\Omega\rangle\ , \quad \mathbf{u} = \{u_1, \dots, u_M\} \tag{7}$$

are called *Bethe states*. Since the B-operators commute, the state $|\mathbf{u}\rangle$ is invariant under the permutations of the elements of the set $\mathbf{u}$.

The Bethe states that are eigenstates of the (twisted) transfer matrix are called 'on-shell'. Their rapidities obey the Bethe Ansatz equations

$$\frac{a(u_j)}{d(u_j)} + \kappa\, \frac{Q_{\mathbf{u}}(u_j + i\varepsilon)}{Q_{\mathbf{u}}(u_j - i\varepsilon)} = 1 \qquad (j = 1, \dots, M). \tag{8}$$

Here and in the following we will use the notation

$$Q_{\mathbf{u}}(v) = \prod_{i=1}^{M}(v-u_i), \quad \mathbf{u} = \{u_1, \dots, u_M\}. \tag{9}$$

The corresponding eigenvalue of the transfer matrix $T(x)$ is

$$t(v) = \frac{Q_{\mathbf{u}}(v-i\varepsilon)}{Q_{\mathbf{u}}(v)} + \kappa\,\frac{d(v)}{a(v)}\frac{Q_{\mathbf{u}}(v+i\varepsilon)}{Q_{\mathbf{u}}(v)}\ . \tag{10}$$

If the rapidities $\mathbf{u}$ are generic, the Bethe state is called 'off-shell'.

In the unitary representations of the RTT-algebra, like the $\mathrm{XXX}_{1/2}$ spin chain, the on-shell states form a complete set in the Hilbert space. The XXX spin chain of length L can be deformed by introducing inhomogeneities $\theta_1, \dots, \theta_L$ associated with the L sites of the spin chain. The eigenvalues of the operators $A(v)$ and $D(v)$ on the vacuum in the inhomogeneous XXX chain are given by

$$a(v) = Q_{\boldsymbol{\theta}}(v+\tfrac{1}{2}i\varepsilon)\ ,\ \ d(v) = Q_{\boldsymbol{\theta}}(v-\tfrac{1}{2}i\varepsilon), \tag{11}$$

where the polynomial $\mathcal{Q}_{\boldsymbol{\theta}}(x)$ is defined as[1]

$$Q_{\boldsymbol{\theta}}(x) = \prod_{l=1}^{L}(x-\theta_l), \quad \boldsymbol{\theta} = \{\theta_1, \dots, \theta_L\}. \tag{12}$$

Any Bethe state is completely characterised by its *pseudo-momentum*, known also under the name of *counting function* [27]

$$2ip(v) = \log\frac{Q_{\mathbf{u}}(v+i\varepsilon)}{Q_{\mathbf{u}}(v-i\varepsilon)} - \log\frac{a(v)}{d(v)} + \log\kappa. \tag{13}$$

The Bethe equations (8) imply that

$$p(u_j) = 2\pi n_j - \pi \quad (j = 1, \dots, M) \tag{14}$$

where the integers n_j are called mode numbers.

3 Determinant formulas for the inner product

In order to expand the states $|\mathbf{v}\rangle$ with given a set of rapidities in the basis of eigenvectors $|\mathbf{u}\rangle$ of the monodromy matrix,

$$|\mathbf{v}\rangle = \sum_{\mathbf{u}\ \text{on shell}} \frac{\langle\,\mathbf{u}|\mathbf{v}\,\rangle}{\langle\,\mathbf{u}|\mathbf{u}\,\rangle}\,|\mathbf{u}\rangle, \tag{1}$$

we need to compute the scalar product $\langle\,\mathbf{v}|\mathbf{u}\,\rangle$ of an off-shell and an on-shell Bethe state. The scalar product is related to the bilinear form

$$(\mathbf{v},\mathbf{u}) = \langle\Omega|\prod_{j=1}^{M}C(v_j)\ \prod_{j=1}^{M}B(u_j)|\Omega\rangle \tag{2}$$

by $(\mathbf{u},\mathbf{v}) = (-1)^M\,\langle\,\mathbf{u}^*|\mathbf{v}\,\rangle$. This follows from the complex Hermitian convention $B(u)^\dagger = -C(u^*)$. The inner product can be computed by commuting the B-operators to the left and the A-operators to the right

[1] This is a particular case of the Drinfeld polynomial $P_1(u)$ [26] when all spins along the chain are equal to $1/2$.

according to the algebra (2), and then applying the relations (5) and (6). The resulting sum formula written down by Korepin [2] works well for small number of magnons but for larger M becomes intractable.

An important observation was made by N. Slavnov [5], who realised that when one of the two states is on-shell, the Korepin sum formula gives the expansion of the determinant of a sum of two $M \times M$ matrices.[2] Although the Slavnov determinant formula does not give obvious advantages for taking the thermodynamical limit, is was used to elaborate alternative determinant formulas, which are better suited for this task [11–14].

Up to a trivial factor, the inner product depends on the functional argument

$$f(v) \equiv \kappa \, \frac{d(v)}{a(v)} \tag{3}$$

and on two sets of rapidities, $\mathbf{u} = \{u_1, \dots, u_M\}$ and $\mathbf{v} = \{v_1, \dots, v_M\}$. Since the rapidities within each of the two sets are not ordered, the inner product has symmetry $S_M \times S_M$, where S_M is the group of permutations of M elements. It came then as a surprise that the inner product can be written [13] [3] as a restriction on the mass shell (for one of the two sets of rapidities) of an expression completely symmetric with respect of the permutations of the union $\mathbf{w} \equiv \{w_1, \dots, w_{2M}\} = \{u_1, \dots, u_M, v_1, \dots, v_M\}$ of the rapidities of the two states:

$$(\mathbf{v}|\mathbf{u}) \underset{\mathbf{u} \to \text{on shell}}{=} \prod_{j=1}^{M} a(v_j) d(u_j) \, \mathscr{A}_{\mathbf{w}}[f] \,, \qquad \mathbf{w} = \mathbf{u} \cup \mathbf{v}, \tag{4}$$

where the functional $\mathscr{A}_{\mathbf{w}}[f]$ is given by the following $N \times N$ determinant ($N = 2M$)

$$\mathscr{A}_{\mathbf{w}}[f] = \det_{jk} \left(w_j^{k-1} - f(w_j) \, (w_j + i\varepsilon)^{k-1} \right) / \det_{jk} \left(w_j^{k-1} \right) . \tag{5}$$

In the $\mathrm{XXX}_{1/2}$ spin chain, the r.h.s. of (4) is proportional to the inner product of an off-shell Bethe state $|\mathbf{w}\rangle$ and a state obtained from the left vacuum by a global $SU(2)$ rotation [13]. Such inner products can be given statistical interpretation as a partial domain-wall partition function (pDWPF) [29]. In this case the identity (4) can be explained with the global $su(2)$ symmetry [13].

Another determinant formula, which is particularly useful for taking the thermodynamical limit, is derived in [14]:

$$\mathscr{A}_{\mathbf{w}} = \det \left(1 - K \right) , \tag{6}$$

where the $N \times N$ matrix K has matrix elements

$$K_{jk} = \frac{Q_j}{w_j - w_k + i\varepsilon} \qquad\qquad (j, k = 1, \dots, N) \, , \tag{7}$$

and the weights Q_j are obtained as the residues of the same function at the roots w_j:

$$Q_j \equiv \operatorname*{Res}_{z \to w_j} \mathcal{Q}(z), \quad \mathcal{Q}(z) \equiv f(z) \, \frac{Q_{\mathbf{w}}(z + i\varepsilon)}{Q_{\mathbf{w}}(z)}. \tag{8}$$

Here $Q_{\mathbf{w}}$ is the Baxter polynomial for the set $\mathbf{w}$, c.f. (9). The determinant formula (6) has the advantage that it exponentiates in a simple way:

$$\log \mathscr{A}_{\mathbf{w}}[f] = -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{j_1, \dots, j_n = 1}^{N} \frac{Q_{j_1}}{w_{j_1} - w_{j_2} + i\varepsilon} \, \frac{Q_{j_2}}{w_{j_1} - w_{j_3} + i\varepsilon} \, \cdots \, \frac{Q_{j_n}}{w_{j_n} - w_{j_1} + i\varepsilon}. \tag{9}$$

[2] This property is particular for the $SU(2)$ model. The the inner product in the $SU(n)$ model is a determinant only for a restricted class of states [28].

[3] The case considered in [13] was that of the periodic inhomogeneous $\mathrm{XXX}_{1/2}$ spin chain of length L, but the proof given there is trivially extended to the generalised $SU(2)$ model.

The identity (6) is the basis for the field-theoretical approach to the computation of the scalar product in the thermodynamical limit.

4 Field theory of the inner product

4.1 The $\mathscr{A}$-functional in terms of free fermions

This determinant on the rhs of (6) can be expressed as a Fock-space expectation value for a Neveu-Schwarz chiral fermion living in the rapidity plane with two-point function

$$\langle 0|\psi(z)\psi^*(u)|0\rangle = \langle 0|\psi^*(z)\psi(u)|0\rangle = \frac{1}{z-u}\,. \tag{1}$$

Representing the matrix K in (6) as

$$K_{jk} = \langle 0|\psi^*(w_j + i\varepsilon)\psi(w_k)|0\rangle \tag{2}$$

it is easy to see that the $\mathscr{A}$-functional is given by the expectation value

$$\mathscr{A}_{\mathbf{w}}[f] = \langle 0|\exp\left(\sum_{j=1}^{N} Q_j\,\psi^*(w_j)\psi(w_j+i\varepsilon)\right)|0\rangle. \tag{3}$$

In order to take the large N limit, we will need reformulate the problem entirely in terms of the meromorphic function $\mathcal{Q}(z)$. The discrete sum of fermion bilinears in the exponent on the rhs of (3) can be written as a contour integral using the fact that the quantities Q_j, defined by (8), are residues of the same function $\mathcal{Q}(z)$ at $z = w_j$. As a consequance, the Fock space representation (3) takes the form

$$\mathscr{A}_{\mathbf{w}}[f] = \langle 0|\exp\left(\oint_{\mathcal{C}_{\mathbf{w}}} \frac{dz}{2\pi i}\mathcal{Q}(z)\,\psi^*(z)\psi(z+i\varepsilon)\right)|0\rangle\,, \tag{4}$$

where the contour $\mathcal{C}_{\mathbf{w}}$ encircles the points $\mathbf{w}$ and leaves outside all other singularities of $\mathcal{Q}$, as shown in Fig. 1. Expanding the exponent and performing the gaussian contractions, one writes the $\mathscr{A}$-functional in the form of a Fredholm determinant

$$\mathscr{A}_{\mathbf{w}}[f] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\oint_{\mathcal{C}_{\mathbf{w}}^{\times n}}\prod_{j=1}^{n}\frac{dz_j\,\mathcal{Q}(z_j)}{2\pi i}\det_{j,k=1}^{n}\frac{1}{z_j - z_k + i\varepsilon}. \tag{5}$$

Since the function $\mathcal{Q}$ has exactly N poles inside the contour $\mathcal{C}_{\mathbf{w}}$, only the first N terms of the series are non-zero. The series exponentiates to

$$\log\mathscr{A}_{\mathbf{w}}[f] = -\sum_{n=1}^{\infty}\frac{1}{n}\oint_{\mathcal{C}_{\mathbf{w}}^{\times n}}\frac{dz_1\dots dz_n}{(2\pi i)^n}\frac{\mathcal{Q}(z_1)}{z_1 - z_2 + i\varepsilon}\cdots\frac{\mathcal{Q}(z_n)}{z_n - z_1 + i\varepsilon}. \tag{6}$$

This is the vacuum energy energy of the fermionic theory, given by the sum of all vacuum loops. The factor (-1) comes from the Fermi statistics and the factor $1/n$ accounts for the cyclic symmetry of the loops. The series (6) can be of course obtained directly from (9).

4.2 Bosonic theory and Coulomb gas

Alternatively, one can express the $\mathscr{A}$-function in term of a chiral boson $\phi(x)$ with two-point function

$$\langle 0|\phi(z)\phi(u)|0\rangle = \log(z-u). \tag{7}$$

After bosonization $\psi(z) \to e^{\phi(z)}$ and $\psi^*(z) \to e^{-\phi(z)}$, where we assumed that the exponents of the gaussian field are normally ordered, the fermion bilinear $\psi^*(z)\psi(z+i\varepsilon)$ becomes, up to a numerical factor, a chiral vertex operator of zero charge

$$\mathcal{V}(z) \equiv e^{\phi(z+i\varepsilon)-\phi(z)}. \tag{8}$$

The coefficient is obtained from the OPE

$$e^{-\phi(z)}\, e^{\phi(u)} \sim \frac{1}{z-u}\, e^{\phi(u)-\phi(z)} \tag{9}$$

with $u = z + i\varepsilon$:

$$\psi^*(z)\psi(z+i\varepsilon) \quad \to \quad e^{-\phi(z)}e^{\phi(z+i\varepsilon)} = -\frac{1}{i\varepsilon}\,\mathcal{V}(z). \tag{10}$$

The bosonized form of the operator representation (4) is therefore

$$\mathscr{A}_{\mathbf{w}}[f] = \langle 0|\exp\left(-\frac{1}{i\varepsilon}\oint_{\mathcal{C}_{\mathbf{w}}}\frac{dz}{2\pi i}\,\mathcal{Q}(z)\,\mathcal{V}(z)\right)|0\rangle, \tag{11}$$

where $|0\rangle$ is the bosonic vacuum state with zero charge. Expanding the exponential and applying the OPE (9) one writes the expectation value as the grand-canonical Coulomb-gas partition function

$$\mathscr{A}_{\mathbf{w}}[f] = \sum_{n=0}^{N}\frac{(-1)^n}{n!}\prod_{j=1}^{n}\oint_{\mathcal{C}_{\mathbf{w}}}\frac{dz_j}{2\pi i}\;\frac{\mathcal{Q}(z_j)}{i\varepsilon}\;\prod_{j<k}^{n}\frac{(z_j-z_k)^2}{(z_j-z_k)^2-i\varepsilon^2}\,. \tag{12}$$

After applying the Cauchy identity, we get back the Fredholm determinant (5).

4.3 The thermodynamical limit

Although the roots $\mathbf{w} = \{w_1, \ldots, w_N\}$ are off-shell, typically they can be divided into two or three on-shell subsets $\mathbf{w}^{(k)}$, each representing a lowest energy solution of the Bethe equations for given (large) magnon number $N^{(k)}$. The Bethe roots for such solution are organised in one of several arrays with spacing $\sim \varepsilon$, called macroscopic Bethe strings, and the distribution of the roots along these arrays is approximated by continuous densities on a collection of contours in the complex rapidity plane [15, 16, 18, 19].

We choose an N-dependent normalisation of the rapidity such that $\varepsilon \sim 1/N$. Then the typical size of the contours and the densities remains finite in the limit $\varepsilon \to 0$.

In order to compute the $\mathscr{A}$-functional in the large N limit, we will follow the method developed on [14] and based on the field-theoretical formulation of the problem, eq. (11). The method involves a coarse-graining procedure, as does the original computation of the quantity $\mathcal{A}$, carried out in [31].

Let us mention that there is a close analogy between the above semiclassical analysis and the computation of the instanton partition functions of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories in the so-called Ω-background, characterised by two deformation parameters, ε_1 and ε_2 [32, 33], in the Nekrasov-Shatashvili limit $\varepsilon_2 \to 0$ [34]. In this limit the result is expressed in terms of the solution of a non-linear integral equation. The derivation, outlined in [34] and explained in great detail in the recent papers [35, 36], is based on the iterated Mayer expansion for a one-dimensional non-ideal gas. Our method is a field-theoretical alternative of the the Mayer expansion of the gas of dipole charges created by the ex-

ponential operators $\mathcal{V}_n$. In our problem the saddle-point of the action (33) also lead to a non-linear integral equation, but the non-linearity disappears when $\varepsilon \to 0$.

Of crucial relevance to our approach is the possibility to deform the contour of integration. In order to take advantage of the contour-integral representation, the original integration contour $\mathcal{C}_{\mathbf{w}}$ surrounding the poles $\mathbf{w}$ of the integrand, should be deformed to a contour $\mathcal{C}$ which remains at finite distance from the singularities of the function $\mathcal{Q}$ when $\varepsilon \to 0$, as shown in Fig. 1. Along the contour $\mathcal{C}$ the function $\mathcal{Q}(z)$ changes slowly at distances $\sim \varepsilon$. In all nontrivial applications the weight function $\mathcal{Q}$ has a dditional poles, which are those of the function f. The contour $\mathcal{C}$ separates the roots $\mathbf{w}$ from the poles of f.

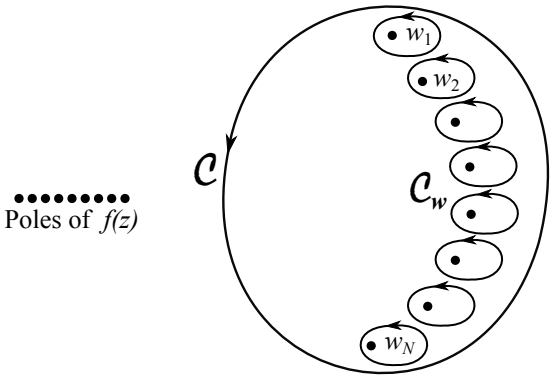

Fig. 1 Schematic representation of the contour $\mathcal{C}_{\mathbf{w}}$ and the deformed contour $\mathcal{C}$.

4.4 Coarse-graining

We would like to compute the ε-expansion of the expectation value (11), with $\mathcal{C}_{\mathbf{w}}$ replaced by $\mathcal{C}$. This is a semi-classical expansion with Planck constant $\hbar = \varepsilon$. As any semi-classical expansion, the perturbative expansion in ε is an asymptotic expansion. Our strategy is to introduce a cutoff Λ, such that

$$\varepsilon \ll \Lambda \ll N\varepsilon \qquad (N\varepsilon \sim 1), \tag{13}$$

integrate the ultra-violet (fast-scale) part of the theory in order to obtain an effective infrared (slow-scale) theory. The splitting of the bosonic field into slow and fast pieces into slow and fast pieces is possible only in the thermodynamical limit $\varepsilon \to 0$. In this limit the dependence on Λ enters through exponenttially small non-perturbative terms and the perturbative expansion in ε does not depend on Λ.

We thus cut the contour $\mathcal{C}$ into segments of length Λ and compute the effective action for the slow piece as the sum of the connected n-point correlators (cumulants) of the vertex operator $\mathcal{V}$. The n-th cumulant $\Xi_n(z)$ is obtained by integrating the OPE of a product of n vertex operators

$$\mathcal{V}(z_1)\dots\mathcal{V}(z_n) = \prod_{j<k} \frac{(z_j - z_k)^2}{(z_j - z_k)^2 + \varepsilon^2} : \mathcal{V}(z_1)\dots\mathcal{V}(z_n) : \tag{14}$$

along a segment of the contour $\mathcal{C}$ of size Λ, containing the point z. Since we want to evaluate the effect of the short-distance interaction due to the poles, we can assume that the rest of the integrand is analytic everywhere. Then the integration can be performed by residues using the Cauchy identity. This computation has been done previously in [32] in a different context. The easiest way to compute the integral is to fix $z_1 = z$ and integrate with respect to $z_2, \dots, z_n$. We expand the numerical factor in (14) as a sum over permutations. The $(n-1)!$ permutations representing maximal cycles of length n give identical contributions to the residue. For the rest of the permutations the contour integral vanishes. We find ($z_{jk} \equiv z_j - z_k$)

$$\begin{aligned}\Xi_n &= \oint \frac{\mathcal{V}(z_1)\dots\mathcal{V}(z_n)}{(-i\varepsilon)^n\, n!}\prod_{k=2}^{n}\frac{dz_k}{2\pi i}\\ &\sim \frac{(n-1)!}{n!}\oint \frac{\prod_{k=2}^{n}\frac{dz_k}{2\pi i}\ :\mathcal{V}(z_1)\cdots\mathcal{V}(z_n):}{(i\varepsilon-z_{12})\dots(i\varepsilon-z_{n-1,n})(i\varepsilon-z_{n,1})}\\ &= -\frac{1}{n^2 i\varepsilon}\,\mathcal{V}_n(z)\,,\end{aligned} \tag{15}$$

where

$$\mathcal{V}_n(z) \equiv :\mathcal{V}(z)\mathcal{V}(z+i\varepsilon)\dots\mathcal{V}(z+ni\varepsilon): = e^{\phi(z+ni\varepsilon)-\phi(z)}\,. \tag{16}$$

The interaction potential of the effective coarse-grained theory therefore contains, besides the original vertex operator $\mathcal{V}\equiv\mathcal{V}_1$, all composite vertex operators $\mathcal{V}_n$ with $n\lesssim\Lambda$. If one repeats the computation (15) with the weights $\mathcal{Q}$, one obtains for the n-th cumulant

$$\Xi_n(z) = -\frac{1}{i\varepsilon}\,\frac{\mathcal{Q}_n(z)\,\mathcal{V}_n(z)}{n^2}\,,\qquad \mathcal{Q}_n(z)=\mathcal{Q}(z)\mathcal{Q}(z+i\varepsilon)\dots\mathcal{Q}(z+in\varepsilon). \tag{17}$$

$$\Xi_n(z) = -\frac{1}{i\varepsilon}\,\frac{\mathcal{Q}_n(z)\,\mathcal{V}_n(z)}{n^2}\,,\qquad \mathcal{Q}_n(z)=\mathcal{Q}(z)\mathcal{Q}(z+i\varepsilon)\dots\mathcal{Q}(z+in\varepsilon)=e^{-\Phi(x)+\Phi(x+ni\varepsilon)}\,. \tag{18}$$

As the spacing $n\varepsilon$ should be smaller than the cut-off length Λ, from the perspective of the effective infrared theory all these particles are point-like. We thus obtained that in the semi-classical limit the $\mathscr{A}$-functional is given, up to non-perturbative terms, by the expectation value

$$\mathscr{A}_{\mathbf{u},\mathbf{z}} \approx \left\langle \exp\left(\frac{1}{\varepsilon}\sum_{n=1}^{\Lambda/\varepsilon}\frac{1}{n^2}\oint_{\mathcal{C}}\frac{dz}{2\pi}\,\mathcal{Q}_n(z)\,\mathcal{V}_n(z)\right)\right\rangle. \tag{19}$$

The effective potential can be given a nice operator form, which will be used to extract the perturbative series in ε. For that it is convenient to represent the function $f(z)$ as the ratio

$$f(z) = \frac{g(z)}{g(z+i\varepsilon)} = g(z)^{-1}\mathbb{D}\,g(z)\,, \tag{20}$$

where we introduced the shift operator

$$\mathbb{D}\equiv e^{i\varepsilon\partial}\,. \tag{21}$$

Then the weight factor $\mathcal{Q}_n$ takes the form

$$\mathcal{Q}_n = e^{-\Phi}\,\mathbb{D}^n\,e^{\Phi},\qquad \Phi(z)=Q_{\mathbf{w}}(z)/g(z)\,, \tag{22}$$

and the series in the exponent in (19) can be summed up to

$$\mathscr{A}_{\mathbf{w}}[f] = \left\langle \exp\left(\frac{1}{\varepsilon}\oint_{\mathcal{C}}\frac{dz}{2\pi}\ :e^{-\Phi(z)-\phi(z)}\,\mathrm{Li}_2(\mathbb{D})\,e^{\Phi(z)+\phi(z)}:\right)\right\rangle, \tag{23}$$

with the operator $\mathrm{Li}_2(\mathbb{D})$ given by the dilogarithmic series

$$\mathrm{Li}_2(\mathbb{D}) = \sum_{n=1}^{\infty}\frac{\mathbb{D}^n}{n^2}. \tag{24}$$

Here we extended the sum over n to infinity, which which can be done with exponential accuracy. The function $\Phi(z)$, which we will refer to as "classical potential", plays the role of classical expectation value for the bosonic field ϕ.

If we specify to the case of the (inhomogeneous, twisted) spin chain, considered in [14], then $f = \kappa\, d/a$ with a, d given by (11). In this case the classical potential is

$$\Phi(z) = \log Q_{\mathbf{w}}(z) - \log Q_{\boldsymbol{\theta}}(z - i\varepsilon/2). \tag{25}$$

Remark. Going back to the fermion representation, we write the result as a Fredholm determinant with different Fredholm kernel,

$$\mathscr{A}_{\mathbf{w}}[f] \approx \langle 0| \exp\left(\oint_{\mathcal{C}} \frac{dz}{2\pi i} e^{-\Phi(z)} \psi^*(z)\, \log(1-\mathbb{D})\, \psi(z) e^{\Phi(z)}\right) |0\rangle = \mathrm{Det}(1-\hat{\mathcal{K}}), \tag{26}$$

where the Fredholm operator $\hat{\mathcal{K}}$ acts in the space of functions analytic in the vicinity of the contour $\mathcal{C}$:

$$\hat{\mathcal{K}}\xi(z) = \oint_{\mathcal{C}} \frac{du}{2\pi i} \hat{\mathcal{K}}(z,u)\xi(u), \qquad \hat{\mathcal{K}}(z,u) = \sum_{n=1}^{\infty} \frac{e^{-\Phi(z)+\Phi(z+i\varepsilon n)}}{z-u+i\varepsilon n}. \tag{27}$$

The expression in terms of a Fredholm determinant can be obtained directly by performing the cumulant expansion for the expression of the $\mathscr{A}$-functional as a product of shift operators [12]

$$\begin{aligned} \mathscr{A}[f] &= \frac{1}{\Psi_{\mathbf{w}}[g]} \prod_{j=1}^{N} (1 - e^{i\varepsilon \partial/\partial w_j}) \prod_{j=1}^{N} \Psi_{\mathbf{w}}[g], \\ \Psi_{\mathbf{w}}[g] &= \frac{\prod_{j<k}(w_j - w_k)}{\prod_{j=1}^{N} g(w_j)}, \quad f(z) = \frac{g(z)}{g(z+i\varepsilon)}. \end{aligned} \tag{28}$$

4.5 The first two orders of the semi-classical expansion

The effective IR theory is compatible with the semi-classical expansion being of the form

$$\log \mathscr{A}_{\mathbf{w}} = \frac{F_0}{\varepsilon} + F_1 + \varepsilon F_2 + \cdots + \mathcal{O}(e^{-\Lambda/\varepsilon}). \tag{29}$$

Below we develop a diagram technique for computing the coefficients in the expansion. First we notice that the ε-expansion of the effective interaction in (23) depends on the field ϕ through the derivatives $\partial\phi, \partial^2\phi$, etc. We therefore consider the first derivative derivative $\partial\phi$ as an independent field

$$\varphi(z) \equiv -\partial\phi(z) \tag{30}$$

with two-point function

$$G(z,u) = \partial_z \partial_u \log(z-u) = \frac{1}{(z-u)^2}. \tag{31}$$

In order to derive the diagram technique, we formulate the expectation value (23) as a path integral for the $(0+1)$-dimensional field $\varphi(x)$ defined on the contour $\mathcal{C}$. The two-point function (31) can be imposed in the standard way by introducing a second field $\rho(x)$ linearly coupled to φ. The path integral reads

$$\mathscr{A}_{\mathbf{w}}[f] = \int [D\varphi\, D\rho]\, e^{-\mathcal{Y}[\varphi,\rho]}\,, \tag{32}$$

with action functional

$$\mathcal{Y}[\varphi,\rho] = -\tfrac{1}{2}\int\limits_{\mathcal{C}\times\mathcal{C}} dzdu\,\frac{\rho(z)\rho(u)}{(z-u)^2} + \oint_{\mathcal{C}} dx\,\rho(z)\varphi(z) + \oint_{\mathcal{C}}\frac{dz}{2\pi}\,W(\varphi,\varphi',\dots)\,. \tag{33}$$

The dependence on ε is through the potential W, obtained by expanding the exponent in (23):

$$\begin{aligned} W(\varphi,\varphi',\dots) &= -\frac{1}{\varepsilon}\,e^{-\Phi(x)-\phi(x)}\,\mathrm{Li}_2(\mathbb{D})\,e^{\Phi(x)+\phi(x)} \\ &= -\frac{1}{\varepsilon}\,\mathrm{Li}_2(\mathcal{Q}) + i\log(1-\mathcal{Q})\varphi - \frac{\varepsilon}{1-\mathcal{Q}}(\varphi^2+\varphi') + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{34}$$

The potential contains a constant term, which gives the leading contribution to the free energy, a tadpole of order 1 and higher vertices that disappears in the limit $\varepsilon \to 0$. The Feynman rules for the effective action $\mathcal{Y}[\varphi,\rho]$ are such that each given order in ε is obtained as a sum of finite number of Feynman graphs. For the first two orders one obtains

$$F_0 = \oint_{\mathcal{C}}\frac{dx}{2\pi}\,\mathrm{Li}_2[\mathcal{Q}(x)]\,, \tag{35}$$

$$F_1 = -\tfrac{1}{2}\oint\limits_{\mathcal{C}\times\mathcal{C}}\frac{dx\,du}{(2\pi)^2}\,\frac{\log\left[1-\mathcal{Q}(x)\right]\,\log\left[1-\mathcal{Q}(u)\right]}{(x-u)^2}. \tag{36}$$

where the double integral is understood as a principal value. The actual choice of the contour $\mathcal{C}$ is a subtle issue and depends on the analytic properties of the function $\mathcal{Q}(x)$. The contour shold be placed in such away that it does not cross the cuts of the integrand.

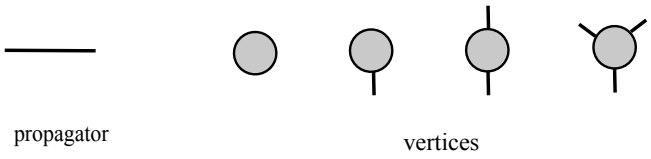

Fig. 2 Feynman rules for the effective field theory

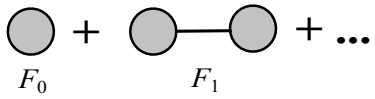

Fig. 3 The leading and the subleading orders of the vacuum energy

Returning to the scalar product and ignoring the trivial factors in (4), we find that the first two coefficients of the semi-classical expansion are given by eqs. (35) and (36) with

$$\mathcal{Q} = e^{ip_{\mathbf{u}}+ip_{\mathbf{v}}}. \tag{37}$$

5 Discussion

In these notes we reviewed the field-theoretical approach to the computation of scalar products of on-shell/off-shell Bethe vectors in the generalised model with $SU(2)$ rational R-matrix, which leads to a systematic procedure for computing the semi-classical expansion. The results reported here represent a slight generalisation if those already reported in [12–14]. We hope that the field-theoretical method could be used to compute scalar products in integrable models associated with higher rank groups, using the fact that the the integrands in the multiple contour integrals of in [7–10] is expressed as products of $\mathscr{A}$-functionals.

The problem considered here is formally similar to the problem of computing the instanton partition functions in $\mathcal{N}=1$ and $\mathcal{N}=2$ SYM [32–34]. As a matter of fact, the scalar product in the form (12) is the grand-canonical version of the partition function of the $\mathcal{N}=1$ SUSY in four dimensions, which was studied in a different large N limit in [38].

Our main motivation was the computation of the three-point function of heavy operators in $\mathcal{N}=4$ four-dimensional SYM. Such operators are dual to classical strings in $AdS_5\times S^5$ and can be compared with certain limit of the string-theory results. For a special class of three-point functions, the semi-classical expansion is readily obtained from that of the scalar product. The leading term F_0 should be obtained on the string theory side as the classical action of a minimal world sheet with three prescribed singularities. The comparison with the recent computation in [37] looks very encouraging. We expect that the meaning of the subleading term on the string theory side is that it takes account of the gaussian fluctuations around the minimal world sheet. In this context it would be interesting to obtain the subleading order of the heavy-heavy-light correlation function in the $su(2)$ sector in string theory [39–41]. In the near-plane-wave limit the subleading order was obtained in [42].

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References

1. M. Gaudin, *La fonction d'onde de Bethe*. Masson, Paris, 1983.
2. V. E. Korepin, "Calculation of norms of Bethe wave functions," *Communications in Mathematical Physics* **86** (1982) 391–418. 10.1007/BF01212176.
3. V. E. Korepin, "Norm of Bethe Wave Function as a Determinant," *ArXiv e-prints* (Nov., 2009) 0911.1881.
4. A. Izergin and V. Korepin, "The quantum inverse scattering method approach to correlation functions," *Communications in Mathematical Physics* **94** (1984), no. 1, 67–92.
5. N. A. Slavnov, "Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe ansatz," *Theoretical and Mathematical Physics* **79** (1989) 502–508. 10.1007/BF01016531.
6. N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, "On the algebraic Bethe Ansatz approach to the correlation functions of the XXZ spin-1/2 Heisenberg chain," *ArXiv High Energy Physics - Theory e-prints* (Apr., 2005) arXiv:hep-th/0505006.
7. S. Z. Pakuliak and S. M. Khoroshkin, "Weight Function for the Quantum Affine Algebra $U_q(\widehat{Sl(3)})$," *Theoretical and Mathematical Physics* **145** (Oct., 2005) 1373–1399, arXiv:math/0610433.
8. L. Frappat, S. Khoroshkin, S. Pakuliak, and É. Ragoucy, "Bethe Ansatz for the Universal Weight Function," *Annales Henri Poincaré* **10** (May, 2009) 513–548, 0810.3135.
9. S. Belliard, S. Pakuliak, and E. Ragoucy, "Universal Bethe Ansatz and Scalar Products of Bethe Vectors," *SIGMA* **6** (Dec., 2010) 94, 1012.1455.
10. M. Wheeler, "Multiple integral formulae for the scalar product of on-shell and off-shell Bethe vectors in SU(3)-invariant models," *ArXiv e-prints* (June, 2013) 1306.0552.
11. I. Kostov, "Classical Limit of the Three-Point Function of N=4 Supersymmetric Yang-Mills Theory from Integrability," *Physical Review Letters* **108** (June, 2012) 261604, 1203.6180.
12. I. Kostov, "Three-point function of semiclassical states at weak coupling," *Journal of Physics A Mathematical General* **45** (Dec., 2012) 4018, 1205.4412.

13. I. Kostov and Y. Matsuo, "Inner products of Bethe states as partial domain wall partition functions," *JHEP10(2012)168* (July, 2012) 1207.2562.
14. E. Bettelheim and I. Kostov, "Semi-classical analysis of the inner product of Bethe states," 1403.0358.
15. B. Sutherland, "Low-Lying Eigenstates of the One-Dimensional Heisenberg Ferromagnet for any Magnetization and Momentum," *Phys. Rev. Lett.* **74** (Jan, 1995) 816–819.
16. A. Dhar and B. Sriram Shastry, "Bloch Walls and Macroscopic String States in Bethe's Solution of the Heisenberg Ferromagnetic Linear Chain," *Phys. Rev. Lett.* **85** (Sep, 2000) 2813–2816.
17. N. Beisert, C. Ahn, L. Alday, Z. Bajnok, J. Drummond, L. Freyhult, N. Gromov, R. Janik, V. Kazakov, T. Klose, G. Korchemsky, C. Kristjansen, M. Magro, T. McLoughlin, J. Minahan, R. Nepomechie, A. Rej, R. Roiban, S. Schäfer-Nameki, C. Sieg, M. Staudacher, A. Torrielli, A. Tseytlin, P. Vieira, D. Volin, and K. Zoubos, "Review of AdS/CFT Integrability: An Overview," *Letters in Mathematical Physics* **99** (2012), no. 1-3, 3–32.
18. N. Beisert, J. A. Minahan, M. Staudacher, and K. Zarembo, "Stringing spins and spinning strings," *JHEP* **09** (2003) 010, hep-th/0306139.
19. V. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, "Classical / quantum integrability in AdS/CFT," *JHEP* **05** (2004) 024, hep-th/0402207.
20. J. Escobedo, N. Gromov, A. Sever, and P. Vieira, "Tailoring three-point functions and integrability," *JHEP* **9** (Sept., 2011) 28, 1012.2475.
21. O. Foda, "$\mathcal{N} = 4$ SYM structure constants as determinants," *Journal of High Energy Physics* **3** (Mar., 2012) 96, 1111.4663.
22. Y. Jiang, I. Kostov, F. Loebbert, and D. Serban, "Fixing the Quantum Three-Point Function," *ArXiv e-prints* (Jan., 2014) 1401.0384.
23. L. A. Takhtajan and L. D. Faddeev, "The Quantum method of the inverse problem and the Heisenberg XYZ model," *Russ. Math. Surveys* **34** (1979) 11–68.
24. L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan, "The Quantum Inverse Problem Method. 1," *Theor. Math. Phys.* **40** (1979), no. 2, 688–706.
25. N. A. Slavnov, "The algebraic Bethe ansatz and quantum integrable systems," *Russian Mathematical Surveys* **62** (2007), no. 4, 727.
26. V. Drinfeld, ""Elliptic modules"," *Matematicheskii Sbornik (Russian)* **94** (1974).
27. H. De Vega, "Yang-Baxter algebras, integrable theories and quantum groups", *International Journal of Modern Physics A* **04** (1989), no. 10, 2371–2463
28. M. Wheeler, "Scalar products in generalized models with SU(3)-symmetry," *ArXiv e-prints* (Apr., 2012) 1204.2089.
29. O. Foda and M. Wheeler, "Partial domain wall partition functions," *eprint arXiv:1205.4400* (May, 2012) 1205.4400.
30. M. Jimbo and T. Miwa, "Solitons and infinite dimensional Lie algebras," *Publ. RIMS, Kyoto Univ* **19** (1983) 943–1001.
31. N. Gromov, A. Sever, and P. Vieira, "Tailoring Three-Point Functions and Integrability III. Classical Tunneling," *ArXiv e-prints* (Nov., 2011) 1111.2349.
32. G.Moore, N.Nekrasov, and S.Shatashvili, "Integrating Over Higgs Branches," *Commun.Math.Phys.* **209** (2000) 97–121, hep-th/9712241.
33. G. W. Moore, N. Nekrasov, and S. Shatashvili, "D particle bound states and generalized instantons," *Commun.Math.Phys.* **209** (2000) 77–95, hep-th/9803265.
34. N. A. Nekrasov and S. L. Shatashvili, "Quantization of Integrable Systems and Four Dimensional Gauge Theories," 0908.4052.
35. C. Meneghelli and G. Yang, "Mayer-Cluster Expansion of Instanton Partition Functions and Thermodynamic Bethe Ansatz," *ArXiv e-prints* (Dec., 2013) 1312.4537.
36. J.-E. Bourgine, "Confinement and Mayer cluster expansions," 1402.1626.
37. Y. Kazama and S. Komatsu, "Three-point functions in the SU(2) sector at strong coupling," *ArXiv e-prints* (Dec., 2013) 1312.3727.
38. V. Kazakov, I. Kostov, and N. A. Nekrasov, "D-particles, matrix integrals and KP hierarchy," *Nucl. Phys.* **B557** (1999) 413–442, hep-th/9810035.
39. K. Zarembo, "Holographic three-point functions of semiclassical states," *Journal of High Energy Physics* **9** (Sept., 2010) 30, 1008.1059.
40. M. S. Costa, R. Monteiro, J. E. Santos, and D. Zoakos, "On three-point correlation functions in the gauge/gravity duality," *JHEP* **1011** (2010) 141, 1008.1070.
41. J. Escobedo, N. Gromov, A. Sever, and P. Vieira, "Tailoring three-point functions and integrability II. Weak/strong coupling match," *Journal of High Energy Physics* **9** (Sept., 2011) 29, 1104.5501.
42. T. Klose and T. McLoughlin, "Comments on World-Sheet Form Factors in AdS/CFT," 1307.3506.