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A POSTERIORI ERROR ESTIMATION FOR THE DISCRETE DUALITY FINITE VOLUME DISCRETIZATION OF THE LAPLACE EQUATION

PASCAL OMNES[†‡], YOHAN PENEL[†‡], AND YANN ROSENBAUM[†‡]

Abstract. An efficient and fully computable *a posteriori* error bound is derived for the discrete duality finite volume discretization of the Laplace equation on very general twodimensional meshes. The main ingredients are the equivalence of this method with a finite element like scheme and tools from the finite element framework. Numerical tests are performed with a stiff solution on highly nonconforming locally refined meshes and with a singular solution on triangular meshes.

Key words. *a posteriori* error estimation, finite volume, discrete duality, nonconforming meshes

AMS subject classifications. 65N15, 65N30

1. Introduction. Let Ω be a twodimensional polygonal domain with boundary Γ such that $\Gamma = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. We are interested in the *a posteriori* error estimation between the exact solution $\hat{\phi} \in H^1(\Omega)$ of the following problem

$$-\Delta\hat{\phi} = f \text{ in } \Omega \tag{1.1}$$

$$\hat{\phi} = \phi_d \text{ on } \Gamma_D \tag{1.2}$$

$$\nabla\hat{\phi} \cdot \mathbf{n} = g \text{ on } \Gamma_N \tag{1.3}$$

and its numerical approximation by the finite volume method (FVM) described in [13] and recalled in section 3. In this introduction, let us only mention that the unknowns of this scheme are located both at the centers and at the vertices of the mesh. Equation (1.1) is then integrated both on the primal mesh, and on a dual mesh, whose cells are centered on the vertices of the primal mesh. Finally, fluxes are computed through the reconstruction of gradients on the so-called “diamond-cells”, which are quadrilateral cells centered on the edges of the mesh. It has been shown that this finite volume method may be written under an equivalent discrete symmetric positive definite variational formulation, and has been named “discrete duality finite volume” (DDFV) method since it can be interpreted in terms of discrete differential gradient and divergence operators which are linked by a discrete Green formula. The main advantage of this scheme is that it may be used on fairly arbitrary meshes with possibly distorted [17, 18] or highly nonconforming primal cells [13]. Another useful feature of this scheme is the reconstruction of both components of the gradients (and not only of its normal component with respect to the cell edges), which makes it easy to use for anisotropic or non-linear (p-Laplacian type) diffusion problems (see, for example, [4, 17, 18]). An extension of this scheme to div-curl problems as well as further definitions and properties of discrete differential operators have been presented in [12]. A priori analysis have been given in [4, 13]. In the linear case, when the solution of (1.1) to (1.3) belongs to $H^2(\Omega)$, it has been proved in [13] that the numerical approximation obtained by the DDFV method tends to the exact solution with the optimal order h in the energy norm. For less regular solutions in $H^{1+s}(\Omega)$, with $s < 1$, a convergence

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with order h^s has been observed in [12]; this motivates the study of *a posteriori* error estimators that could efficiently drive an adaptive refinement strategy.

For the system (1.1)–(1.3), *a posteriori* error estimations for conforming Lagrange finite element methods (FEM) are now very common. The reader is referred to, e.g., [2, 5, 27] in which several types of estimators are detailed. In the residual based estimators, the main terms are inter-element jumps of the normal components of the gradients of the computed solution, weighted by constants whose explicit computation was performed in [7] and [28]. Efficiencies of the estimators obtained in [7] vary, according to the problems, between 30 and 70, and between 1.5 and 3.5 if one numerically evaluates eigenvalues of some vertex centered local problems, as reported in [8]. References for non-conforming FEM may be found in [3] and for mixed FEM in [29].

The case of cell-centered FVM has been less studied, on the one hand because of their more recent use for elliptic problems, and, on the other hand, because they generally lack a discrete variational formulation. For the basic "four point" scheme on so-called "admissible" triangular meshes (see [14, 16]), Agouzal and Oudin [1] have used the connection of this scheme with mixed finite elements to derive an *a posteriori* estimator for the L^2 norm of the error; this estimator is not an upper bound for the error, but is asymptotically exact under mild hypothesis. A second estimator for this scheme has been given by Nicaise in [20]. This estimator is shown to be equivalent to the (broken) energy norm of the difference between the exact solution and an elementwise second order polynomial (globally discontinuous) reconstructed numerical solution. Then, in [21], Nicaise extends his ideas to the so-called "diamond-cell" FVM (as described in [10]) and proposes an *a posteriori* error estimator which may be used if the cells of the mesh are triangles or rectangles (or tetrahedrons in dimension three). This estimator is completely computable (no unknown constant) and its efficiency is around 7 for the tests performed in [21]. Finally, Nicaise has extended his work to diffusion-convection-reaction equations in [22]. More recently, Vohralík [31] has also proposed a fully computable *a posteriori* error estimator for numerical approximations of diffusion-convection-reaction equations by cell-centered FVM on general meshes. The main improvement over [21, 22] is the asymptotic exactness of the error bound which, like in [21], measures the energy norm of the difference between the exact solution and a reconstructed, globally discontinuous, elementwise second order polynomial numerical solution. Note that in [22] the reconstructed numerical solution is globally continuous and may involve higher order polynomials on each element.

Since the computations of the estimators in [21, 31] only require fluxes on the edges and values of the unknowns at the centers of the primal cells (quantities which are usually the output of FVM), we may apply them to the DDFV method. However, the treatment of meshes as general as those we employ with the DDFV scheme, like in particular the non-conforming meshes of section 6, is impossible with the techniques of [21] and require extra computational work with those of [31].

Let us finally mention some related results in the context of vertex-centered finite volume (element) methods [6, 9, 19, 23, 24, 25].

In the present work, we use the equivalent discrete variational formulation of the DDFV method and tools developed in the FEM framework to obtain a fully computable *a posteriori* bound for the L^2 norm of the error in the computed gradient: hence, no kind of postprocessing or solution reconstruction like in [21, 31] is needed in our approach. This error estimator is efficient under classical geometrical constraints on a subtriangulation of the primal mesh. The main two difficulties encountered are, on the one hand, that the basis functions on which the discrete variational formulation rely are non-conforming, and, on the other hand, that the DDFV scheme uses two dual meshes. The first difficulty is dealt with through a Helmholtz-Hodge decomposition of the error, an argument which is classical when the discrete solution does not belong to $H^1(\Omega)$ (see [3, 11, 21]). The conforming part of this decomposition is treated rather classically and involves the normal jumps of the gradients through neighboring diamond-cells. The nonconforming part of the error is treated thanks to the orthogonality property which links the discrete gradients and curls, as shown in [12], and involves the tangential jumps of the gradients. The second difficulty results in the total estimator being a sum of local estimators on both the primal and dual cells, before we distribute each dual estimator on the primal cells which intersect the considered dual cell. Throughout all the calculations, we tried to obtain the best possible bounds, with the objective that the resulting estimator be fully computable, and that the efficiency be as small as possible. The constants which are involved in the computations are explicitly evaluated thanks to the expressions found in [7, 21, 28], and there is a free parameter in the bounds, with respect to which the estimators are numerically minimized. The resulting tests show that the efficiency of the proposed estimator varies most of the time between 5 and 10.

The remainder of this article is organized as follows. Section 2 sets some notations and definitions related to the meshes, to discrete differential operators and to discrete functions. In section 3, a slightly modified version of the DDFV scheme is presented and its equivalent discrete variational formulation is recalled. In section 4, a representation of the error is elaborated. This is used in section 5 to find a computable upper bound of this error. We also verify the local efficiency of the error estimators. Section 6 is devoted to numerical tests with a regular but stiff solution and with a singular solution. Conclusions are drawn in section 7.

2. Notations and definitions. The following notations are summarized in Fig. 2.1 and 2.2. Let the domain Ω be covered by a primal mesh with polygonal cells denoted by T_i, with $i \in [1, I]$. With any T_i, we associate a point G_i located in the interior of T_i. This point is not necessarily the centroid of T_i. With any vertex S_k, with $k \in [1, K]$, we associate a dual cell P_k by joining points G_i associated with the primal cells surrounding S_k to the midpoints of the edges of which S_k is a node.

Remark 2.1. *The present construction of the dual cells slightly differs from that given in [12, 13]. It ensures that a dual cell P_k is star-shaped with respect to the associated node S_k, and also that when $T_i \cap P_k \neq \emptyset$, the segment $[G_i S_k]$ belongs to $T_i \cap P_k$. It also ensures that the dual cells form a partition of Ω. These facts are crucial in the application of the Poincaré type and trace inequalities in section 5 and when summing the contributions of all dual cells into the global a posteriori bound.*

With any primal edge A_j with $j \in [1, J]$, we associate a diamond-cell D_j obtained by joining the vertices $S_{k_1(j)}$ and $S_{k_2(j)}$ of A_j to the points $G_{i_1(j)}$ and $G_{i_2(j)}$ associated with the primal cells that share A_j as a part of their boundaries. When A_j is a boundary edge (there are J^Γ such edges), the associated diamond-cell is a flat quadrilateral (i.e. a triangle) and we denote by $G_{i_2(j)}$ the midpoint of A_j (thus, there are J^Γ such additional points G_i). The unit normal vector to A_j is $\mathbf{n}_j$ and points from $G_{i_1(j)}$ to $G_{i_2(j)}$. We denote by A'_{j1} (resp. A'_{j2}) the segment joining $G_{i_1(j)}$ (resp. $G_{i_2(j)}$) and the midpoint of A_j. Its associated unit normal vector, pointing from $S_{k_1(j)}$ to $S_{k_2(j)}$, is denoted by $\mathbf{n}'_{j1}$ (resp. $\mathbf{n}'_{j2}$). In the case of a boundary diamond-cell, A'_{j2} reduces to $\{G_{i_2(j)}\}$ and does not play any role. Finally, for any diamond-cell D_j, we shall denote by $M_{i_\alpha k_\beta}$ the midpoint of $[G_{i_\alpha(j)} S_{k_\beta(j)}]$, with $(\alpha, \beta) \in \{1; 2\}^2$. With $\mathbf{n}_j$, $\mathbf{n}'_{j1}$ and $\mathbf{n}'_{j2}$, we associate orthogonal unit vectors $\boldsymbol{\tau}_j$, $\boldsymbol{\tau}'_{j1}$ and $\boldsymbol{\tau}'_{j2}$, such that the corresponding

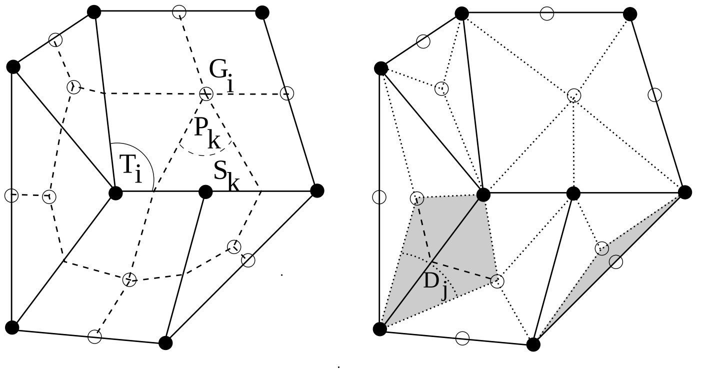

FIG. 2.1. *A primal mesh and its associated dual mesh (left) and diamond-mesh (right).*

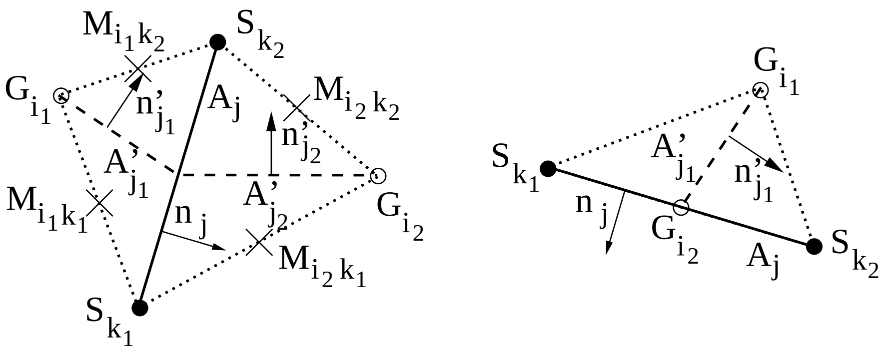

FIG. 2.2. *Notations for an inner diamond-cell (left) and a boundary diamond-cell (right).*

orthonormal bases are positively oriented. For any primal T_i such that $A_j \subset \partial T_i$, we shall define $\mathbf{n}_{ji} := \mathbf{n}_j$ if $i = i_1(j)$ and $\mathbf{n}_{ji} := -\mathbf{n}_j$ if $i = i_2(j)$, so that $\mathbf{n}_{ji}$ is always exterior to T_i. With $\mathbf{n}_{ji}$, we associate $\boldsymbol{\tau}_{ji}$ such that $(\mathbf{n}_{ji}, \boldsymbol{\tau}_{ji})$ is positively oriented. Similarly, when A'_{j1} and A'_{j2} belong to ∂P_k, we define $(\mathbf{n}'_{jk1}, \boldsymbol{\tau}'_{jk1})$ and $(\mathbf{n}'_{jk2}, \boldsymbol{\tau}'_{jk2})$ so that $\mathbf{n}'_{jk1}$ and $\mathbf{n}'_{jk2}$ are orthogonal to A'_{j1} and A'_{j2} and exterior to P_k.

For the sake of simplicity, the boundary Γ_N is supposed to be simply connected, but this hypothesis is in no matter restrictive. By a slight abuse of notations, we shall write $k \in \overset{\circ}{\Gamma}_D$ (resp. $\bar{\Gamma}_D$, $\overset{\circ}{\Gamma}_N$ and $\bar{\Gamma}_N$) if the vertex S_k belongs to the interior of Γ_D, relatively to Γ (resp. to the closure of Γ_D, to the interior of Γ_N and to the closure of Γ_N). Identically, we shall write $i \in \Gamma_D$ (resp. $i \in \Gamma_N$, $j \in \Gamma_D$ and $j \in \Gamma_N$) if $G_i \in \Gamma_D$ (resp. $G_i \in \Gamma_N$, $A_j \subset \Gamma_D$ and $A_j \subset \Gamma_N$).

In the DDFV scheme, we associate scalar unknowns to the points G_i and S_k and twodimensional vector fields to the diamond-cells. Hence the following definitions

DEFINITION 2.2. *Let $\phi = (\phi_i^T, \phi_k^P)$ and $\psi = (\psi_i^T, \psi_k^P)$ be in $\mathbb{R}^I \times \mathbb{R}^K$. Let $\mathbf{u} = (\mathbf{u}_j)$ and $\mathbf{v} = (\mathbf{v}_j)$ be in $(\mathbb{R}^2)^J$. We define the following scalar products*

$$(\phi, \psi)_{T,P} := \frac{1}{2}\Big(\sum_{i \in [1,I]} |T_i| \, \phi_i^T \, \psi_i^T + \sum_{k \in [1,K]} |P_k| \, \phi_k^P \, \psi_k^P \Big), \tag{2.1}$$

$$(\mathbf{u}, \mathbf{v})_D := \sum_{j \in [1,J]} |D_j| \, \mathbf{u}_j \cdot \mathbf{v}_j. \tag{2.2}$$

We shall also need the following trace operator and boundary scalar product

Definition 2.3. *Let $\phi = (\phi_i^T, \phi_k^P)$ be in $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$. For any boundary edge A_j, with the notations of Fig. 2.2, we define $\tilde{\phi}_j$ as the trace of ϕ over A_j by*

$$\tilde{\phi}_j = \frac{1}{4}\left(\phi^P_{k_1(j)} + 2\phi^T_{i_2(j)} + \phi^P_{k_2(j)}\right). \tag{2.3}$$

Let $\phi = (\phi_i^T, \phi_k^P)$ be in $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ and let $w = (w_j)$ be defined (at least) on the boundary Γ_D, or on Γ_N or on Γ. We define the following boundary scalar products

$$(w, \tilde{\phi})_{\Gamma_D,h} = \sum_{j\in\Gamma_D} |A_j|\, w_j\, \tilde{\phi}_j \quad , \quad (w, \tilde{\phi})_{\Gamma_N,h} = \sum_{j\in\Gamma_N} |A_j|\, w_j\, \tilde{\phi}_j, \tag{2.4}$$
$$(w, \tilde{\phi})_{\Gamma,h} := (w, \tilde{\phi})_{\Gamma_D,h} + (w, \tilde{\phi})_{\Gamma_N,h}.$$

We recall here the discrete differential operators which have been constructed on fairly general two dimensional meshes, and some of their properties. For more details and for the proofs, see [12, 13].

Definition 2.4. *Let $\mathbf{u} = (\mathbf{u}_j)$ be in $(\mathbb{R}^2)^J$. We define its divergence and (scalar) curl on the primal and dual cells by*

$$\left(\nabla_h^T \cdot \mathbf{u}\right)_i := \frac{1}{|T_i|} \sum_{j\in\partial T_i} |A_j| \mathbf{u}_j \cdot \mathbf{n}_{ji},$$
$$\left(\nabla_h^P \cdot \mathbf{u}\right)_k := \frac{1}{|P_k|}\left(\sum_{j\in\partial P_k} \left(|A'_{j1}|\mathbf{u}_j \cdot \mathbf{n}'_{j1k} + |A'_{j2}|\mathbf{u}_j \cdot \mathbf{n}'_{j2k}\right) + \sum_{j\in\partial P_k\cap\Gamma} \frac{|A_j|}{2}\mathbf{u}_j \cdot \mathbf{n}_j\right),$$
$$\left(\nabla_h^T \times \mathbf{u}\right)_i := \frac{1}{|T_i|} \sum_{j\in\partial T_i} |A_j| \mathbf{u}_j \cdot \boldsymbol{\tau}_{ji},$$
$$\left(\nabla_h^P \times \mathbf{u}\right)_k := \frac{1}{|P_k|}\left(\sum_{j\in\partial P_k} \left(|A'_{j1}|\mathbf{u}_j \cdot \boldsymbol{\tau}'_{j1k} + |A'_{j2}|\mathbf{u}_j \cdot \boldsymbol{\tau}'_{j2k}\right) + \sum_{j\in\partial P_k\cap\Gamma} \frac{|A_j|}{2}\mathbf{u}_j \cdot \boldsymbol{\tau}_j\right).$$

We stress that $\partial P_k \cap \Gamma$ is non-empty if and only if $S_k \in \Gamma$.

Definition 2.5. *Let $\phi = (\phi_i^T, \phi_k^P)$ be in $\mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$; its discrete gradient $\nabla_h^D \phi$ and (vector) curl $\nabla_h^D \times \phi$ are defined by their values on the cells D_j by*

$$(\nabla_h^D \phi)_j := \frac{1}{2\,|D_j|}\left\{\left[\phi^P_{k_2} - \phi^P_{k_1}\right]\left(|A'_{j1}|\mathbf{n}'_{j1} + |A'_{j2}|\mathbf{n}'_{j2}\right) + \left[\phi^T_{i_2} - \phi^T_{i_1}\right]|A_j|\mathbf{n}_j\right\},$$
$$(\nabla_h^D \times \phi)_j := -\frac{1}{2\,|D_j|}\left\{\left[\phi^P_{k_2} - \phi^P_{k_1}\right]\left(|A'_{j1}|\boldsymbol{\tau}'_{j1} + |A'_{j2}|\boldsymbol{\tau}'_{j2}\right) + \left[\phi^T_{i_2} - \phi^T_{i_1}\right]|A_j|\boldsymbol{\tau}_j\right\}.$$

We recall that the formulae in Def. 2.5 are exact for affine functions.

Proposition 2.6. *For $\mathbf{u} \in (\mathbb{R}^2)^J$ and $\phi = (\phi^T, \phi^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$, the following discrete Green formulae hold:*

$$(\mathbf{u}, \nabla_h^D \phi)_D = -(\nabla_h^{T,P} \cdot \mathbf{u}, \phi)_{T,P} + (\mathbf{u}\cdot\mathbf{n}, \tilde{\phi})_{\Gamma,h} \tag{2.5}$$
$$(\mathbf{u}, \nabla_h^D \times \phi)_D = (\nabla_h^{T,P} \times \mathbf{u}, \phi)_{T,P} - (\mathbf{u}\cdot\boldsymbol{\tau}, \tilde{\phi})_{\Gamma,h}. \tag{2.6}$$

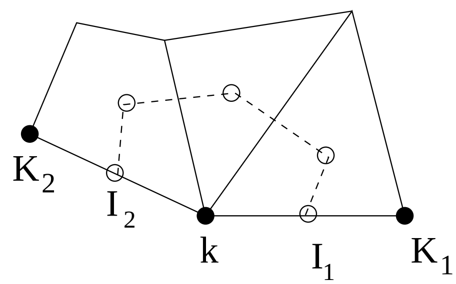

FIG. 2.3. *Notations for a boundary dual cell in formula (2.9)*

PROPOSITION 2.7. *For all* $\phi = (\phi_i^T, \phi_k^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$, *there holds*

$$\left(\nabla_h^T \cdot (\nabla_h^D \times \phi)\right)_i = 0 \ \ \forall i \in [1, I] \quad \text{and} \quad \left(\nabla_h^P \cdot (\nabla_h^D \times \phi)\right)_k = 0 \ \ \forall k \notin \Gamma, \tag{2.7}$$

$$\left(\nabla_h^T \times (\nabla_h^D \phi)\right)_i = 0 \ \ \forall i \in [1, I] \quad \text{and} \quad \left(\nabla_h^P \times (\nabla_h^D \phi)\right)_k = 0 \ \ \forall k \notin \Gamma. \tag{2.8}$$

In addition, for $k \in \Gamma$, *the following equality holds (see Fig. 2.3 for the notations)*

$$\left(\nabla_h^P \times (\nabla_h^D \phi)\right)_k = \frac{1}{|P_k|} \left[\left(\phi_{I_2}^T - \phi_{I_1}^T\right) + \frac{1}{2}\left(\phi_{K_1}^P - \phi_{K_2}^P\right) \right]. \tag{2.9}$$

DEFINITION 2.8. *For* $\phi = (\phi_i^T, \phi_k^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$, *we define the function* ϕ_h *by*

$$(\phi_h)_{|D_j} \in P^1(D_j), \ \ \forall j \in [1, J],$$
$$\phi_h(M_{i_\alpha(j)\, k_\beta(j)}) = \frac{1}{2}(\phi^T_{i_\alpha(j)} + \phi^P_{k_\beta(j)}), \ \ \forall j \in [1, J], \ \ \forall (\alpha\, \beta) \in \{1; 2\}^2 \,.$$

REMARK 2.9. *Though the definition of a* P^1 *function by its values in four different points is in general not possible, existence and uniqueness of the function* ϕ_h *are ensured in the present case because* $\phi_h(M_{i_1k_1}) + \phi_h(M_{i_2k_2}) = \phi_h(M_{i_1k_2}) + \phi_h(M_{i_2k_1})$ *and since the quadrilateral* $(M_{i_1k_1} M_{i_1k_2} M_{i_2k_2} M_{i_2k_1})$ *is a parallelogram. Moreover, the function* ϕ_h *is continuous only at the midpoints of the diamond-cell edges.*

PROPOSITION 2.10. *Elementary calculations show that*

$$\nabla(\phi_h)_{|D_j} = (\nabla_h^D \phi)_j, \tag{2.10}$$

$$\nabla \times (\phi_h)_{|D_j} = (\nabla_h^D \times \phi)_j. \tag{2.11}$$

DEFINITION 2.11. *In the sequel of the present work, we shall note by* $\nabla_h \phi_h$ *the* $(L^2(\Omega))^2$ *function whose restriction to each cell* D_j *is equal to* $\nabla(\phi_h)_{|D_j} = (\nabla_h^D \phi)_j$.

3. The finite volume scheme on general meshes. We recall the finite volume scheme used for the numerical approximation of Eq. (1.1)-(1.2)-(1.3). This scheme is constructed on the basis of the discrete operators defined in section 2.

$$-(\nabla_h^T \cdot (\nabla_h^D \phi))_i = (\bar{f})_i^T \ \ \forall i \in [1, I], \tag{3.1}$$

$$-(\nabla_h^P \cdot (\nabla_h^D \phi))_k = (\bar{f})_k^P \ \ \forall k \notin \bar{\Gamma}_D, \tag{3.2}$$

in which $(\bar{f})_i^T$ and $(\bar{f})_k^P$ are the mean values of f over T_i and P_k, respectively:

$$(\bar{f})_i^T = \frac{1}{|T_i|} \int_{T_i} f(\mathbf{x})\, d\mathbf{x} \ \text{ and } \ (\bar{f})_k^P = \frac{1}{|P_k|} \int_{P_k} f(\mathbf{x})\, d\mathbf{x}. \tag{3.3}$$

Dirichlet boundary conditions are discretized by

$$\phi_k^P = \phi_d(S_k)\,,\ \forall k \in \bar{\Gamma}_D \ \text{ and } \ \phi_i^T = \frac{1}{2}\left(\phi_{k_1}^P + \phi_{k_2}^P\right)\,,\ \forall i \in \Gamma_D, \tag{3.4}$$

where in the second equality, it is understood that $G_i \in \Gamma_D$ is the midpoint of $[S_{k_1} S_{k_2}] \subset \Gamma_D$. Note that there is a slight modification in the last boundary conditions in (3.4) with respect to those proposed in [13]. The reason for this will appear in section 4. Neumann boundary conditions are discretized by

$$(\nabla_h^D \phi)_j \cdot \mathbf{n}_j = \bar{g}_j\ ,\ \forall j \in \Gamma_N, \tag{3.5}$$

where $\bar{g}_j$ is the mean value of g over the corresponding segment A_j

$$\bar{g}_j = \frac{1}{|A_j|}\int_{A_j} g(\sigma)\,d\sigma. \tag{3.6}$$

LEMMA 3.1. *The scheme (3.1), (3.2), (3.4), (3.5) has a unique solution.*

Proof. Although the scheme (3.1), (3.2), (3.4), (3.5) is not exactly the same as that proposed in [13], as stated above, the proof of this lemma may be easily adapted from the proof of existence and uniqueness given in [13, Proposition 3.2]. □

PROPOSITION 3.2. *Let $\phi = (\phi_i^T, \phi_k^P)$ be the solution of the scheme (3.1)–(3.6). Let $\psi = (\psi_i^T, \psi_k^P)$ be such that $\psi_k^P = 0\,, \forall k \in \bar{\Gamma}_D$ and $\psi_i^T = 0\,, \forall i \in \Gamma_D$. Let ϕ_h and ψ_h be the functions associated to ϕ and ψ by Def. 2.8. Let us set in addition*

$$\psi_h^*(\mathbf{x}) := \frac{1}{2}\left(\sum_{i\in[1,I]} \psi_i^T \theta_i^T(\mathbf{x}) + \sum_{k\in[1,K]} \psi_k^P \theta_k^P(\mathbf{x})\right) \tag{3.7}$$

$$\tilde{\psi}_h(\sigma) := \sum_{j\in\Gamma} \tilde{\psi}_j \theta_j(\sigma), \tag{3.8}$$

where θ_i^T, θ_k^P and θ_j are respectively the characteristic functions of the cells T_i and P_k and of the edge $A_j \subset \Gamma$. Then, there holds

$$\sum_j \int_{D_j} \nabla\phi_h \cdot \nabla\psi_h(\mathbf{x})\,d\mathbf{x} = \int_\Omega f\,\psi_h^*(\mathbf{x})\,d\mathbf{x} + \int_{\Gamma_N} g\,\tilde{\psi}_h(\sigma)\,d\sigma\,. \tag{3.9}$$

Proof. From (3.1), (3.2) and the fact that ψ_k^P vanishes for $k \in \bar{\Gamma}_D$, it follows that

$$-(\nabla_h^{T,P} \cdot (\nabla_h^D \phi), \psi)_{T,P} = (\bar{f}, \psi)_{T,P}. \tag{3.10}$$

Using the discrete Green formula (2.5), the fact that ψ vanishes over $\bar{\Gamma}_D$ and taking (3.5) into account, we may transform (3.10) into

$$(\nabla_h^D \phi, \nabla_h^D \psi)_D = (\bar{f}, \psi)_{T,P} + (\bar{g}, \tilde{\psi})_{\Gamma_N,h}. \tag{3.11}$$

Evaluating the left-hand side in Eq. (3.11) with (2.2) and (2.10), and the right-hand side with (2.1), (2.4), (3.3), (3.6), (3.7) and (3.8) leads to (3.9). □

4. A representation of the error in the energy norm. Let us first recall that the solution of system (1.1)-(1.2)-(1.3) verifies

$$\int_\Omega \nabla\hat{\phi}\cdot\nabla\psi\,(\mathbf{x})\,d\mathbf{x} = \int_\Omega f\,\psi\,(\mathbf{x})\,d\mathbf{x} + \int_{\Gamma_N} g\,\psi\,(\sigma)\,d\sigma \tag{4.1}$$

for all $\psi \in H^1_D := \{\psi \in H^1(\Omega)\,/\,\psi = 0 \text{ on}\,\Gamma_D\}$. We seek to measure the broken H^1 semi norm of the error between the exact solution $\hat{\phi}$ and the function ϕ_h associated to the solution of the DDFV scheme. For this, we shall define

$$e = \left(\sum_j \int_{D_j} \left|\nabla\hat{\phi} - \nabla_h\phi_h\right|^2(\mathbf{x})\,d\mathbf{x}\right)^{1/2} \tag{4.2}$$

and we follow a now classical strategy, employed as soon as the discrete solution does not belong to $H^1(\Omega)$ (see [3, 11, 21]). Since $\nabla\hat{\phi} - \nabla_h\phi_h$ belongs to $(L^2(\Omega))^2$, we may write its discrete Helmholtz-Hodge decomposition in the following way

$$\nabla\hat{\phi} - \nabla_h\phi_h = \nabla\hat{\Phi} + \nabla\times\hat{\Psi} \tag{4.3}$$

with $\hat{\Phi} \in H^1_D(\Omega)$ and $\hat{\Psi} \in H^1_N := \left\{\hat{\Psi} \in H^1(\Omega)\,,\ \nabla\hat{\Psi}\cdot\boldsymbol{\tau} = 0 \ \text{ over }\ \Gamma_N\right\}$. This decomposition is orthogonal and there holds

$$\nabla\hat{\Psi}\cdot\boldsymbol{\tau} = 0 \text{ on } \Gamma_N \Leftrightarrow \exists\, c_N \in \mathbb{R} \text{ s.t. } \hat{\Psi}_{|\Gamma_N} = c_N\,. \tag{4.4}$$

If Γ_N were multiply connected, then there would exist one constant c_q for each component $\Gamma_{N,q}$ of Γ_N. Then, there holds

$$\begin{aligned} e^2 &= \left\|\nabla\hat{\Phi}\right\|^2_{0,\Omega} + \left\|\nabla\times\hat{\Psi}\right\|^2_{0,\Omega} \\ &= \sum_j \int_{D_j} (\nabla\hat{\phi} - \nabla_h\phi_h)\cdot\nabla\hat{\Phi}\,(\mathbf{x})\,d\mathbf{x} + \sum_j \int_{D_j} (\nabla\hat{\phi} - \nabla_h\phi_h)\cdot\nabla\times\hat{\Psi}\,(\mathbf{x})\,d\mathbf{x} \\ &:= i_1 + i_2\,. \end{aligned} \tag{4.5}$$

In order to find a suitable representation of i_1 and i_2, we need the following definitions

DEFINITION 4.1. *The boundary ∂D_j of any diamond-cell D_j is composed of the four segments $\left[G_{i_\alpha(j)}S_{k_\beta(j)}\right]$ with $(\alpha,\beta) \in \{1;2\}$. (see Fig. 2.2). Let us denote by S the set of these edges when j runs over the whole set of diamond-cells and $\overset{\circ}{S}$ those edges in S that do not lie on the boundary Γ. Each $s \in S$ is thus a segment that we shall denote by $\left[G_{i(s)}S_{k(s)}\right]$. We shall also write $s \in \overset{\circ}{T_i}$ (resp. $s \in \overset{\circ}{P_k}$) if $s \subset T_i$ (resp. $s \subset P_k$) and $s \not\subset \Gamma$. Finally, we shall denote by $\mathbf{n}_s$ one of the two unit normal vectors to s, arbitrarily chosen among the two possible choices but then fixed for the sequel, and $\left[\nabla_h\phi_h\cdot\mathbf{n}_s\right]_s$, the jump of the normal component of $\nabla_h\phi_h$ through s.*

PROPOSITION 4.2. *Let $\phi = (\phi^T_i, \phi^P_k)$ be the solution of the scheme (3.1)–(3.6) and ϕ_h the function associated to ϕ by Def. 2.8. Let $\hat{\Phi}$ be defined in Eq. (4.3). Let $\Phi = (\Phi^T_i, \Phi^P_k) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ be such that*

$$\Phi^P_k = 0\,,\ \forall k \in \bar{\Gamma}_D \ \ \textit{and}\ \ \Phi^T_i = 0\,,\ \forall i \in \Gamma_D. \tag{4.6}$$

The following representation holds

$$
\begin{aligned}
i_1 = {} & \frac{1}{2} \sum_{i \in [1,I]} \int_{T_i} f \left(\hat{\Phi} - \Phi_i^T \right) (\mathbf{x}) \, d\mathbf{x} + \frac{1}{2} \sum_{k \in [1,K]} \int_{P_k} f \left(\hat{\Phi} - \Phi_k^P \right) (\mathbf{x}) \, d\mathbf{x} \\
& + \sum_{A_j \subset \Gamma_N} \int_{A_j} (g - \bar{g}_j) \left(\hat{\Phi} - \tilde{\Phi}_h \right) (\sigma) \, d\sigma \\
& - \frac{1}{2} \sum_{i \in [1,I]} \sum_{s \subset \overset{\circ}{T_i}} \int_s [\nabla_h \phi_h \cdot \mathbf{n}_s]_s \left(\hat{\Phi} - \Phi_i^T \right) (\sigma) \, d\sigma \\
& - \frac{1}{2} \sum_{k \in [1,K]} \sum_{s \subset \overset{\circ}{P_k}} \int_s [\nabla_h \phi_h \cdot \mathbf{n}_s]_s \left(\hat{\Phi} - \Phi_k^P \right) (\sigma) \, d\sigma.
\end{aligned}
\tag{4.7}
$$

Proof. Since $\hat{\Phi} \in H_D^1$, and for any Φ verifying (4.6), formulae (4.1) and (3.9) lead to

$$
\begin{aligned}
i_1 &= \sum_j \int_{D_j} \nabla \hat{\phi} \cdot \nabla \hat{\Phi} \, (\mathbf{x}) \, d\mathbf{x} - \sum_j \int_{D_j} \nabla_h \phi_h \cdot \nabla \hat{\Phi} \, (\mathbf{x}) \, d\mathbf{x} \\
&= \int_\Omega f \left(\hat{\Phi} - \Phi_h^* \right) (\mathbf{x}) \, d\mathbf{x} + \int_{\Gamma_N} g \left(\hat{\Phi} - \tilde{\Phi}_h \right) (\sigma) \, d\sigma \\
&\quad - \sum_j \int_{D_j} \nabla_h \phi_h \cdot \left(\nabla \hat{\Phi} - \nabla_h \Phi_h \right) (\mathbf{x}) \, d\mathbf{x}.
\end{aligned}
\tag{4.8}
$$

Since $\nabla_h \phi_h$ is a constant over a given diamond-cell D_j, and since Φ_h belongs to P^1 and equals $\frac{1}{2} \left(\Phi_{i(s)}^T + \Phi_{k(s)}^P \right)$ at the midpoint of any diamond edge s, we may write, using Green's formula and the midpoint rule

$$
\int_{D_j} \nabla_h \phi_h \cdot \left(\nabla \hat{\Phi} - \nabla_h \Phi_h \right) (\mathbf{x}) \, d\mathbf{x} = \int_{\partial D_j} \nabla_h \phi_h \cdot \mathbf{n}_{\partial D_j} \left[\hat{\Phi} - \frac{1}{2} \left(\Phi_{i(s)}^T + \Phi_{k(s)}^P \right) \right] (\sigma) \, d\sigma \, ,
$$

where $\mathbf{n}_{\partial D_j}$ is the unit normal vector exterior to D_j on its boundary. Each $s \in \overset{\circ}{S}$ contributes twice to the sum of integrals contained in the last line of (4.8), since each interior s is located at the interface of two diamond-cells D_j. Moreover, since $\hat{\Phi} \in H^1(\Omega)$, the jump of this function through s vanishes. On the other hand, for any diamond-cell D_j whose boundary intersects Γ, one may easily remark that

$$
\int_{\partial D_j \cap \Gamma} \nabla_h \phi_h \cdot \mathbf{n}_{\partial D_j} \left[\hat{\Phi} - \frac{1}{2} \left(\Phi_{i(s)}^T + \Phi_{k(s)}^P \right) \right] (\sigma) \, d\sigma = \int_{A_j} \nabla_h \phi_h \cdot \mathbf{n}_j \left(\hat{\Phi} - \tilde{\Phi}_h \right) (\sigma) \, d\sigma \, .
\tag{4.9}
$$

On the boundary Γ_D, $\hat{\Phi}$ and Φ, and thus $\tilde{\Phi}_h$, vanish. On the boundary Γ_N, the value of $\nabla_h \phi_h \cdot \mathbf{n}_j$ is known thanks to (3.5). With all these remarks, we may write

$$
\begin{aligned}
& \sum_j \int_{D_j} \nabla_h \phi_h \cdot \left(\nabla \hat{\Phi} - \nabla_h \Phi_h \right) (\mathbf{x}) \, d\mathbf{x} = \\
& \sum_{s \in \overset{\circ}{S}} \int_s [\nabla_h \phi_h \cdot \mathbf{n}_s]_s \left[\hat{\Phi} - \frac{1}{2} \left(\Phi_{i(s)}^T + \Phi_{k(s)}^P \right) \right] (\sigma) \, d\sigma \\
& + \sum_{A_j \subset \Gamma_N} \int_{A_j} \bar{g}_j \left(\hat{\Phi} - \tilde{\Phi}_h \right) (\sigma) \, d\sigma \, .
\end{aligned}
\tag{4.10}
$$

Then, we may write $\hat{\Phi} - \frac{1}{2}\left(\Phi^T_{i(s)} + \Phi^P_{k(s)}\right) = \frac{1}{2}\left[\left(\hat{\Phi} - \Phi^T_{i(s)}\right) + \left(\hat{\Phi} - \Phi^P_{k(s)}\right)\right]$. Summing in the right-hand side of (4.10) the various contributions of Φ^T_i for a fixed i and the various contributions of Φ^P_k for a fixed k, we obtain the following formula

$$\begin{aligned}\sum_j \int_{D_j} \nabla_h \phi_h \cdot \left(\nabla\hat{\Phi} - \nabla_h \Phi_h\right)(\mathbf{x})\, d\mathbf{x} = {} & \frac{1}{2}\sum_{i\in[1,I]} \sum_{s\subset \overset{\circ}{T}_i} \int_s [\nabla_h \phi_h \cdot \mathbf{n}_s]_s \left(\hat{\Phi} - \Phi^T_i\right)(\sigma) d\sigma \\ & + \frac{1}{2}\sum_{k\in[1,K]} \sum_{s\subset \overset{\circ}{P}_k} \int_s [\nabla_h \phi_h \cdot \mathbf{n}_s]_s \left(\hat{\Phi} - \Phi^P_k\right)(\sigma) d\sigma \\ & + \sum_{A_j\subset\Gamma_N} \int_{A_j} \bar{g}_j \left(\hat{\Phi} - \tilde{\Phi}_h\right)(\sigma)\, d\sigma.\end{aligned} \tag{4.11}$$

Finally, according to (4.8) and the definition (3.7) of Φ^*_h, we obtain (4.7). ∎

Before we turn to a representation formula for i_2 in (4.5), we need some technical lemmas related to the $L^2(\Omega)$ scalar product of discrete gradients and curls.

Lemma 4.3. *Let $\phi = (\phi^T_i, \phi^P_k)$ be the solution of (3.1)–(3.6). There holds*

$$(\nabla^{T,P}_h \times (\nabla^D_h \phi), \Psi)_{T,P} = 0 \tag{4.12}$$

for any $\Psi = (\Psi^T_i, \Psi^P_k) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ such that

$$\Psi^P_k = c_N\,,\ \forall k \in \bar{\Gamma}_N \quad \text{and} \quad \Psi^T_i = c_N\,,\ \forall i \in \Gamma_N\,. \tag{4.13}$$

Proof. According to Eq. (2.8), there holds

$$(\nabla^T_h \times (\nabla^D_h \phi))_i = 0,\ \forall i \in [1,I] \quad \text{and} \quad (\nabla^P_h \times (\nabla^D_h \phi))_k = 0\,,\ \forall\, k \notin \Gamma. \tag{4.14}$$

On the other hand, since the solution of the discrete problem verifies (3.4), there holds $\phi^T_{I_1} = \frac{1}{2}(\phi^P_k + \phi^P_{K_1})$ and $\phi^T_{I_2} = \frac{1}{2}(\phi^P_k + \phi^P_{K_2})$ for $k \in \overset{\circ}{\Gamma}_D$, with the notations of Fig. 2.3. This implies, thanks to Eq. (2.9)

$$(\nabla^P_h \times (\nabla^D_h \phi))_k = 0,\ \forall\, k \in \overset{\circ}{\Gamma}_D\,. \tag{4.15}$$

With the definition (2.1) and the choice (4.13), Eqs. (4.14) and (4.15) imply that

$$(\nabla^{T,P}_h \times (\nabla^D_h \phi), \Psi)_{T,P} = \frac{1}{2} c_N \sum_{k\in\bar{\Gamma}_N} |P_k|\, (\nabla^P_h \times (\nabla^D_h \phi))_k\,. \tag{4.16}$$

Now, if Γ_N is a closed path, formula (2.9) imply that the sum in (4.16) vanishes since every ϕ^T_i and every ϕ^P_k in that sum have two contributions that cancel. On the other hand, if Γ_N is not a closed path, then let us consider the boundary Γ_N represented on Fig. 4.1. The points $k \in \bar{\Gamma}_N$ are the points $K_2, \cdots, K_{N-1}$. We have also displayed the neighboring edges $[K_1, K_2]$ and $[K_{N-1}, K_N]$ located on the Dirichlet boundaries which are neighboring to Γ_N. According to formula (2.9), in which the notations of Fig. 2.3 are used, we obtain

$$\begin{aligned}\sum_{k\in\bar{\Gamma}_N} |P_k|\, (\nabla^P_h \times (\nabla^D_h \phi))_k &= \sum_{p\in[1,N-2]} (\phi_{I_{p+1}} - \phi_{I_p}) + \frac{1}{2}(\phi_{K_p} - \phi_{K_{p+2}}) \\ &= -\phi_{I_1} + \frac{1}{2}(\phi_{K_1} + \phi_{K_2}) + \phi_{I_{N-1}} - \frac{1}{2}(\phi_{K_{N-1}} + \phi_{K_N}).\end{aligned} \tag{4.17}$$

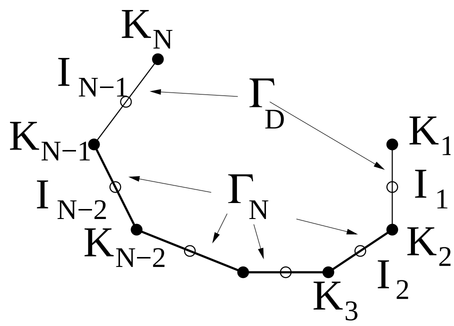

FIG. 4.1. *Notations for the Neumann boundary*

Since we have chosen the solution ϕ of the discrete problem such that (3.4) is verified on the Dirichlet boundaries, then $\phi_{I_1} = \frac{1}{2}(\phi_{K_1} + \phi_{K_2})$ and $\phi_{I_{N-1}} = \frac{1}{2}(\phi_{K_{N-1}} + \phi_{K_N})$. The sum in (4.16) thus vanishes. ▯

LEMMA 4.4. *Let $\phi = (\phi_i^T, \phi_k^P)$ and $\Psi = (\Psi_i^T, \Psi_k^P)$ be like in lemma 4.3. Let ϕ_h and Ψ_h be their associated functions through Def. 2.8. There holds*

$$\sum_j \int_{D_j} \nabla_h \phi_h \cdot \nabla_h \times \Psi_h(\mathbf{x}) d\mathbf{x} = -\int_{\Gamma_D} \nabla \phi_d \cdot \boldsymbol{\tau}\, \tilde{\Psi}_h(\sigma) d\sigma - c_N \int_{\Gamma_N} \nabla \hat{\phi} \cdot \boldsymbol{\tau}(\sigma) d\sigma. \tag{4.18}$$

Proof. Applying (2.10), (2.11) and the discrete Green formula (2.6), there holds

$$\sum_j \int_{D_j} \nabla_h \phi_h \cdot \nabla_h \times \Psi_h\,(\mathbf{x})\, d\mathbf{x} = (\nabla_h^{T,P} \times (\nabla_h^D \phi), \Psi)_{T,P} - (\nabla_h^D \phi \cdot \boldsymbol{\tau}, \tilde{\Psi})_{\Gamma,h}\,. \tag{4.19}$$

The first term in the right-hand side of Eq. (4.19) vanishes thanks to lemma 4.3 and the second term may be split into a contribution over Γ_D and a contribution over Γ_N. Now, for any $j \in \Gamma$, Def. 2.5 and the fact that for boundary diamond-cells $|A'_{j2}| = 0$ and $2|D_j| = |A_j|\,|A'_{j1}|\,\mathbf{n}'_{j1} \cdot \boldsymbol{\tau}_j$, imply that $(\nabla_h^D \phi)_j \cdot \boldsymbol{\tau}_j = \frac{1}{|A_j|}(\phi_{k_2(j)} - \phi_{k_1(j)})$. In particular, on the boundary Γ_D, the boundary conditions (3.4) imply

$$(\nabla_h^D \phi)_j \cdot \boldsymbol{\tau}_j \tilde{\Psi}_j = \frac{1}{|A_j|}(\phi_d(S_{k_2(j)}) - \phi_d(S_{k_1(j)}))\tilde{\Psi}_j = \frac{1}{|A_j|} \int_{A_j} \nabla \phi_d \cdot \boldsymbol{\tau}\, \tilde{\Psi}_h(\sigma) d\sigma \tag{4.20}$$

since $\tilde{\Psi}_h$ is a constant equal to $\tilde{\Psi}_j$ on A_j. This implies that

$$(\nabla_h^D \phi \cdot \boldsymbol{\tau}, \tilde{\Psi})_{\Gamma_D,h} = \sum_{j\in\Gamma_D} |A_j|\ (\nabla_h^D \phi)_j \cdot \boldsymbol{\tau}_j\, \tilde{\Psi}_j = \int_{\Gamma_D} \nabla \phi_d \cdot \boldsymbol{\tau}\, \tilde{\Psi}_h\,(\sigma)\, d\sigma\,. \tag{4.21}$$

As far as the contribution over Γ_N is concerned, we infer from (4.13) that $\tilde{\Psi}_j = c_N$ for all $j \in \Gamma_N$. Hence $(\nabla_h^D \phi \cdot \boldsymbol{\tau}, \tilde{\Psi})_{\Gamma_N,h} = c_N \sum_{j\in\Gamma_N} (\phi_{k_2(j)} - \phi_{k_1(j)}) = c_N(\phi_{K_2} - \phi_{K_{N-1}})$ where the notations of figure 4.1 are used. If Γ_N is a closed path, then $S_{K_2} = S_{K_{N-1}}$ and this sum vanishes and is thus equal to $c_N \int_{\Gamma_N} \nabla\hat{\phi} \cdot \boldsymbol{\tau}\,(\sigma)\, d\sigma$ which also vanishes. On the other hand, if Γ_N is not a closed path, then K_2 and K_{N-1} are on $\bar{\Gamma}_D$ and the values of ϕ at those points are imposed to be the values of ϕ_d by (3.4), that is to say the values of $\hat{\phi}$ at those points, which means

$$(\nabla_h^D \phi \cdot \boldsymbol{\tau}, \tilde{\Psi})_{\Gamma_N,h} = c_N(\hat{\phi}(S_{K_2}) - \hat{\phi}(S_{K_{N-1}})) = c_N \int_{\Gamma_N} \nabla\hat{\phi} \cdot \boldsymbol{\tau}\,(\sigma)\, d\sigma\,. \tag{4.22}$$

Eqs. (4.21) and (4.22), together with (4.19) lead to (4.18). ∎

PROPOSITION 4.5. *Let $\phi = (\phi_i^T, \phi_k^P)$ be the solution of the scheme (3.1)–(3.6) and ϕ_h its associated function. Let $\hat{\Psi}$ be defined in Eq. (4.3). Let $\Psi = (\Psi_i^T, \Psi_k^P) \in \mathbb{R}^{I+J^\Gamma} \times \mathbb{R}^K$ be such that (4.13) holds and let Ψ_h be its associated function. Let $t := \nabla\phi_d \cdot \boldsymbol{\tau}$ be defined on the boundary Γ_D. Then, the following representation holds*

$$\begin{aligned} i_2 = &- \sum_{A_j \subset \Gamma_D} \int_{A_j} (t - \bar{t}_j) \left(\hat{\Psi} - \tilde{\Psi}_h\right)(\sigma)\, d\sigma \\ &+ \frac{1}{2} \sum_{i\in[1,I]} \sum_{s \subset \overset{\circ}{T}_i} \int_s [\nabla_h \phi_h \cdot \boldsymbol{\tau}_s]_s \left(\hat{\Psi} - \Psi_i^T\right)(\sigma)\, d\sigma \\ &+ \frac{1}{2} \sum_{k\in[1,K]} \sum_{s \subset \overset{\circ}{P}_k} \int_s [\nabla_h \phi_h \cdot \boldsymbol{\tau}_s]_s \left(\hat{\Psi} - \Psi_k^P\right)(\sigma)\, d\sigma\,. \end{aligned} \tag{4.23}$$

Proof. From (4.5), there holds

$$\begin{aligned} i_2 = &\int_\Omega \nabla\hat{\phi} \cdot \nabla \times \hat{\Psi}\,(\mathbf{x})\, d\mathbf{x} - \sum_j \int_{D_j} \nabla_h \phi_h \cdot \nabla_h \times \Psi_h\,(\mathbf{x})\, d\mathbf{x} \\ &- \sum_j \int_{D_j} \nabla_h \phi_h \cdot \left(\nabla \times \hat{\Psi} - \nabla_h \times \Psi_h\right)(\mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{4.24}$$

By application of the continuous Green formula and taking into account the boundary condition (1.2) and Eq. (4.4) there holds

$$\int_\Omega \nabla\hat{\phi} \cdot \nabla \times \hat{\Psi}\,(\mathbf{x})\, d\mathbf{x} = -\int_{\Gamma_D} \nabla\phi_d \cdot \boldsymbol{\tau}\, \hat{\Psi}\,(\sigma)\, d\sigma - c_N \int_{\Gamma_N} \nabla\hat{\phi} \cdot \boldsymbol{\tau}\,(\sigma)\, d\sigma\,. \tag{4.25}$$

Using (4.25) and (4.18), formula (4.24) may be rewritten as

$$\begin{aligned} i_2 = &-\int_{\Gamma_D} \nabla\phi_d \cdot \boldsymbol{\tau} \left(\hat{\Psi} - \tilde{\Psi}_h\right)(\sigma)\, d\sigma \\ &- \sum_j \int_{D_j} \nabla_h \phi_h \cdot \left(\nabla \times \hat{\Psi} - \nabla_h \times \Psi_h\right)(\mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{4.26}$$

We may now compute the second term in the right-hand side of Eq. (4.26) just like we computed the last term in the right-hand side of Eq. (4.8). Considering separately inner and boundary edges, we may write

$$\begin{aligned} &\sum_j \int_{D_j} \nabla_h \phi_h \cdot \left(\nabla \times \hat{\Psi} - \nabla_h \times \Psi_h\right)(\mathbf{x})\, d\mathbf{x} = \\ &- \sum_{s \in \overset{\circ}{S}} \int_s [\nabla_h \phi_h \cdot \boldsymbol{\tau}_s]_s \left[\hat{\Psi} - \frac{1}{2}\left(\Psi_{i(s)}^T + \Psi_{k(s)}^P\right)\right](\sigma)\, d\sigma \\ &- \sum_{j\in\Gamma} \int_{\partial D_j \cap \Gamma} \nabla_h \phi_h \cdot \boldsymbol{\tau}_{\partial D_j} \left[\hat{\Psi} - \frac{1}{2}\left(\Psi_{i(s)}^T + \Psi_{k(s)}^P\right)\right](\sigma)\, d\sigma\,. \end{aligned} \tag{4.27}$$

As far as boundary edges are concerned, a formula analogous to (4.9) holds:

$$\int_{\partial D_j \cap \Gamma} \nabla_h \phi_h \cdot \boldsymbol{\tau}_{\partial D_j} \left[\hat{\Psi} - \frac{1}{2}\left(\Psi_{i(s)}^T + \Psi_{k(s)}^P\right)\right](\sigma)\, d\sigma = \int_{A_j} \nabla_h \phi_h \cdot \boldsymbol{\tau}_j \left[\hat{\Psi} - \tilde{\Psi}_h\right](\sigma) d\sigma. \tag{4.28}$$

Now, if $j \in \Gamma_N$, then $\hat{\Psi}$ is a constant on A_j whose value is c_N; the same holds for $\tilde{\Psi}_h$ thanks to (4.13). The contribution over the boundary Γ_N thus vanishes in the above expression. As far as $\nabla_h \phi_h \cdot \boldsymbol{\tau}_j$ is concerned, we have seen thanks to (4.20), that its value over $\partial D_j \cap \Gamma_D$ is $\nabla_h \phi_h \cdot \boldsymbol{\tau}_j = \bar{t}_j := \frac{1}{|A_j|} \int_{A_j} \nabla \phi_d \cdot \boldsymbol{\tau}\,(\sigma)\, d\sigma$. We thus have

$$\sum_{j \in \Gamma} \int_{\partial D_j \cap \Gamma} \nabla_h \phi_h \cdot \boldsymbol{\tau}_j \left[\hat{\Psi} - \frac{1}{2} \left(\Psi^T_{i(s)} + \Psi^P_{k(s)} \right) \right] (\sigma) d\sigma = \int_{\Gamma_D} \bar{t} \left[\hat{\Psi} - \tilde{\Psi}_h \right] (\sigma) d\sigma, \tag{4.29}$$

where $\bar{t}$ is the piecewise constant function defined over each segment $A_j \subset \Gamma_D$ by $\bar{t}(\sigma) := \bar{t}_j \theta_j(\sigma)$. If we compute the first term in the right-hand side of (4.27) just like we computed the first term in the right-hand side of (4.10), and taking (4.29) into account, Eq. (4.26) leads to (4.23). □

5. A computable error bound and its efficiency. Before stating the main results of this article, we recall some Poincaré-type inequalities and a trace inequality that will be useful in the derivation of the error bound, and we state an hypothesis under which the local error estimators are efficient.

5.1. Preliminaries. LEMMA 5.1. *Let ω be an open bounded set which is star-shaped with respect to one of its points. Let $u \in H^1(\omega)$ and let $\bar{u}_\omega$ be the mean-value of u over ω. Then,*

$$\exists C(\omega), \text{ s.t. } \|u - \bar{u}_\omega\|_{L^2(\omega)} \leq C(\omega)\, \mathrm{diam}(\omega)\, \|\nabla u\|_{L^2(\omega)} . \tag{5.1}$$

Note that when ω is convex, a universal constant $C(\omega)$ is given by $\frac{1}{\pi}$. When ω is not convex, we may use explicitly computable formulas given, for example, by [7, 28].

LEMMA 5.2. *Let ω be an open polygonal set such that $\bar{\omega}$ is star-shaped with respect to one of its vertices z located on a part γ_D (with non vanishing measure) of the boundary $\gamma = \partial\omega$. Let us suppose that at least one of the edges s included in $\partial\omega$ is such that the considered point z is a vertex of s and such that $s \subset \gamma_D$. Then,*

$$\exists C(\omega, \gamma_D)\, \text{s.t. } \|u\|_{L^2(\omega)} \leq C(\omega, \gamma_D)\, |\omega|^{1/2}\, \|\nabla u\|_{L^2(\omega)} . \tag{5.2}$$

for any function $u \in H^1(\omega)$, such that $u_{|\gamma_D} = 0$.

We may precise $C(\omega, \gamma_D)$ by using formula (3.2) of reference [7].

REMARK 5.3. *In (5.1) and (5.2), the constants $C(\omega)$ and $C(\omega, \gamma_D)$ do not depend on the diameter of ω, but only on its shape.*

LEMMA 5.4. *Let T be a triangle and let E be one of its edges. Then, for any function $u \in H^1(T)$, such that $\int_E u(\sigma) d\sigma = 0$, there holds*

$$\|u\|_{L^2(E)} \leq \frac{\alpha}{\sqrt{2}\hat{\rho}} \left(\frac{|E|}{|T|} \right)^{1/2} \mathrm{diam}(T)\, \|\nabla u\|_{L^2(T)} , \tag{5.3}$$

where $\alpha \approx 0.730276$ and $\hat{\rho} = 1 - \frac{\sqrt{2}}{2}$ are given by formula (23) of [21].

Finally, the trace inequality given by Theorem 4.1 and Remark 4.1 in [7] for functions in $W^{1,p}, p > 1$ may be improved for $p \geq 2$ and provides

LEMMA 5.5. *Let T be a triangle and let E be one of its edges; let ρ be the distance from E to the vertex of T opposite to E, and let σ be the length of the longest among the two other sides of T. Let $\varepsilon > 0$ be an arbitrary real-valued number; then for all $u \in H^1(T)$, there holds*

$$\|u\|^2_{L^2(E)} \leq \frac{1}{\rho} \left((2 + \varepsilon^{-2})\, \|u\|^2_{L^2(T)} + \varepsilon^2 \sigma^2\, \|\nabla u\|^2_{L^2(T)} \right) . \tag{5.4}$$

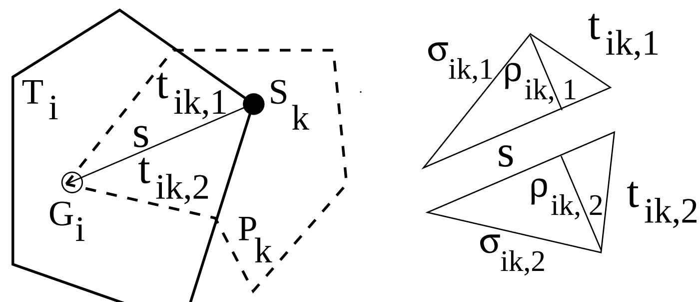

FIG. 5.1. *For a primal cell T_i and its vertex S_k, $T_i \cap P_k$ is split in two triangles $t_{ik,1}$ and $t_{ik,2}$.*

HYPOTHESIS 5.6. *We assume that the subtriangulation of Ω composed of all the triangles $t_{ik,\alpha}$ (see Fig. 5.1) is regular in the sense that the minimum angles in those triangles are bounded by below independently of the mesh.*

5.2. Statement of the main results. THEOREM 5.7. *Let $h_i^T := \text{diam}(T_i)$ and $h_k^P := \text{diam}(P_k)$. Let $\bar{f}_i^T$ (resp. $\bar{f}_k^P$) be the mean-value of f over T_i (resp. over P_k). Let $\bar{g}_j$ (resp. $\bar{t}_j$) be the mean-value of g (resp. of $\nabla\phi_d \cdot \boldsymbol{\tau}$) over the Neumann (resp. Dirichlet) boundary segment A_j. Let $C(T_i)$, $C(P_k)$ and $C(P_k, \partial P_k \cap \Gamma_D)$, $C(P_k, \partial P_k \cap \Gamma_N)$ be the computable constants respectively involved in (5.1) and (5.2). Let α be the constant involved in (5.3). Let us define, by analogy with [3],*

$$osc\,(f,T,\Omega) = \left(\sum_{i\in[1,N]} \left(C(T_i)\,h_i^T\right)^2 \|f - \bar{f}_i^T\|^2_{L^2(T_i)} \right)^{1/2}, \tag{5.5}$$

$$osc\,(f,P,\Omega) = \left(\sum_{k\notin\bar{\Gamma}_D} \left(C(P_k)h_k^P\right)^2 \|f - \bar{f}_k^P\|^2_{L^2(P_k)} \right)^{1/2}, \tag{5.6}$$

$$str\,(f,P,\Gamma_D,\Omega) = \left(\sum_{k\in\bar{\Gamma}_D} \left(C(P_k, \partial P_k \cap \Gamma_D)\right)^2 |P_k|\, \|f\|^2_{L^2(P_k)} \right)^{1/2}, \tag{5.7}$$

$$osc(g,\Gamma_N) = \alpha(1+\sqrt{2})\left(\sum_{j\in\Gamma_N} \frac{\text{diam}^2(D_j)}{|D_j|} |A_j|\, \|g - \bar{g}_j\|^2_{L^2(A_j)} \right)^{1/2}, \tag{5.8}$$

$$osc(t,\Gamma_D) = \alpha(1+\sqrt{2})\left(\sum_{j\in\Gamma_D} \frac{\text{diam}^2(D_j)}{|D_j|} |A_j|\, \|t - \bar{t}_j\|^2_{L^2(A_j)} \right)^{1/2}. \tag{5.9}$$

Moreover, for any $\mu > 0$, let us define

$$\chi_i(\mu) = \left(C(T_i)h_i^T\right)^2 + \mu \tag{5.10}$$

$$\chi_k(\mu) = \begin{cases} \left(C(P_k)h_k^P\right)^2 + \mu & \text{if} \quad k \notin \bar{\Gamma}_D \\ C^2(P_k, \Gamma_D \cap \partial P_k)\,|P_k| + \mu & \text{if} \quad k \in \bar{\Gamma}_D \end{cases}, \tag{5.11}$$

$$\chi'_k(\mu) = \begin{cases} \left(C(P_k)h_k^P\right)^2 + \mu & \text{if} \quad k \notin \bar{\Gamma}_N \\ C^2(P_k, \Gamma_N \cap \partial P_k)\,|P_k| + \mu & \text{if} \quad k \in \bar{\Gamma}_N \end{cases}, \tag{5.12}$$

For any primal cell T_i and any dual cell P_k such that $T_i \cap P_k \neq \emptyset$, let $s = [G_i S_k]$ and $t_{ik,1}$ and $t_{ik,2}$ be the triangles defined in Fig. 5.1 such that $t_{ik,1} \cup t_{ik,2} = T_i \cap P_k$. Let $\rho_{ik,\alpha}$ be the distance from s to the vertex of $t_{ik,\alpha}$ opposite to s and $\sigma_{ik,\alpha}$ be the length of the longest among the two other edges of $t_{ik,\alpha}$. For any $\mu > 0$, let us define

$$C_s(\mu) = \frac{\left(1+\sqrt{1+\frac{\sigma_{ik,1}^2}{\mu}}\right)\left(1+\sqrt{1+\frac{\sigma_{ik,2}^2}{\mu}}\right)}{\left(1+\sqrt{1+\frac{\sigma_{ik,1}^2}{\mu}}\right)\rho_{ik,2}+\left(1+\sqrt{1+\frac{\sigma_{ik,2}^2}{\mu}}\right)\rho_{ik,1}}. \tag{5.13}$$

We define the local and global error estimators by

$$\left(\eta_i^T\right)^2 = \inf_{\mu>0}\left[\chi_i(\mu)\sum_{s\in \overset{\circ}{T_i}} C_s(\mu)\,\|[\nabla_h\phi_h\cdot\mathbf{n}_s]_s\|^2_{L^2(s)}\right] \textit{ and } \left(\eta^T\right)^2 = \sum_i \left(\eta_i^T\right)^2, \tag{5.14}$$

$$\left(\eta'^T_i\right)^2 = \inf_{\mu>0}\left[\chi_i(\mu)\sum_{s\in \overset{\circ}{T_i}} C_s(\mu)\,\|[\nabla_h\phi_h\cdot\boldsymbol{\tau}_s]_s\|^2_{L^2(s)}\right] \textit{ and } \left(\eta'^T\right)^2 = \sum_i \left(\eta'^T_i\right)^2, \tag{5.15}$$

$$\left(\eta_k^P\right)^2 = \inf_{\mu>0}\left[\chi_k(\mu)\sum_{s\in \overset{\circ}{P_k}} C_s(\mu)\,\|[\nabla_h\phi_h\cdot\mathbf{n}_s]_s\|^2_{L^2(s)}\right] \textit{ and } \left(\eta^P\right)^2 = \sum_k \left(\eta_k^P\right)^2, \tag{5.16}$$

$$\left(\eta'^P_k\right)^2 = \inf_{\mu>0}\left[\chi'_k(\mu)\sum_{s\in \overset{\circ}{P_k}} C_s(\mu)\,\|[\nabla_h\phi_h\cdot\boldsymbol{\tau}_s]_s\|^2_{L^2(s)}\right] \textit{ and } \left(\eta'^P\right)^2 = \sum_k \left(\eta'^P_k\right)^2. \tag{5.17}$$

Then, the following a posteriori error estimate holds

$$\left(\sum_j \int_{D_j} \left|\nabla\hat{\phi} - \nabla_h\phi_h\right|^2(\mathbf{x})\,d\mathbf{x}\right)^{1/2} \leq \tag{5.18}$$

$$\frac{1}{2}\Bigg(\Big[osc(f,T,\Omega) + \left(osc^2(f,P,\Omega) + str^2(f,P,\Gamma_D,\Omega)\right)^{1/2} + 2osc(g,A,\Gamma_N) + \eta^T + \eta^P\Big]^2$$
$$+ \left[2osc(t,A,\Gamma_D) + \eta'^T + \eta'^P\right]^2\Bigg)^{1/2}.$$

Moreover, under Hyp. 5.6, there exists a constant C independent of the mesh such that

$$\left(\eta_i^T\right)^2 \leq C\left(\left\|\nabla_h\phi_h - \nabla\hat{\phi}\right\|^2_{L^2(T_i)} + \left(h_i^T\right)^2\left\|f - (\bar{f})_i^T\right\|^2_{L^2(T_i)}\right), \forall i\in[1,I], \tag{5.19}$$

$$\left(\eta'^T_i\right)^2 \leq C\left\|\nabla_h\phi_h - \nabla\hat{\phi}\right\|^2_{L^2(T_i)}, \forall i\in[1,I], \tag{5.20}$$

$$\left(\eta_k^P\right)^2 \leq C\left(\left\|\nabla_h\phi_h - \nabla\hat{\phi}\right\|^2_{L^2(P_k)} + \left(h_k^P\right)^2\left\|f - (\bar{f})_k^P\right\|^2_{L^2(P_k)}\right), \forall k\in[1,K], \tag{5.21}$$

$$\left(\eta'^P_k\right)^2 \leq C\left\|\nabla_h\phi_h - \nabla\hat{\phi}\right\|^2_{L^2(P_k)}, \forall k\in[1,K]. \tag{5.22}$$

Proof. The proof of (5.18) is based on (4.5), (4.7), (4.23) and on propositions 5.9, 5.11 and 5.13 below. The proof of (5.19)–(5.22) is postponed to subsection 5.5. ◻

The proofs of propositions 5.9, 5.11 and 5.13 are based on special choices for the values of (Φ_i^T, Φ_k^P) and (Ψ_i^T, Ψ_k^P), since in the expressions (4.7) of i_1 and (4.23) of i_2, these values are arbitrary, except for boundary values given by (4.6) and (4.13).

Definition 5.8. *Since $\hat{\Phi}$ and $\hat{\Psi}$ are not necessarily more regular than $H^1(\Omega)$, we choose their interpolations to be their L^2 projections on the primal and dual cells*

$$\Phi_i^T = \frac{1}{|T_i|}\int_{T_i} \hat{\Phi}(\mathbf{x})\, d\mathbf{x} \quad \forall i \in [1, I]\ ,\ \Phi_k^P = \frac{1}{|P_k|}\int_{P_k} \hat{\Phi}(\mathbf{x})\, d\mathbf{x} \quad \forall k \notin \bar{\Gamma}_D\,. \tag{5.23}$$

$$\Psi_i^T = \frac{1}{|T_i|}\int_{T_i} \hat{\Psi}(\mathbf{x})\, d\mathbf{x} \quad \forall i \in [1, I]\ ,\ \Psi_k^P = \frac{1}{|P_k|}\int_{P_k} \hat{\Psi}(\mathbf{x})\, d\mathbf{x} \quad \forall k \notin \bar{\Gamma}_N\,. \tag{5.24}$$

Once the values of Φ_k^P (resp. Ψ_k^P) at the vertices have been chosen by (4.6) or (5.23) (resp. (4.13) or (5.24)), we complete the definitions of (Φ_i^T, Φ_k^P) (resp. (Ψ_i^T, Ψ_k^P)), by requiring that boundary values Φ_i^T for $i \in \Gamma_N$ (resp. Ψ_i^T for $i \in \Gamma_D$) be chosen through formulae analogous to (2.3) in a way such that $\tilde{\Phi}_j$ (resp. $\tilde{\Psi}_j$) are the mean-values of $\hat{\Phi}$ (resp. $\hat{\Psi}$) over the corresponding boundary edges A_j.

5.3. Bounds for the higher-order terms. Since the *a priori* estimations obtained in [13] show that the error norm e behaves like $O(h)$ when the solution $\hat{\phi}$ is sufficiently regular, any contribution in i_1 or i_2 that behaves like $O(h^{1+\alpha})$ with $\alpha > 0$ will be asymptotically negligible in the error estimation. Contrarily to what is often done, (see, e.g., [27]), we shall include these higher-order terms (HOT) in our estimator since our purpose is to obtain a guaranteed upper bound for the error.

Proposition 5.9. *Let the definitions of Theorem 5.7 hold. Then, there holds*

$$\left|\sum_i \int_{T_i} f\, \left(\hat{\Phi} - \Phi_i^T\right)(\mathbf{x}) d\mathbf{x}\right| \le osc(f, T, \Omega) \left\|\nabla\hat{\Phi}\right\|_{L^2(\Omega)}, \tag{5.25}$$

$$\left|\sum_k \int_{P_k} f\left(\hat{\Phi} - \Phi_k^P\right)(\mathbf{x}) d\mathbf{x}\right| \le \left(osc^2(f, P, \Omega) + str^2(f, P, \Gamma_D, \Omega)\right)^{1/2} \left\|\nabla\hat{\Phi}\right\|_{L^2(\Omega)}, \tag{5.26}$$

$$\left|\sum_{j\in\Gamma_N} \int_{A_j} (g - \bar{g}_j)\left(\hat{\Phi} - \tilde{\Phi}_h\right)(\sigma) d\sigma\right| \le osc(g, \Gamma_N) \left\|\nabla\hat{\Phi}\right\|_{L^2(\Omega)}, \tag{5.27}$$

$$\left|\sum_{j\in\Gamma_D} \int_{A_j} (t - \bar{t}_j)\left(\hat{\Psi} - \tilde{\Psi}_h\right)(\sigma) d\sigma\right| \le osc(t, \Gamma_D) \left\|\nabla\hat{\Psi}\right\|_{L^2(\Omega)}. \tag{5.28}$$

Proof. Since Φ_i^T was chosen as the mean value of $\hat{\Phi}$ over T_i (see (5.23)), we have $\int_{T_i} f\left(\hat{\Phi} - \Phi_i^T\right)(\mathbf{x})\, d\mathbf{x} = \int_{T_i}\left(f - \bar{f}_i^T\right)\left(\hat{\Phi} - \Phi_i^T\right)(\mathbf{x})\, d\mathbf{x}$. The Cauchy-Schwarz inequality, formula (5.1) (since $\hat{\Phi} \in H^1(T_i)$) and the discrete Cauchy-Schwarz inequality lead to (5.25). As far as (5.26) is concerned, we may proceed in the same way but we have to distinguish whether $k \notin \bar{\Gamma}_D$ or not. Indeed, for $k \notin \bar{\Gamma}_D$, we have chosen Φ_k^P as the mean-value of $\hat{\Phi}$ over P_k. On the other hand, when $k \in \bar{\Gamma}_D$, we have set Φ_k^P to 0 by (4.6) and the associated dual cells P_k have a part of their boundary located on Γ_D, on which $\hat{\Phi}$ vanishes, so that we may apply (5.2). Setting $\Omega_D^P = \bigcup_{k\in\bar{\Gamma}_D} P_k$, we obtain

$$\left|\sum_{k\notin\bar{\Gamma}_D} \int_{P_k} f\, \left(\hat{\Phi} - \Phi_k^P\right)(\mathbf{x})\, d\mathbf{x}\right| \le osc\,(f, P, \Omega)\, \left\|\nabla\hat{\Phi}\right\|_{L^2(\Omega\setminus\Omega_D^P)}, \tag{5.29}$$

$$\left|\sum_{k\in\bar{\Gamma}_D} \int_{P_k} f\, \left(\hat{\Phi} - \Phi_k^P\right)(\mathbf{x})\, d\mathbf{x}\right| \le str(f, P, \Gamma_D, \Omega) \left\|\nabla\hat{\Phi}\right\|_{L^2(\Omega_D^P)}. \tag{5.30}$$

Inequalities (5.29) and (5.30) lead to (5.26). As far as (5.27) is concerned, $\tilde{\Phi}_j$ has been chosen in Def. 5.8 so that the function $\hat{\Phi} - \tilde{\Phi}_j$ has a vanishing mean-value over A_j, which is an edge of the triangle D_j. Thus, Cauchy-Schwarz inequalities, together with (5.3), lead to (5.27). Inequality (5.28) is obtained like (5.27). □

REMARK 5.10. *For $\varepsilon > 0$, the quantities (5.5) to (5.9) are HOT as soon as* $(f,g,t) \in H^{\varepsilon}(\Omega) \times H^{1/2+\varepsilon}(\Gamma_N) \times H^{1/2+\varepsilon}(\Gamma_D)$. Indeed, in that case, $\left\| f - \bar{f}_i^T \right\|_{L^2(T_i)}$ is of order $\left(h_i^T\right)^{\min(1,\varepsilon)} \|f\|_{H^{\varepsilon}(T_i)}$, so that $osc\,(f,T,\Omega)$ is of order $h^{1+\min(1,\varepsilon)}$. This is also the case for the term (5.6). Moreover, since Ω_D^P is the union of the dual cells whose associated vertex S_k lies on $\bar{\Gamma}_D$, it is included in a stripe of width h along Γ_D. Ilin's inequality (see, e.g., [9]), ensures that $\|f\|_{L^2(\Omega_D^P)}$ is of order $h^{\min(\frac{1}{2},\varepsilon)}$, which implies that $str\,(f,P,\Gamma_D,\Omega)$ is of order $h^{1+\min(\frac{1}{2},\varepsilon)}$. Finally, $\|g - \bar{g}_j\|_{L^2(A_j)}$ and $\|t - \bar{t}_j\|_{L^2(A_j)}$ are of order $\frac{1}{2}+\varepsilon$, and thus $osc(g,\Gamma_N)$ and $osc(t,\Gamma_D)$ are of order $1+\varepsilon$ provided that the quantity $\frac{\mathrm{diam}^2(D_j)}{|D_j|}$ is bounded independently of h on the whole boundary Γ.

5.4. Bounds for the main terms. PROPOSITION 5.11. *Let the definitions of Theorem 5.7 hold. Then, there holds*

$$\left| \sum_{i\in[1,I]} \sum_{s\in \overset{\circ}{T}_i} \int_s [\nabla_h \phi_h \cdot \mathbf{n}_s]_s \left(\hat{\Phi} - \Phi_i^T\right)(\sigma)\, d\sigma \right| \le \eta^T \left\|\nabla\hat{\Phi}\right\|_{L^2(\Omega)}, \tag{5.31}$$

$$\left| \sum_{i\in[1,I]} \sum_{s\in \overset{\circ}{T}_i} \int_s [\nabla_h \phi_h \cdot \boldsymbol{\tau}_s]_s \left(\hat{\Psi} - \Psi_i^T\right)(\sigma)\, d\sigma \right| \le {\eta'}^T \left\|\nabla\hat{\Psi}\right\|_{L^2(\Omega)}. \tag{5.32}$$

Proof. We shall only give the proof of (5.31), since the proof of (5.32) exactly follows the same lines. By application of the Cauchy-Schwarz inequality on each of the edges $s \in \overset{\circ}{T}_i$, and by the weighted discrete Cauchy-Schwarz inequality, we obtain for any set of strictly positive real-valued numbers C_s^T

$$\begin{aligned} &\left| \sum_{s\in \overset{\circ}{T}_i} \int_s [\nabla_h \phi_h \cdot \mathbf{n}_s]_s \left(\hat{\Phi} - \Phi_i^T\right)(\sigma) d\sigma \right| \le \\ &\left(\sum_{s\in \overset{\circ}{T}_i} C_s^T \left\| [\nabla_h \phi_h \cdot \mathbf{n}_s]_s \right\|^2_{L^2(s)} \right)^{1/2} \left(\sum_{s\in \overset{\circ}{T}_i} \frac{1}{C_s^T} \left\| \hat{\Phi} - \Phi_i^T \right\|^2_{L^2(s)} \right)^{1/2}. \end{aligned} \tag{5.33}$$

Now, for each segment s, we may apply the trace inequality (5.4) on each of the two triangles $t_{ik,1}$ and $t_{ik,2}$. A convex combination with weights κ_s for $\alpha = 1$ and $(1-\kappa_s)$ for $\alpha = 2$ of the resulting two inequalities leads to

$$\begin{aligned} \sum_{s\in \overset{\circ}{T}_i} \frac{1}{C_s^T} \left\| \hat{\Phi} - \Phi_i^T \right\|^2_{L^2(s)} \le \sum_{s\in \overset{\circ}{T}_i} \Bigg[&\frac{\kappa_s}{C_s^T} \left(C_{1,s,1} \left\| \hat{\Phi} - \Phi_i^T \right\|^2_{L^2(t_{ik,1})} + C_{2,s,1} \left\| \nabla\hat{\Phi} \right\|^2_{L^2(t_{ik,1})} \right) \\ &+ \frac{(1-\kappa_s)}{C_s^T} \left(C_{1,s,2} \left\| \hat{\Phi} - \Phi_i^T \right\|^2_{L^2(t_{ik,2})} + C_{2,s,2} \left\| \nabla\hat{\Phi} \right\|^2_{L^2(t_{ik,2})} \right) \Bigg]. \end{aligned}$$

for all strictly positive $\varepsilon_{ik,\alpha}$, with $C_{1,s,\alpha} = \frac{(2+\varepsilon_{ik,\alpha}^{-2})}{\rho_{ik,\alpha}}$ and $C_{2,s,\alpha} = \frac{\varepsilon_{ik,\alpha}^2 \sigma_{ik,\alpha}^2}{\rho_{ik,\alpha}}$. If we give an equal weight to the various contributions of the triangles $t_{ik,\alpha}$ in the above sum, we can sum them up into a norm over T_i. This may be obtained by fixing μ_i in T_i

independently of s and choosing $\varepsilon_{ik,\alpha}$ for each $s \in \overset{\circ}{T_i}$ and $\alpha \in \{1;2\}$ so that

$$\varepsilon_{ik,\alpha}^2 = \frac{\mu_i + \sqrt{\mu_i^2 + \mu_i \sigma_{ik,\alpha}^2}}{\sigma_{ik,\alpha}^2} \Longleftrightarrow C_{2,s,\alpha} = \mu_i C_{1,s,\alpha} \ , \ \forall s \in \overset{\circ}{T_i} \ , \forall \alpha \in \{1;2\}. \tag{5.34}$$

Then κ_s and C_s^T are chosen such that $\kappa_s C_{1,s,1} = (1-\kappa_s) C_{1,s,2} = C_s^T$. It is readily checked that this leads to $C_s^T = C_s(\mu_i)$ (see definition (5.13)). Then, there holds

$$\sum_{s \in \overset{\circ}{T_i}} \frac{1}{C_s^T} \left\| \hat{\Phi} - \Phi_i^T \right\|_{L^2(s)}^2 \leq \sum_{s \in \overset{\circ}{T_i}} \sum_{\alpha \in \{1;2\}} \left(\left\| \hat{\Phi} - \Phi_i^T \right\|_{L^2(t_{ik,\alpha})}^2 + \mu_i \left\| \nabla \hat{\Phi} \right\|_{L^2(t_{ik,\alpha})}^2 \right).$$

Summing up these norms into norms over T_i and applying (5.1), we get

$$\sum_{s \in \overset{\circ}{T_i}} \frac{1}{C_s^T} \left\| \hat{\Phi} - \Phi_i^T \right\|_{L^2(s)}^2 \leq \left[\left(C(T_i)\, h_i^T \right)^2 + \mu_i \right] \left\| \nabla \hat{\Phi} \right\|_{L^2(T_i)}^2 . \tag{5.35}$$

With formulae (5.10), (5.33), (5.35) and the discrete Cauchy-Schwarz inequality, we are lead to (5.31) after minimizing separately over each μ_i. □

REMARK 5.12. *This minimization is performed numerically when we effectively compute the estimators. However, we may already get an idea of the behaviour of* η_i^T *by bounding it by the value of the function in (5.14) for* $\mu = \left(h_i^T\right)^2$*, for example. By definition of* $\sigma_{ik,\alpha}$*, this length is lower than the diameter of* T_i*, which implies*

$$C_s \left(\left(h_i^T \right)^2 \right) \leq \frac{(1+\sqrt{2})^2}{2(\rho_{ik,1} + \rho_{ik,2})}. \tag{5.36}$$

If we assume that the ratios $\frac{\rho_{ik,\alpha}}{h_i^T}$ *are all greater than the same constant, independently of the mesh, we obtain the following bound, for a constant* K *independent of the mesh*

$$(\eta_i^T)^2 \leq K h_i^T \sum_{s \in \overset{\circ}{T_i}} \left\| [\nabla_h \phi_h \cdot \mathbf{n}_s]_s \right\|_{L^2(s)}^2 .$$

The same remark holds for the choice of μ_i'.

As far as dual cells are concerned, we have the following result

PROPOSITION 5.13. *Let the definitions of Theorem 5.7 hold. Then, there holds*

$$\left| \sum_{k \in [1,K]} \sum_{s \subset \overset{\circ}{P_k}} \int_s [\nabla_h \phi_h \cdot \mathbf{n}_s]_s \left(\hat{\Phi} - \Phi_k^P \right)(\sigma)\, d\sigma \right| \leq \eta^P \left\| \nabla \hat{\Phi} \right\|_{L^2(\Omega)}, \tag{5.37}$$

$$\left| \sum_{k \in [1,K]} \sum_{s \subset \overset{\circ}{P_k}} \int_s [\nabla_h \phi_h \cdot \boldsymbol{\tau}_s]_s \left(\hat{\Psi} - \Psi_k^P \right)(\sigma)\, d\sigma \right| \leq \eta'^P \left\| \nabla \hat{\Psi} \right\|_{L^2(\Omega)} . \tag{5.38}$$

Proof. We proceed like on the primal cells, but we have to distinguish those dual cells whose boundary does not intersect Γ_D (resp. Γ_N), on which Φ_k^P (resp. Ψ_k^P) are the mean values of $\hat{\Phi}$ (resp. $\hat{\Psi}$) and for which we may thus apply (5.1), and those whose boundary intersects Γ_D (resp. Γ_N), for which we have to apply (5.2) since $(\hat{\Phi} - \Phi_k^P)$ (resp. $(\hat{\Psi} - \Psi_k^P)$) vanishes over $\Gamma_D \cap \partial P_k$ (resp. $\Gamma_N \cap \partial P_k$), see Eqs. (4.3) and (4.6) (resp. Eqs. (4.4) and (4.13)). □

5.5. Efficiency of the estimators. Here, we prove inequalities (5.19)–(5.22) under Hypothesis 5.6. In what follows, the letter C designates quantities whose values do not depend on the mesh. We start by two lemmas of which we skip the proof

LEMMA 5.14. *Under Hyp. 5.6, there exists a constant C independent of the mesh such that for primal and dual cells T_i and P_k such that $T_i \cap P_k \neq \emptyset$*

$$\left(h_i^T\right)^2 |T_i \cap P_k|^{-1} \leq C \quad \text{and} \quad \left(h_k^P\right)^2 |T_i \cap P_k|^{-1} \leq C.$$

LEMMA 5.15. *Under Hyp. 5.6, the constants $C(P_k)$, $C(P_k, \Gamma_D \cap \partial P_k)$ and $C(P_k, \Gamma_N \cap \partial P_k)$ in Theorem 5.7 are bounded by a constant C independent of the mesh.*
We start by proving (5.19). Since the estimator η_i^T involves jumps of $\nabla_h \phi_h$ through the common edge $s = [G_i S_k]$ of two neighboring diamond-cells, we shall use functions with a support included in the triangles $t_{ik,\alpha}$, with $\alpha = 1$ or 2, defined in Figure 5.1. Since we consider a fixed s in what follows, we simplify the notations into t_1 and t_2. For any triangle t in $\{t_1, t_2\}$, we denote by $\lambda_{t,\beta}$ the barycentric coordinates associated with the three vertices of t, with $\beta \in \{1, 2, 3\}$. We suppose that the vertices of t_1 and t_2 are locally numbered so that the two nodes of the edge s are the vertices 1 and 2 of each of the triangles t_1 and t_2. We define the following bubble functions

$$b_t = 27\lambda_{t,1}\lambda_{t,2}\lambda_{t,3} \quad \text{for} \quad t = t_1 \quad \text{or} \quad t = t_2, \tag{5.39}$$

$$b_s = \begin{cases} 4\lambda_{t_\alpha,1}\lambda_{t_\alpha,2} & \text{on} \quad t_\alpha,\ \alpha \in \{1; 2\} \\ 0 & \text{elsewhere} \end{cases}. \tag{5.40}$$

There holds $\omega_t := supp(b_t) \subset t$ and $\omega_s := supp(b_s) = T_i \cap P_k = t_1 \cup t_2$. In the following proposition, proved, e.g., in [27], the constant $C > 0$ only depends on the minimal angle in (t_1, t_2) so that, under Hyp. 5.6, it is independent of the mesh.

PROPOSITION 5.16. *For $t = t_1$ or $t = t_2$ and $h_t = \text{diam}(t)$, there holds*

$$0 \leq b_t \leq 1\ ,\ 0 \leq b_s \leq 1, \tag{5.41}$$

$$\int_s b_s(\sigma) d\sigma = \frac{2}{3}|s|, \tag{5.42}$$

$$C^{-1} h_t^2 \leq \int_t b_t(\mathbf{x}) d\mathbf{x} = \frac{9}{20}|t| \leq C\, h_t^2, \tag{5.43}$$

$$C^{-1}|s|^2 \leq \int_t b_s(\mathbf{x}) d\mathbf{x} = \frac{1}{3}|t| \leq C|s|^2, \tag{5.44}$$

$$\|\nabla b_t\|_{L^2(t)} \leq C h_t^{-1} \|b_t\|_{L^2(t)}\,, \tag{5.45}$$

$$\|\nabla b_s\|_{L^2(t)} \leq C|s|^{-1} \|b_s\|_{L^2(t)}\,. \tag{5.46}$$

Let us consider a primal cell T_i and an edge s in $\overset{\circ}{T_i}$. By definition, such an edge s does not belong to Γ. Then, the function $w_s = [\nabla_h \phi_h \cdot \mathbf{n}_s]_s b_s$ belongs to H_D^1 and we may thus apply (4.1), which, taking into account the support of w_s, reduces to

$$\int_{\omega_s} \nabla\hat{\phi} \cdot \nabla w_s\,(\mathbf{x})\, d\mathbf{x} = \int_{\omega_s} f\, w_s\,(\mathbf{x})\, d\mathbf{x}. \tag{5.47}$$

Moreover, since ϕ_h belongs to $P^1(D_j)$ and w_s vanishes on Γ, the application of the Green formula on each D_j implies

$$\int_\Omega \nabla_h \phi_h \cdot \nabla w_s(\mathbf{x}) d\mathbf{x} = \sum_j \int_{\partial D_j} \nabla \phi_h \cdot \mathbf{n}_{\partial D_j} w_s(\sigma) d\sigma = \sum_i \sum_{s' \subset \overset{\circ}{T_i}} \int_{s'} [\nabla_h \phi_h \cdot \mathbf{n}_{s'}]_{s'} w_s(\sigma) d\sigma.$$

The only non-zero terms in the above double sum is that corresponding to $s' = s$, so that, taking into account the definition of w_s and property (5.42)

$$\int_\Omega \nabla_h \phi_h \cdot \nabla w_s(\mathbf{x})\, d\mathbf{x} = |[\nabla_h \phi_h \cdot \mathbf{n}_s]_s|^2 \int_s b_s(\sigma)\, d\sigma = C\, \|[\nabla_h \phi_h \cdot \mathbf{n}_s]_s\|^2_{L^2(s)} . \tag{5.48}$$

Eq. (5.48) implies, taking into account (5.47) and the support of w_s in ω_s

$$\begin{aligned}\|[\nabla_h \phi_h \cdot \mathbf{n}_s]_s\|^2_{L^2(s)} &= C \left[\int_{\omega_s} \left(\nabla_h \phi_h - \nabla \hat{\phi}\right) \cdot \nabla w_s(\mathbf{x})\, d\mathbf{x} + \int_{\omega_s} f\, w_s\,(\mathbf{x})\, d\mathbf{x} \right] \\ &\leq C \left(\left\|\nabla_h \phi_h - \nabla \hat{\phi}\right\|_{L^2(\omega_s)} \|\nabla w_s\|_{L^2(\omega_s)} + \|f\|_{L^2(\omega_s)} \|w_s\|_{L^2(\omega_s)} \right) . \end{aligned} \tag{5.49}$$

Let us now bound $\|\nabla w_s\|_{L^2(\omega_s)}$ and $\|w_s\|_{L^2(\omega_s)}$. There holds, thanks to (5.46),

$$\|\nabla w_s\|_{L^2(\omega_s)} = |[\nabla_h \phi_h \cdot \mathbf{n}_s]_s|\, \|\nabla b_s\|_{L^2(\omega_s)} \leq |[\nabla_h \phi_h \cdot \mathbf{n}_s]_s|\, C|s|^{-1}\, \|b_s\|_{L^2(\omega_s)} . \tag{5.50}$$

$$\|w_s\|_{L^2(\omega_s)} = |[\nabla_h \phi_h \cdot \mathbf{n}_s]_s|\, \|b_s\|_{L^2(\omega_s)} . \tag{5.51}$$

Since (5.41) implies that $b_s^2 \leq b_s$, then using (5.44) we get $\|b_s\|_{L^2(\omega_s)} \leq C\,|s|$. This, with (5.49)–(5.51) and since $|[\nabla_h \phi_h \cdot \mathbf{n}_s]_s| = |s|^{-1/2}\, \|[\nabla_h \phi_h \cdot \mathbf{n}_s]_s\|_{L^2(s)}$ leads to

$$\|[\nabla_h \phi_h \cdot \mathbf{n}_s]_s\|_{L^2(s)} \leq C \left(|s|^{-1/2} \left\|\nabla_h \phi_h - \nabla \hat{\phi}\right\|_{L^2(\omega_s)} + |s|^{1/2}\, \|f\|_{L^2(\omega_s)} \right) . \tag{5.52}$$

One usually expresses $\|f\|_{L^2(\omega_s)}$ as a function of $\left\|\nabla_h \phi_h - \nabla \hat{\phi}\right\|_{L^2(\omega_s)}$ and of HOT. For this, let $t = t_1$ or t_2, and let us denote by $\bar{f}_t$ the mean value of f over t. Then, consider $w_t = \bar{f}_t b_t$, where b_t is defined by (5.39). The function w_t belongs to H^1_D. Thus, taking into account the support of b_t, Eq. (4.1) reduces to

$$\int_t \nabla \hat{\phi} \cdot \nabla w_t\,(\mathbf{x}) d\mathbf{x} = \int_t f w_t(\mathbf{x}) d\mathbf{x}. \tag{5.53}$$

Moreover, since $\nabla_h \phi_h$ is a constant over t, and since w_t vanishes on ∂t, there holds

$$\int_t \nabla_h \phi_h \cdot \nabla w_t\,(\mathbf{x}) d\mathbf{x} = 0. \tag{5.54}$$

Since $\bar{f}_t$ is a constant over t, there holds, thanks to (5.43), (5.53) and (5.54),

$$\begin{aligned}\left\|\bar{f}_t\right\|^2_{L^2(t)} &= |t| \left(\bar{f}_t\right)^2 = C \left(\bar{f}_t\right)^2 \int_t b_t(\mathbf{x}) d\mathbf{x} = C \int_t \bar{f}_t w_t(\mathbf{x}) d\mathbf{x} \\ &= C \left[\int_t \left(\bar{f}_t - f\right) w_t(\mathbf{x}) d\mathbf{x} + \int_t \left(\nabla \hat{\phi} - \nabla_h \phi_h\right) \cdot \nabla w_t\,(\mathbf{x}) d\mathbf{x} \right] \\ &\leq C \left(\left\|\bar{f}_t - f\right\|_{L^2(t)} \|w_t\|_{L^2(t)} + \left\|\nabla \hat{\phi} - \nabla_h \phi_h\right\|_{L^2(t)} \|\nabla w_t\|_{L^2(t)} \right) . \end{aligned} \tag{5.55}$$

Let us now bound $\|w_t\|_{L^2(t)}$ and $\|\nabla w_t\|_{L^2(t)}$. With (5.45), there holds

$$\|w_t\|_{L^2(t)} = \left|\bar{f}_t\right| \|b_t\|_{L^2(t)} \quad \text{and} \quad \|\nabla w_t\|_{L^2(t)} \leq \left|\bar{f}_t\right| C h_t^{-1}\, \|b_t\|_{L^2(t)} \tag{5.56}$$

Since (5.41) implies that $b_t^2 \le b_t$, then using (5.43) we get $|\bar f_t| \, \|b_t\|_{L^2(t)} \le C \left\|\bar f_t\right\|_{L^2(t)}$ Combining this and (5.55)–(5.56), we finally get

$$\|\bar f_t\|_{L^2(t)} \le C \left(\left\|\bar f_t - f\right\|_{L^2(t)} + h_t^{-1} \left\| \nabla\hat\phi - \nabla_h \phi_h \right\|_{L^2(t)} \right).$$

Since s is an edge of t, there holds $|s| \le h_t$; applying the triangle inequality, we obtain

$$\|f\|_{L^2(t)} \le C \left(\left\|\bar f_t - f\right\|_{L^2(t)} + |s|^{-1} \left\| \nabla\hat\phi - \nabla_h \phi_h \right\|_{L^2(t)} \right).$$

Thus, taking into account that $\omega_s = t_1 \cup t_2$, the above inequality implies

$$\begin{aligned} \|f\|_{L^2(\omega_s)} &\le \|f\|_{L^2(t_1)} + \|f\|_{L^2(t_2)} \\ &\le C \left(\left\|\bar f_{\omega_s} - f\right\|_{L^2(\omega_s)} + C|s|^{-1} \left\| \nabla\hat\phi - \nabla_h \phi_h \right\|_{L^2(\omega_s)} \right). \end{aligned} \tag{5.57}$$

In the last inequality, we have used the fact that $\bar f_t$ minimizes $\|c - f\|_{L^2(t)}$ when c runs over $\mathbb{R}$; in particular, $\left\|\bar f_t - f\right\|_{L^2(t)} \le \left\|\bar f_{\omega_s} - f\right\|_{L^2(t)}$, where $\bar f_{\omega_s}$ is the mean value of f over ω_s. Combining (5.52) and (5.57), we obtain

$$\|[\nabla_h\phi_h \cdot \mathbf{n}_s]_s\|_{L^2(s)} \le C \left(|s|^{-1/2} \left\| \nabla_h\phi_h - \nabla\hat\phi \right\|_{L^2(\omega_s)} + |s|^{1/2} \left\| f - \bar f_{\omega_s} \right\|_{L^2(\omega_s)} \right). \tag{5.58}$$

By definition, the local quantity $(\eta_i^T)^2$ is lower than the value taken by the function in (5.14) in $\mu = \left(h_i^T\right)^2$. Since the primal cells have been supposed to be convex, we may bound $C(T_i)$ by $1/\pi$. With (5.36) and (5.58), we obtain

$$\left(\eta_i^T\right)^2 \le C \left(h_i^T\right)^2 \sum_{s \in \overset{\circ}{T_i}} \frac{1}{\rho_{ik,1} + \rho_{ik,2}} \left(|s|^{-1} \left\| \nabla_h\phi_h - \nabla\hat\phi \right\|^2_{L^2(\omega_s)} + |s| \left\| f - \bar f_{\omega_s} \right\|^2_{L^2(\omega_s)} \right).$$

Using lemma 5.14, and since by definition $|T_i \cap P_k| = \frac{1}{2}|s|\,(\rho_{ik,1} + \rho_{ik,2})$ and $|s| \le h_i^T$, the above inequality leads to (5.19). As far as (5.20) is concerned, let us consider the function $v_s = [\nabla_h\phi_h \cdot \boldsymbol{\tau}_s]_s b_s$. There obviously holds

$$\int_\Omega \nabla\hat\phi \cdot \nabla \times v_s(\mathbf{x})\, d\mathbf{x} = \int_{\omega_s} \nabla\hat\phi \cdot \nabla \times v_s(\mathbf{x})\, d\mathbf{x} = 0\,. \tag{5.59}$$

Eq. (5.59) and the calculations that previously led to (5.48) may be used to yield

$$\begin{aligned} \|[\nabla_h\phi_h \cdot \boldsymbol{\tau}_s]_s\|^2_{L^2(s)} = C \int_{\omega_s} \nabla_h\phi_h \cdot \nabla \times v_s(\mathbf{x}) d\mathbf{x} &= C \int_{\omega_s} \left(\nabla_h\phi_h - \nabla\hat\phi\right) \cdot \nabla \times v_s(\mathbf{x}) d\mathbf{x} \\ &\le C \left\| \nabla_h\phi_h - \nabla\hat\phi \right\|_{L^2(\omega_s)} \|\nabla v_s\|_{L^2(\omega_s)}\,. \end{aligned} \tag{5.60}$$

Just like (5.49) led to (5.52) and then to (5.19), the inequality (5.60) leads to (5.20). The dual inequalities (5.21) and (5.22) may be obtained in the same way. The only difference is in the bounds of χ_k and χ_k' defined by (5.11) and (5.12), where lemma 5.15 is used, and where $|P_k|$ is sometimes used in place of $\left(h_k^P\right)^2$. But since the cell P_k is star-shaped with respect to S_k, it is included in the ball of radius h_k^P centred on S_k. Thus, there holds $|P_k| \le \pi \left(h_k^P\right)^2$, which allows us to conclude.

REMARK 5.17. *The second terms in (5.19) and (5.21) are of higher order as soon as f is more regular than $L^2(\Omega)$.*

6. Numerical results. We shall now consider two tests. The first has a stiff but regular ($C^\infty(\Omega)$) solution. A uniform mesh refinement will thus asymptotically give the optimal order of convergence in $O(h)$, or, equivalently, in $O\left(N^{-1/2}\right)$, where N is the number of primal cells in the mesh. We shall verify that the adaptive strategy will give the same asymptotic order of convergence, but with lower errors. The second test has a less regular solution since it belongs to $H^{1+s}(\Omega)$, with $s < 2/3$. A uniform mesh refinement will provide a convergence order in $O(h^{2/3})$, which means in $O\left(N^{-1/3}\right)$. The adaptive strategy will recover the optimal order in $O\left(N^{-1/2}\right)$. In both cases, we shall be interested in the efficiency of the estimator.

In order to apply a mesh refinement strategy, it is necessary to rewrite the total estimator given by (5.18) into a sum over the primal cells; indeed, it is on the primal mesh that one usually has some kind of control, either through some meshing software or through an appropriate refinement of a coarse mesh. Rewriting (5.18) is an easy task, since we may split each dual cell into its intersections with various primal cells, and since we may assess each boundary term to the primal cell whose boundary includes the considered boundary edge. In the sequel, we shall denote by η_i this aggregated local estimator.

6.1. Adaptivity for a stiff but regular solution. We start with a problem inspired by [15], in which the authors consider the following multiscale problem. Let $\Omega =]-1,1[^2$. We set $r = \sqrt{x^2+y^2}$ and $\chi(r) = 1$ if $r \leq \varepsilon$, while $\chi(r) = 0$ if $r > \varepsilon$. Homogeneous Dirichlet boundary conditions are imposed in (1.2) and the function f in (1.1) is chosen so that the exact solution $\hat{\phi}$ is given by

$$\hat{\phi} = \cos(k\pi x)\cos(k\pi y) + \eta\chi(r)\ \exp(1/\varepsilon^2)\ \exp[-1/(\varepsilon^2 - r^2)],$$

in which we impose $k = \frac{1}{2}$, $\eta = 10$ and $\varepsilon = \frac{1}{4}$. This solution is thus in $C^\infty(\Omega)$, but displays a very strong peak in the neighborhood of $(0,0)$. We shall use a family of meshes with possibly nonconforming square cells. More precisely, like in [15], we consider $\omega = [-1/4, 1/4]^2$ and $\Omega \setminus \omega$ is uniformly meshed with squares of size h, while ω is uniformly meshed with squares of size $h_0 = h/2^p$. For $p \geq 1$, the mesh is thus nonconforming. The mesh corresponding to $h = 1/4$ and $h/h_0 = 4$ is displayed on Fig. 6.1. Then, the following refinement strategy is employed: we start with a conforming coarse mesh $h_0 = h = 1/4$, and for any given mesh of this family let $\eta_{ext}^2 := \sum_{T_i \subset \Omega\setminus\omega} \eta_i^2$ and $\eta_{int}^2 := \sum_{T_i\subset\omega} \eta_i^2$, and N_{ext} and N_{int} respectively represent the number of primal cells in $\Omega \setminus \omega$ and in ω. We expect the total error to behave like $e \approx C(N_{ext} + N_{int})^{-1/2}$. We may also roughly expect η_{ext} (respectively η_{int}) to behave proportionally to $N_{ext}^{-1/2}$ (resp. $N_{int}^{-1/2}$), so that:

- if we refine ω only, the total error will roughly be $(\eta_{ext}^2 + \eta_{int}^2/4)^{1/2}$ with $(N_{ext} + 4N_{int})$ cells.
- if we refine $\Omega \setminus \omega$ only, the total error will roughly be $(\eta_{ext}^2/4 + \eta_{int}^2)^{1/2}$ with $(4N_{ext} + N_{int})$ cells.
- if we refine both ω and $\Omega \setminus \omega$, the total error will roughly be $\frac{1}{2}(\eta_{ext}^2 + \eta_{int}^2)^{1/2}$ with $4(N_{ext} + N_{int})$ cells.

Then, after each computation, we compare $C_i := (\eta_{ext}^2 + \eta_{int}^2/4)^{1/2}(N_{ext} + 4N_{int})^{1/2}$, $C_e := (\eta_{ext}^2/4 + \eta_{int}^2)^{1/2}(4N_{ext} + N_{int})^{1/2}$ and $C_{ie} := (\eta_{ext}^2 + \eta_{int}^2)^{1/2}(N_{ext} + N_{int})^{1/2}$ and the mesh is refined

- in ω only if $C_i = \min(C_i, C_e, C_{ie})$,
- in $\Omega \setminus \omega$ only if $C_e = \min(C_i, C_e, C_{ie})$,
- in both ω and $\Omega \setminus \omega$ if $C_{ie} = \min(C_i, C_e, C_{ie})$.

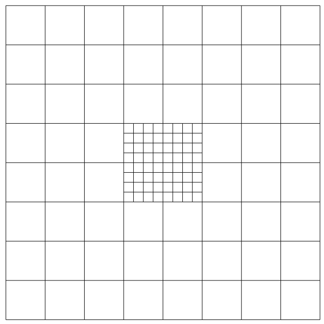

FIG. 6.1. *Nonconforming mesh with $h = 1/4$ and $p = 2$.*

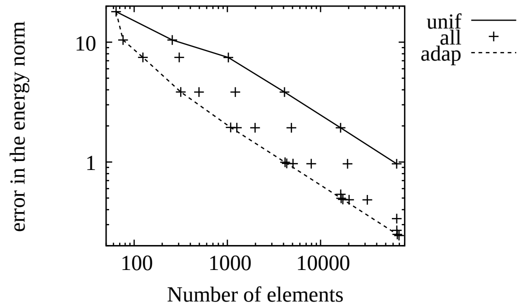

FIG. 6.2. *Errors for the multiscale problem for the uniform (upper curve) and adaptive (lower curve) refinements and all other possible meshes.*

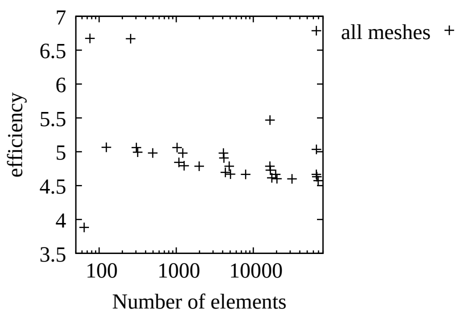

FIG. 6.3. *Efficiencies for the multiscale problem.*

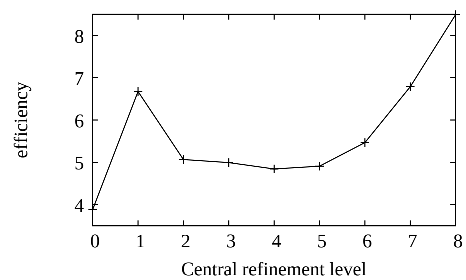

FIG. 6.4. *Efficiencies for the multiscale problem as a function of the central refinement ratio p.*

We present in Figure 6.2 a cloud of points corresponding to the true errors as a function of the total number of primal cells, for all possible choices of couples (h, h_0) (with $h_0 \leq h$) that are so that the number of primal cells is lower than 70000. We have also plotted the curve corresponding to a uniform mesh refinement ($h = h_0$) and the curve corresponding to the above described refinement strategy; we remark that the latter curve is always below the cloud of points, and we may thus consider that, in the present test, this strategy is optimal. This strategy leads to refine only in ω until $h/h_0 = 16$, with $h = 1/4$, and then to refine on the whole mesh. Note that this corresponds to the observation in [13]. However, this mesh refinement is now driven by the error estimator, while in [13] we needed the exact error! As far as the efficiency of the estimator is concerned, Fig. 6.3 displays all the ratios of the estimators over the true errors for all the meshes used in the previous computations. For these tests, the efficiency of the estimator is mostly around 5, and always between 3.5 and 7. Fig. 6.4 displays the efficiencies for a fixed coarse grid $h = 1/4$ and for various refinement ratios, with p up to 8. We remark that the efficiency is rather constant around 5, until $p = 6$, and starts to deteriorate for $p \geq 7$. This is however robust enough for our purposes here, since the optimal p was found to be 4. This deterioration was expected, since it was proved in section 5.5 that the efficiency of the estimator depends on the regularity of the subtriangulation $t_{ik,\alpha}$ (see Figure 5.1). The fact that these triangles degenerate at the boundary between the fine and coarse meshes when p grows larger explains the observed worse efficiency. Note however that, as proved in [13], the *a priori* error estimation does not degenerate with this refinement ratio.

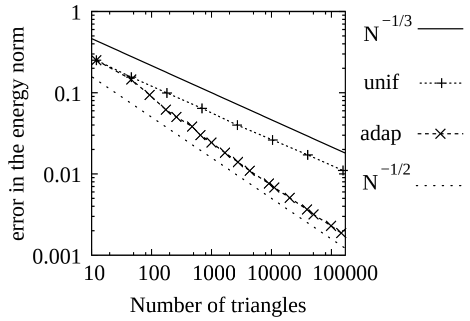

FIG. 6.5. *Errors for the singular solution for a uniform and an adaptive refinement.*

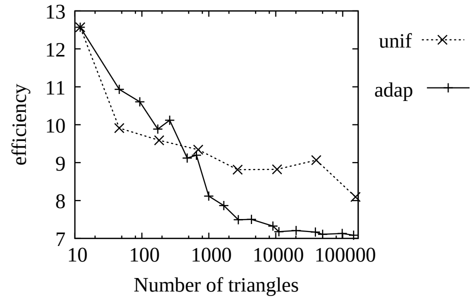

FIG. 6.6. *Efficiencies for the singular solution for a uniform and an adaptive refinement.*

6.2. Adaptivity for a singular solution. This test case is rather classical. The domain Ω is chosen to be $]-1;1[\times]-1;1[\setminus]0;1[\times]-1;0[$. The exact solution is $\hat{\phi}(r,\theta) = r^{2/3}\sin(2\theta/3)$, expressed in cylindrical coordinates (r,θ) centered on $(0,0)$. We use the Triangle mesh generator described in [26]. On a given mesh, we compute the aggregated estimators η_i and ask to refine a given T_i by a factor 4 in terms of area if $\eta_i \geq (\max_j \eta_j)/2$. The Triangle mesh generator will not exactly refine a given T_i into 4 similar sub-triangles, but will arrange so that the areas of the triangles near the former T_i will be lower than or equal to $|T_i|/4$. In Figure 6.5, we have plotted the curves of the true errors for a uniform and for an adaptive refinement, as a function of the number of triangles in the primal mesh. The curve corresponding to the uniform mesh refinement is, as expected, parallel to the $N^{-1/3}$ curve, while the curve corresponding to the adaptive mesh refinement is parallel to the $N^{-1/2}$ curve, which means the optimal convergence is recovered. Finally, we plot in Fig. 6.6 the efficiency curves for the uniform refinement and for the adaptive refinement. The efficiency varies roughly between 10 and 8 (except for the very coarse mesh) in the former case, and seems to tend to 7 in the latter.

7. Conclusion. We have applied tools from the Finite Element framework to derive a fully computable and efficient error bound for the DDFV discretization of the Laplace equation in two dimensions. We have applied this theory to the adaptive simulation on nonconforming meshes of a regular but stiff problem, and to the adaptive simulation of a problem with a singular solution. On these tests, the efficiency of the estimator varies most of the time between 5 and 10. Based on ideas developed for example in [30], further work is under progress to obtain an estimator with a better efficiency for more general diffusion equations discretized by the DDFV method.

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