Error estimates for a finite volume method for the Laplace equation in dimension one through discrete Green functions.

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Pascal Omnes[†⋆]

[†]*CEA, DEN, DM2S-SFME,*

F-91191 Gif-sur-Yvette Cedex, France.

pascal.omnes@cea.fr

[⋆]*Université Paris 13, LAGA, CNRS UMR 7539, Institut Galilée,*

99, Avenue J.-B. Clément F-93430 VILLETANEUSE Cedex, France.

Abstract

The cell-centered finite volume approximation of the Laplace equation in dimension one is considered. An exact expression of the error between the exact and numerical solutions is derived through the use of continuous and discrete Green functions. This allows to discuss convergence of the method in the L^∞ and L^2 norms with respect to the choice of the control points in the cells and with respect to the regularity of the data. Well-known second-order convergence results are recovered if those control points are properly chosen and if the data belongs to H^1. Counterexamples are constructed to show that second-order may be lost if these conditions are not met.

Key words : finite volumes, Laplace equation, error estimates, Green functions

1 Introduction and statement of the problem

The finite volume method (FVM) is a popular technique used to compute approximate solutions of various engineering problems encountered for example in fluid mechanics and heat and mass transfer, to cite only a few of its many applications. Simplicity, robustness and local conservativity are some of the features which are at the basis of the popularity of this method.

As far as the numerical analysis of the FVM is concerned, there have been many works devoted to the proof of convergence of such schemes, and to the derivation of error estimates. We refer for example to [16] for a review of such results covering elliptic, parabolic and hyperbolic equations.

In this paper, we shall be interested in a very specific problem, namely the numerical solution of the Laplace equation in dimension one by means of cell-centered finite volumes:

Let $\Omega =]0;1[$ be the domain of computation; let f be a given function in $L^2(\Omega)$ and let $\hat{\phi}$ be the exact solution of the following problem:

$$\left\{ \begin{array}{l} -\phi'' = f \quad \text{in } \Omega\,, \\ \phi(0) = \phi(1) = 0\,. \end{array} \right. \tag{1}$$

We recall that $\hat{\phi} \in \mathrm{H}_0^1(\Omega)$ verifies the weak formulation of Eq. (1):

$$\left(\hat{\phi}', \psi'\right)_\Omega = (f, \psi)_\Omega \quad , \quad \forall \psi \in H_0^1(\Omega)\,, \tag{2}$$

where we have used the following definition

Definition 1.1 The standard continuous L^2 scalar product over a generic domain K will be denoted by $(\cdot,\cdot)_K$, while the associated L^2 norm will be denoted by $||\cdot||_{0,K}$. The H_0^1 semi-norm and the H^1 norm over K will be respectively denoted by $|\cdot|_{1,K}$ and $||\cdot||_{1,K}$.

For the finite volume method, we split $\bar{\Omega}$ into N segments named K_i and defined by $K_i := [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ for $i \in [1,N]_{\mathbb{N}}$ with $x_{\frac{1}{2}} = 0 < x_{\frac{3}{2}} < \ldots < x_{N-\frac{1}{2}} < x_{N+\frac{1}{2}} = 1$. We shall denote by

$$h_i = |K_i| = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$$

the length of K_i and set

$$h = \sup_{i \in [1,N]_{\mathbb{N}}} h_i. \tag{3}$$

In each of these intervals, we choose a point x_i, *which is not necessarily equal to the midpoint of* K_i. Then, we build dual cells $K_{i+\frac{1}{2}}$ in the following way: $K_{i+\frac{1}{2}} = [x_i, x_{i+1}]$ for $i \in [0,N]_{\mathbb{N}}$, with, by convention, $x_0 = 0$ and $x_{N+1} = 1$. We set

$$h_{i+\frac{1}{2}} = \left|K_{i+\frac{1}{2}}\right| = x_{i+1} - x_i$$

and remark that since $x_i \in K_i$ and $x_{i+1} \in K_{i+1}$, there holds $h_{i+\frac{1}{2}} \leq h_{i+1} + h_i$, which, due to the definition (3), leads to

$$h_{i+\frac{1}{2}} \leq 2h\;,\;\forall i \in [0,N]_{\mathbb{N}}. \tag{4}$$

The cell-centered FVM defined by Eqs. (5.8)–(5.11) of [16] amounts to find a set of values $(\phi_i)_{i\in[0,N+1]_{\mathbb{N}}}$ such that:

$$\left\{ \begin{array}{rcl} -\frac{1}{h_i}\left(F_{i+1/2} - F_{i-1/2}\right) & = & f_i \;\;,\;\; \forall i \in [1,N]_{\mathbb{N}}\,, \\ F_{i+1/2} & = & \frac{1}{h_{i+1/2}}\left(\phi_{i+1} - \phi_i\right) \;\;,\;\; \forall i \in [0,N]_{\mathbb{N}}\,, \\ \phi_0 & = & \phi_{N+1} = 0\,, \end{array} \right. \tag{5}$$

where f_i is the average value of f on K_i:

$$f_i = \frac{1}{h_i}\int_{K_i} f(x)\,dx\,. \tag{6}$$

The numerical analysis of this FVM is reviewed in [16, Theorem 6.1, Remarks 6.2, 6.3 and 6.4]. Additional useful references are [19, 20], while [24] treats a slightly different scheme in which f_i is "a linear combination of the values of f evaluated at certain points". A second-order convergence result in the L^∞ norm ([17]) and in the H_0^1 norm ([20]) is proved provided that the exact solution is regular enough (C^4 in [17] and H^3 in [20]) and provided that the points x_i are chosen to be the midpoints of the cells K_i. If these conditions are not met, the only proved result is first-order convergence both in the L^∞ and the (discrete) H_0^1 norms.

Additionally, we mention that the authors of [30] have studied the same FVM but for the case of homogeneous Neumann boundary conditions. This latter case is much simpler than that studied here, since the first equation in (5) with the initial condition $F_{1/2} = 0$ obviously leads to the exact value of $F_{i+1/2}$ for all i; then ϕ is recovered by the second equation in (5) and is easily shown to be a second-order approximation of $\hat{\phi}(x_i)$ through Taylor expansions if x_j is the midpoint of K_j for all j or if $x_{j+1/2}$ is the midpoint of $K_{j+1/2}$ for all $j \in [1, N-1]_{\mathbb{N}}$.

Our main goal in the present work is to discuss the convergence order of the solution of the FVM (5) with respect to the regularity of f and with respect to the choice of the set of points (x_i), keeping in mind possible extensions of this FVM to dimensions greater than one. Let us now detail these two points.

1) In dimension one, the fact that f belongs to $H^m(\Omega)$, $m \in \mathbb{N}$, is equivalent to $\hat{\phi}$ being in $H^{m+2}(\Omega)$, and thus, discussing the dependence of the convergence order of the FVM with respect to the regularity of $\hat{\phi}$ is equivalent to discussing it with respect to the regularity of f. This is however not the case in dimensions greater than one. Indeed, since $-\Delta\hat{\phi} = f$, then $H^{m+2}(\Omega)$ regularity of $\hat{\phi}$ obviously implies $H^m(\Omega)$ regularity of f; on the other hand, $H^m(\Omega)$ regularity of f implies $H^{m+2}(\Omega)$ regularity of $\hat{\phi}$ only under sufficient regularity of the boundary $\partial\Omega$. In the related context of *vertex-centered finite volume element methods* (FVEM), on (primal) triangular meshes, this has led to distinguish regularity conditions over $\hat{\phi}$ on the one hand, and over f on the other hand, in the convergence order results in the L^2 norm: second-order convergence is obtained provided that $\hat{\phi}$ belongs to $H^2(\Omega)$, that f belongs to $H^1(\Omega)$ and that the dual mesh associated to the method is the barycentric one (see [9, 14]). The H^1 regularity condition over f comes from the fact that the error depends on the L^2 norm of $(f - \bar{f})$, where $\bar{f}$ is the L^2 projection of f on the cells of the mesh. This is in contrast to the linear finite element method (FEM) (both conforming or of Crouzeix-Raviart type), where second-order convergence in the L^2 norm is proved under the sole condition that f belongs to $L^2(\Omega)$ (in dimension one, and in dimension two under supplementary conditions on $\partial\Omega$, e.g. it suffices that Ω is polygonal convex). Indeed, in the FEM, the Aubin-Nitsche trick uses Galerkin orthogonality, which has no equivalent form in the FVM. A discrete version of this trick was proposed in [20] for the error analysis in the L^2 norm of cell-centered FVM for convection-diffusion associated to non-homogeneous boundary conditions in dimension two over square grids, *under the assumption that the right-hand side in the convection-diffusion equation vanishes*. The authors obtain second-order convergence provided the exact solution is in $H^2(\Omega)$ and the boundary data in $H^{3/2}(\Gamma)$, but the above-mentioned assumption circumvents the discussion on the regularity of the data in the convection-diffusion equation.

2) In dimension one, the natural choice for the point x_i is the midpoint of K_i. Therefore, one might wonder why there should be any interest in considering a different choice. The reason stands in the generalization of this FVM to dimensions higher than one; for the sake of simplicity, let us restrict the discussion to the dimension two. In such a case, the FVM may be used on meshes made up of polygonal convex cells which may have an arbitrary number of edges, provided the choice of the set of points (x_i) satisfies some orthogonality conditions given in the definition of so-called "admissible meshes" (see [16, Definition 9.1]). So what is the equivalent of the "midpoint" for a polygonal convex cell? A reasonable answer to this question seems to be the barycenter of the cell; however the above-mentioned orthogonality conditions lead to choose the circumcenters of the triangles, when the mesh is a Delaunay triangulation (see [16, Example 9.1]). Note that even in that particular case of admissible meshes, the order of convergence of the FVM in the L^2 norm is still an open issue, although, based on numerical evidence, some authors believe it to be two [3]. However, the only proved result in this context is the order one, see [16, Theorem 9.4] and [1, Proposition 5]. This first-order convergence result given by [16] also holds for another important example of admissible mesh, namely the Voronoi diagram associated with a given set of points (x_i). In that case, from the geometrical point of view, a point x_i may hardly be considered as "the center" of its associated Voronoi cell K_i. Note that, as recalled above, the second-order convergence in the L^2 norm proved in [9, 14] requires the dual mesh to be the barycentric one, so that this result does not apply to (dual) Voronoi meshes associated to (primal) Delaunay triangulations. These two examples justify the interest in choosing points x_i which are not the midpoints of the segments K_i in dimension one.

In this paper, we give exact expressions of the errors $\hat{\phi}(x_i) - \phi_i$, through the introduction of continuous and discrete Green functions. While this is a pretty common tool in the FEM, Galerkin or Petrov-Galerkin approximations, (see e.g. [8, 10, 11, 12, 22, 26, 28, 29]) and in the Finite Difference context (see e.g. [2, 4, 5, 6, 7, 13, 18, 25]), it has been rarely used, to our knowledge, in the FVM context. As far as the vertex-centered FVM is concerned, the authors of [23] consider a non-compact discretization of convection-diffusion problems and prove pointwise error estimates thanks to estimations of the discrete Green functions associated to the scheme. In [21], a finite volume scheme for convection-reaction-diffusion equations is recast into a finite difference form and bounds on the associated Green functions are derived to prove first-order pointwise error estimates. As far as the cell-centered FVM is concerned, the only reference to discrete Green functions we are aware of, is a Master Lecture by R. Eymard [15], where this tool is used on regular grids with the point x_i being the midpoint of K_i. On more general grids, this tool will allow us to discuss the convergence order of the FVM with respect to the regularity of f and with respect to the choice of the points x_i in K_i. It turns out of our investigations that if $f \in H^1(\Omega)$ and if either x_i lies within $O(h^2)$ of the midpoint of K_i for all $i \in [1, N]$ or $x_{i+1/2}$ lies within $O(h^2)$ of the midpoint of $K_{i+1/2}$ for all $i \in [1, N-1]$, then second-order convergence is obtained by the FVM in the L^∞ norm (and thus in the L^2 norm). On the other hand, it is shown by counterexamples that if f is not in $H^1(\Omega)$ or if the points x_i are not properly chosen, we may have a reduced order of convergence.

This paper is organized as follows: in section 2 we give some definitions and notations, while in section 3, we define and give the expressions of continuous and discrete Green functions associated to the equation (1) and to the scheme (5). This enables us to deduce the exact expression of the error $\hat{\phi}(x_i) - \phi_i$. In section 4, we provide sufficient conditions for second-order convergence of the FVM and give in section 5 counterexamples in less favorable cases. We give our conclusions in section 6.

2 Definitions and notations

Definition 2.1 Like in [30], we define an operator named "primal discrete derivative", denoted by ∇_h^P, on the cells K_i by the following formula:

$$\begin{array}{ccc} \mathbb{R}^{N+1} & \to & \mathbb{R}^N \\ \left(u_{i+\frac{1}{2}}\right)_{i\in[0,N]_{\mathbb{N}}} & \mapsto & \left(\nabla_h^P u\right)_i = \dfrac{1}{h_i}\left(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}\right) \quad , \quad \forall i \in [1,N]_{\mathbb{N}} \,. \end{array}$$

Definition 2.2 In the same way, we define an operator named "dual discrete derivative", denoted by ∇_h^D on the cells $K_{i+\frac{1}{2}}$ by the following formula:

$$\begin{array}{ccc} \mathbb{R}^{N+2} & \to & \mathbb{R}^{N+1} \\ (\psi_i)_{i\in[0,N+1]_{\mathbb{N}}} & \mapsto & \left(\nabla_h^D \psi\right)_{i+\frac{1}{2}} = \dfrac{1}{h_{i+1/2}}\left(\psi_{i+1} - \psi_i\right) \quad , \quad \forall i \in [0,N]_{\mathbb{N}} \,. \end{array} \tag{7}$$

Proposition 2.3 These operators are linked by the discrete Green formula:

$$\forall u = \left(u_{i+\frac{1}{2}}\right)_{i\in[0,N]_{\mathbb{N}}} \in \mathbb{R}^{N+1} \,, \forall \psi = (\psi_i)_{i\in[0,N+1]_{\mathbb{N}}} \in \mathbb{R}^{N+2},$$

$$\left(\nabla_h^P u, \psi\right)_P = -\left(u, \nabla_h^D \psi\right)_D + \left[u_{N+\frac{1}{2}}\psi_{N+1} - u_{\frac{1}{2}}\psi_0\right] \,, \tag{8}$$

where we defined the following primal and dual discrete scalar products

Definition 2.4

$$\begin{aligned} (\phi,\psi)_P &= \sum_{i=1}^{N} h_i \phi_i \psi_i \\ (u,v)_D &= \sum_{i=0}^{N} h_{i+1/2} u_{i+\frac{1}{2}} v_{i+\frac{1}{2}} \,. \end{aligned}$$

With these definitions, the FVM (5) reads: find $(\phi_i)_{i\in[0,N+1]_{\mathbb{N}}}$ such that:

$$\left\{ \begin{array}{rcl} -\left(\nabla_h^P \nabla_h^D \phi\right)_i &=& f_i \quad , \quad \forall i \in [1,N]_{\mathbb{N}} \,, \\ \phi_0 &=& \phi_{N+1} = 0 \,. \end{array} \right. \tag{9}$$

3 Continuous and discrete Green functions

The definition of continuous and discrete Green functions will be useful to obtain error estimates.

DEFINITION 3.1 Let K_i be a cell, with $i \in [1,N]_{\mathbb{N}}$. Let us define the Green function g^i associated with the point x_i:

$$\begin{cases} -\left(g^i\right)'' &= \delta_{x_i} \text{ in } \mathcal{D}'(\Omega)\,, \\ g^i(0) &= g^i(1) = 0\,. \end{cases}$$

PROPOSITION 3.2 It is straightforward to check that the expression of g^i is given by the following formula

$$g^i(x) = \begin{cases} (1-x_i)x & \text{if} \quad x \le x_i \\ x_i(1-x) & \text{if} \quad x \ge x_i \end{cases} . \tag{10}$$

Since $g^i \in H_0^1(\Omega)$, we have by Eq. (2),

$$(f, g^i)_\Omega = \left(\hat{\phi}', \left(g^i\right)'\right)_\Omega = (1-x_i)\int_0^{x_i} \hat{\phi}'(x)dx - x_i \int_{x_i}^1 \hat{\phi}'(x)dx = \hat{\phi}\left(x_i\right), \tag{11}$$

by direct integration and because $\hat{\phi}$ vanishes in $x=0$ and $x=1$.

DEFINITION 3.3 Let us also define the discrete Green function $\left(G_j^i\right)_{j\in[0,N+1]_{\mathbb{N}}}$ associated with K_i in the following way, where δ_j^i is the standard Kronecker symbol:

$$\begin{aligned} -\left(\nabla_h^P \nabla_h^D G^i\right)_j &= \frac{\delta_j^i}{h_i}\ , \quad \forall j \in [1,N]_{\mathbb{N}}\,, \\ G_0^i &= G_{N+1}^i = 0\,. \end{aligned} \tag{12}$$

PROPOSITION 3.4 Let us check that the expression of G^i is given by the following formula:

$$G_j^i = g^i(x_j)\,,\ \forall j \in [0, N+1]_{\mathbb{N}}\,, \tag{13}$$

where the function g^i is defined by (10).

Indeed, the application of (12) for j in $[1,i-1]_{\mathbb{N}}$ on the one hand, and for j in $[i+1,N]_{\mathbb{N}}$ on the other hand leads to

$$\left(\nabla_h^D G^i\right)_{j+\frac{1}{2}} = \left(\nabla_h^D G^i\right)_{j-\frac{1}{2}}\ ,\ \forall j \in [1,i-1]_{\mathbb{N}}$$

and

$$\left(\nabla_h^D G^i\right)_{j+\frac{1}{2}} = \left(\nabla_h^D G^i\right)_{j-\frac{1}{2}}\ ,\ \forall j \in [i+1,N]_{\mathbb{N}}\ .$$

Let us denote by a_L the common value of $\left(\nabla_h^D G^i\right)_{j+\frac{1}{2}}$ for all $j \in [0,i-1]_{\mathbb{N}}$, and by a_R the common value of $\left(\nabla_h^D G^i\right)_{j+\frac{1}{2}}$ for all $j \in [i,N]_{\mathbb{N}}$. Then, according to the definition of $\left(\nabla_h^D G^i\right)_{j+\frac{1}{2}}$ (see formula (7)) and since $h_{i+1/2} = x_{i+1} - x_i$ and $G_0^i = 0$, $x_0 = 0$, $G_{N+1}^i = 0$ and $x_{N+1} = 1$, there holds

$$G_j^i = a_L x_j \ , \ \forall j \in [0, i]_{\mathbb{N}} \ , \tag{14}$$

$$G_j^i = -a_R(1 - x_j) \ , \ \forall j \in [i, N+1]_{\mathbb{N}} \ . \tag{15}$$

Applying these two equalities for $j = i$, we get

$$a_L x_i = -a_R(1 - x_i) \ . \tag{16}$$

On the other hand, the application of (12) for $j = i$ leads to

$$-(a_R - a_L) = 1 \ . \tag{17}$$

Equations (16) and (17) lead to $a_L = 1 - x_i$ and $a_R = -x_i$, which, together with (14) and (15) and the definition of g^i by (10), leads to (13).

Proposition 3.5 The following exact expression of the error $\hat{\phi}(x_i) - \phi_i$ holds

$$\hat{\phi}\,(x_i) - \phi_i = \left(f, g^i - g_*^i\right)_\Omega \ , \tag{18}$$

where the piecewise constant function g_*^i is defined by

$$g_*^i(x) = G_j^i = g^i(x_j) \ \ , \forall x \in K_j \, , \forall j \in [1, N]_{\mathbb{N}} \, .$$

Since both G^i and ϕ vanish on the boundary, we have, by double application of the discrete Green formula (8):

$$\phi_i = \left(-\nabla_h^P \, \nabla_h^D G^i, \phi\right)_P = \left(\nabla_h^D G^i, \nabla_h^P \phi\right)_D = \left(G^i, -\nabla_h^P \, \nabla_h^D \phi\right)_P \ .$$

Next, by definition of the scheme, Eq. (5), and by definition of the mean values f_j, Eq. (6), there holds

$$\phi_i = \left(G^i, f\right)_P = \sum_{j=1}^{N} h_j f_j G_j^i = \left(g_*^i, f\right)_\Omega \ .$$

The result follows from the previous equality and from (11).

4 Second order convergence

Proposition 3.5 enables us to study the errors at the points x_i and also the error in the discrete L^2 norm defined by

$$e_i := \left|\hat{\phi}(x_i) - \phi_i\right| \ \ , \ \ e = \left(\sum_{i=1}^{N} h_i \left|\hat{\phi}\,(x_i) - \phi_i\right|^2\right)^{\frac{1}{2}} \ .$$

We can state the following results:

Proposition 4.1 Let x_j^* be the midpoint of K_j. If $f \in \mathrm{H}^1(\Omega)$ and if there exists a constant C, independent of the grid, such that $\forall j \in [1,N]_{\mathbb{N}}$, $\left|x_j - x_j^*\right| \leq Ch^2$, then there is a constant K, depending only on f, such that

$$\begin{aligned} \forall i \in [1,N]_{\mathbb{N}},\, e_i &\leq Kh^2\,, && (19)\\ e &\leq Kh^2\,. && (20) \end{aligned}$$

Indeed, equation (18) may also be written

$$\hat{\phi}\left(x_i\right) - \phi_i = \left(f, g^i - g_*^i\right)_\Omega = \left(f - \Pi f, g^i - g_*^i\right)_\Omega + \left(\Pi f, g^i - g_*^i\right)_\Omega\,, \tag{21}$$

where Πf is the L^2 projection of f on the grid:

$$(\Pi f)(x) = f_j\,,\ \forall x \in K_j\,,\ \forall j \in [1,N]_{\mathbb{N}}\,.$$

Since $f \in \mathrm{H}^1(\Omega)$, a standard result (see, e.g. [27]) states that there exists a constant K, which does not depend on the grid such that

$$\|f - \Pi f\|_{0,K_j} \leq K\, h_j\, |f|_{1,K_j} \leq K h^{\frac{1}{2}} h_j^{\frac{1}{2}}\, |f|_{1,K_j} \quad,\quad \forall j \in [1,N]_{\mathbb{N}}\,. \tag{22}$$

In addition, for all $x \in K_j$,

$$\left|\left(g^i - g_*^i\right)(x)\right| \leq |x - x_j| \sup(x_i, 1 - x_i) \leq |x - x_j| \leq h\,. \tag{23}$$

This implies of course that

$$\left\|g^i - g_*^i\right\|_{0,K_j} \leq h^{\frac{3}{2}}\,. \tag{24}$$

By the Cauchy-Schwarz formula there holds

$$\left|\left(f - \Pi f, g^i - g_*^i\right)_\Omega\right| \leq \sum_{j=1}^{N} \|f - \Pi f\|_{0,K_j} \left\|g^i - g_*^i\right\|_{0,K_j} \leq Kh^2 \sum_{j=1}^{N} h_j^{\frac{1}{2}}\, |f|_{1,K_j} \tag{25}$$

thanks to Eq. (22) and (24). Finally, since $\sum_{j=1}^{N} h_j = |\Omega| = 1$, the discrete Cauchy-Schwarz formula yields

$$\left|\left(f - \Pi f, g^i - g_*^i\right)_\Omega\right| \leq Kh^2\, |f|_{1,\Omega}\,. \tag{26}$$

In addition, the function $g^i - g_*^i$ being linear on each K_j (for $j \neq i$), its integral can be evaluated by the midpoint rule, and we have

$$\begin{aligned} \left|\left(\Pi f, g^i - g_*^i\right)_\Omega\right| \;\leq\;& \left|\sum_{j<i} f_j h_j (1 - x_i)\left(x_j^* - x_j\right) + \sum_{j>i} f_j h_j x_i \left(x_j - x_j^*\right)\right| \\ +\;& \left|\frac{1}{2} f_i \left[\left(x_i - x_{i-\frac{1}{2}}\right)^2 (1 - x_i) + \left(x_{i+\frac{1}{2}} - x_i\right)^2 x_i\right]\right|\,. \end{aligned} \tag{27}$$

Since x_i and $(1-x_i)$ are both bounded by 1, and since

$$|f_j|\, h_j = \left| \int_{K_j} f(x)dx \right| \le h_j^{\frac{1}{2}} \, \|f\|_{0,K_j} \ ,$$

the assumption $\left| x_j - x_j^* \right| \le Ch^2$ leads to

$$\left| \sum_{j<i} f_j h_j (1-x_i) \left(x_j^* - x_j\right) + \sum_{j>i} f_j h_j x_i \left(x_j - x_j^*\right) \right| \le Ch^2 \sum_{j=1}^{N} h_j^{\frac{1}{2}} \, \|f\|_{0,K_j} .$$

By the discrete Cauchy-Schwarz inequality, this implies that the first line in (27) is bounded by $Ch^2 \|f\|_{0,\Omega}$. As far as the second line in (27) is concerned, it is bounded by $Ch^2 |f_i|$. But since f is in H^1 (and since we are in dimension 1), this function is bounded on Ω by $C \|f\|_{1,\Omega}$, where the constant C depends only on Ω. Finally, we can thus write

$$\left| \left(\Pi f, g^i - g_*^i\right)_\Omega \right| \le Ch^2 \, \|f\|_{1,\Omega} \ . \tag{28}$$

The result (19) follows from (21), (26) and (28), and the result (20) immediately from (19).

Proposition 4.2 If $f \in \mathrm{H}^1(\Omega)$ and if there exists a constant C, independent of the grid, such that $\forall j \in [1, N-1]_{\mathbb{N}}$, $\left| x_{j+\frac{1}{2}} - \frac{x_j + x_{j+1}}{2} \right| \le Ch^2$, then there is a constant K, depending only on f, such that

$$\forall i \in [1,N]_{\mathbb{N}},\ e_i \ \le \ Kh^2 \, , \tag{29}$$

$$e \ \le \ Kh^2 \, . \tag{30}$$

The preceding calculations still hold if we consider the mean-values of f on the dual cells $K_{j+\frac{1}{2}}$ instead of its mean-values on the primal cells K_j. We define

$$(\mathrm{P} f)(x) = f_{j+\frac{1}{2}} := \frac{1}{h_{j+1/2}} \int_{K_{j+\frac{1}{2}}} f(y)\, dy \quad \forall x \in K_{j+\frac{1}{2}} \ , \forall j \in [0,N]_{\mathbb{N}} \, .$$

Then, we have

$$\hat{\phi}\,(x_i) - \phi_i = \left(f, g^i - g_*^i\right)_\Omega = \left(f - \mathrm{P} f, g^i - g_*^i\right)_\Omega + \left(\mathrm{P} f, g^i - g_*^i\right)_\Omega \ . \tag{31}$$

For $f \in \mathrm{H}^1(\Omega)$, there exists a constant K, which does not depend on the grid such that

$$\|f - \mathrm{P} f\|_{0,K_{j+\frac{1}{2}}} \le K \, diam(K_{j+\frac{1}{2}}) \, |f|_{1,K_{j+\frac{1}{2}}} \le 2Kh \, |f|_{1,K_{j+\frac{1}{2}}} \quad , \quad \forall j \in [0,N]_{\mathbb{N}} \, ,$$

thanks to (4). In addition, for all $x \in K_{j+\frac{1}{2}}$,

$$\left| \left(g^i - g_*^i\right)(x) \right| \le \sup\left(|x - x_j| \, , |x - x_{j+1}|\right) \sup(x_i, 1 - x_i) \le 2h \, .$$

Thus, by the Cauchy-Schwarz inequality and then the discrete Cauchy-Schwarz inequality, there holds

$$\left|\left(f-\mathrm{P}f, g^i-g^i_*\right)_\Omega\right| \leq Kh^2 \left|f\right|_{1,\Omega} . \tag{32}$$

In addition,

$$\begin{aligned}\left(\mathrm{P}f, g^i-g^i_*\right)_\Omega &= -f_{\frac{1}{2}} h_{1/2}(1-x_i)\frac{x_1}{2} - f_{N+\frac{1}{2}} h_{N+1/2} x_i \frac{(1-x_N)}{2} \\ &+ \sum_{j=1}^{N-1} f_{j+\frac{1}{2}} \int_{K_{j+\frac{1}{2}}} \left(g^i-g^i_*\right)(x)\, dx .\end{aligned} \tag{33}$$

For $1 \leq j \leq i-1$, we have

$$\begin{aligned}\int_{K_{j+\frac{1}{2}}} \left(g^i-g^i_*\right)(x)\, dx &= (1-x_i)\Bigg[(x_{j+1}-x_j)\,\frac{x_{j+1}+x_j}{2} \\ &\qquad -x_j\left(x_{j+\frac{1}{2}}-x_j\right) - x_{j+1}\left(x_{j+1}-x_{j+\frac{1}{2}}\right)\Bigg] \\ &= (1-x_i)\left(x_{j+\frac{1}{2}} - \frac{x_{j+1}+x_j}{2}\right) h_{j+1/2} .\end{aligned}$$

In the same way, for $i \leq j \leq N-1$, we have

$$\int_{K_{j+\frac{1}{2}}} \left(g^i-g^i_*\right)(x)\, dx = -x_i\left(x_{j+\frac{1}{2}} - \frac{x_{j+1}+x_j}{2}\right) h_{j+1/2} .$$

Using the Cauchy-Schwarz inequality, and the assumption on $\left|x_{j+\frac{1}{2}} - \frac{x_{j+1}+x_j}{2}\right|$, these equalities enable us to bound the second line of (33) by $Kh^2 \left\|f\right\|_{0,\Omega}$. As for the first line of this expression, it can be bounded by taking into account the following inequalities

$$|x_1| \leq h \;\;,\;\; |1-x_N| \leq h \;\;,\;\; \left|K_{\frac{1}{2}}\right| \leq h \;\;,\;\; \left|K_{N+\frac{1}{2}}\right| \leq h \;\;,$$

$$\left|f_{\frac{1}{2}}\right| \leq C\left\|f\right\|_{1,\Omega} \;\;,\;\; \left|f_{N+\frac{1}{2}}\right| \leq C\left\|f\right\|_{1,\Omega} \;\;,$$

where the constant C depends only on Ω for reasons evoked in the proof of proposition 4.1. Finally, we have

$$\left|\left(\mathrm{P}f, g^i-g^i_*\right)_\Omega\right| \leq Kh^2 \left\|f\right\|_{1,\Omega} . \tag{34}$$

The result (29) follows from (31), (32) and (34), and the result (30) immediately from (29).

5 Counterexamples in less favorable cases

We will now check that for less regular functions f, the error can be of an order lower than 2, even if the grid verifies the assumptions of proposition 4.1. Since in that case $g^i - g_*^i$ changes sign at the midpoint of K_i, the idea is to choose a function f that will systematically be "much" greater on the first half of K_j than on its second half, so that Eq. (18) will imply an accumulation of errors.

PROPOSITION 5.1 Let the grid be made up of N identical segments of length $h = \frac{1}{N}$ and let the points x_j be chosen as the midpoints of the segments

$$x_j = (j - 1/2)\, h \ , \ 1 \leq j \leq N \, .$$

Let in addition $\alpha \in]0;1/2[$ and let us choose $f(x) = x^{-\alpha}$. (Of course, $f \in \mathrm{L}^2(\Omega)$ but $f \notin \mathrm{H}^1(\Omega)$). Let in addition $x_* \in]0;1[$, be fixed independently of the grid and let us denote by N_* the integer such that $N_* h \leq x_* < (N_* + 1)h$. We will suppose h sufficiently small so that $N_* \geq 2$. Then, there exists $K > 0$ depending only on α and x_*, such that for h sufficiently small

$$e_i \geq K h^{2-\alpha} \ , \ \forall i \in [2, N_*]_{\mathbb{N}} \ , \tag{35}$$
$$e \geq K h^{2-\alpha}. \tag{36}$$

Let indeed i be fixed in $[2, N_*]_{\mathbb{N}}$. Formula (18) enables us to write

$$\hat{\phi}\,(x_i) - \phi_i \leq \int_0^{(i-1)h} \left(g^i(x) - g_*^i(x)\right) x^{-\alpha}\, dx + \int_{ih}^1 \left(g^i(x) - g_*^i(x)\right) x^{-\alpha}\, dx \tag{37}$$

because the function $g^i - g_*^i$ is negative over $[(i-1)h; ih]$ since on this interval $g_*^i(x) = g^i(x_i) \geq g^i(x)$. Then, let us set, for $1 \leq j \leq i - 1$,

$$A_j = \int_{(j-1)h}^{jh} \left(g^i(x) - g_*^i(x)\right) x^{-\alpha}\, dx = (1 - x_i) \int_{(j-1)h}^{jh} (x - x_j) x^{-\alpha}\, dx \, .$$

By carrying out the change of variable $x = x_j - s$ on $[(j-1)h; x_j]$ and $x = x_j + s$ on $[x_j; jh]$, we obtain the following expression, since $x_j = (j - 1/2)h$

$$A_j = (1 - x_i) \int_0^{h/2} \left[(x_j + s)^{-\alpha} - (x_j - s)^{-\alpha}\right] s\, ds \, .$$

We can write

$$(x_j - s)^{-\alpha} = x_j^{-\alpha} \left(1 - \frac{s}{x_j}\right)^{-\alpha} \geq x_j^{-\alpha} \left(1 + \alpha \frac{s}{x_j}\right) \ , \forall s \in [0; h/2] \, .$$

In addition,

$$(x_j + s)^{-\alpha} \leq x_j^{-\alpha} \ , \ \ \forall s \in [0; h/2] \, .$$

Thus,

$$A_j \leq -(1 - x_i) \alpha x_j^{-1-\alpha} \int_0^{h/2} s^2 ds \, .$$

By taking into account the equality $x_j = (j-1/2)\,h$, we finally obtain

$$A_j \leq -(1-x_i)\frac{\alpha}{24}\frac{h^{2-\alpha}}{(j-1/2)^{1+\alpha}}\,.$$

Then, we can bound the first term of (37):

$$\int_0^{(i-1)h} \left(g^i(x)-g^i_*(x)\right) x^{-\alpha} \leq -(1-x_i)\frac{\alpha}{24}h^{2-\alpha}\sum_{j=1}^{i-1}\frac{1}{(j-1/2)^{1+\alpha}}\,. \tag{38}$$

The sum in (38) comprises at least a term, since we consider $i \geq 2$. Thus, we have

$$\int_0^{(i-1)h} \left(g^i(x)-g^i_*(x)\right) x^{-\alpha} \leq -(1-x_i)\frac{2^{1+\alpha}\alpha}{24}h^{2-\alpha}\,. \tag{39}$$

Let us set in addition, for $i+1 \leq j \leq N$,

$$B_j = \int_{(j-1)h}^{jh} \left(g^i(x)-g^i_*(x)\right) x^{-\alpha}\,dx = x_i \int_{(j-1)h}^{jh} (x_j - x)x^{-\alpha}\,dx\,.$$

By carrying out the same changes of variable as previously, we obtain

$$B_j = x_i \int_0^{h/2} \left[(x_j-s)^{-\alpha}-(x_j+s)^{-\alpha}\right] s\,ds\,.$$

We can write

$$(x_j-s)^{-\alpha}-(x_j+s)^{-\alpha} \leq (x_j-h/2)^{-\alpha}-(x_j+h/2)^{-\alpha} \quad , \quad \forall s\in[0,h/2]\,,$$

and thus

$$B_j \leq x_i\frac{h^2}{8}\left\{[(j-1)h]^{-\alpha}-(jh)^{-\alpha}\right\}\,.$$

Then, we can bound the second term of (37):

$$\begin{aligned}\int_{ih}^1 \left(g^i(x)-g^i_*(x)\right) x^{-\alpha}\,dx &\leq x_i\frac{h^2}{8}\left[(ih)^{-\alpha}-(Nh)^{-\alpha}\right] \\ &\leq x_i\left(x_i^{-\alpha}-1\right)\frac{h^2}{8}\end{aligned}$$

because $Nh = 1$ and $x_i \leq ih$. Since $1 > \alpha > 0$ and $0 \leq x_i \leq 1$, we have $x_i\left(x_i^{-\alpha}-1\right) = x_i^{1-\alpha}\left(1-x_i^{\alpha}\right) \leq 1$, which implies

$$\int_{ih}^1 \left(g^i(x)-g^i_*(x)\right) x^{-\alpha}\,dx \leq \frac{h^2}{8}\,. \tag{40}$$

By gathering (37), (39) and (40), we obtain

$$\hat{\phi}\,(x_i)-\phi_i \leq -(1-x_i)\frac{2^{1+\alpha}\alpha}{24}h^{2-\alpha}+\frac{h^2}{8}\,.$$

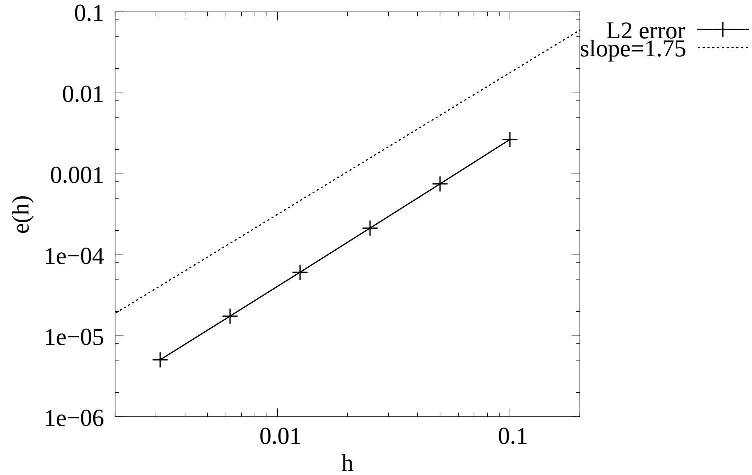

Figure 1: Discrete L^2 error for the counterexample $f(x) = x^{-\alpha}$ with $\alpha = 1/4$.

Now, if we want the result not to depend on x_i, we note that $(1 - x_i) \geq (1 - x_*)$ since $i \leq N_*$, so that

$$\hat{\phi}(x_i) - \phi_i \leq -(1 - x_*)\frac{2^{1+\alpha}\alpha}{24}h^{2-\alpha} + \frac{h^2}{8}\,. \tag{41}$$

For h sufficiently small, (for example such that $\frac{h^2}{8} \leq (1 - x_*)\frac{2^{1+\alpha}\alpha}{48}h^{2-\alpha}$), the result (35) is obtained by taking the absolute value of the two sides of this inequality. The result (36) is obtained, starting from (35), by writing

$$\begin{aligned} e^2 \geq \sum_{i=2}^{N_*} h \left|\hat{\phi}(x_i) - \phi_i\right|^2 &\geq (N_* - 1)h\, K\,(1 - x_*)^2 h^{4-2\alpha} \\ &\geq (x_* - 2h)\, K\,(1 - x_*)^2 h^{4-2\alpha} \\ &\geq K\, h^{4-2\alpha}\,, \end{aligned}$$

where the constant K, for h sufficiently small, does not depend on the grid. Figure 1 displays the observed discrete L^2 norm of the error, $e(h)$, as a function of the meshstep size h for $\alpha = 1/4$, as well as a reference curve with slope $2 - \alpha = 7/4$, which confirms that the order of convergence in the L^2 norm behaves like $2 - \alpha$.

We shall now take into consideration a case where the assumptions of propositions 4.1 and 4.2 concerning the grid are not verified. Keeping in mind formula (18), the idea is to choose f as a constant function, and points (x_i) so that $f(g^i - g^i_*)$ will be positive on a much greater part of Ω than the part of Ω on which it will be negative. We can state the following result

Proposition 5.2 On general grids, and even if f is regular, we may not get a better estimate than

$$F(x_i)h \leq e_i \leq Kh\,, \tag{42}$$

$$C_1 h \leq e \leq C_2 h\,, \tag{43}$$

where the constants K, C_1 and C_2 and the function F are independent of the grid.

First of all, formula (18) together with estimate (23), coupled with the Cauchy-Schwarz inequality, allow us to write that for all $i \in [1, N]_{\mathbb{N}}$,

$$\left|\hat{\phi}\left(x_i\right) - \phi_i\right| \leq \|f\|_{0,\Omega}\, h\,,$$

which immediately implies

$$e \leq \|f\|_{0,\Omega}\, h\,.$$

These two inequalities lead to the inequalities in the right-hand sides of (42) and (43). To show the left-hand side inequalities, let us consider now the following example: the grid consists of $N = 2P$ identical segments of length $h = \frac{1}{2P}$ and we choose the points x_j defined by

$$x_j = \begin{cases} \left(j - \frac{3}{4}\right) h & \text{if} \quad j \leq P \\ \left(j - \frac{1}{4}\right) h & \text{if} \quad j > P \end{cases}.$$

In addition, the function f is chosen so that

$$f(x) = 1 \quad \forall x \in \Omega\,,$$

and we thus have $\hat{\phi}(x) = \frac{1}{2}(1-x)x$. Setting $v_j = \phi_{j+1} - \phi_j$ and taking into account (5) and the size of the various dual cells, we obtain the following system

$$\begin{cases} v_1 - 4v_0 &= -h^2 & \\ v_j - v_{j-1} &= -h^2 & \forall j \text{ s.t. } 2 \leq j \leq P-1 \\ \frac{2}{3}v_P - v_{P-1} &= -h^2 & \\ v_{P+1} - \frac{2}{3}v_P &= -h^2 & \\ v_j - v_{j-1} &= -h^2 & \forall j \text{ s.t. } P+2 \leq j \leq 2P-1 \\ 4v_{2P} - v_{2P-1} &= -h^2 & \end{cases}.$$

This allows us to get the following expressions

$$\begin{cases} v_j &= 4v_0 - jh^2 & \forall j \text{ s.t. } 1 \leq j \leq P-1 \\ v_P &= 6v_0 - \frac{3P}{2}h^2 & \\ v_j &= 4v_0 - jh^2 & \forall j \text{ s.t. } P+1 \leq j \leq 2P-1 \\ v_{2P} &= v_0 - \frac{P}{2}h^2 & \end{cases}.$$

In addition, $\sum_{j=0}^{2P} v_j = \phi_{2P+1} - \phi_0 = 0$, which implies $v_0 = \frac{P}{4}h^2$. We thus have

$$\begin{cases} v_0 &= \frac{P}{4}h^2 & \\ v_j &= (P-j)h^2 & \forall j \text{ s.t. } 1 \leq j \leq 2P-1 \\ v_{2P} &= -\frac{P}{4}h^2 & \end{cases}.$$

This allows us to write

$$\begin{cases} \phi_0 &= 0 \\ \phi_i &= \frac{P}{4}h^2 + (i-1)\left(P - \frac{i}{2}\right) h^2 \;, \quad \forall i \in [1, 2P]_{\mathbb{N}} \\ \phi_{2P+1} &= 0 \end{cases}.$$

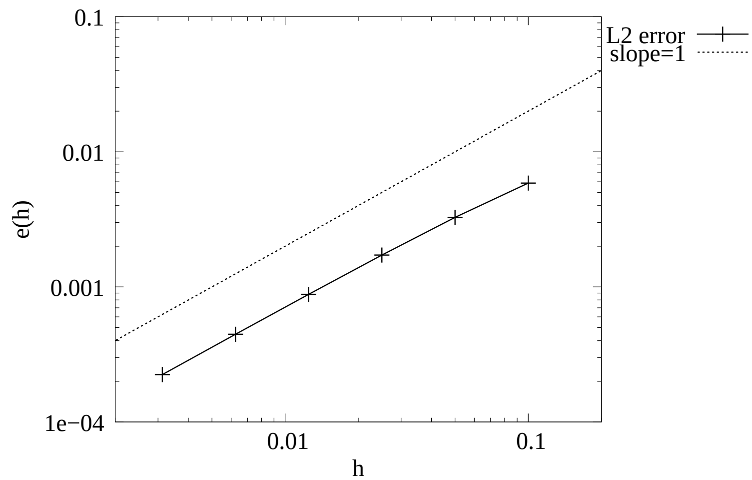

Figure 2: Discrete L^2 error for the counterexample in which the control points associated to the control volumes are badly chosen.

In addition, since $2Ph = 1$, we can write

$$\phi_i = \frac{h}{8} + \frac{1}{2}\left(1 - \frac{i}{2P}\right) \times \frac{(i-1)}{2P} \ , \ \forall i \in [1, 2P]_{\mathbb{N}} \ .$$

But for $1 \leq i \leq P$ we have by definition

$$x_i = \frac{(i - \frac{3}{4})}{2P} = \frac{i}{2P} - \frac{3}{4}h = \frac{(i-1)}{2P} + \frac{1}{4}h \, .$$

Thus, we find the expression of ϕ_i to be

$$\begin{aligned} \phi_i &= \frac{1}{2}(1 - x_i)x_i - \frac{h}{4}x_i + \frac{3}{32}h^2 \\ &= \hat{\phi}(x_i) - \frac{h}{4}x_i + \frac{3}{32}h^2 \ , \ \forall i \in [1, P]_{\mathbb{N}}. \end{aligned}$$

That is to say

$$e_i = x_i h \left| \frac{1}{4} - \frac{3}{8(4i-3)} \right| \geq x_i \frac{h}{8} \ , \ \forall i \in [1, P]_{\mathbb{N}}.$$

In the same way,

$$\begin{aligned} \phi_i &= \frac{1}{2}(1 - x_i)x_i - \frac{h}{4}(1 - x_i) + \frac{3}{32}h^2 \\ &= \hat{\phi}(x_i) - \frac{h}{4}(1 - x_i) + \frac{3}{32}h^2 \ , \ \forall i \in [P+1, 2P]_{\mathbb{N}} \ , \end{aligned}$$

which implies

$$\left| \hat{\phi}(x_i) - \phi_i \right| \geq (1 - x_i)\frac{h}{8} \ , \ \forall i \in [P+1, 2P]_{\mathbb{N}}.$$

This allows to obtain the left inequality in (42), and then easily the left inequality in (43). Figure 2 displays the observed discrete L^2 norm of the error, $e(h)$, as a function of the meshstep size h, as well as a reference curve with slope 1, which confirms that the order of convergence in the L^2 norm is exactly one in this case.

6 Conclusions

Through the introduction of discrete Green functions, we have computed the exact error between the numerical solution of the one dimensional Laplace equation discretized by cell-centered finite volumes and its exact solution evaluated at the points associated to the control volumes. We have recovered the well-known second-order accuracy result under the conditions that these points are chosen as (or $O(h^2)$ close to) the centers of the control volumes, and that the right-hand side of the Laplace equation belongs to $H^1(\Omega)$. This result also holds if the endpoints of the control volumes are ($O(h^2)$ close to) the centers of the dual cells. However, if either the points associated to the control volumes are not properly chosen, or if the right-hand side of the Laplace equation does not belong to $H^1(\Omega)$, second-order accuracy may be lost. This gives us indications on what to expect in higher dimensions: not properly chosen families of admissible meshes may display a reduced order of convergence, and, like in the FVEM, H^1 regularity of the right-hand side seems to be necessary to get second-order convergence.

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