Nonparametric estimation of the mark density of Exponential Shot-Noise

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Introduction

Shot-noise processes - or Filtered Marked Point Processes - are natural models for phenomenons that can be represented as a linear filtration of sums of independent jumps (marks) $(Y_i)_{i\in\mathbb{Z}}$ arrived at random times $(T_i)_{i\in\mathbb{Z}}$. There are particularly adapted to model data transfers in telecommunication or interacting particles in physics.Based on a low-frequency sample of the Shot-Noise $(X_1,\cdots,X_n)$, we introduce a novel **nonparametric estimator of the mark density** when arrival times follow a Poisson Process with intensity λ and the filter is of the form $h(t)=e^{-\alpha t}$ using a "plug-in" method with the empirical characteristic function $\hat\varphi_n(u)\triangleq n^{-1}\sum_{k=1}^n e^{iuX_k}$.

Problem

We consider the stationary process $(X_t)_{t\geq 0}$ defined by:

$$X_t=\sum_{T_k\leq t} Y_k e^{-\alpha(t-T_k)} \qquad (1)$$

where:

1. $\sum_k \delta_{T_k,Y_k}$ is a Poisson random measure on $\mathbb{R}^2$ with intensity measure $\lambda Leb\otimes F$
2. $\lambda,\alpha>0$
3. Y_0 has a p.d.f. θ which belong to the class:
   $\Theta_{C,K,L,a,\beta}=\{\theta:\theta \text{ p.d.f. },\int_{\mathbb{R}} x^8\theta(x)dx\leq K,\int_{\mathbb{R}}|u|^\beta|\varphi_Y(u)|du\leq L,|\varphi_Y(u)|\geq Ce^{-au^2}\}$

From its definition and under Assumptions 1-3, the characteristic function φ_{X_0} of X_0 is non zero,differentiable and given by:

$$\varphi_{X_0}(u)=\exp\left(\lambda\int_0^\infty\left(\varphi_Y(ue^{-\alpha s})-1\right)ds\right)\quad,\quad u\in\mathbb{R}$$

Synthetic scheme

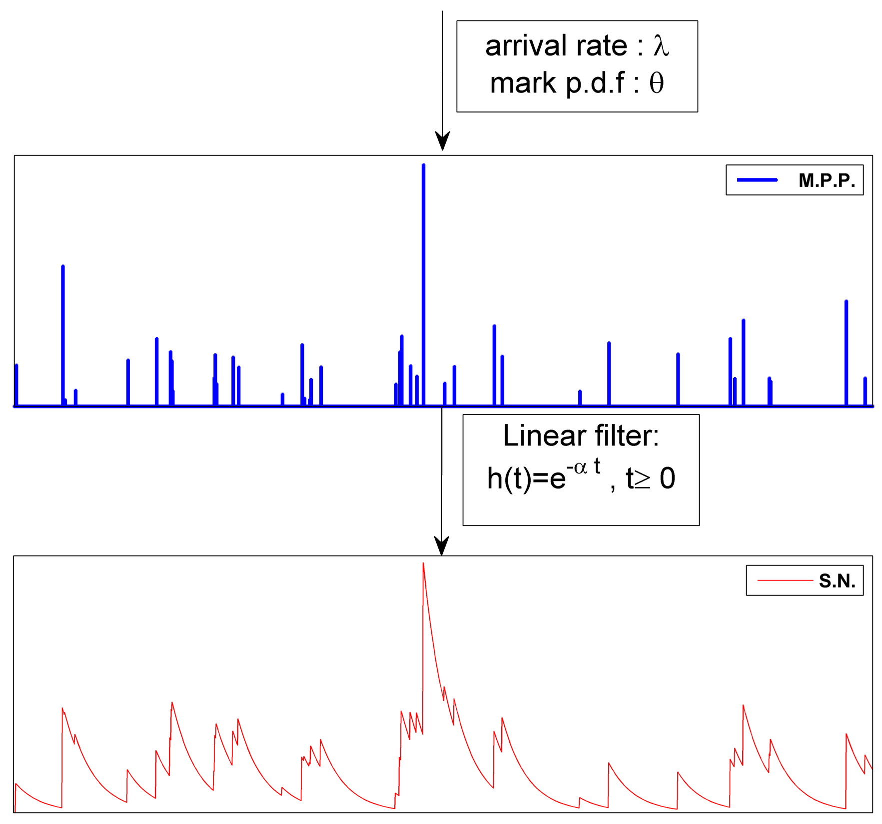

Figure 1: Simulation of a Shot-Noise sample path

Inverse problem and estimator

Inversion formula:
Differentiating the characteristic function φ_{X_0} gives: $\varphi_Y(u)=1+u\frac{\alpha}{\lambda}\frac{\varphi'_{X_0}(u)}{\varphi_{X_0}(u)}\quad,\quad u\in\mathbb{R}$

Estimation issues:
Tempted to construct a "plug-in" estimator using the empirical characteristic (ECF) function but:

- $\hat\varphi_n(u)$ might take arbitrary small values
- The "pluged-in" integrand might diverge

"Plug-in" estimator
We then propose a mark density estimator $\hat\theta_n$ given by:

$$\hat\theta_n(x)\triangleq 2\max\left(\mathcal{R}\left(\int_0^{h_n^{-1}}e^{-ixu}\left(1+u\frac{\alpha}{\lambda}\frac{\hat\varphi'_n(u)}{\hat\varphi_n(u)}1_{|\hat\varphi_n(u)|\geq\kappa_n}\right)\right),0\right) \qquad (2)$$

where

$$h_n=\left(\frac{6a}{\log(n)}\right)^{1/2}\text{ and }\kappa_n=\frac{C}{2}e^{-ah_n^{-2}}=\frac{C}{2}n^{-1/6}$$

Theoritical result :

$$\boxed{\sup_{\theta\in\Theta_{C,K,L,a,\beta}}\mathbb{E}_\theta\left[\left|\hat\theta_n-\theta\right|_\infty^2\right]\lesssim\log(n)^{-\beta}}$$

Numerical Results

Illustration of our algorithm with two numerical examples with $\lambda=4.10^5$, $\alpha=3,5.10^5$:

- Marks follows a gaussian mixture $\sum_{i=1}^3 p_i\mathcal{N}_{\mu_i,\sigma_i^2}(x)$ with:

$p=[0.3\ 0.5\ 0.2]\quad,\quad\mu=[4\ 12\ 22]\quad,\quad\sigma=[1\ 1\ 0.5]$

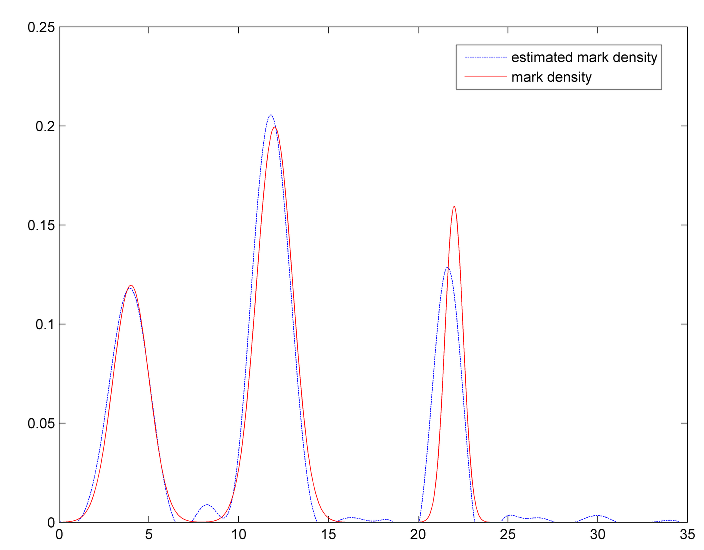

Figure 2: Gaussian mixture case

- Marks follows a gamma mixture $\sum_{i=1}^3 p_iG_{a,\beta_i}(x)$ with :

$a=2\quad,\quad p=[0.2\ 0.3\ 0.5]\quad,\quad\beta=[3\ 6\ 12]$

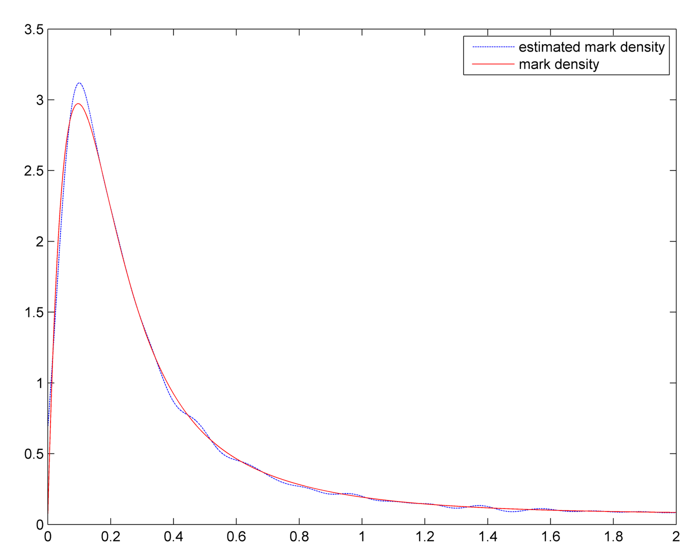

Figure 3: Gamma mixture case

Algorithm

Algorithm 1: Nonparametric mark density estimator

Input: Shot-noise observations $X_1,\cdots,X_n$

1. Choose $\Delta>0$.
2. Compute the histogram $(H_i)_{i\in\mathbb{Z}}$ of observations with bins of size Δ :
   $$H_i=\{X_k\in[\Delta i;\Delta(i+1)]\}/n$$
3. Set $(dH_i)_{i\in\mathbb{Z}}$ by $dH_i=\Delta iH_i$
4. Set $N\triangleq 2^{16}$ and compute $X\phi=\text{FFT}(H,N)$, $dX\phi=\text{FFT}(dH,N)$
   It gives for $k=1,\cdots,N$, $X\phi_k\simeq\hat\varphi_n\left(-\frac{2\pi(k-1)}{N\Delta}\right)$ and $dX\phi_k\simeq-i\hat\varphi'_n\left(-\frac{2\pi(k-1)}{N\Delta}\right)$
5. Compute the ECF estimator :$\hat\varphi_{Y\,n}\left(\frac{2\pi(k-1)}{N\Delta}\right)=1+\frac{2\pi\alpha(k-1)}{N\Delta\lambda}\frac{dX\phi_k}{dX\phi_k}$
6. Compute the density estimator by : $\hat\theta_n=\frac{2}{N\Delta}FFT(\hat\varphi_{Y\,n},N)$

Remarks

- The rate of convergence depends on the smoothness of the mark distribution
- The distribution F_X of X_0 is regularly varying at 0 with index λ/α
  $\hookrightarrow$ One can use Hill's estimator to estimate λ/α
- When the mark distribution is a mixture of exponential laws, a parametric approach is possible

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