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Nonparametric estimation of the mark density of Exponential Shot-Noise

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INTRODUCTION

Shot-noise processes - or Filtered Marked Point Processes - are natural models for phenomena that can be represented as a linear filtration of sums of independent jumps (marks) $(Y_i)_{i \in \mathbb{Z}}$ arrived at random times $(T_i)_{i \in \mathbb{Z}}$. There are particularly adapted to model data transfers in telecommunication or interacting particles in physics. Based on a low-frequency sample of the Shot-Noise $(X_1, \dots, X_n)$, we introduce a novel **nonparametric estimator of the mark density** when arrival times follow a Poisson Process with intensity λ and the filter is of the form $h(t) = e^{-\alpha t}$ using a "plug-in" method with the empirical characteristic function $\hat{\varphi}_n(u) \triangleq n^{-1} \sum_{k=1}^n e^{iuX_k}$.

PROBLEM

We consider the stationary process $(X_t)_{t \geq 0}$ defined by:

$$X_t = \sum_{T_k \leq t} Y_k e^{-\alpha(t-T_k)} \quad (1)$$

where:

1. $\sum_k \delta_{T_k, Y_k}$ is a Poisson random measure on $\mathbb{R}^2$ with intensity measure $\lambda \text{Leb} \otimes F$
2. $\lambda, \alpha > 0$
3. Y_0 has a p.d.f. θ which belong to the class:
 $\Theta_{C,K,L,a,\beta} = \{\theta : \theta \text{ p.d.f.}, \int_{\mathbb{R}} x^8 \theta(x) dx \leq K, \int_{\mathbb{R}} |u|^\beta |\varphi_Y(u)| du \leq L, |\varphi_Y(u)| \geq C e^{-au^2}\}$

From its definition and under Assumptions 1-3, the characteristic function φ_{X_0} of X_0 is non zero, differentiable and given by:

$$\varphi_{X_0}(u) = \exp\left(\lambda \int_0^\infty (\varphi_Y(u e^{-\alpha s}) - 1) ds\right), \quad u \in \mathbb{R}$$

INVERSE PROBLEM AND ESTIMATOR

Inversion formula:

Differentiating the characteristic function φ_{X_0} gives: $\varphi_Y(u) = 1 + u \frac{\alpha}{\lambda} \frac{\varphi'_{X_0}(u)}{\varphi_{X_0}(u)}, \quad u \in \mathbb{R}$

Estimation issues:

Tempted to construct a "plug-in" estimator using the empirical characteristic (ECF) function but:

- $\hat{\varphi}_n(u)$ might take arbitrary small values
- The "plugged-in" integrand might diverge

"Plug-in" estimator

We then propose a mark density estimator $\hat{\theta}_n$ given by:

$$\hat{\theta}_n(x) \triangleq 2 \max \left(\mathcal{R} \left(\int_0^{h_n^{-1}} e^{-ixu} \left(1 + u \frac{\alpha}{\lambda} \frac{\hat{\varphi}'_n(u)}{\hat{\varphi}_n(u)} 1_{|\hat{\varphi}_n(u)| \geq \kappa_n} \right) du \right), 0 \right) \quad (2)$$

where

$$h_n = \left(\frac{6a}{\log(n)} \right)^{1/2} \text{ and } \kappa_n = \frac{C}{2} e^{-ah_n^{-2}} = \frac{C}{2} n^{-1/6}$$

Theoretical result :

$$\sup_{\theta \in \Theta_{C,K,L,a,\beta}} \mathbb{E}_\theta \left[\left| \hat{\theta}_n - \theta \right|_\infty^2 \right] \lesssim \log(n)^{-\beta}$$

ALGORITHM

Algorithm 1: Nonparametric mark density estimator

Input: Shot-noise observations $X_1, \dots, X_n$

1. Choose $\Delta > 0$.
2. Compute the histogram $(H_i)_{i \in \mathbb{Z}}$ of observations with bins of size Δ :

$$H_i = \{X_k \in [\Delta i; \Delta(i+1)]\} / n$$

3. Set $(dH_i)_{i \in \mathbb{Z}}$ by $dH_i = \Delta i H_i$
4. Set $N \triangleq 2^{16}$ and compute $X\phi = \text{FFT}(H, N)$, $dX\phi = \text{FFT}(dH, N)$
It gives for $k = 1, \dots, N$, $X\phi_k \simeq \hat{\varphi}_n \left(-\frac{2\pi(k-1)}{N\Delta} \right)$ and $dX\phi_k \simeq -i\hat{\varphi}'_n \left(-\frac{2\pi(k-1)}{N\Delta} \right)$
5. Compute the ECF estimator: $\hat{\varphi}_{Y_n} \left(\frac{2\pi(k-1)}{N\Delta} \right) = 1 + \frac{2\pi\alpha(k-1)}{N\Delta\lambda} \frac{dX\phi_k}{dX\phi_k}$
6. Compute the density estimator by: $\hat{\theta}_n = \frac{2}{N\Delta} \text{FFT}(\hat{\varphi}_{Y_n}, N)$

REMARKS

- The rate of convergence depends on the smoothness of the mark distribution
- The distribution F_X of X_0 is regularly varying at 0 with index λ/α
 $\hookrightarrow$ One can use Hill's estimator to estimate λ/α
- When the mark distribution is a mixture of exponential laws, a parametric approach is possible

SYNTHETIC SCHEME

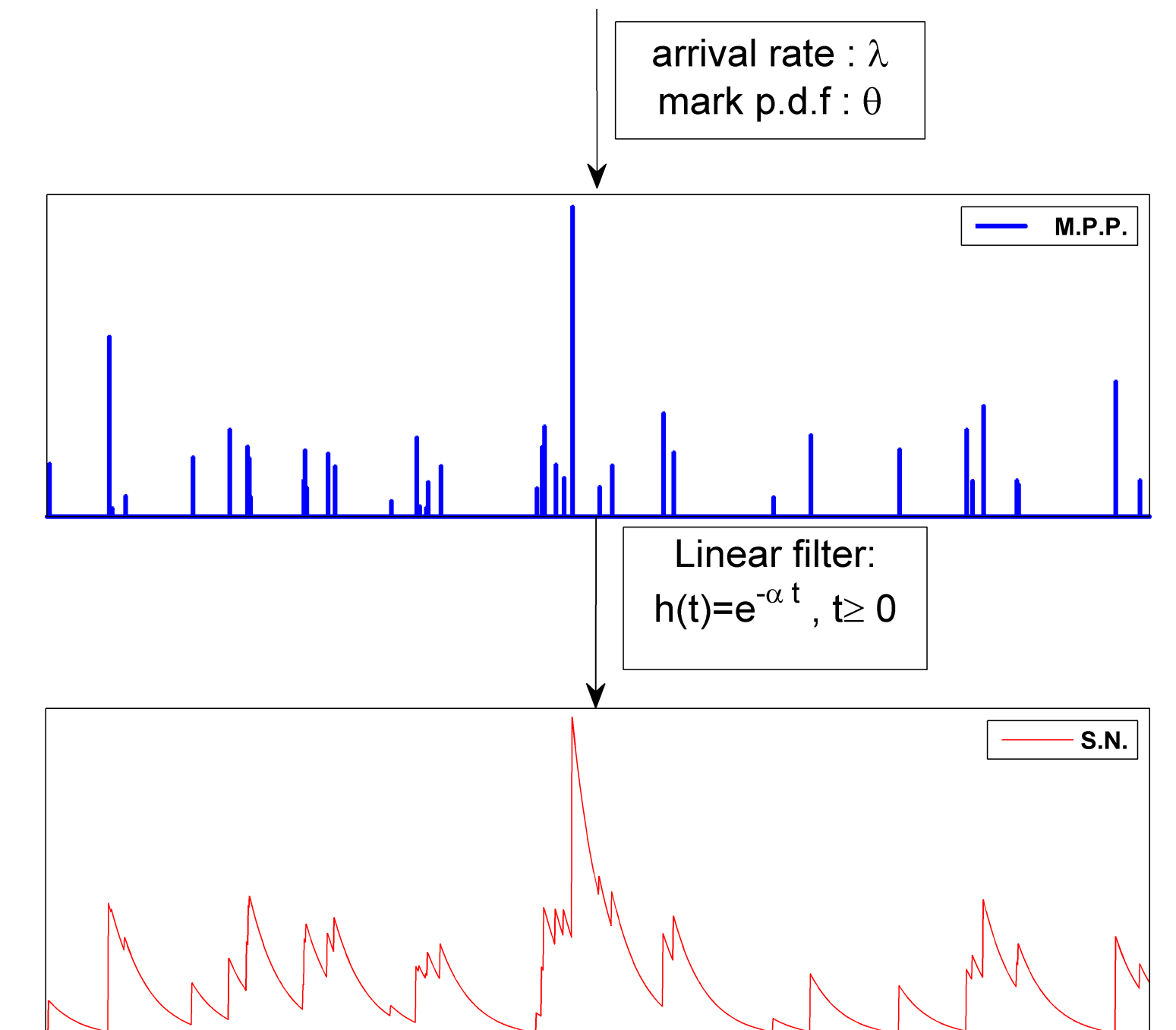


Figure 1: Simulation of a Shot-Noise sample path

NUMERICAL RESULTS

Illustration of our algorithm with two numerical examples with $\lambda = 4 \cdot 10^5$, $\alpha = 3, 5 \cdot 10^5$:

- Marks follows a gaussian mixture $\sum_{i=1}^3 p_i \mathcal{N}_{\mu_i, \sigma_i^2}(x)$ with:

$$p = [0.3 \ 0.5 \ 0.2], \quad \mu = [4 \ 12 \ 22], \quad \sigma = [1 \ 1 \ 0.5]$$

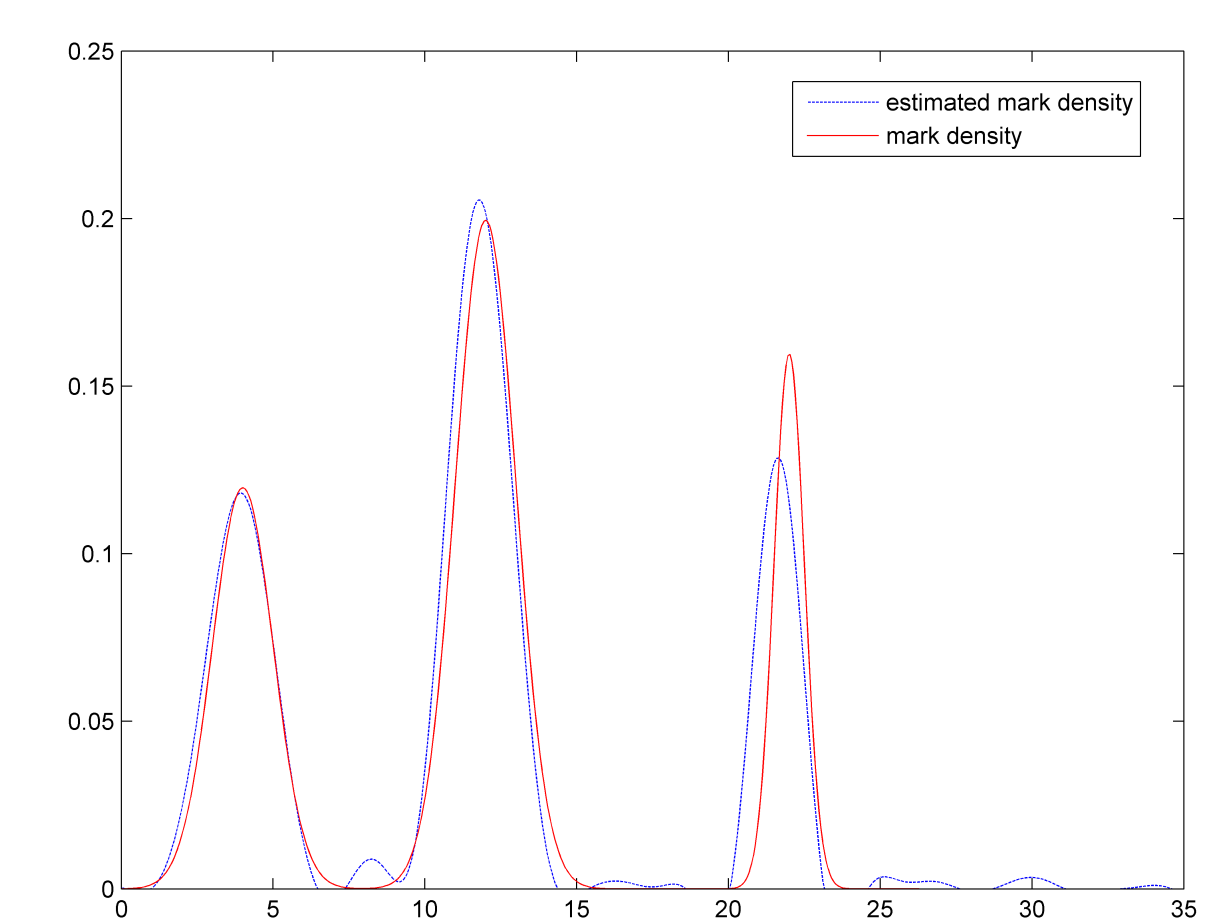


Figure 2: Gaussian mixture case

- Marks follows a gamma mixture $\sum_{i=1}^3 p_i G_{a, \beta_i}(x)$ with:

$$a = 2, \quad p = [0.2 \ 0.3 \ 0.5], \quad \beta = [3 \ 6 \ 12]$$

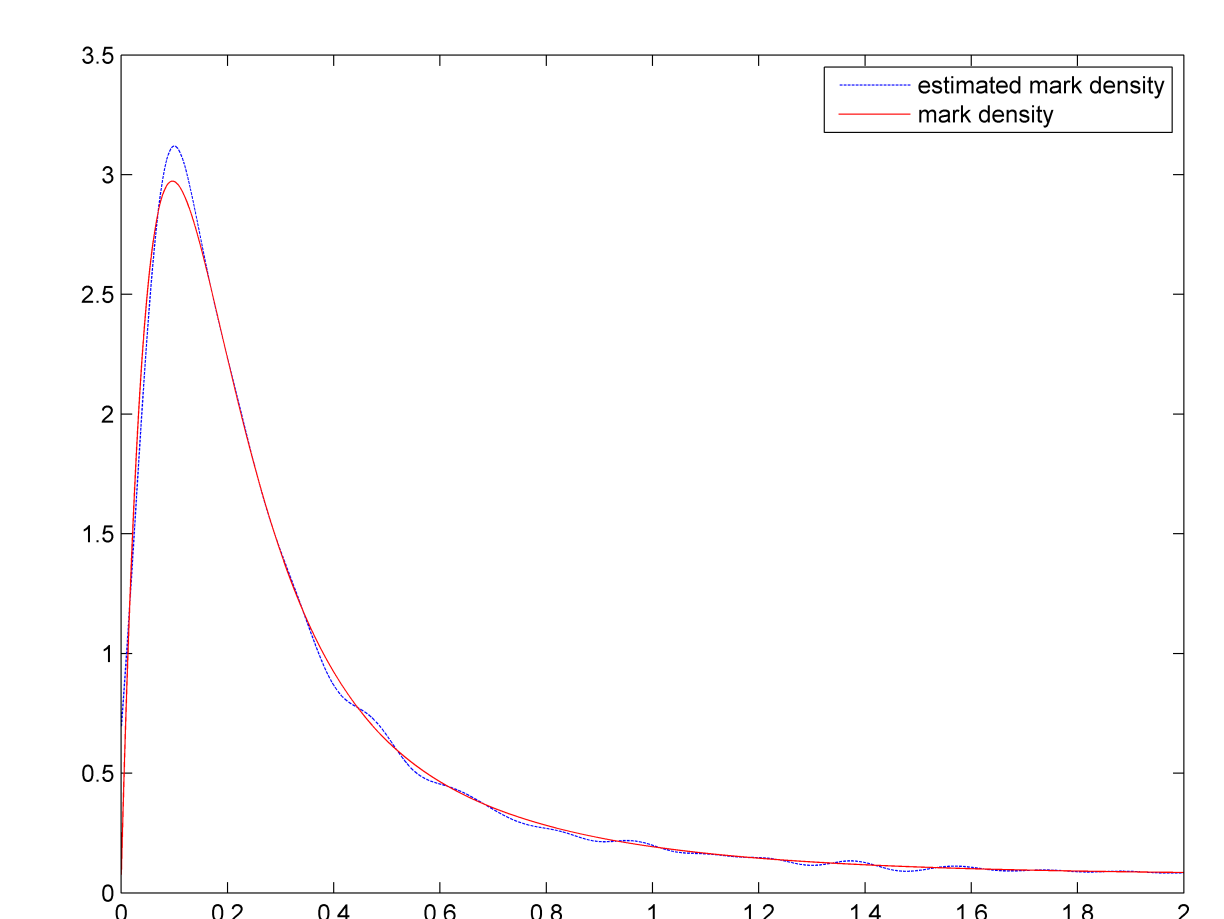


Figure 3: Gamma mixture case

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